A closed-form approximation of the exact unbiased inverse of the Anscombe variance-stabilizing transformation

Markku Mäkitalo and Alessandro Foi

***Abstract*—In [1], we presented an exact unbiased inverse of the Anscombe variance-stabilizing transformation and showed that when applied to Poisson image denoising, the combination of variance stabilization and state-of-the-art Gaussian denoising algorithms is competitive with some of the best Poisson denoising algorithms. We also provided a Matlab implementation of our method, where the exact unbiased inverse transformation appears in non-analytical form.**

Here we propose a closed-form approximation of the exact unbiased inverse, in order to facilitate the use of this inverse. The proposed approximation produces results equivalent to those obtained with the accurate (non-analytical) exact unbiased inverse, and thus notably better than one would get with the asymptotically unbiased inverse transformation, which is commonly used in applications.

***Index Terms*—Denoising, photon-limited imaging, Poisson noise, variance stabilization.**

I. Introduction

The removal of Poisson noise is of great importance in many imaging applications. A common approach combines variance stabilization with Gaussian denoising algorithms. First, the noise variance is stabilized by applying the Anscombe root transformation [2] $f\,:\,z\longmapsto 2\sqrt{z+\frac{3}{8}}$ to the data. In the transformed signal, the noise can be treated as additive Gaussian with unitary variance. Thus, the noise can be removed with a conventional denoising algorithm designed for additive white Gaussian noise. Finally, an inverse transformation is applied to the denoised signal in order to obtain the final estimate of the signal.

As the forward Anscombe transformation is nonlinear, it generally leads to biased estimates if one uses either its direct algebraic inverse $\mathcal{I}_A=f^{-1}$ or the asymptotically unbiased inverse [2] $\mathcal{I}_B=\mathcal{I}_A+\frac{1}{4}$,

$$\mathcal{I}_A(D)=\left(\frac{D}{2}\right)^2-\frac{3}{8},\quad \mathcal{I}_B(D)=\left(\frac{D}{2}\right)^2-\frac{1}{8},\tag{1}$$

where D is the signal obtained by denoising the transformed data $f(z)$. As shown in [1], when using $\mathcal{I}_B$ the bias is especially significant for low intensities of the signal.

In [1] we showed that the problem of bias can be solved by finding the exact unbiased inverse

$$\mathcal{I}_C: E\left\{f(z)\mid y\right\}\longmapsto E\left\{z\mid y\right\},\tag{2}$$

where $E\left\{z\mid y\right\}=y\geq 0$ is the true value of the signal. This involves computing the infinite sum

$$E\{f(z)\mid y\}=2\sum_{z=0}^{+\infty}\left(\sqrt{z+\frac{3}{8}}\cdot\frac{y^z e^{-y}}{z!}\right)\tag{3}$$

and then constructing the inverse of the mapping $y\longmapsto E\left\{f(z)\mid y\right\}$. We remark that $E\left\{f(z)\mid y\right\}\geq 2\sqrt{\frac{3}{8}}$ for all $y\geq 0$, and for $D<2\sqrt{\frac{3}{8}}$ we define $\mathcal{I}_C(D)=0$. In low-count Poisson image denoising, the results obtained with the exact unbiased inverse $\mathcal{I}_C$ (2) and a state-of-the-art Gaussian denoising algorithm are dramatically better than those obtained with the asymptotically unbiased inverse $\mathcal{I}_B$, and better than what is achieved with currently existing methods specifically designed for Poisson noise removal. These good results can be justified by the fact [1] that $\mathcal{I}_C$ can be interpreted as a maximum likelihood inverse.

We provided a Matlab implementation[1] of this inverse including pre-computed values of (3), which are used for the interpolation of $\mathcal{I}_C$ at arbitrary values of D. However, it would be desirable to have a closed-form expression for this inverse transformation, or at least a closed-form expression that approximates the inverse.

