A Remez-Type Algorithm for Designing Digital Filters Composed All-Pass Sections Based on Phase Approximations

Tapio Saramäki and Markku Renfors

Department of Information Technology
Tampere University of Technology
P. O. Box 553, FIN–33101 Tampere, Finland
E-Mails: ts@cs.tut.fi, mr@cs.tut.fi

Tranfer Function

- Nth-order **unstable** all-pass filter:

$$A_N(z) = \sum_{k=0}^{N} a_k z^{-(N-k)} / \sum_{k=0}^{N} a_k z^{-k}, \quad a_0 = 1.$$

- This transfer function is factorizable as

$$A_N(z) = B_L(z)/C_{N-L}(z).$$

  - $B_L(z)$ and $C_{N-L}(z)$ are **stable** Lth-order and $(N-L)$th-order allpass filters.

- The phase response of $A_N(z)$ is

$$\phi_A(\omega) = \phi_B(\omega) - \phi_C(\omega).$$

  - $\phi_B(\omega)$ and $\phi_C(\omega)$ are the phase responses of $B_L(z)$ and $C_{N-L}(z)$.

Approximation Problem

- **Given** the desired function $D(\omega)$ and a positive weighting function $W(\omega)$ on a closed subset X of $[0, \pi]$,

- **Find** the coefficients of $A_N(z)$ to minimize

$$\epsilon = \max_{\omega \in X} |E(\omega)|,$$

where

$$E(\omega) = W(\omega)[\phi_A(\omega) - D(\omega)].$$

Characterization of the Best Solution

- **If** there is a solution satisfying:
  - There exist at least $N+1$ points $\omega_1, \omega_2, \cdots, \omega_{N+1}$ in X such that

$$\omega_1 < \omega_2 < \cdots < \omega_N < \omega_{N+1}$$

$$E(\omega_{r+1}) = -E(\omega_r), \quad r = 1, 2, \cdots, N$$

$$|E(\omega_r)| = \epsilon, \quad r = 1, 2, \cdots, N+1.$$

- **then** this solution is the best unique solution minimizing ϵ.

- This **equiripple** performance together with the recurrence formulas of Henk form the basis for constructing an efficient **Remez** algorithm for finding this solution.

Solvable Problems: A Parallel Connection of Two All-Pass Filters

- Transfer function:

$$H(z) = \frac{1}{2}[B_L(z) + z^{-M}C_{N-L}(z)].$$

- Amplitude response:

$$|H(e^{j\omega})| = |\cos\frac{1}{2}[\phi_B(\omega) - \phi_C(\omega) + M\omega]|.$$

- **Find** the coefficients of $B_L(z)$ and $C_{N-L}(z)$ such that

$$1 - \delta_p(\omega) \leq |H(e^{j\omega})| \leq 1 \quad \text{for} \quad \omega \in X_p = \bigcup_{l=1}^{J_p} x_p^{(l)}$$

$$|H(e^{j\omega})| \leq \delta_s(\omega) \quad \text{for} \quad \omega \in X_s = \bigcup_{l=1}^{J_s} x_s^{(l)}$$

  - $x_p^{(l)} = [\omega_{2l-1}^{(p)}, \omega_{2l}^{(p)}]$ and $x_s^{(l)} = [\omega_{2l-1}^{(s)}, \omega_{2l}^{(s)}]$ are passband and stopband regions, which alternate.
  - $x_p^{(1)}$ is the first band.

Equivalent Problem in Terms of the Phase of $A_N(z) = B_L(z)/C_{N-L}(z)$, $\phi_A(\omega) = \phi_B(\omega) - \phi_C(\omega)$

- The given criteria are met if

$$\epsilon = \max_{\omega \in X} |E(\omega)| \leq 1,$$

  where

$$E(\omega) = W(\omega)[\phi_A(\omega) - D(\omega)].$$

- $X = X_p \cup X_s$.

- $D(\omega)$ is given by

$$D(\omega) = \begin{cases} -M\omega + 2\rho(l-1)\pi & \text{for } \omega \in x_p^{(l)} \\ -M\omega + \rho(2l-1)\pi & \text{for } \omega \in x_s^{(l)}. \end{cases}$$

  - $\rho = 1$ or $\rho = -1$.

