Starting Point for Generating Wavelet Banks: A Perfect-Reconstruction Two-Channel Filter Bank

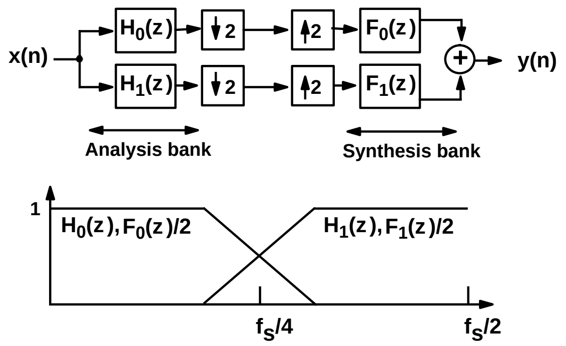


Figure 1. Two-channel filter bank.

- The above figure shows a two-channel filter bank consisting of analysis and synthesis parts.
- The processing unit is omitted in the figure since this bank is used as an intermediate step for generating a wavelet bank.
- The role of the analysis part is to split the overall signal into lowpass and highpass parts using a lowpass–highpass filter pair with transfer functions $H_0(z)$ and $H_1(z)$. These filtering operations are followed by downsampling by a factor of two.
- The role of the synthesis lowpass–highpass filter pair with transfer functions $F_0(z)$ and $F_1(z)$ is to reconstruct the original signal with a small delay. Before using these filters, the

decimated outputs of $H_0(z)$ and $H_1(z)$ are upsampled by a factor of two (one zero-valued sample is inserted between the existing samples).

- A special case guaranteeing that $y(n) = x(n-K)$, that is, the output is the input delayed by K samples with K being odd, is achieved by the following conditions:

1. $F_0(z) = 2H_1(-z)$.
2. $F_1(z) = -2H_0(-z)$.
3. $E(z) = H_0(z)H_1(-z)$ is the transfer function of a linear-phase half-band FIR filter of order $2K$.

- Conditions 1 and 2 guarantee that there is no aliasing at the output.
- Condition 3 implies that $E(z) = H_0(z)H_1(-z) = \sum_{n=0}^{2K} e(n)z^{-n}$ satisfies

1. $e(2K - n) = e(n)$ for $n=0, 1, \cdots, K$.
2. $e(K) = 1/2$.
3. $e(K \pm 2r) = e(n)$ for $r=1, 2, \cdots, (K-1)/2$.

- Here, $H_1(z) = \sum_{n=0}^{N_1} h_1(n)z^{-n}$ and $G_1(z) \equiv H_1(-z) = \sum_{n=0}^{N_1} g_1(n)z^{-n}$ are high-pass and low-pass filter transfer functions with the impulse-response values being related via $g_1(n) = (-1)^n h_1(n)$ or $h_1(n) = (-1)^n g_1(n)$ for $n=0, 1, \cdots, N_1$.
- The corresponding frequency and amplitude responses are related through $G_1(e^{j\omega}) = H_1(e^{j(\omega+\pi)})$ or $H_1(e^{j\omega}) =$

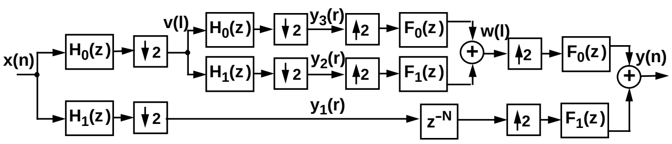
$G_1(e^{j(\omega+\pi)})$; and $|G_1(e^{j\omega})| = |H_1(e^{j(\pi-\omega)})|$ or $|H_1(e^{j\omega})| = |G_1(e^{j(\pi-\omega)})|$, respectively.

Comments

- The overall design problem is to find $E(z)$, the transfer function of a linear-phase half-band FIR filter being factorizable into the terms $H_0(z)$ and $H_1(-z)$, such that $H_0(z)$, $H_1(z)$, $F_0(z)$, and $F_1(z)$ provide the desired performance for the overall system of Figure 1.
- Note that after determining $H_0(z)$ and $H_1(-z)$, $H_0(z)$, $H_1(z)$, $F_0(z)$, and $F_1(z)$ are also uniquely determined.
- Before considering the desired performance, the next task is to generate the discrete-time wavelet bank based on use of the system of Figure 1.

How to Generate Two-Level Wavelet Banks?

- When generating wavelet banks, the first step is use a two-channel filter bank after the decimated lowpass filtered signal in the original filter bank shown in Figure 1.
- This bank is exactly the same as the original one, as shown in Figure 2.



Equivalent structure:

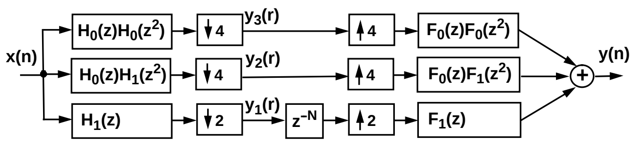


Figure 2. Two-level wavelet bank.

- The input-output relation for this bank is $w(l) = v(l-K)$, showing that there is again a delay of K samples.
- Therefore, in order to make the system of Figure 2 perfect, the decimated high-pass filtered signal, denoted by $y_1(r)$, has also to be delayed by K samples.

- Since these delays of K samples are between the down-sampling and up-sampling by a factor of 2, the overall extra delay in terms of the input and output sampling rates of the overall system of Figure 2 is $2K$ samples.
- The overall delay is thus $3K$ samples and $y(n) = x(n-3K)$.
- Using the identities considered in Part II of this course (Pages 17 and 18), the overall system is expressible using the equivalent structure also shown in Figure 2.

How to Generate Multi-Level Wavelet Banks?

- In order to generate a multilevel wavelet bank, the last decimated lowpass filtered signal of Figure 2 is treated in the same manner.
- This process can be repeated several times. The number of steps depends on the application.
- Figure 3 shows the structure for the analysis part in the case where the band splitting has been performed five times.
- This figure shows also the equivalent structure where the input data is filtered by six filters followed by decimation by different factors.
- Figure 4 shows the corresponding synthesis part.
- Finally, the overall system is depicted in Figure 5.
- In Figure 5, extra delays are included in order to make the delays through all the branches the same and to generate a perfect-reconstruction system.
- This is achieved when the delays of $z^{-2N_1}E_1(z)G_1(z)$, $z^{-4N_2}E_2(z)G_2(z)$, $z^{-8N_3}E_3(z)G_3(z)$, $z^{-16N_4}E_4(z)G_4(z)$, $z^{-32N_5}E_5(z)G_5(z)$, and $z^{-32N_6}E_6(z)G_6(z)$ are equal.
- When $H_0(z)$ and $H_1(-z)$, $H_0(z)$, $H_1(z)$, $F_0(z)$, and $F_1(z)$ are constructed according to the previous discussion, then $N_5 = N_6 = 0$, $N_4 = K$, $N_3 = 3K$, $N_2 = 7K$, and $N_1 = 15K$.
- In the wavelet bank of Figure 5, the signals denoted by $y_k(r)$ for $k = 1, 2, \cdots, 6$ are called the **wavelet coefficients**.

- In the processing unit, these signals can be treated in several ways depending on the applications. This produces the processed coefficients $\hat{y}_k(r)$. Typical example applications are signal compression and de-noising.
- If the processed wavelet coefficients satisfy $\hat{y}_k(r) = y_k(r)$ for $k = 1, 2, \cdots, 6$, then $y(n) = x(n-31K)$, that is, the output signal is the input signal delayed by $31K$ samples.
- The above procedure can be extended in a straightforward manner to wavelet banks having more than 5 levels.

Figure 3. Five-Level Wavelet Banks: Analysis Part.

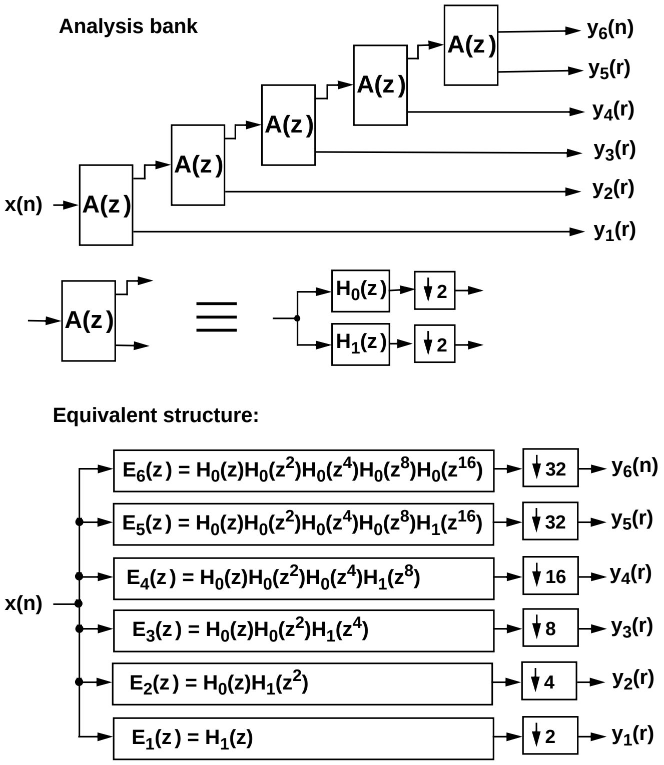


Figure 4. Five-Level Wavelet Bank: Synthesis Part.

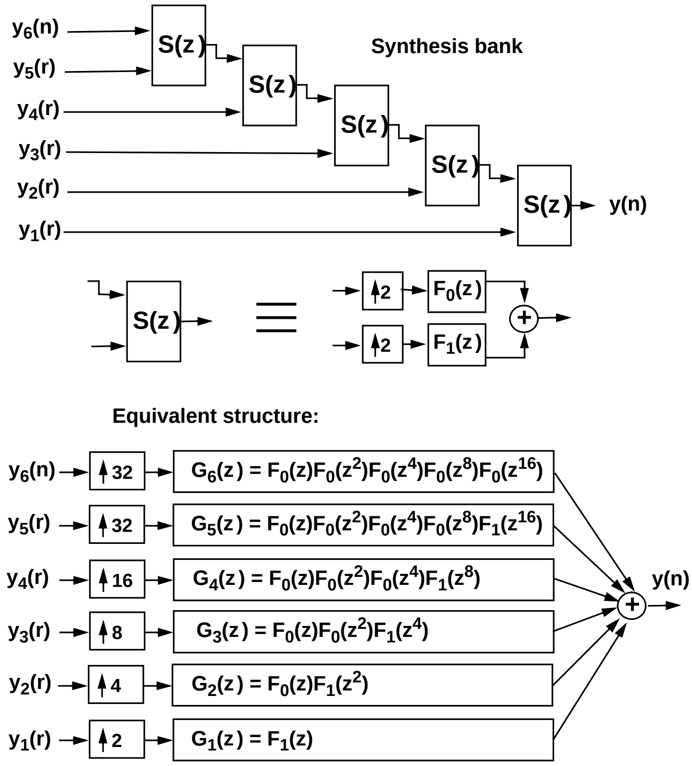
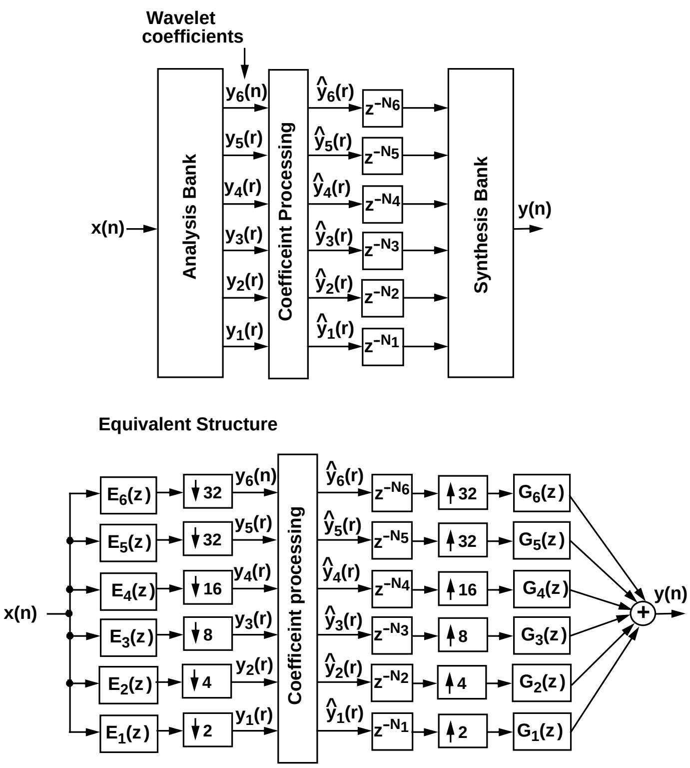


Figure 5. Overall Five-Level Wavelet Bank.



