# A Class of Approximately Linear Phase Digital Filters Composed of Allpass Subfilters

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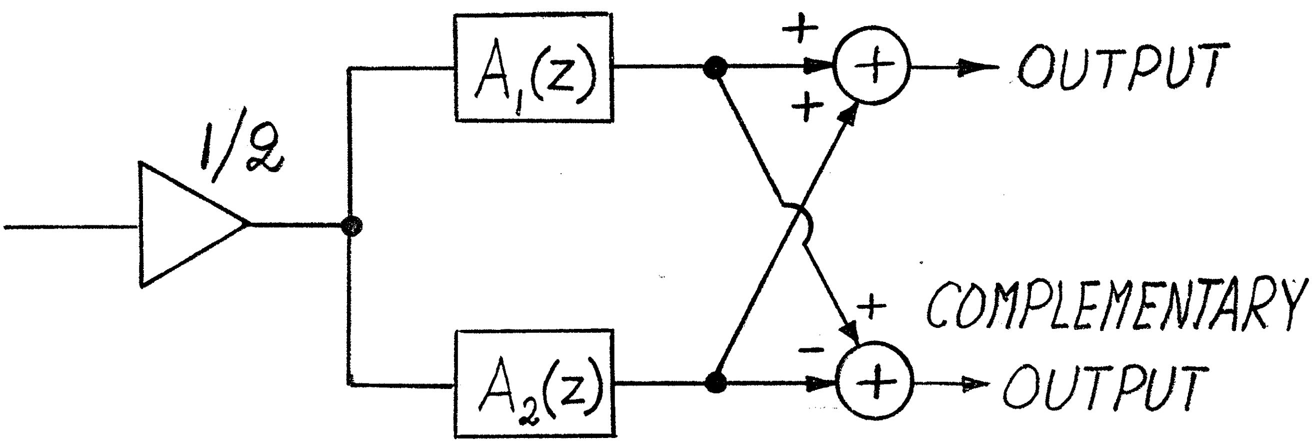
**The purpose of this talk is:**

1. To introduce a new class of approximately linear phase IIR filters composed of a parellel connection of pure delay term and an allpass section
2. To describe an efficient algorithm for the design of this kind of filters
3. To use this kind of filters for constructing an efficient filter for processing an ECG-signal

# Introduction

Recently, filters composed as a parellel connection of two allpass filters have been studied intensively.




We consider special filters with transfer function

$$H(z) = \frac{1}{2}[z^{-M_1} \pm A_{M_2}(z)]$$

where $A_{M_2}(z)$ is an allpass filter of order $M_2$.

For lowpass and highpass designs, $M_2 = M_1 + 1$

For bandpass and bandstop designs, $M_2 = M_1 + 2$

## Approximation Problem

Specifications

$$1 \le |H(e^{j\omega})| \le 1 - \delta_p(\omega) \quad \text{for} \quad \omega \in [0, \omega_p]$$

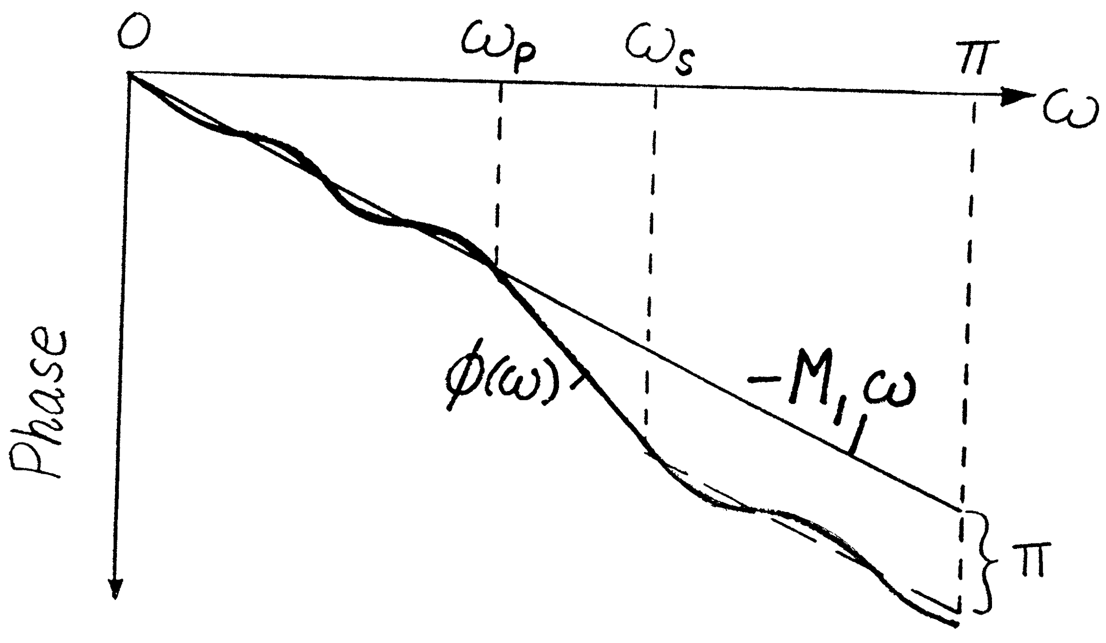
$$|H(e^{j\omega})| \le \delta_s(\omega) \quad \text{for} \quad \omega \in [\omega_s, \pi]$$