- $W(\omega)$ is given by

$$W(\omega) = \begin{cases} \dfrac{1}{2\cos^{-1}[1-\delta_p(\omega)]} & \text{for } \omega \in x_p^{(l)} \\ \dfrac{1}{2\sin^{-1}[\delta_s(\omega)]} & \text{for } \omega \in x_s^{(l)}. \end{cases}$$

- $\rho = -1$ and R bands: $L = N - L + M + R - 1$; $\rho = 1$: $L = N - L + M - R + 1$.

Other Solvable Problems

2) Phase and phase delay equalization:

- If the phase response to be equalized on X is $\psi(\omega)$, find the coefficients of a stable $A_N(z)$ as well as an integer l and τ_p to minimize on X the maximum absolute value of

$$E(\omega) = \frac{1}{\omega}[\phi_A(\omega) + \psi(\omega) - \tau_p\omega - l \cdot 2\pi]$$

or

$$E(\omega) = [\phi_A(\omega) + \psi(\omega) - \tau_p\omega - l \cdot 2\pi].$$

3) Filters with an arbitrary noninteger delay $M + \alpha$, $0 < \alpha < 1$ on $X = [0,\ \pi - \omega_1]$:

- $D(\omega) = -(M + \alpha)\omega$, $N = M + 1$.

4) Approximately linear-phase Hilbert transformers:

- $D(\omega) = -M\omega - \pi/2$, $N = M + 1$, $X = [\omega_1,\ \pi - \omega_2]$.

Remez Algorithm

1) Select initial set of extremal points $\Omega^{(1)} = \{\omega_1^{(1)}, \omega_2^{(1)}, \ldots, \omega_{N+1}^{(1)}\}$ on X. Set $k = 1$.

2) Solve the following system of $N+1$ equations

$$E^{(k)}(\omega_r^{(k)}) = W(\omega_r^{(k)})[\phi_A^{(k)}(\omega_r^{(k)}) - D(\omega_r^{(k)})] = (-1)^r \epsilon^{(k)},$$
$$r = 1, 2, \ldots, N+1$$

for the N filter coefficients $a_1^{(k)}, a_2^{(k)}, \ldots, a_N^{(k)}$ of $A_N^{(k)}(z)$ and for $\epsilon^{(k)}$.

3) Locate on X the $N+1$ local extrema of $E^{(k)}(\omega)$. Store the abscissae of the extrema into $\Omega^{(k+1)} = \{\omega_1^{(k+1)}, \omega_2^{(k+1)}, \ldots, \omega_{N+1}^{(k+1)}\}$.

4) If $|\,\omega_r^{(k)} - \omega_r^{(k+1)}\,| \leq \alpha$ for $r = 1, 2, \ldots, N+1$ (α is a small number), then go to the next step. Otherwise set $k = k+1$ and go to Step 2.

5) Locate the poles of the resulting $A_N(z) \equiv A_N^{(k)}(z)$. For L stable poles, form $B_L(z)$ from these poles and $C_{N-L}(z)$ from the $N-L$ unstable poles by replacing them by their reciprocal values.

Solution to the system of nonlinear equations

- Rewrite the system as

$$W(\omega_r^{(k)})[\phi_A^{(k)}(\omega_r^{(k)}) - D(\omega_r^{(k)})] - (-1)^r \epsilon^{(k)} = 0,$$
$$r = 1, 2, \ldots, N+1.$$

- The first N equations define the coefficients $a_1^{(k)}$, $a_2^{(k)}$, $\ldots$, $a_N^{(k)}$ implicitly as functions of $\epsilon^{(k)}$.

- For a fixed value of $\epsilon^{(k)}$, the coefficients can be obtained through the recurrence formulas of Henk.

- The remaining equation may then be written as

$$G(\epsilon^{(k)}, a_1^{(k)}(\epsilon^{(k)}), a_2^{(k)}(\epsilon^{(k)}), \ldots, a_N^{(k)}(\epsilon^{(k)})) \equiv G(\epsilon^{(k)}) = 0$$

  and may be solved by any technique for the solution of the equation $G(x) = 0$.