Maximally-Flat Half-Band FIR Filters

- Mathematicians developing the theory for useful wavelet banks found out that maximally-flat half-band FIR filters are good starting points for developing proper discrete-time wavelet banks.
- It should be pointed out that these filters are special cases of maximally-flat FIR filters introduced by Herrmann already in 1971.
- For a maximally-flat half-band filter of order $2K$ with $K = 2L - 1$, the transfer function has the following closed-form expression:

$$E(z) = \left[\frac{1+z^{-1}}{2}\right]^{2L} \sum_{n=0}^{L-1} (-1)^n d(n) z^{-(L-1-n)} \left[\frac{1-z^{-1}}{2}\right]^{2n},$$

where

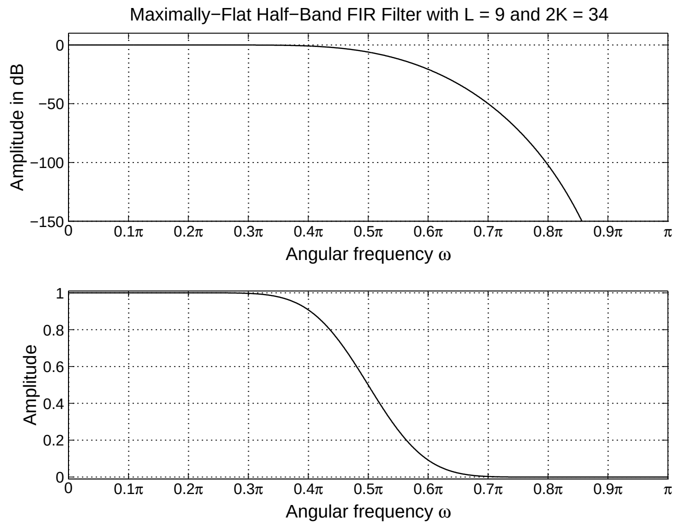
$$d(n) = \frac{(L-1+n)!}{(L-1)!\,n!}.$$

- This linear-phase FIR filter has $2L$ zeros at $z = -1$ and $2(L-1)$ zeros off the unit circle.
- We consider in more details the following two cases:
- Case A: $L = 9$ and $2K = 34$.
- Case B: $L = 8$ and $2K = 30$.

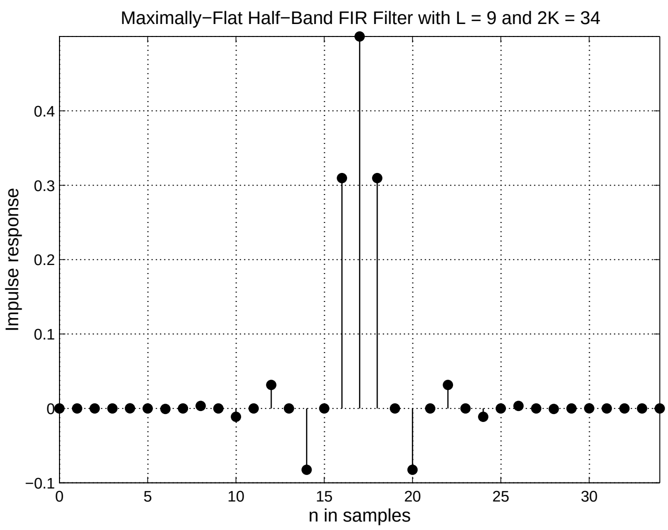
Responses for Case A: $L = 9$ and $2K = 34$

- The following three pages give the amplitude response, the impulse response, and the zero-plot for Case A.
- In this case, $E(z)$ has $2L = 18$ zeros at $z = -1$ and four zero quadruplets at $z = r_k \exp(\pm j\theta_k)$, $(1/r_k)\exp(\pm j\theta_k)$ for $k = 1, 2, 3, 4$.
- $r_1 = 0.37196$, $\theta_1 = 0.04357\pi$, $r_2 = 0.38943$, $\theta_2 = 0.13239\pi$, $r_3 = 0.43036$, $\theta_3 = 0.22743\pi$, $r_4 = 0.51567$, $\theta_4 = 0.34001\pi$.

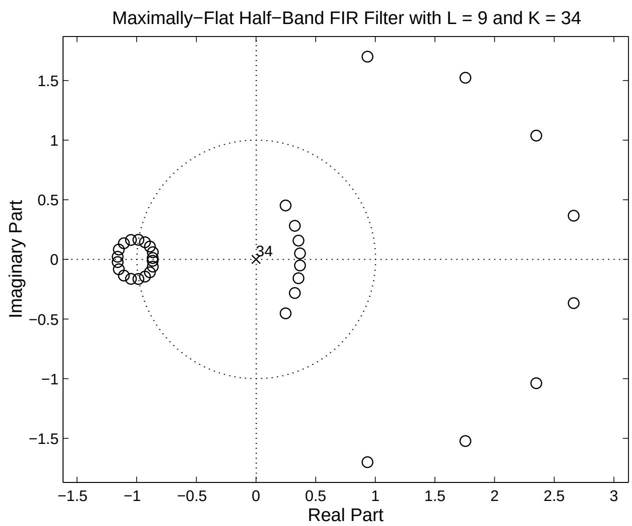
Amplitude Response for Case A: $L = 9$ and $2K = 34$



Impulse Response for Case A: $L = 9$ and $2K = 34$



Zero-plot for Case A: $L = 9$ and $2K = 34$

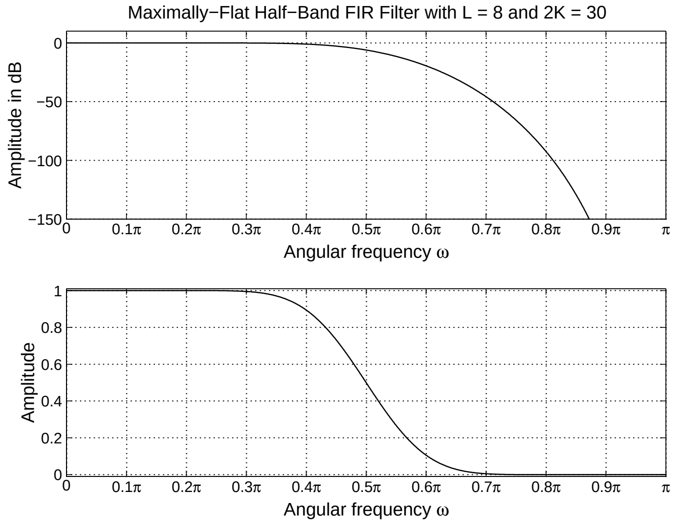


- Note that because of an error in the MATLAB routine, all the 9 zeros are not located at $z = -1$.

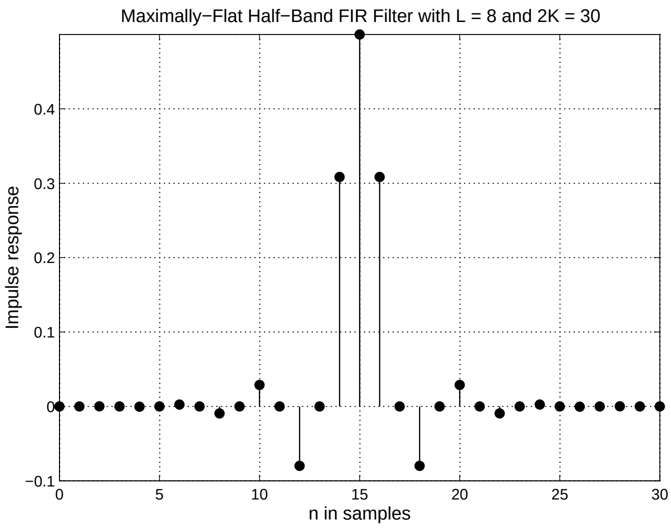
Responses for Case B: $L = 8$ and $2K = 30$

- The following three pages give the amplitude response, the impulse response, and the zero-plot for Case B.
- In this case, $E(z)$ has $2L = 16$ zeros at $z = -1$ and three zero quadruplets at $z = r_k \exp(\pm j\theta_k)$, $(1/r_k)\exp(\pm j\theta_k)$ for $k = 1, 2, 3$ as well as one reciprocal zero pair at $z = R, 1/R$.
- $r_1 = 0.37606$, $\theta_1 = 0.09975\pi$, $r_2 = 0.41244$, $\theta_2 = 0.20501\pi$, $r_3 = 0.49557$, $\theta_3 = 32802\pi$, $R = 0.36540$.

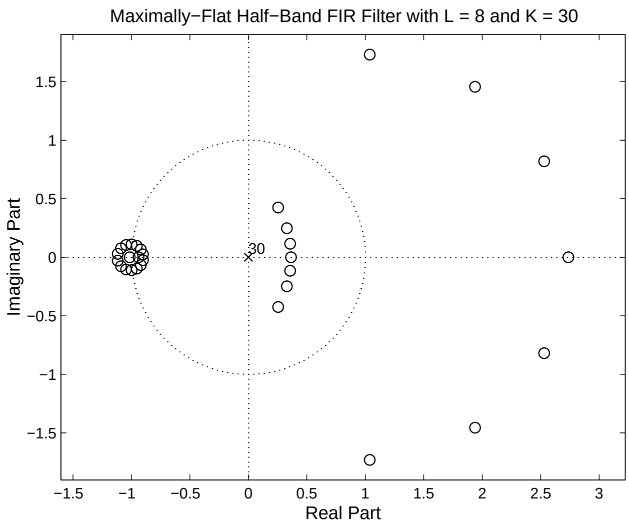
Amplitude Response for Case B: $L = 8$ and $2K = 30$



Impulse Response for Case A: $L = 8$ and $2K = 30$



Zero-plot for Case A: $L = 8$ and $2K = 30$



- Note that because of an error in the MATLAB routine, all the 8 zeros are not located at $z = -1$.

Orthogonal (Paraunitary) Wavelet Banks Derived from Maximally-Flat Half-Band FIR Filters

- In this case,

$$H_0(z)=\sum_{n=0}^{N_0} h_0(n)z^{-n} \text{ and } G_1(z)\equiv H_1(-z)=\sum_{n=0}^{N_1} g_1(n)z^{-n}$$

satisfy

1. $N_0 = N_1 = K = 2L-1$.
2. $E(z) = H_0(z)H_1(-z)=\sum_{n=0}^{2K} e(n)z^{-n}$ is the transfer function of a linear-phase maximally-flat half-band FIR filter of order $2K$.
3. $g_1(n)=h_0(K-n)$ for $n=0,1,\cdots,K$.

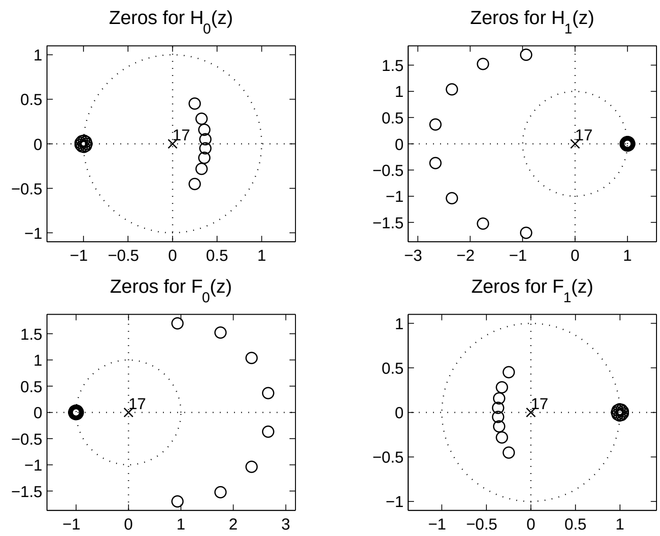
- Here, $E(z)$ can be factorized into minimum-phase and maximum-phase terms $H_0(z)$ and $H_1(-z)$ or mixed-phase terms.
- This gives two types of solutions. In the second case, the impulse responses of $H_0(z)$ and $H_1(-z)$ are desired to make rather linear.
- We start with the case where the terms are minimum- and maximum-phase FIR filters.
- Then, the second case will be considered.