With the aid of the phase response $\phi(\omega)$ of $A_{M_2}(z)$,

$$|H(e^{j\omega})| = \left|\frac{1}{2}\left[e^{-jM_1\omega} + e^{j\phi(\omega)}\right]\right| = \left|\cos\frac{1}{2}[\phi(\omega) + M_1\omega]\right|$$

**1.** On $[0, \omega_p]$, the difference of the phases $\phi(\omega)$ and $-M_1\omega$ has to approximate zero $\Rightarrow$ Good phase response $(\phi(\omega) - M_1\omega)/2$.

**2.** On $[\omega_s, \pi]$, the difference has to approximate $-\pi$.



With the aid of $\phi(\omega)$, the approximation problem is to minimize the maximum absolute value of

$$E(\omega) = W(\omega)[\phi(\omega) - D(\omega)],$$

where

$$W(\omega) = \begin{cases} \dfrac{1}{2\cos^{-1}(1-\delta_p(\omega))} & \text{for } \omega \in [0, \omega_p] \\ \dfrac{1}{2\sin^{-1}(\delta_s(\omega))} & \text{for } \omega \in [\omega_s, \pi] \end{cases}$$

and

$$D(\omega) = \begin{cases} -M_1\omega & \text{for } \omega \in [0, \omega_p] \\ -M_1\omega - \pi & \text{for } \omega \in [\omega_s, \pi]. \end{cases}$$

The given criteria are met if the maximum absolute value $\leq 1$.

$E(\omega)$ equiripple $\Longrightarrow$ $|H(e^{j\omega})|$ equiripple

**Remez-Type Algorithm:**

1) Select initial extremal points $\Omega = \{\omega_1, \omega_2, \ldots, \omega_{M_2+1}\}$ on $[0, \omega_p] \bigcup [\omega_s, \pi]$.

2) Solve the system of $M_2 + 1$ equations

$$E(\omega_j) = W(\omega_j)[\phi(\omega_j) - D(\omega_j)] = (-1)^j \delta,$$

$$j = 1, 2, \ldots, M_2 + 1$$

for the $M_2$ filter coefficients $a_1, a_2, \ldots, a_{M_2}$ of $A_{M2}(z)$ and for $\delta$.

3) Find on $[0, \omega_p] \bigcup [\omega_s, \pi]$ the $M_2 + 1$ local extrema of $E(\omega)$. Store the abscissae of the extrema into $\Omega' = \{\omega'_1, \omega'_2, \ldots, \omega'_{M_2+1}\}$.

4) If $|\,\omega_j - \omega'_j\,| \le \epsilon$ for $j = 1, 2, \ldots, M_2 + 1$, then stop. Otherwise, set $\Omega = \Omega'$ and go to step 2.

## Extensions of the Algorithm

**I.** Design of filters with phase delay constraints

In many cases, e. g., when filtering periodic signals, the phase delay variation in the passband is a better measure of the phase distortion caused by a filter than the traditionally used group delay variation.