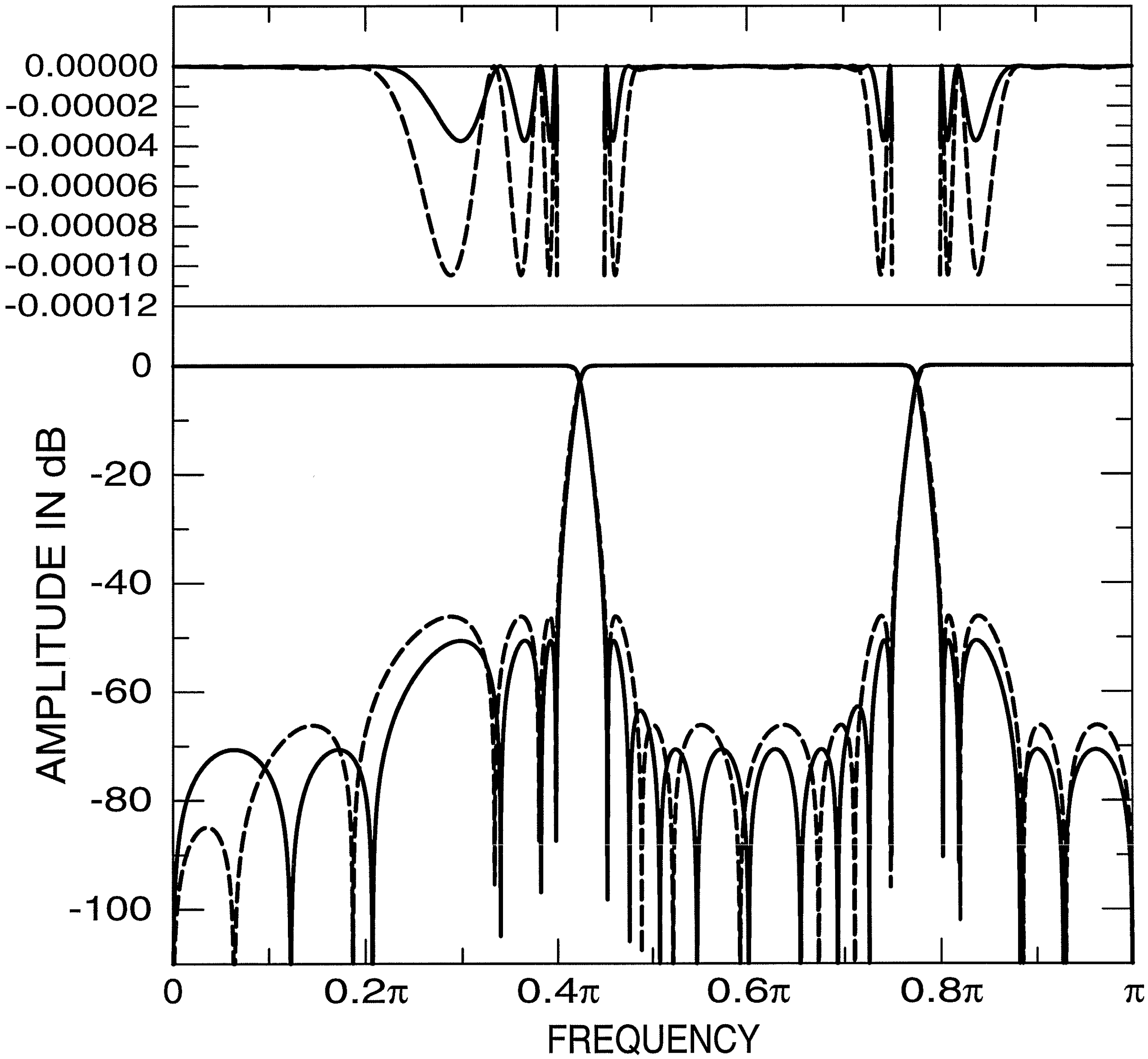
- We use the secant method.

EXAMPLE 1: A bandpass-bandstop filter pair $\frac{1}{2}[B(z) \pm C(z)]$ with arbitrary specifications

- **Bandstop filter:** Stopband is $[0.45\pi,\ 0.75\pi]$.
  - Minimum attenuation on $[0.5\pi,\ 0.7\pi]$ is 60 dB
  - Minimum attenuation on $[0.45\pi,\ 0.5\pi]$ and $[0.5\pi,\ 0.75\pi]$ is 40 dB.
- **Bandpass filter:** Stopbands are $[0,\ 0.4\pi]$ and $[0.8\pi,\ \pi]$.
  - Minimum attenuation on $[0,\ 0.1\pi]$ and $[0.9\pi,\ \pi]$ is 60 dB
  - Minimum attenuation on $[0.1\pi,\ 0.2\pi]$ and $[0.8\pi,\ 0.9\pi]$ is 40 dB.
- The minimum even order to meet these criteria is 20: $B(z)$ is of order 9 and $C(z)$ is of order 11.
- $C(z)$ has three rather unefficient poles, which can be moved to the origin.
- $C(z)$ is a cascade of z^{-3} and an eighth-order allpass section.