Orthogonal Wavelet Banks Based on the Use of Maximum-Phase and Minimum-Phase Components of a Maximally-Flat Half-Band FIR

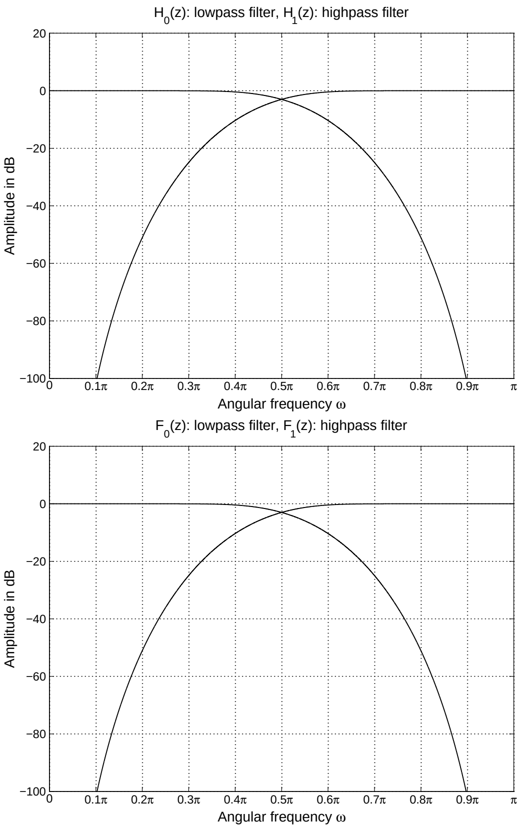
- In this case, both $H_0(z)$ and $H_1(-z)$ contain L zeros $z=-1$, whereas $H_0(z)$ $[H_1(-z)]$ possesses the $L-1$ zeros of $E(z)$ lying inside (outside) the unit circle.
- The following set of pages shows various responses for the five-level wavelet bank in Case A, that is, $L=9$ and $2K=34$.
- They include the characteristics of the building-block two-channel filter bank as well as the responses of the filters $E_k(z)$ and $F_k(z)$ for $k=1,2,\cdots,6$ in the equivalent structure of Figure 6.
- Also the analysis and synthesis scaling functions and wavelets, to be defined in more details in the end of this pile of lecture notes, are shown.
- In drawing these responses, $H_0(z)$, $H_1(z)$, $F_0(z)$, and $F_1(z)$ have been normalized such that $H_0(1)=F_0(1)=1$ and $H_1(-1)=F_1(-1)=1$. This means that the amplitude responses of $H_0(z)$ and $F_0(z)$ $[H_0(z)$ and $F_0(z)]$ take on the value of unity at $\omega=0$ $[\omega=\pi]$.
- In the case of two-channel filter banks, the normalization constant is 2 for $F_0(z)$ and $F_1(z)$, due to the interpolation by a factor of two.

- Typically, for wavelet banks, the normalization constant is $\sqrt{2}$ for $H_0(z)$, $H_1(z)$, $F_0(z)$, and $F_1(z)$ although this results in overflows if fixed-point arithmetic is used?
- It should be pointed out that the wavelets resulting using the above procedure are called Daubechies wavelets.
- In the MATLAB Wavelet Toolbox manual, the wavelet corresponding to our case is denoted by db9 with 9 indicating that $L=9$.
- It is seen that the frequency selectivities provided by the $E_k(z)$'s and $F_k(z)$'s are very poor.
- This means that if the original wavelet coefficients $y_k(r)$ and the processed coefficients $\hat{y}_k(r)$ are very different in Figure 6, the aliased terms are not cancelled very well.
- However, this does not matter and is, in fact, beneficial if we are studying images or waveforms of one-dimensional signal. In this case, our eyes are the 'referees'.
- If our ears are the 'referees', then we hate especially sinusoidal components jumping to a wrong frequency range.
- In this case, we need very selective filter banks.
- Therefore, there is a room for both multirate wavelet banks with poor selectivity and selective multirate filter banks!

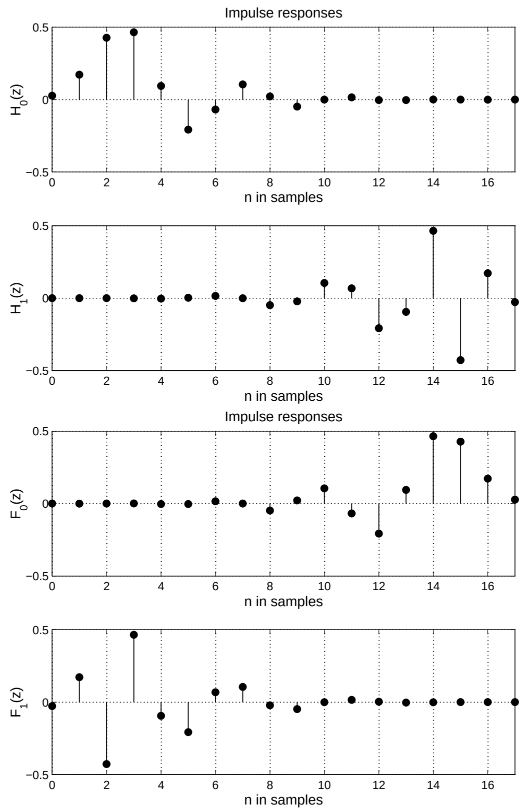
Zero Plots for $H_0(z)$, $H_1(z)$, $F_0(z)$, and $F_1(z)$



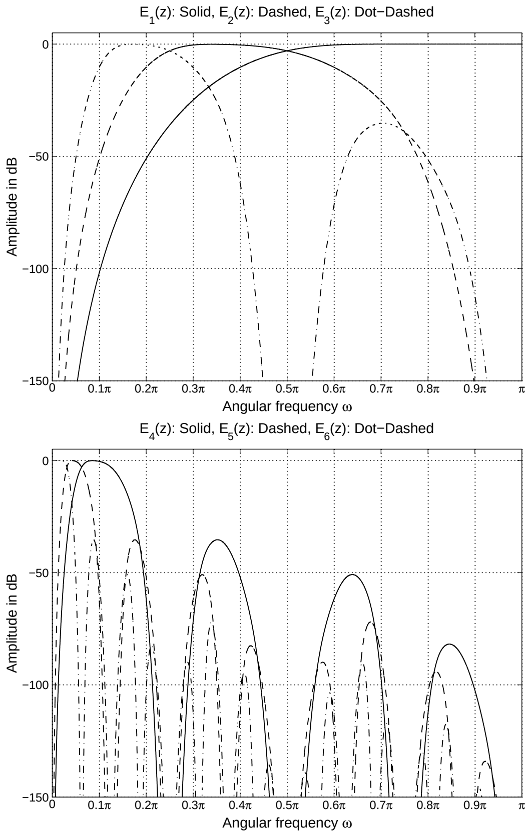
Amplitude Responses for $H_0(z)$, $H_1(z)$, $F_0(z)$, and $F_1(z)$. $F_0(z)$ [$F_1(z)$] has been normalized such that it achieves the value of unity at $\omega = 0$ [$\omega = \pi$].



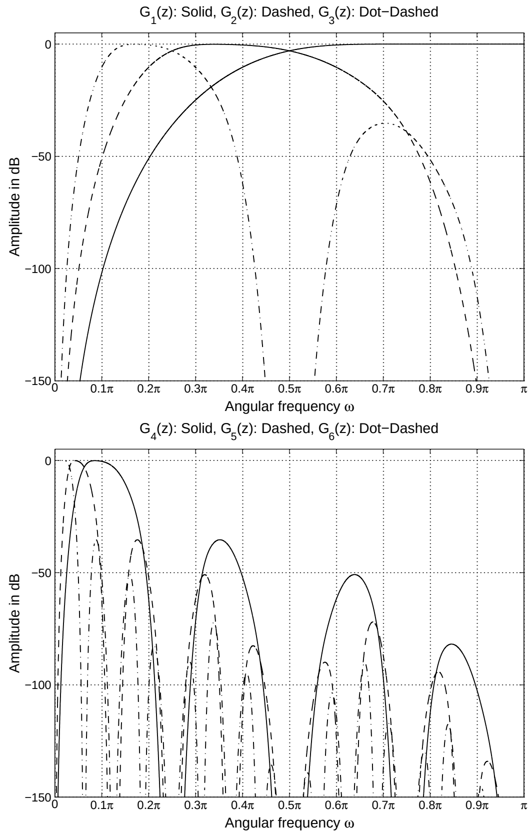
Impulse Responses for $H_0(z)$, $H_1(z)$, $F_0(z)$, and $F_1(z)$. $F_0(z)$ [$F_1(z)$] has been normalized such that it achieves the value of unity at $\omega = 0$ [$\omega = \pi$].



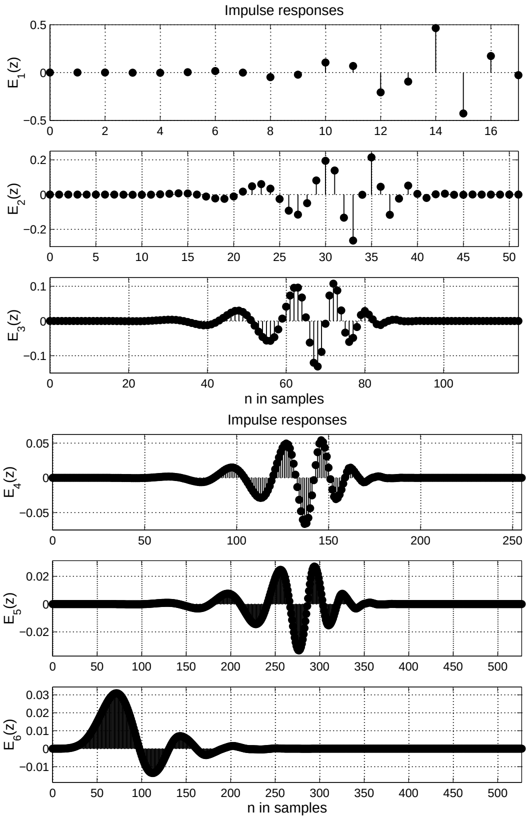
Amplitude Responses for the Resulting Six Analysis Transfer Functions $E_k(z)$ in Figures 3 and 5



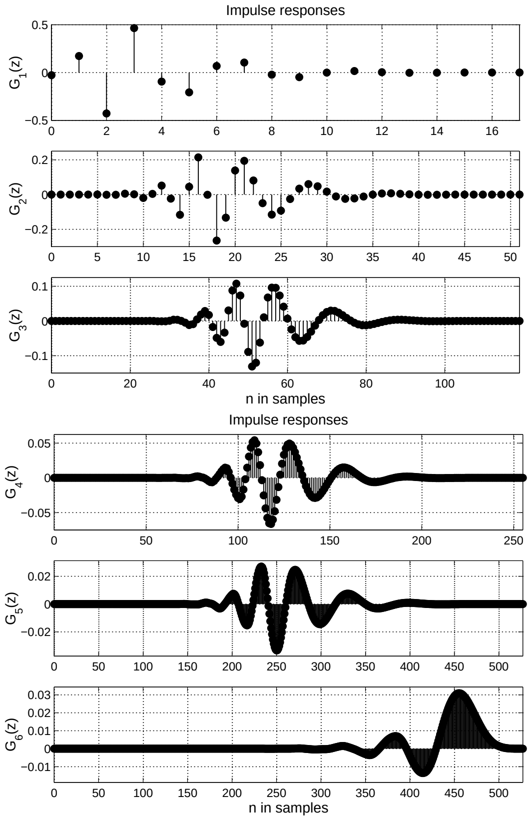
Amplitude Responses for the Resulting Six Synthesis Transfer Functions $G_k(z)$ in Figures 4 and 5



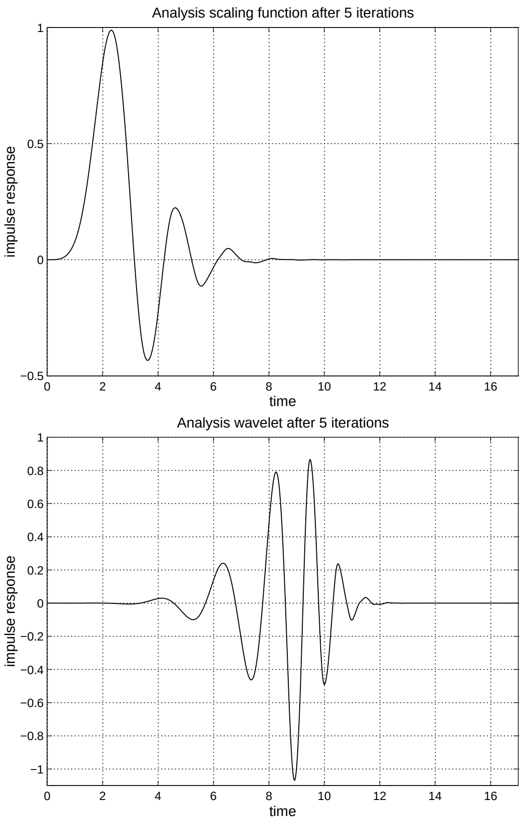
Impulse Responses for the Resulting Six Analysis Transfer Functions $E_k(z)$ in Figures 3 and 5