II. Closed-form approximation of the exact unbiased inverse transformation

In this letter, we construct a closed-form approximation $\widetilde{\mathcal{I}}_C$ of the exact unbiased inverse $\mathcal{I}_C$ by first considering the asymptotically unbiased inverse $\mathcal{I}_B$ (1), and then subtracting a non-constant correction term from it. Note that $\mathcal{I}_C\left(2\sqrt{3/8}\right)=\mathcal{I}_A\left(2\sqrt{3/8}\right)=\mathcal{I}_B\left(2\sqrt{3/8}\right)-\frac{1}{4}=0$, and that $\mathcal{I}_C(D)-\mathcal{I}_B(D)\to 0$, as $D\to\infty$. Thus, our correction term should equal $\frac{1}{4}$ at $2\sqrt{\frac{3}{8}}$, and approach zero as D increases. It is easy to verify that the inverse proposed below satisfies these conditions:

$$\begin{aligned}\widetilde{\mathcal{I}}_C(D)\;=\;&\left(\frac{D}{2}\right)^2-\frac{1}{8}-\frac{1}{4}\left[\alpha\left(D\sqrt{\frac{2}{3}}\right)^{-1}+\right.\\&\left.\beta\left(D\sqrt{\frac{2}{3}}\right)^{-2}+(1-\alpha-\beta)\left(D\sqrt{\frac{2}{3}}\right)^{-3}\right],\end{aligned}\tag{4}$$

where $D\geq 2\sqrt{\frac{3}{8}}$ and $\alpha,\beta\in\mathbb{R}$. As with $\mathcal{I}_C$, for $D<2\sqrt{\frac{3}{8}}$ we set also $\widetilde{\mathcal{I}}_C(D)=0$. We choose the values of α and β by minimizing the integral criterion

$$\Phi=\int_0^{+\infty}\frac{\left(\widetilde{\mathcal{I}}_C\left(E\left\{f(z)\mid y\right\}\right)-y\right)^2}{y^2}\,dy,\tag{5}$$

which can be interpreted as a normalized quadratic fit between $\mathcal{I}_C$ and $\widetilde{\mathcal{I}}_C$. The minimization yields $\alpha\approx -1.0008$ and $\beta\approx 3.6634$, with the corresponding value of $\Phi\approx 7.4368\cdot 10^{-5}$. As small variations in the coefficient values do not greatly affect the fit, it seems reasonable to choose $\alpha=-1$ and $\beta=\frac{11}{3}$ ($\Phi\approx 7.4658\cdot 10^{-5}$), and thus simplify (4) into

$$\widetilde{\mathcal{I}}_C(D)=\frac{1}{4}D^2+\frac{1}{4}\sqrt{\frac{3}{2}}D^{-1}-\frac{11}{8}D^{-2}+\frac{5}{8}\sqrt{\frac{3}{2}}D^{-3}-\frac{1}{8}.\tag{6}$$

III. Experiments and conclusions

We evaluated the impact on the denoising quality due to substituting the exact unbiased inverse (2) with the proposed closed-form approximation (6), over the set of experiments considered in Table I of [1]. We observed that in terms of normalized mean integrated square error (NMISE), the results of the two inverses are in all cases within 0.5% of each other. These experiments confirm that the proposed closed-form approximation $\widetilde{\mathcal{I}}_C$ is practically equivalent to the exact unbiased inverse $\mathcal{I}_C$. Thus, (6) can be successfully used as a direct replacement of the traditional inverses (1), for improving the quality of the results in low-count Poisson image denoising.

References

[1] Mäkitalo, M., and A. Foi, "Optimal inversion of the Anscombe transformation in low-count Poisson image denoising", *IEEE Trans. Image Process.*, vol. 20, no. 1, pp. 99–109, Jan. 2011. http://dx.doi.org/10.1109/TIP.2010.2056693

[2] Anscombe, F.J., "The transformation of Poisson, binomial and negative-binomial data", *Biometrika*, vol. 35, no. 3/4, pp. 246–254, Dec. 1948. http://dx.doi.org/10.1093/biomet/35.3-4.246

This work was supported by the Academy of Finland (project no. 213462, Finnish Programme for Centres of Excellence in Research 2006-2011, project no. 118312, Finland Distinguished Professor Programme 2007-2010, and project no. 129118, Postdoctoral Researcher's Project 2009-2011) and by Tampere Doctoral Programme in Information Science and Engineering (TISE).

The authors are with the Department of Signal Processing, Tampere University of Technology, P.O. Box 553, 33101 Tampere, Finland (e-mail: firstname.lastname@tut.fi).

[1] Available online at http://www.cs.tut.fi/~foi/invansc/.