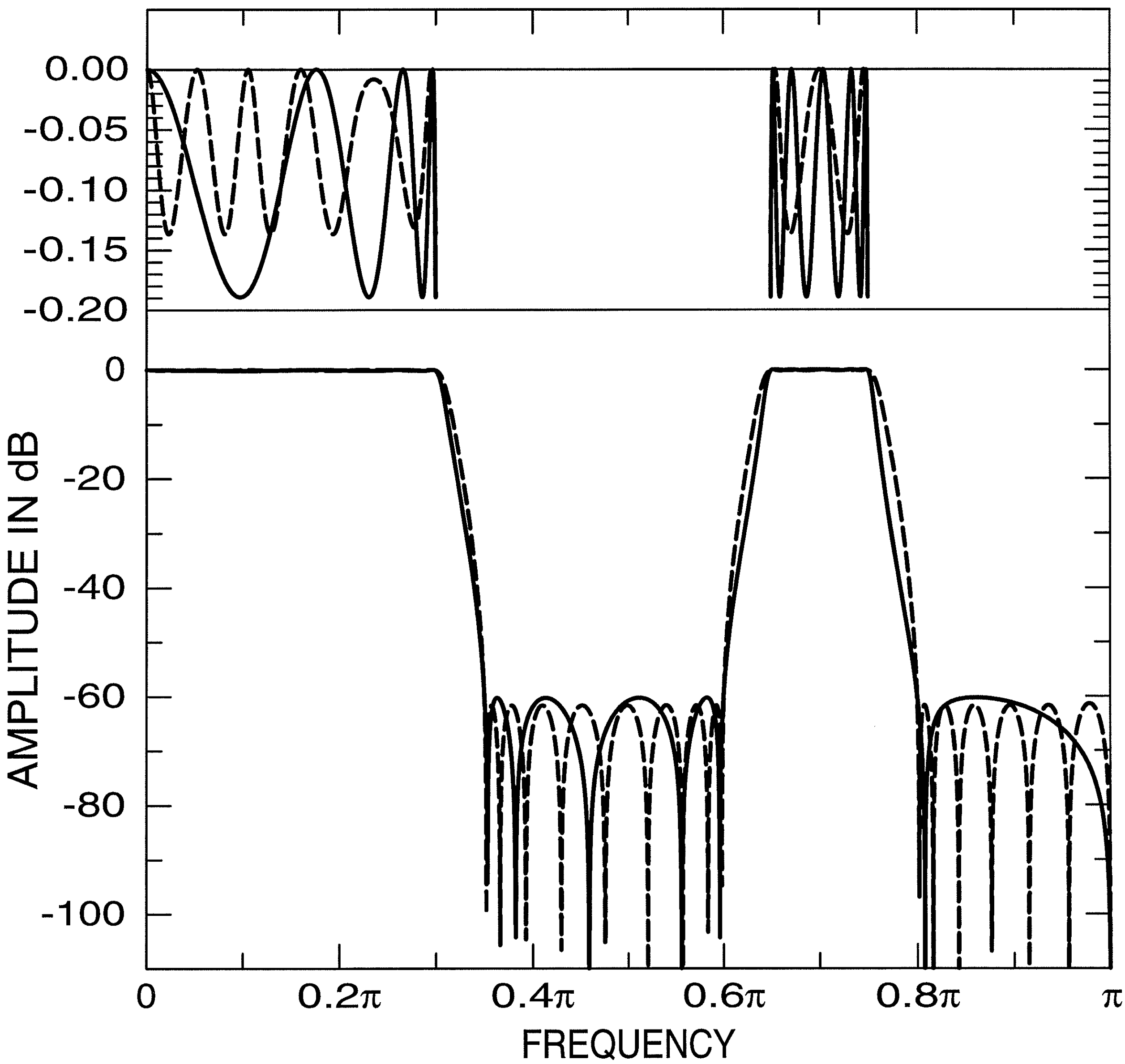
Optimized responses

- Solid line: Original design.
- Dashed line: Three poles transferred to the origin.