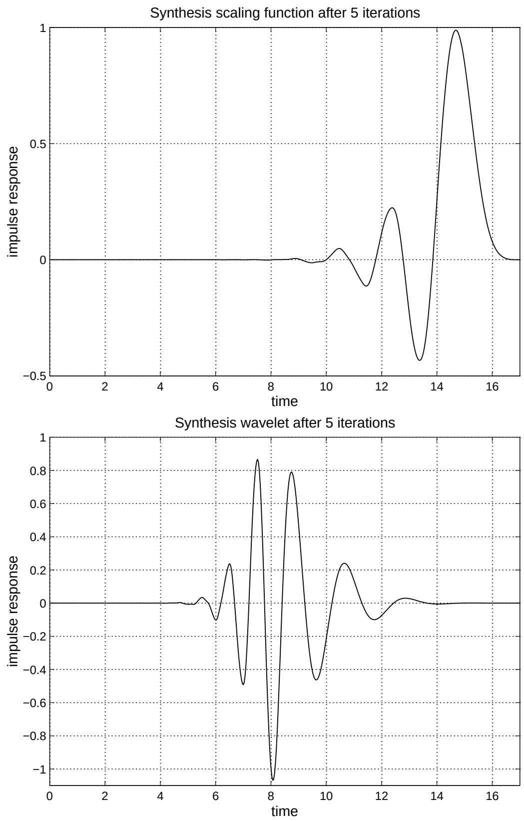
Impulse Responses for the Resulting Six Synthesis transfer functions $G_k(z)$ in Figures 4 and 5



Analysis Scaling Function and Wavelet after 5 Iterations



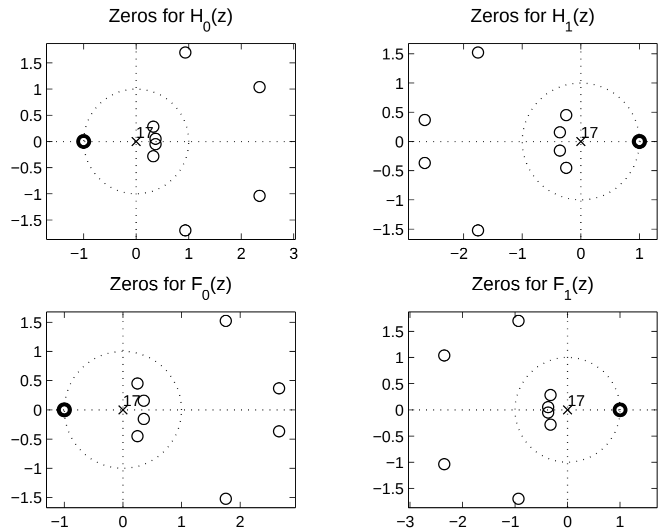
Synthesis Scaling Function and Wavelet after 5 Iterations



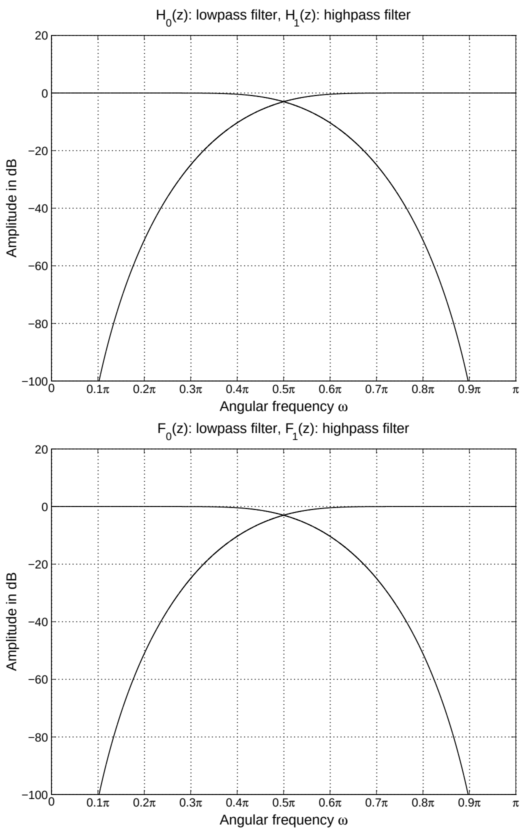
Orthogonal Wavelet Banks Based on the Use of Mixed-Phase Components of a Maximally-Flat Half-Band FIR

- In this case, both $H_0(z)$ and $H_1(-z)$ again contain L zeros $z=-1$. In order to make the impulse responses rather linear in both the analysis and synthesis banks, $H_0(z)$ and $H_1(-z)$ are selected to be mixed-phase designs.
- In Case A ($L = 9$ and $2K = 34$), a good result is obtained by selecting $H_0(z)$ $[H_1(-z)]$ to contain the zeros at $z = r_1\exp(\pm j\theta_1)$, $z = (1/r_2)\exp(\pm j\theta_2)$, $z = r_3\exp(\pm j\theta_3)$, and $z = (1/r_4)\exp(\pm j\theta_4)$ $[z = (1/r_1)\exp(\pm j\theta_1), z = r_2\exp(\pm j\theta_2)$, $z = (1/r_3)\exp(\pm j\theta_3)$, and $z = r_4\exp(\pm j\theta_4)]$.
- Here the zero quadruplets have been sorted according to the increasing angle θ.
- In Case B ($L = 8$ and $2K = 30$), a good result is obtained by selecting $H_0(z)$ $[H_1(-z)]$ to contain the zeros at $z = (1/r_1)\exp(\pm j\theta_1)$, $z = r_2\exp(\pm j\theta_2)$, and $z = (1/r_3)\exp(\pm j\theta_3)$ $[z = r_1\exp(\pm j\theta_1)$, $z = (1/r_2)\exp(\pm j\theta_2)$, $z = r_3\exp(\pm j\theta_3)]$.
- The following set of pages shows the resulting responses for the five-level wavelet bank in Case A.
- The corresponding wavelets are called symlets.
- In the MATLAB Wavelet Toolbox manual, the wavelet corresponding to our case is denoted by sym9 with 9 indicating that $L = 9$.

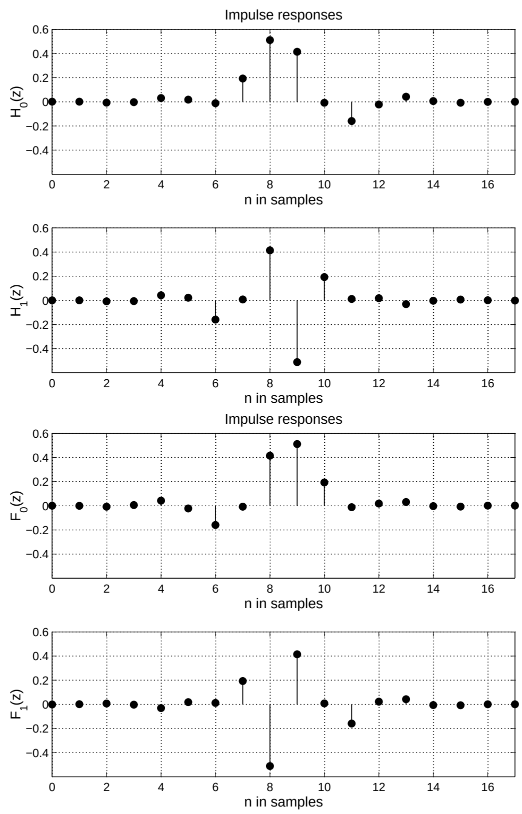
Zero Plots for $H_0(z)$, $H_1(z)$, $F_0(z)$, and $F_1(z)$



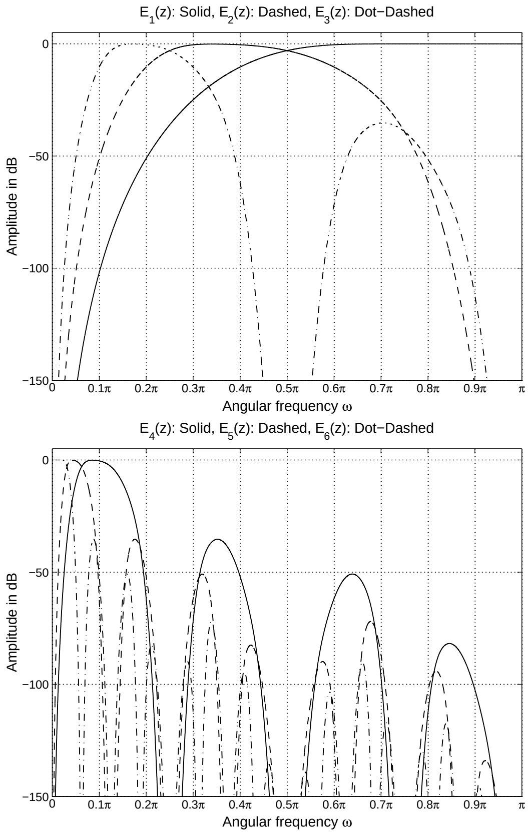
Amplitude Responses for $H_0(z)$, $H_1(z)$, $F_0(z)$, and $F_1(z)$. $F_0(z)$ $[F_1(z)]$ has been normalized such that it achieves the value of unity at $\omega = 0$ $[\omega = \pi]$.



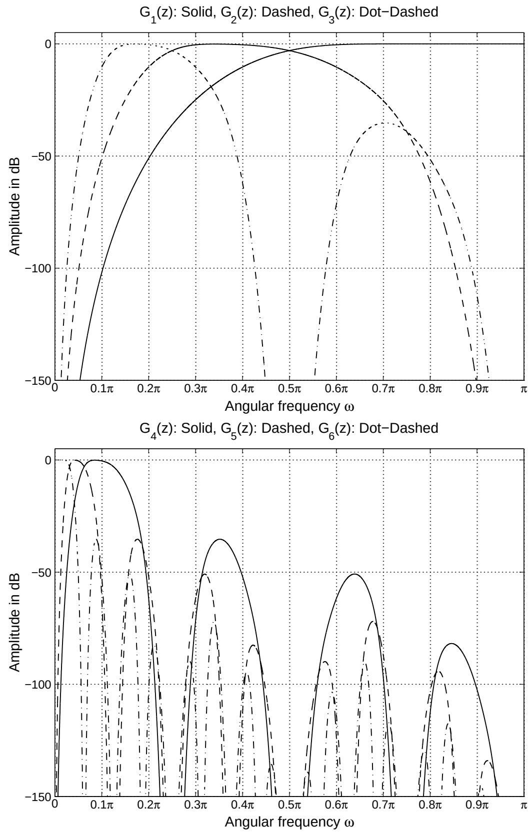
Impulse Responses for $H_0(z)$, $H_1(z)$, $F_0(z)$, and $F_1(z)$. $F_0(z)$ $[F_1(z)]$ has been normalized such that it achieves the value of unity at $\omega = 0$ $[\omega = \pi]$.