If the maximum allowable phase delay ripple in the passband is $\Delta\tau_p(\omega)$, then we use in the above algorithm

$$W(\omega) = \begin{cases} W_p(\omega) & \text{for } \omega \in [0, \omega_p] \\ W_s(\omega) & \text{for } \omega \in [\omega_s, \pi] \end{cases}$$

$$W_p(\omega) = \frac{1}{\min\{2\cos^{-1}(1-\delta_p(\omega)), \omega\Delta\tau_p(\omega)\}}$$

$$W_s(\omega) = \frac{1}{2\sin^{-1}(\delta_s(\omega))}$$

**II.** Other extensions

The algorithm is also readily extendable to

**1.** Design of multiband filters (with phase delay constraints)

**2.** Design of phase delay equalizers

**Example 1:** A complementary filter pair with

edge angles of $0.6\pi$ and $0.75\pi$

at least 40 dB attenuations

**New Filter Pair:** $M_1 = 8$, $M_2 = 9$: **9** multipliers

Without phase delay constraints, attenuations are 41.7 dB and phase delay ripple is 0.028 samples.

Optimizing the phase delay ripple, while maintaining 40 dB attenuations, gives a ripple of 0.013 samples

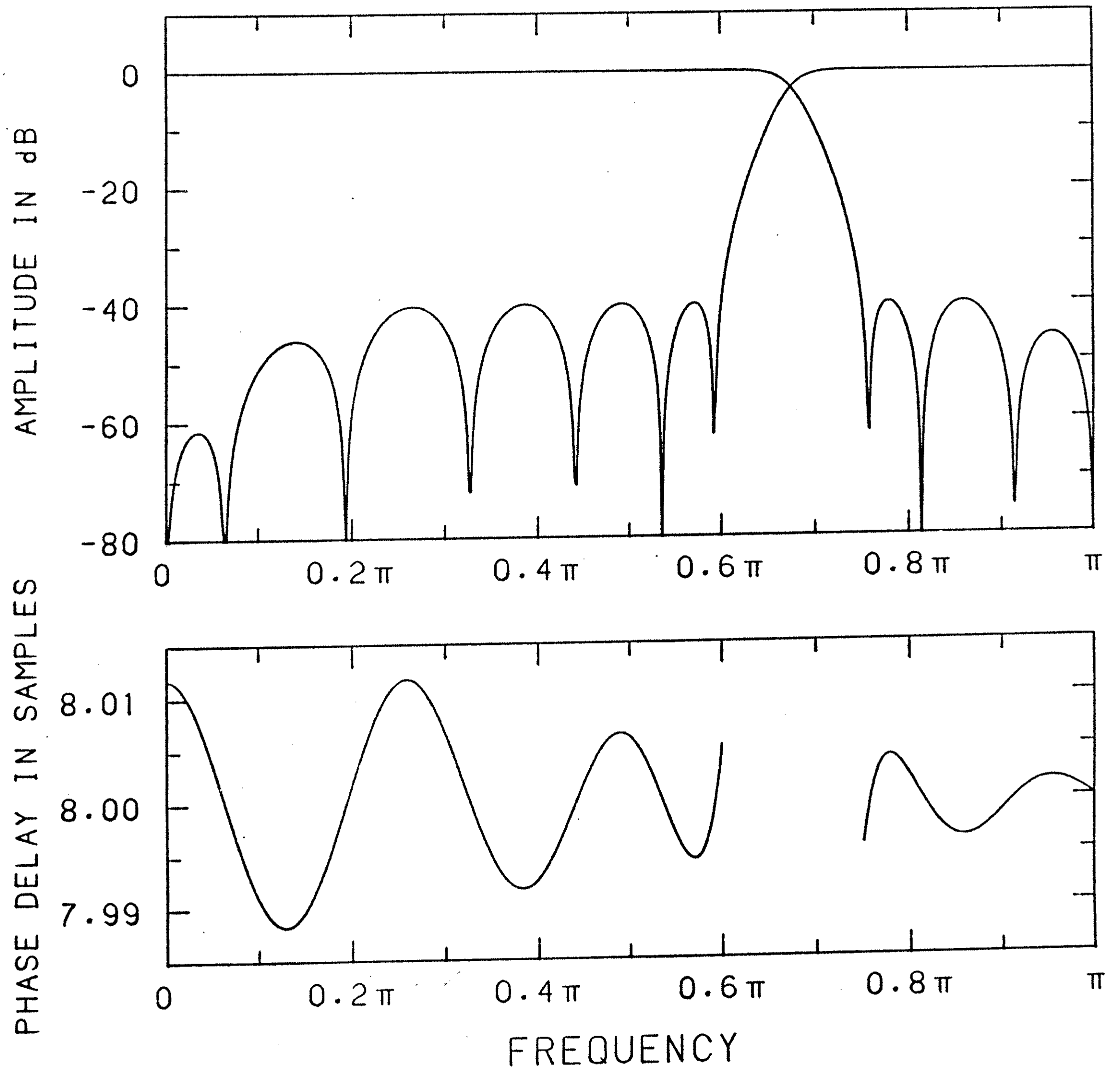
**Elliptic Filter Pair:**

Without phase delay equalization, 7 multipliers

Phase delay equalization (ripple $<$ 0.025 samples) requires equalizers of orders 2 and 4: **13** multipliers

**Linear Phase FIR:** order=28, **15** multipliers

# Responses for the optimized new filter



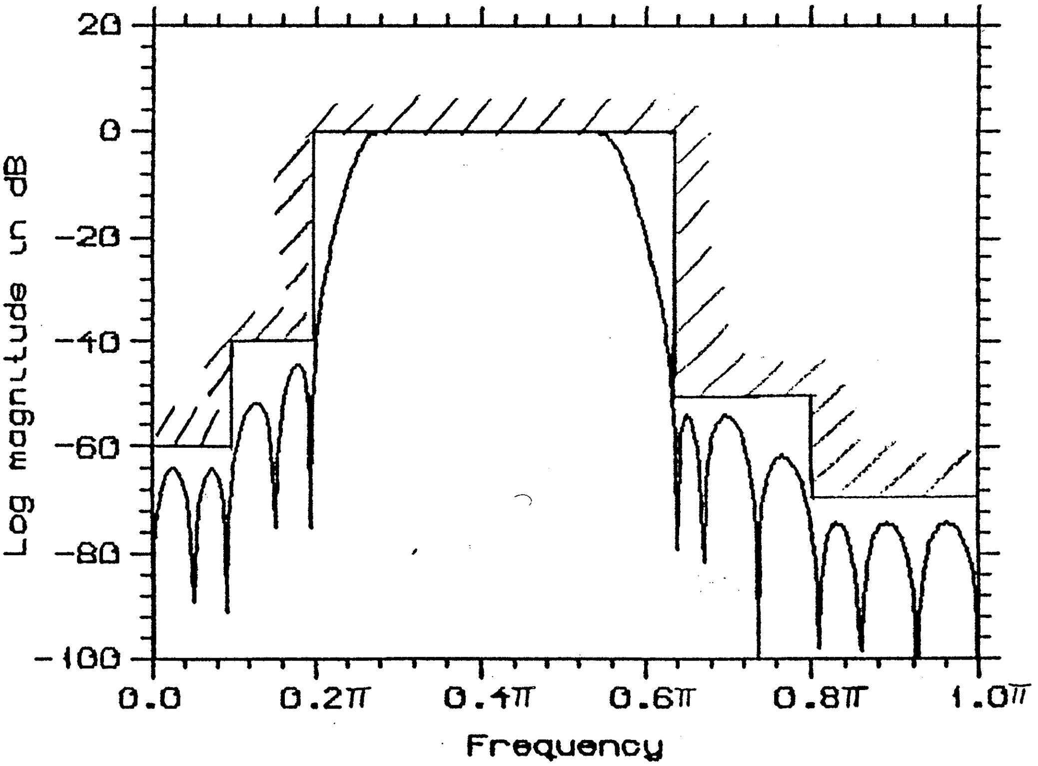
**Example 2:**

Bandpass filter:

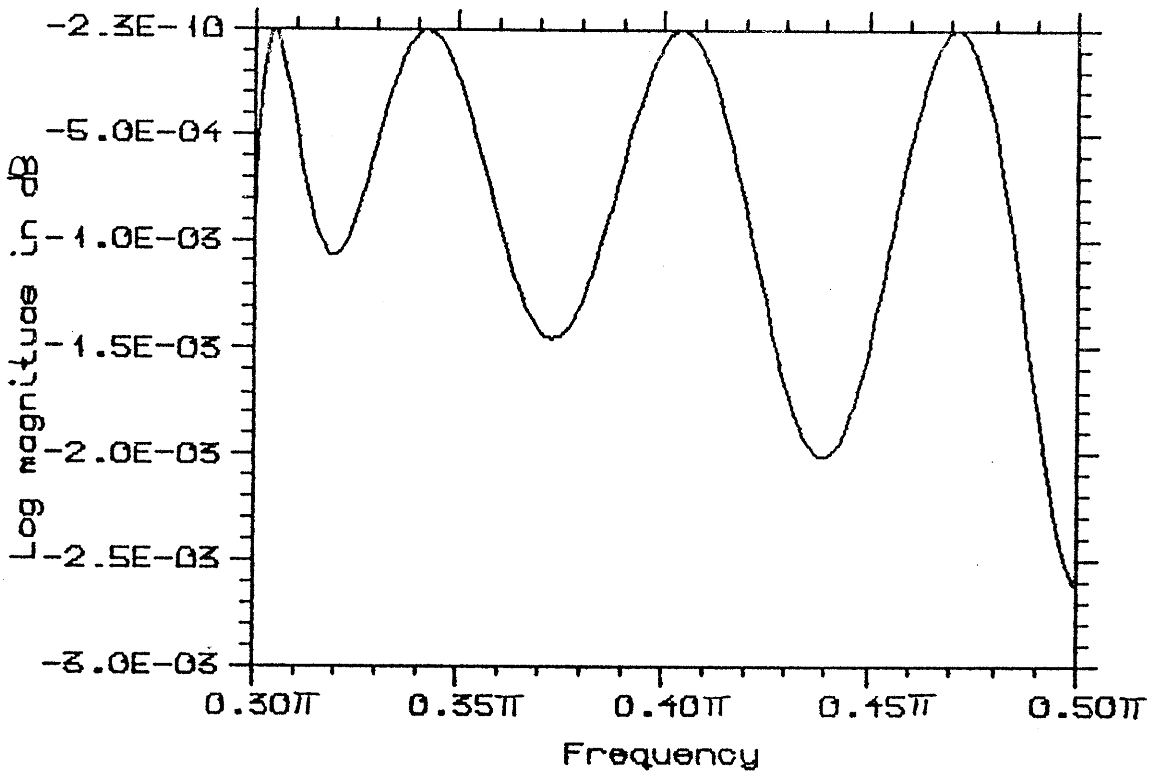
1. Stopband specifications: 60 dB attenuation on $[0, .1\pi]$, 40 dB attenuation on $[0.1\pi, 0.2\pi]$, 50 dB on $[0.65\pi, 0.8\pi]$, 70 dB on $[0.8\pi, \pi]$
2. Passband specifications: Phase delay ripple on $[0.3\pi, 0.5\pi]$ at most 0.02 samples.