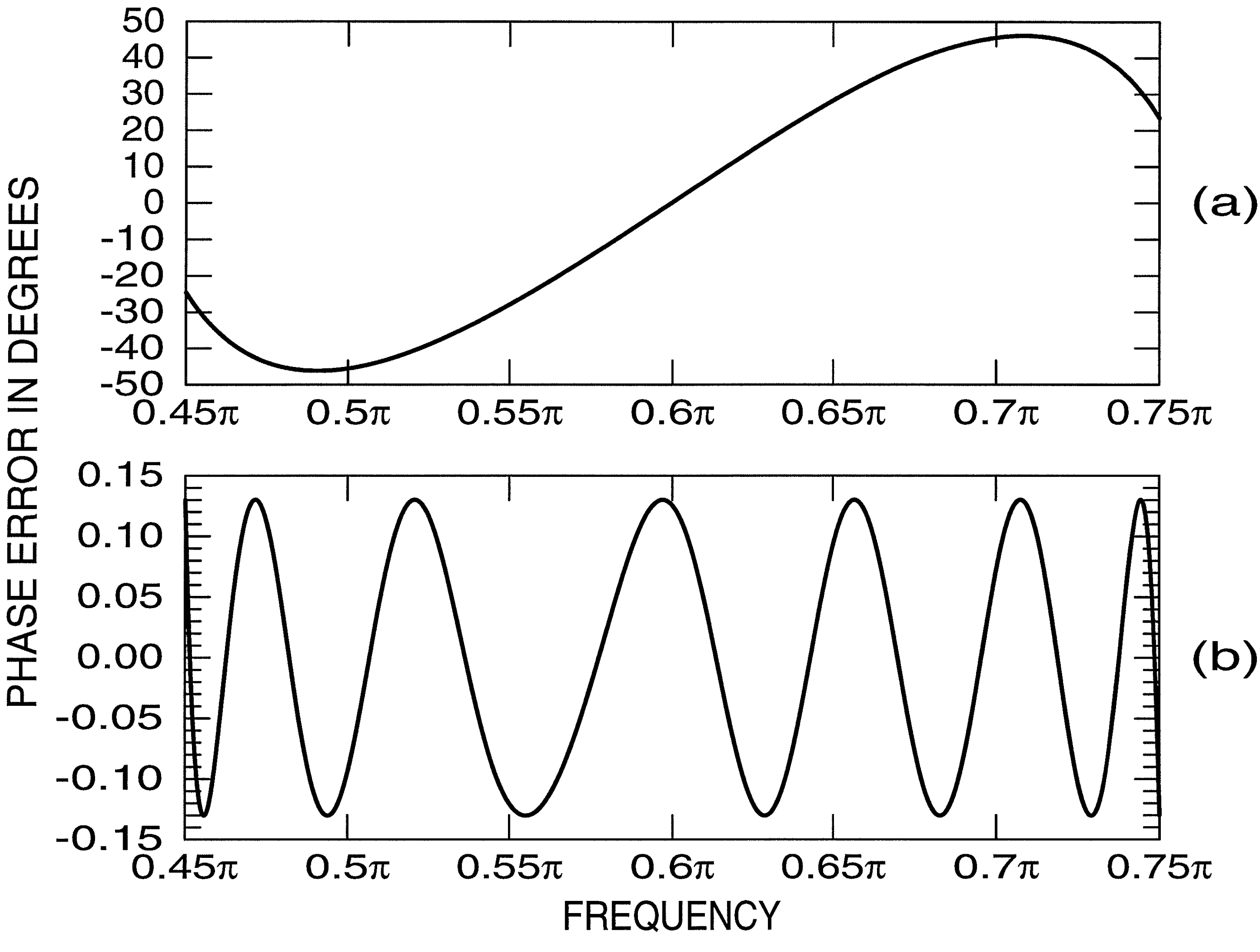
EXAMPLE 2: A multiband filter $\frac{1}{2}[B(z) + C(z)]$

- Two passbands $[0,\ 0.4\pi]$ and $[0.65\pi,\ 0.75\pi]$ with a 0.2-dB ripple.
- Two stopbands $[0.35\pi,\ 0.6\pi]$ and $[0.8\pi,\ \pi]$ with a 60-dB attenuation.
- Solid line: $B(z)$ of order 7, $C(z)$ of order 10.
- Dahed line: $B(z) = z^{-20}$, $C(z)$ of order 23.