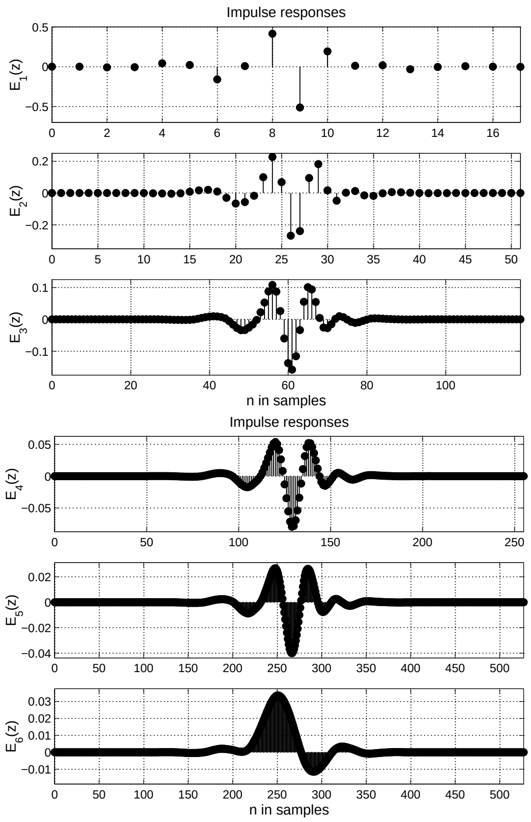
Amplitude Responses for the Resulting Six Analysis Transfer Functions $E_k(z)$ in Figures 3 and 5



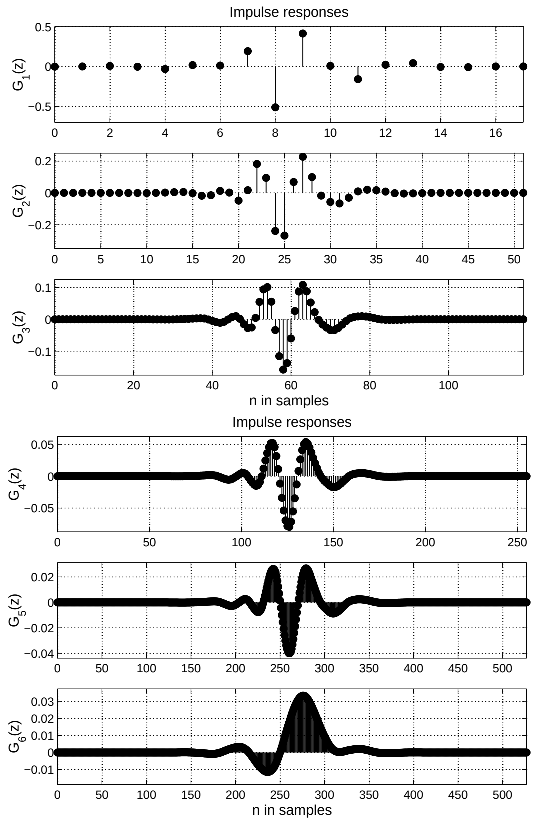
Amplitude Responses for the Resulting Six Synthesis Transfer Functions $G_k(z)$ in Figures 4 and 5



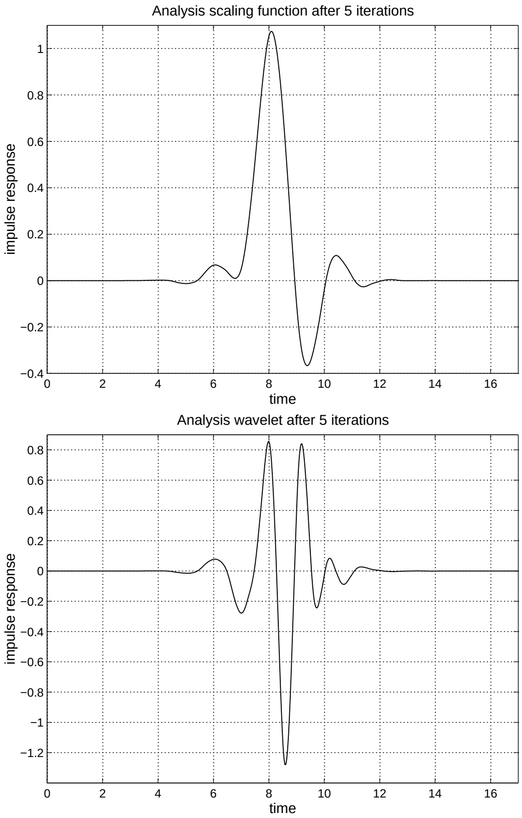
Impulse Responses for the Resulting Six Analysis Transfer Functions $E_k(z)$ in Figures 3 and 5



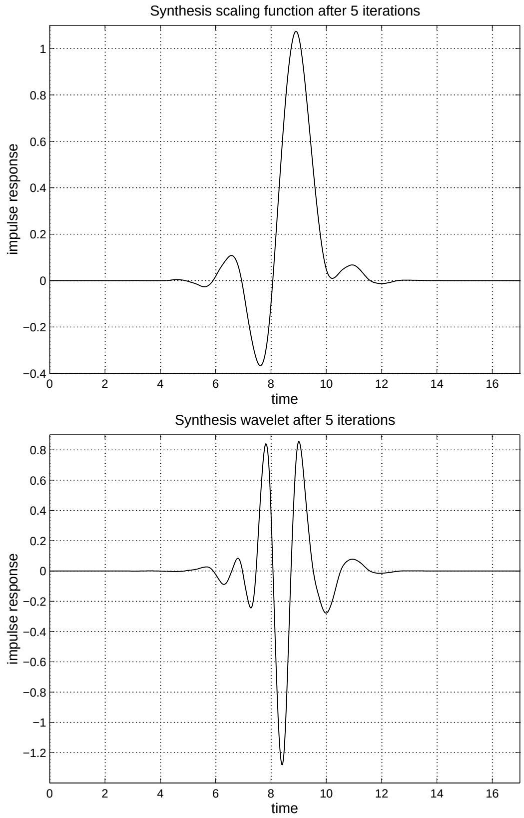
Impulse Responses for the resulting six synthesis transfer functions $G_k(z)$ in Figures 4 and 5



Analysis Scaling Function and Wavelet after 5 Iterations



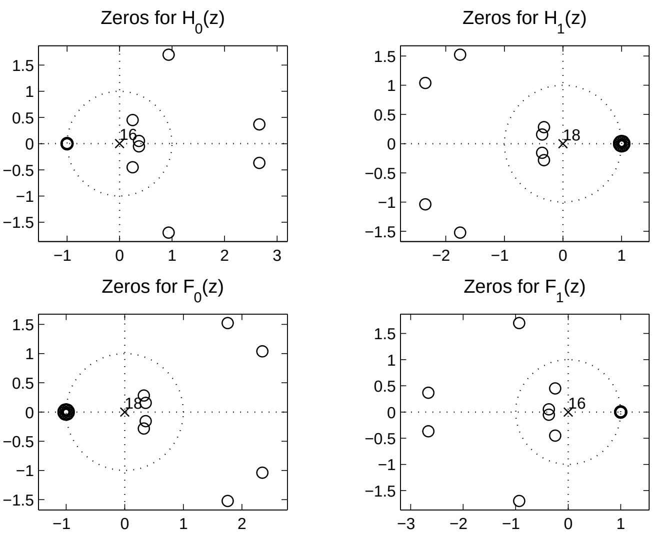
Synthesis Scaling Function and Wavelet after 5 Iterations



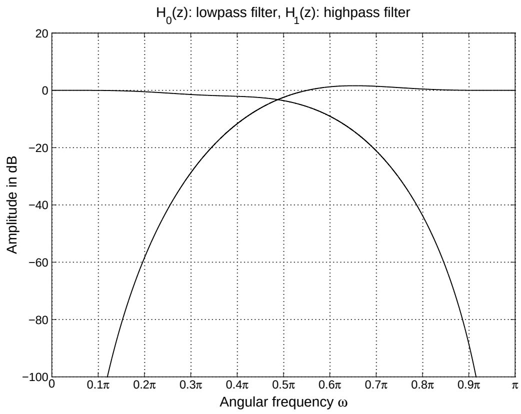
Biorthogonal Wavelet Banks Based on Factorizing a Maximally-Flat Half-Band FIR into Linear-Phase FIR Components.

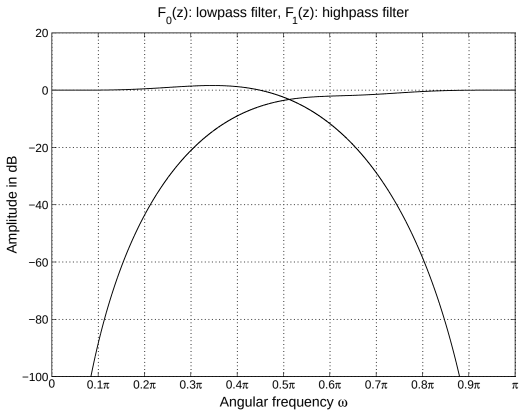
- In this case it is desired that both components $H_0(z)$ and $H_1(-z)$ are linear-phase FIR filters.
- There are several ways of sharing the zeros of a maximally-flat half-band filter between $H_0(z)$ and $H_1(-z)$. All what is needed is that $H_0(z)$ and $H_1(-z)$ contain the overall quad-ruplet or a reciprocal zero pair on the real axis.
- Otherwise, the factorization can be performed arbitrarily. For instance, the zeros $z = -1$ can be arbitrarily shared be-tween $H_0(z)$ and $H_1(-z)$.
- A good result in Case A is obtained by forming $H_0(z)$ to contain 8 zeros at $z = -1$ and the first and fourth quadruplets.
- Correspondingly, $H_1(-z)$ contains 10 zeros at $z = -1$and the second and third quadruplets.
- The following set of pages shows the responses for the re-sulting five-level wavelet bank.

Zero Plots for $H_0(z)$, $H_1(z)$, $F_0(z)$, and $F_1(z)$

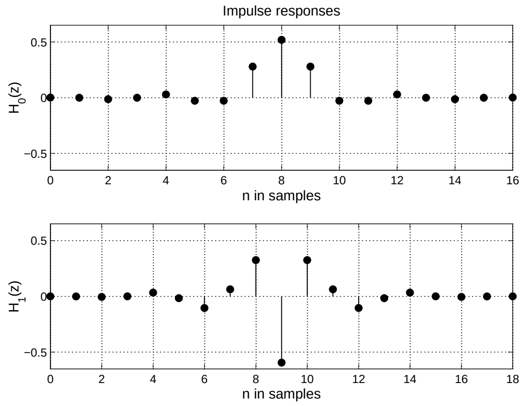


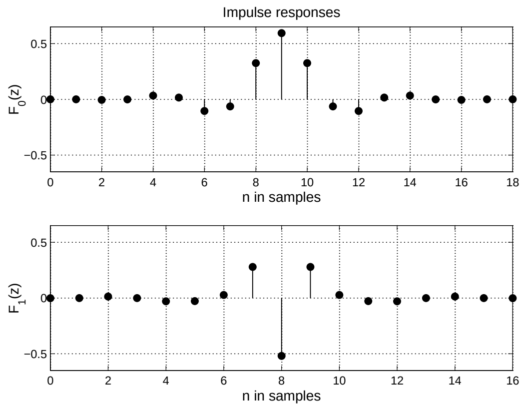
Amplitude Responses for $H_0(z)$, $H_1(z)$, $F_0(z)$, and $F_1(z)$. $F_0(z)$ [$F_1(z)$] has been normalized such that it achieves the value of unity at $\omega = 0$ [$\omega = \pi$].



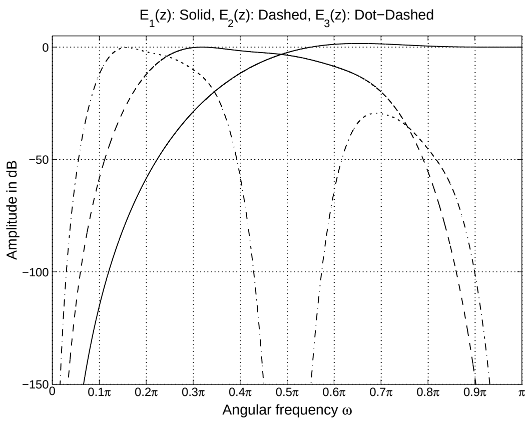


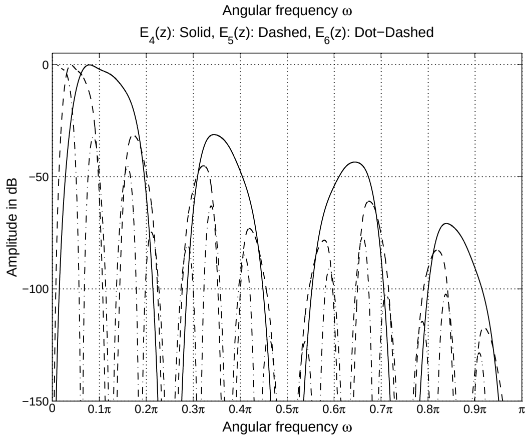
Impulse Responses for $H_0(z)$, $H_1(z)$, $F_0(z)$, and $F_1(z)$. $F_0(z)$ [$F_1(z)$] has been normalized such that it achieves the value of unity at $\omega = 0$ [$\omega = \pi$].