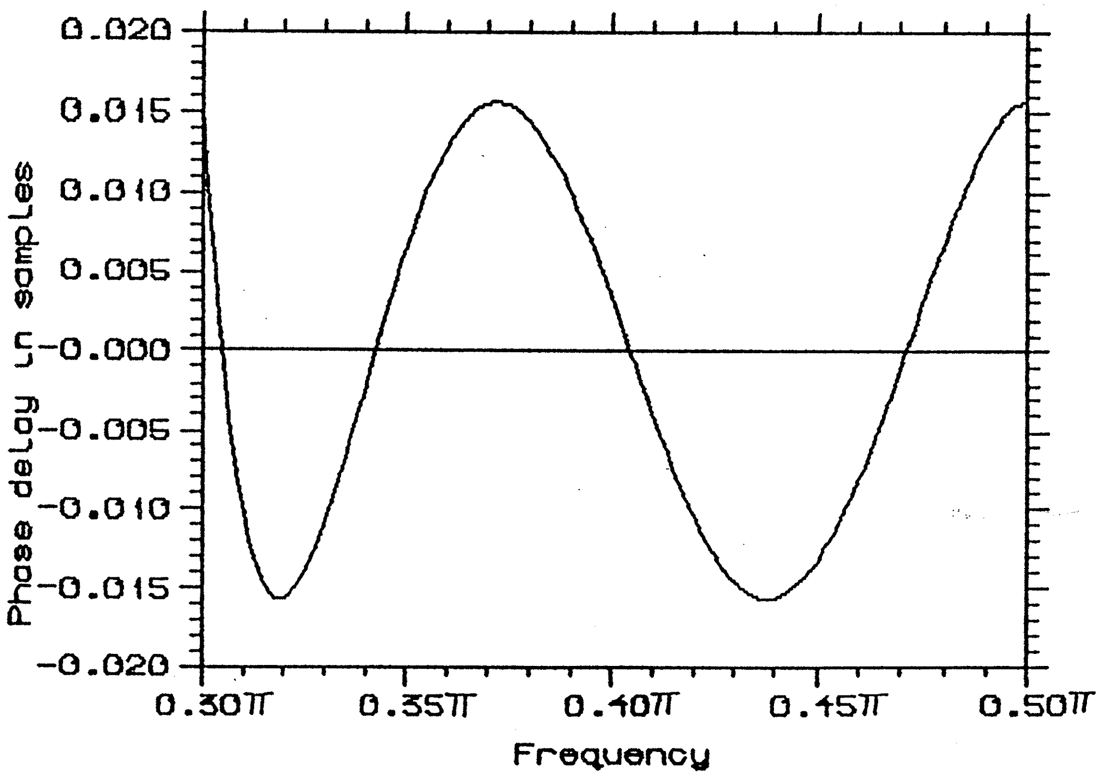
The given criteria are met by $H(z) = \frac{1}{2}[z^{-13} + A_{15}(z)]$

Stopband response



# Passband responses



## Example 3:

Design of a filter for processing an ECG-signal:

1. 40 dB attenuation for $0 \leq f \leq 1/3$ Hz
2. Amplitude response ripple less than 0.05 in the pass-band starting at $2/3$ Hz
3. Notches at 50 Hz and 100 Hz

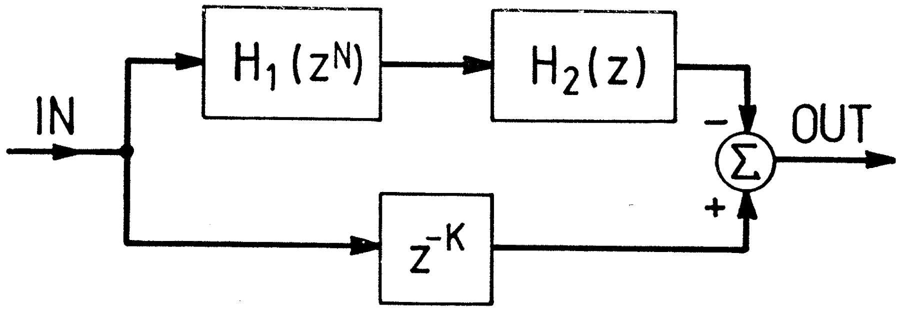
Sampling rate = 200 Hz

Overall transfer function $H_{\text{overall}} = H(z^4)$

Requirements for $H(z)$:

1. $|H(e^{j\omega})| \leq 0.01$ for $\omega \in [0, 4\pi/300]$
2. $0.95 \leq |H(e^{j\omega})| \leq 1.05$ for $\omega \in [4\pi/300, \pi]$
3. Phase delay ripple $\leq 1/200$ seconds $\Longrightarrow$ Phase delay ripple of $H(z) \leq 0.25$ samples.

Proposed filter structure:




$K = 77$, $N = 6$, $H_1(z)$ proposed filter with $M_2 = 13$, $M_1 = 12$, and $H_2(z)$ consists of two running sum filters of length 6.

14 multipliers, delay = 1.54 seconds

If $H(z)$ is a linear-phase FIR filter, 116 multipliers are required and delay is 2.3 seconds.

# Composite filter responses