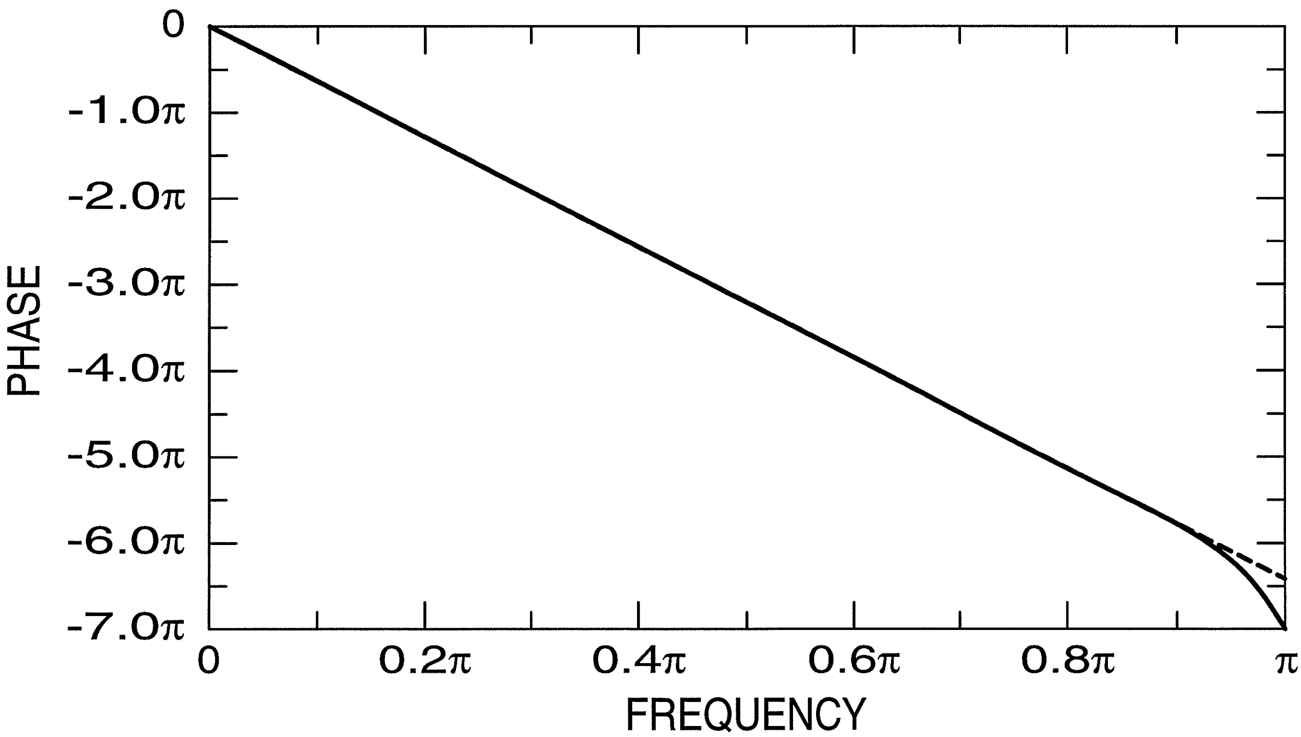
EXAMPLE 3: Phase equalization of the passband phase of the second filter in Example 1

- Without equalization, the maximum deviation of the phase from $-9.75935\omega - 4\pi$ is **46.19** degrees [Fig. (a)].
- With a twelfth-order phase equalizer, the deviation from $-36.6825\omega + 6\pi$ is **0.1302** degrees [Fig. (b)].



EXAMPLE 4: An allpass filter providing a delay of $5+\sqrt{2}$ on $[0,\ 0.9\pi]$

- $D(\omega) = -(5+\sqrt{2})\omega$ (dashed line).
- Seventh-order allpass filter (nearest integer greater than $5 + \sqrt{2}$) provides the solution with error of 0.004027π (solid line).



EXAMPLE 5: Approximately linear-phase Hilbert transform design

- Eighteenth-order all-pass section approximating $D(\omega) = -17\omega - \pi/2$ on $[0.05\pi,\ 0.95\pi]$.