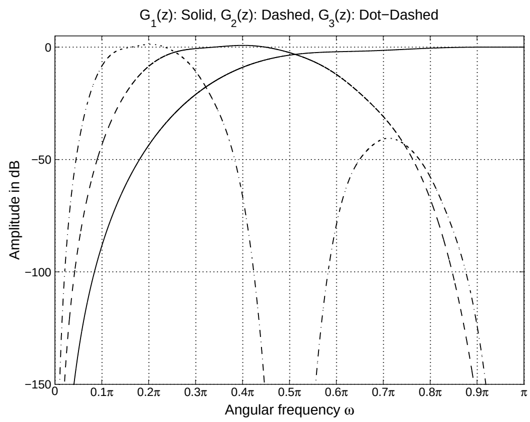


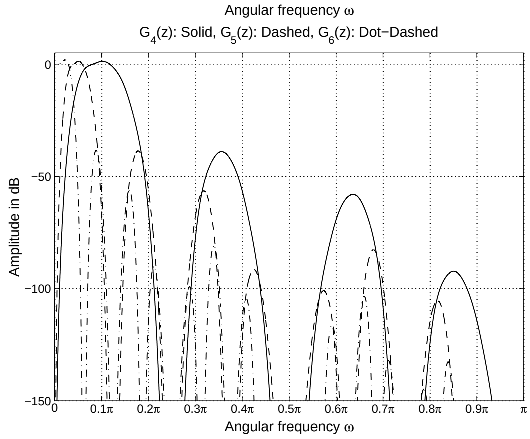
Amplitude Responses for the Resulting Six Analysis Transfer Functions $E_k(z)$ in Figures 3 and 5



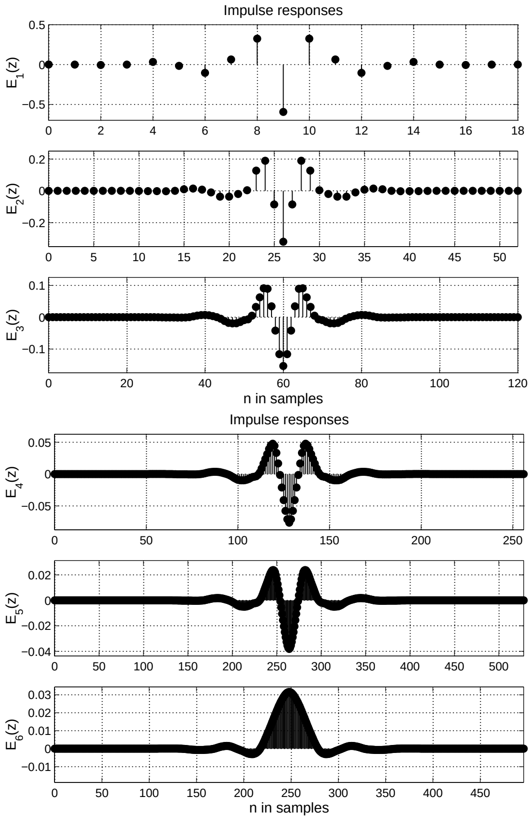


Amplitude Responses for the Resulting Six Synthesis Transfer Functions $G_k(z)$ in Figures 4 and 5

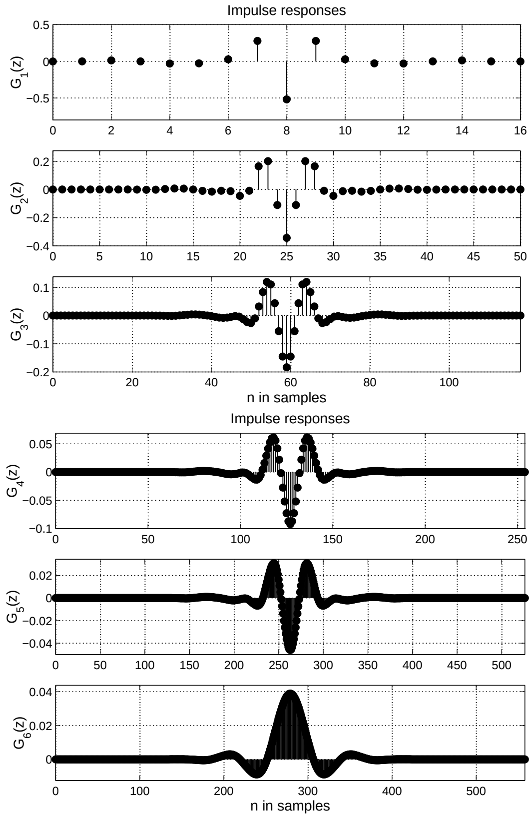




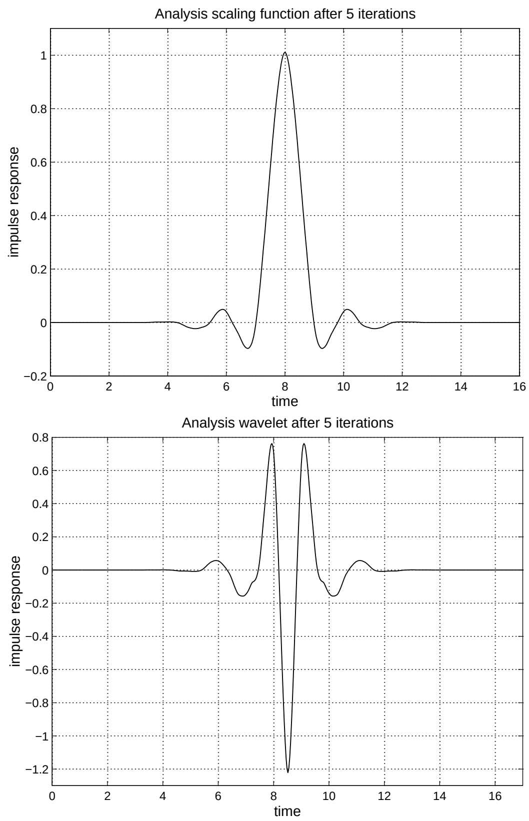
Impulse Responses the Resulting Six Analysis Transfer Functions $E_k(z)$ in Figures 3 and 5



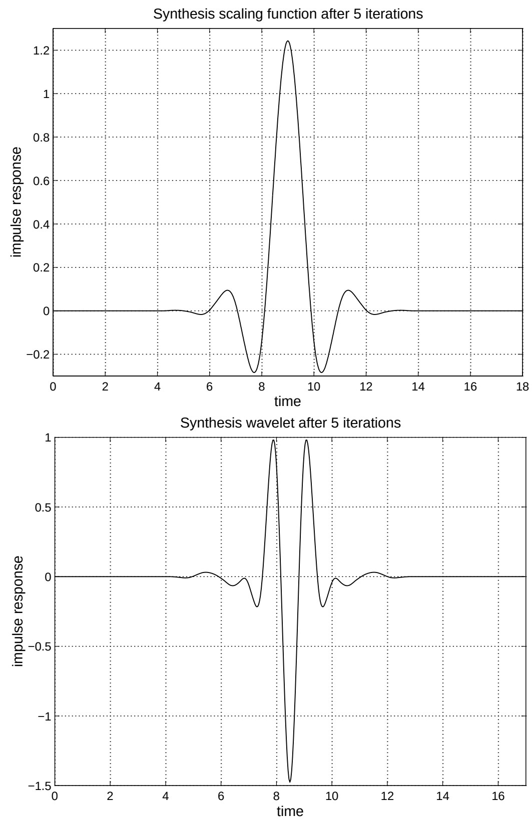
Impulse Responses for Resulting Six Synthesis Transfer Functions $G_k(z)$ in Figures 4 and 5



Analysis Scaling Function and Wavelet after 5 Iterations



Synthesis Scaling Function and Wavelet after 5 Iterations



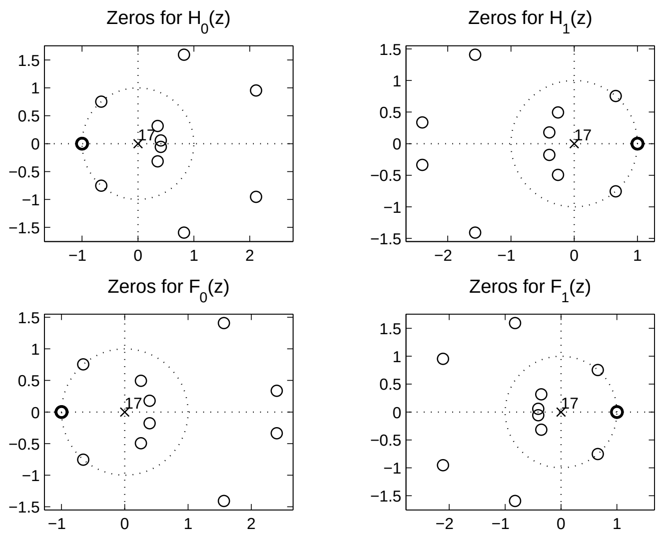
Generalized Orthogonal Wavelet Banks

- In the above, we considered the case where the starting-point half-band maximally-flat FIR filter with transfer function $E(z) = H_0(z)H_1(-z)$ had $2L$ zeros at $z = -1$ and $2(L - 1)$ zeros off the unit circle.
- In this case, the filter order is $2K$ with $K = 2L - 1$.
- In the most general case of orthogonal wavelet banks, our starting-point half-band filter can have M double zero pairs on the unit circle and $2(L - 2M)$ zeros at $z = -1$.
- This is because in the orthogonal case the zeros on the unit circle must be the same for both $H_0(z)$ and $H_1(-z)$.
- After fixing the zeros on the unit circle, the $2(L - 1)$ zeros off the unit circle zeros off the unit circle can be determined in a straightforward manner in such a way that the overall transfer function $E(z) = H_0(z)H_1(-z)$ becomes that of a half-band filter.
- The author of these lecture notes has generated a MATLAB file for this purpose (not well commented, but available).
- The following set of pages shows the responses for an approximately symmetric five-level wavelet bank in the case where $M = 1$ and $L = 9$.
- The number of zeros at $z = -1$ is 7 and the double zero pair on the unit is located at $z = \exp(\pm j0.728\pi)$.
- When comparing the filter responses to the earlier case where there were no zeros on the unit circle outside the point

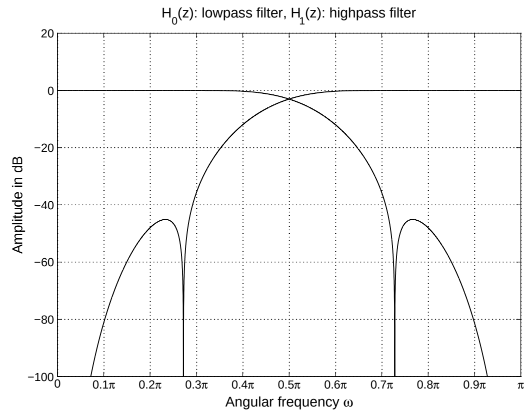
$z = -1$, it is observed that the zero pair on the unit circle improves the frequency selectivities of the filters in the bank.

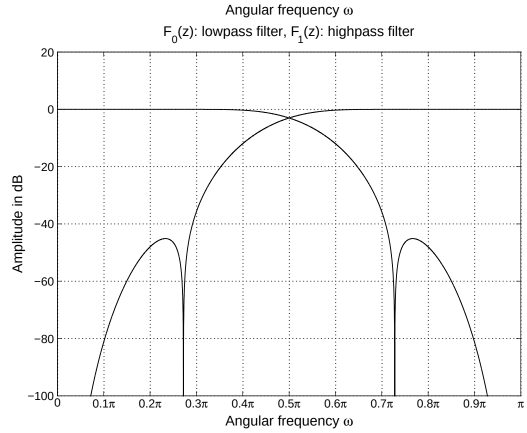
- If more zeros are moved from the point $z = -1$, then the selectivities can be further improved.
- However, the smoothness of the impulse responses becomes worse.
- It should be emphasized that the smoothness of the impulse responses of the filters in the wavelet banks is very crucial in many applications.

Zero Plots for $H_0(z)$, $H_1(z)$, $F_0(z)$, and $F_1(z)$

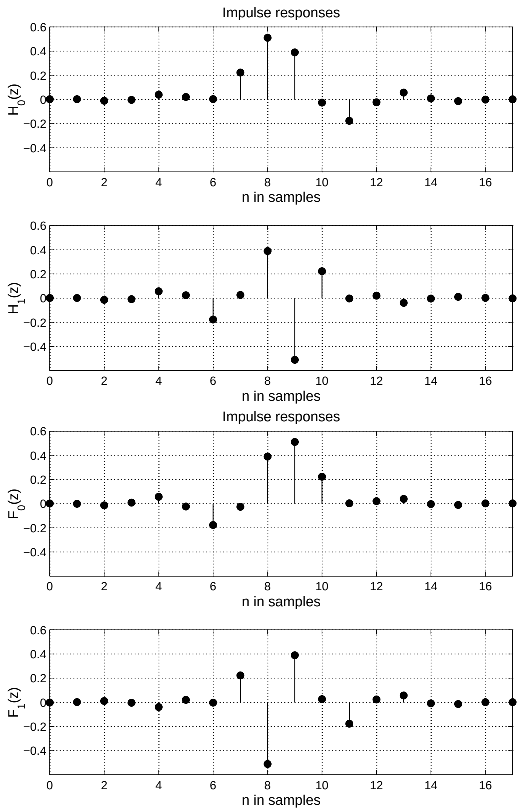


Amplitude Responses for $H_0(z)$, $H_1(z)$, $F_0(z)$, and $F_1(z)$. $F_0(z)$ [$F_1(z)$] has been normalized such that it achieves the value of unity at $\omega = 0$ [$\omega = \pi$].

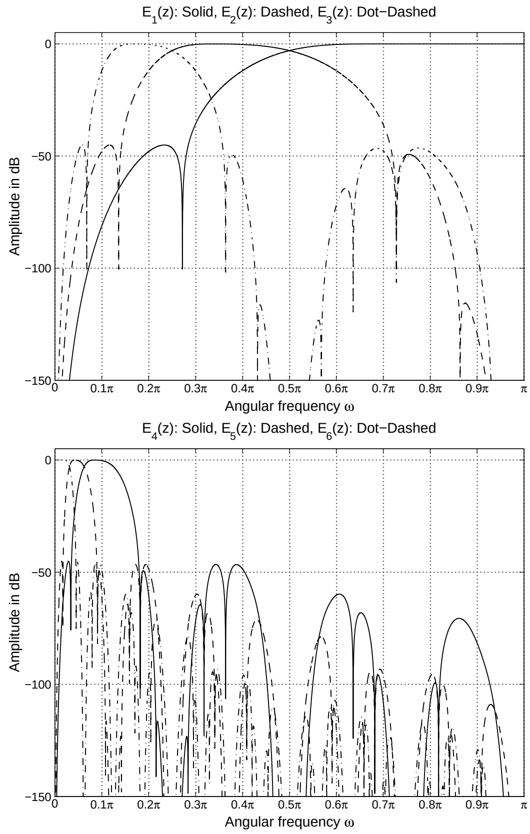




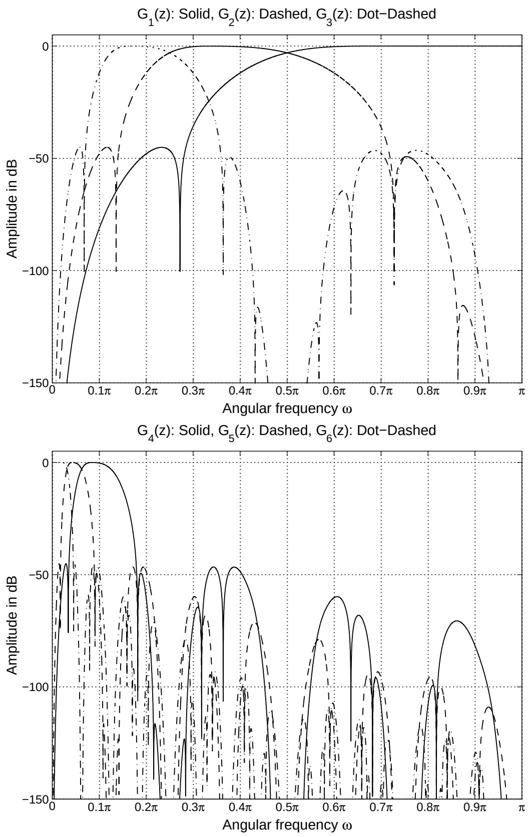
Impulse Responses for $H_0(z)$, $H_1(z)$, $F_0(z)$, and $F_1(z)$. $F_0(z)$ [$F_1(z)$] has been normalized such that it achieves the value of unity at $\omega = 0$ [$\omega = \pi$].



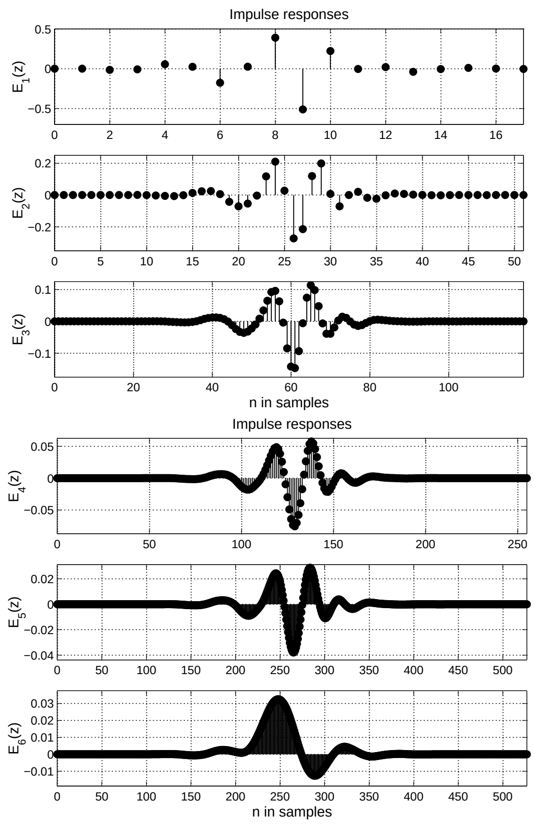
Amplitude Responses for the Resulting Six Analysis Transfer Functions $E_k(z)$ in Figures 3 and 5



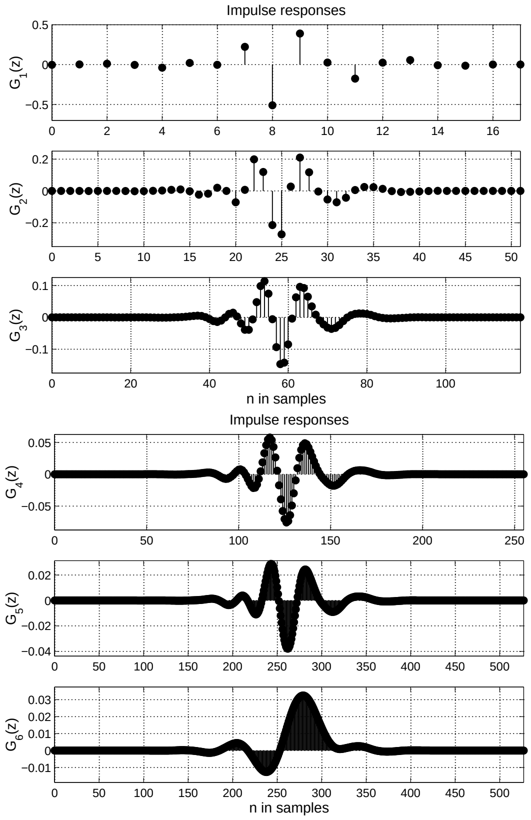
Amplitude Responses for the Resulting Six Synthesis Transfer Functions $G_k(z)$ in Figures 4 and 5



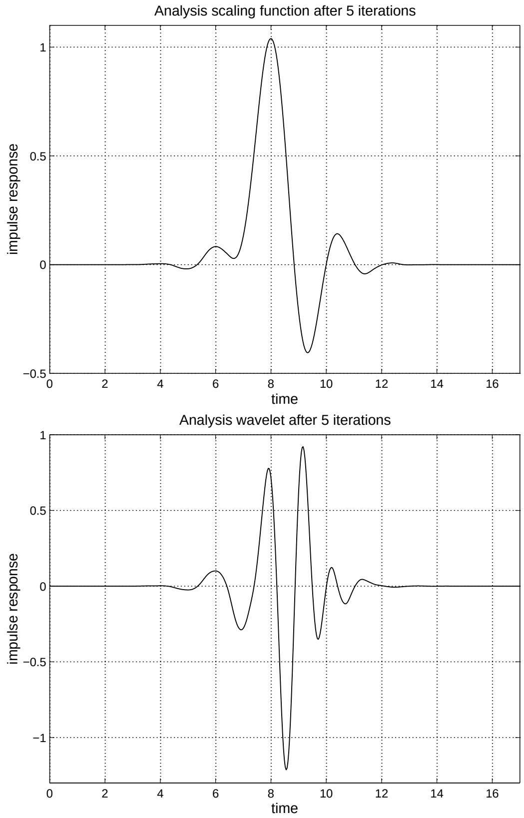
Impulse Responses for the Resulting Six Analysis Transfer Functions $E_k(z)$ in Figures 3 and 5



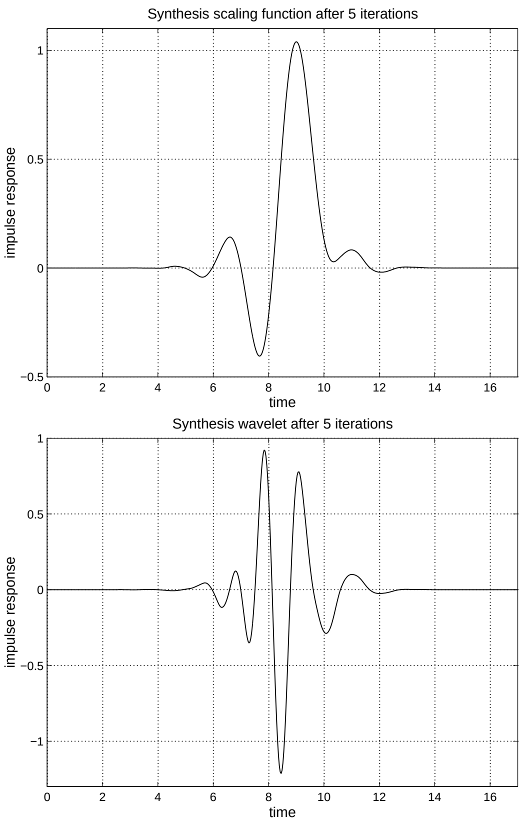
Impulse Responses for the Resulting Six Synthesis Transfer Functions $G_k(z)$ in Figures 4 and 5



Analysis Scaling Function and Wavelet after 5 Iterations



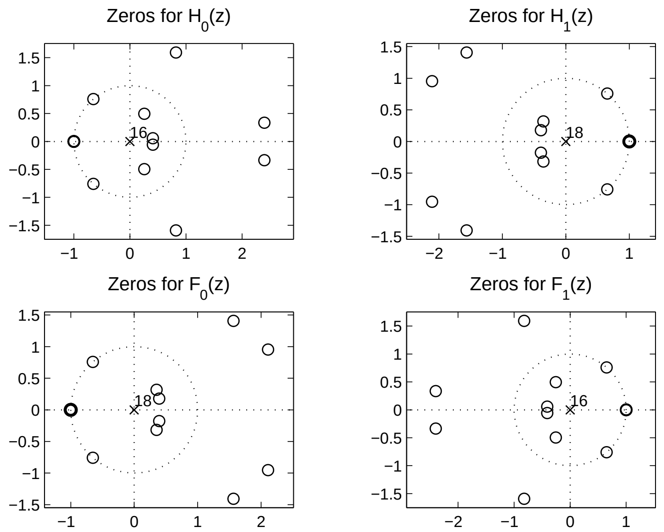
Synthesis Scaling Function and Wavelet after 5 Iterations



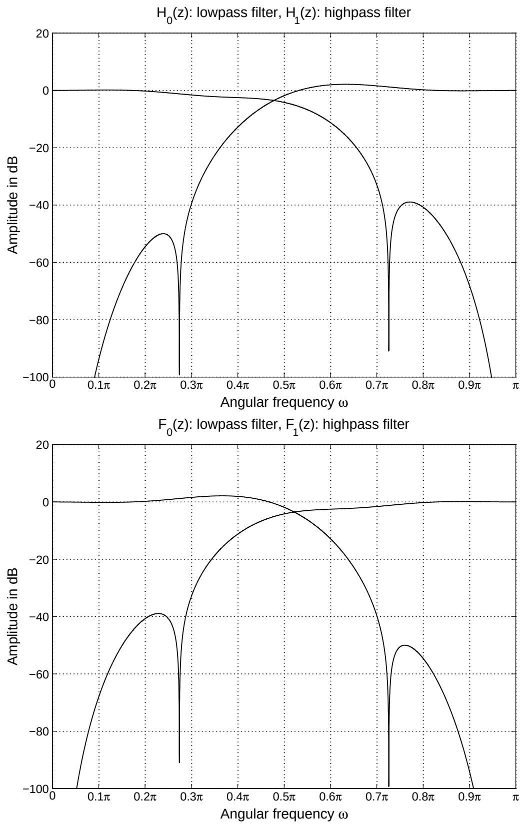
Generalized Biorthogonal Wavelet Banks

- In this case, both $H_0(z)$ and $H_1(-z)$ may have their own zero pairs on the unit circle.
- The following set of pages show responses in the biorthogonal case considered above with the exception that now two zeros of both $H_0(z)$ and $H_1(-z)$ have been moved from $z = -1$ to a zero pair at $z = \exp(\pm j0.726\pi)$.
- The resulting $H_0(z)$ and $H_1(-z)$ have now 6 and 8 zeros at $z = -1$, respectively.
- When comparing the filter responses to the earlier case, it is again observed that the frequency selectivities are increased.

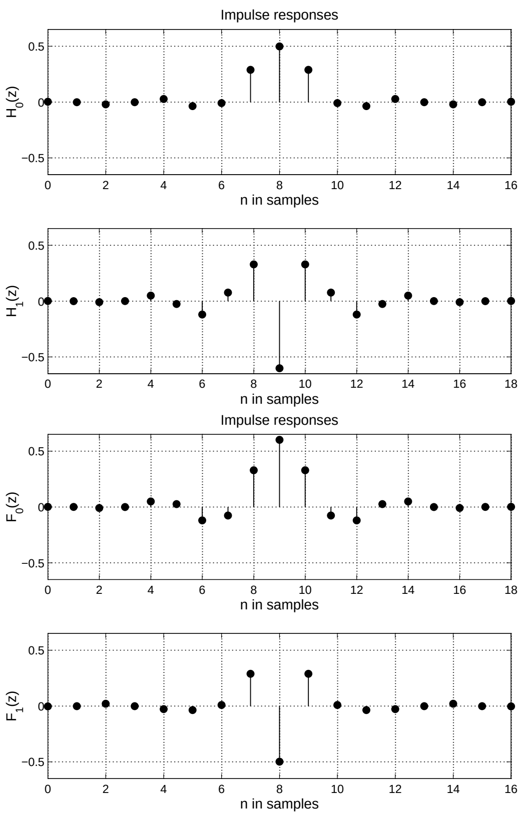
Zero Plots for $H_0(z)$, $H_1(z)$, $F_0(z)$, and $F_1(z)$



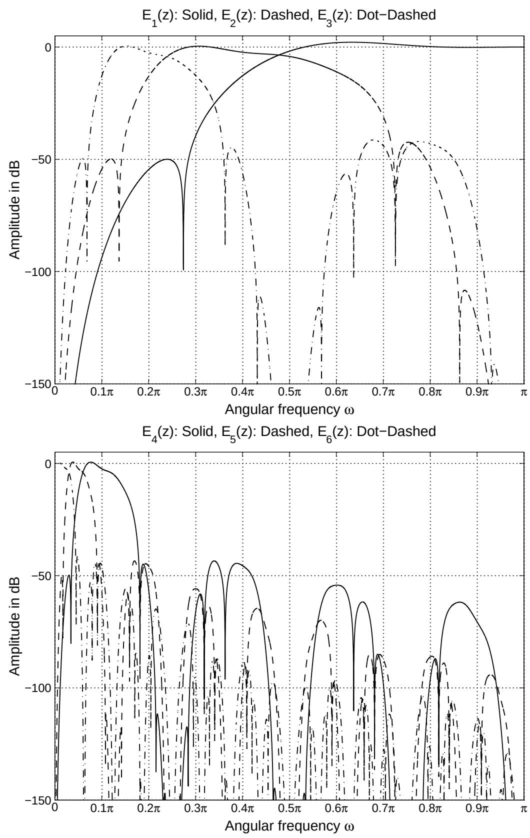
Amplitude Responses for $H_0(z)$, $H_1(z)$, $F_0(z)$, and $F_1(z)$. $F_0(z)$ [$F_1(z)$] has been normalized such that it achieves the value of unity at $\omega = 0$ [$\omega = \pi$].



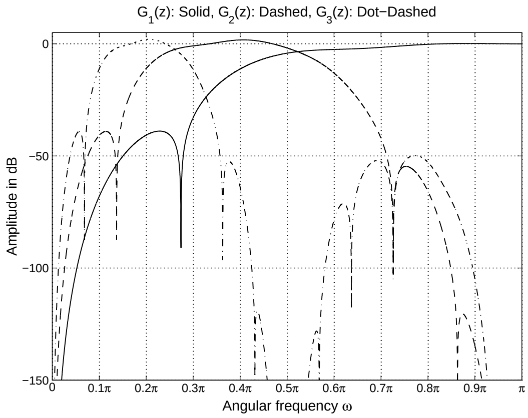
Impulse Responses for $H_0(z)$, $H_1(z)$, $F_0(z)$, and $F_1(z)$. $F_0(z)$ [$F_1(z)$] has been normalized such that it achieves the value of unity at $\omega = 0$ [$\omega = \pi$].

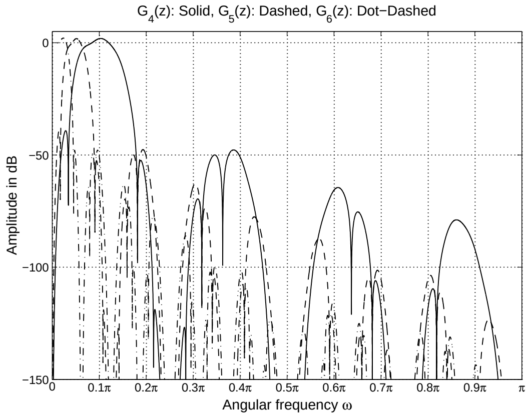


Amplitude Responses for the Resulting Six Analysis Transfer Functions $E_k(z)$ in Figures 3 and 5

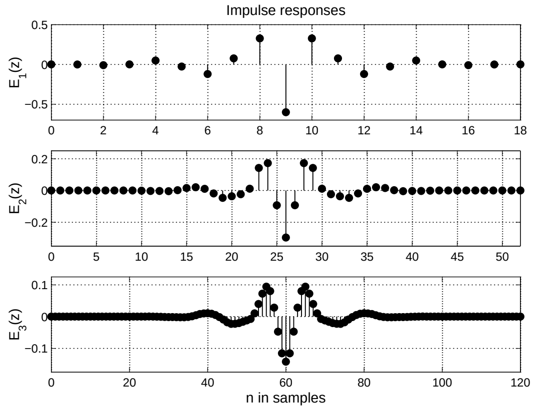


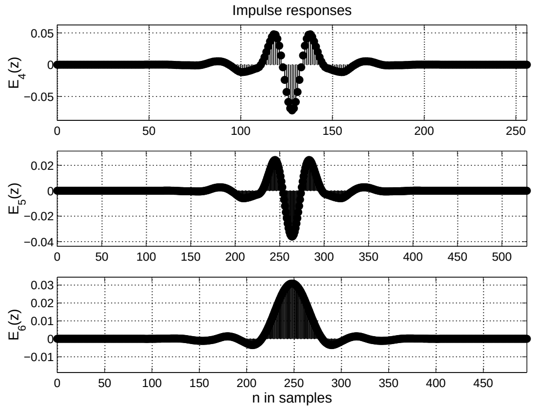
Amplitude Responses for the Resulting Six Synthesis Transfer Functions $G_k(z)$ in Figures 4 and 5



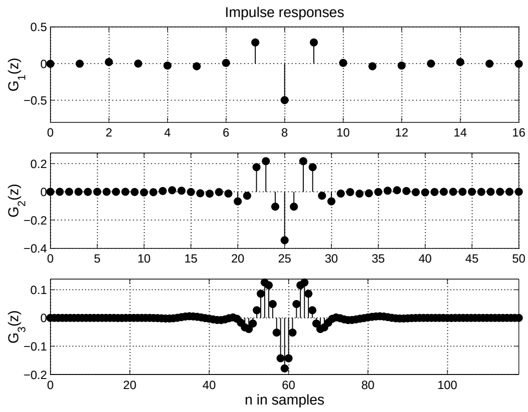


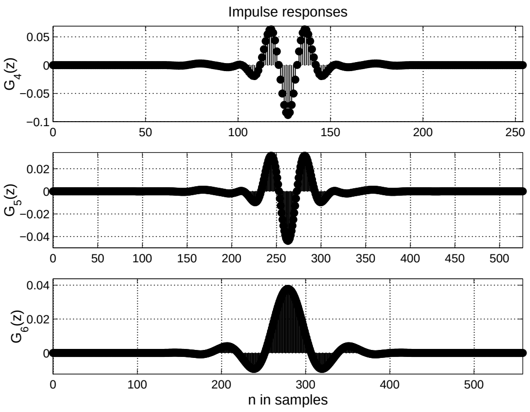
Impulse Responses for Resulting Six Analysis Transfer Functions $E_k(z)$ in Figures 3 and 5



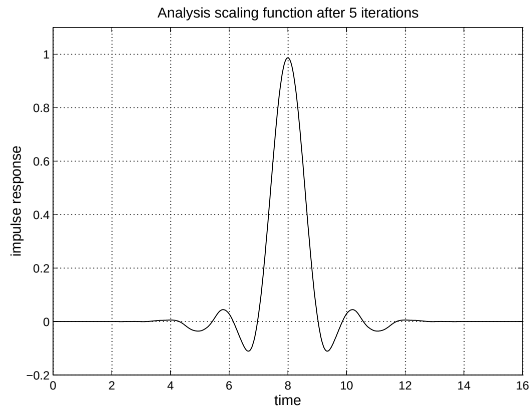


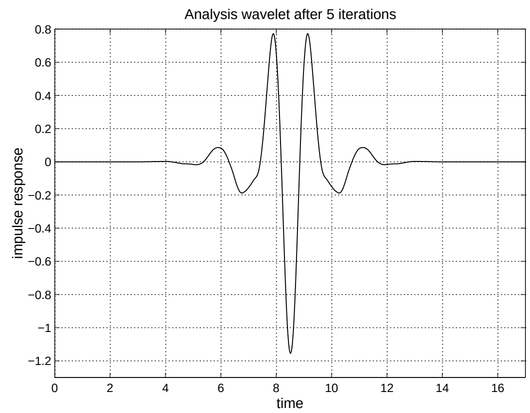
Impulse Responses for the Resulting Six Synthesis Transfer Functions $G_k(z)$ in Figures 4 and 5



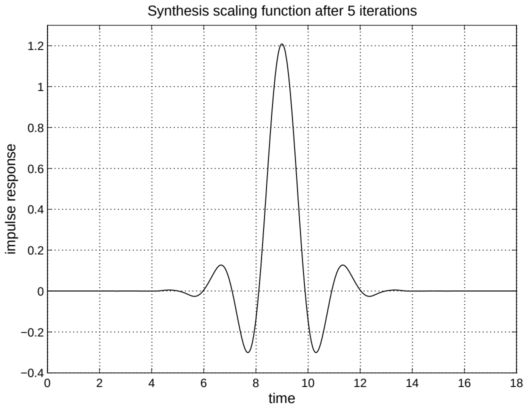


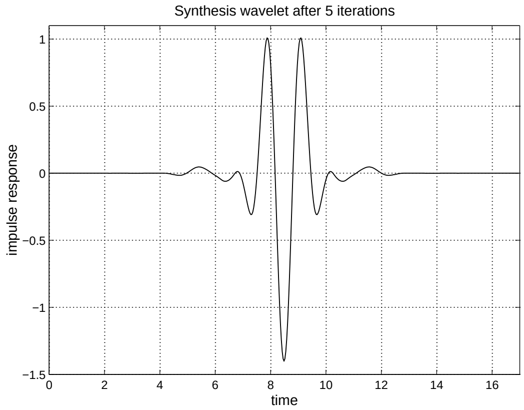
Analysis Scaling Function and Wavelet after 5 Iterations





Synthesis Scaling Function and Wavelet after 5 Iterations





How to Measure the "Goodness" of a Wavelet Bank

- There exist several factors being crucial for the applicability of a wavelet bank. These include:

I. Lengths of the impulse responses.

- It is desired to keep the orders of $H_0(z)$ and $H_1(z)$ as small as possible to still provide the required performance for the overall wavelet bank.

II. Phase linearity.

- In many applications, it is desired that $H_0(z)$ and $H_1(z)$ are linear-phase or approximately linear-phase FIR filters.

III. Number of vanishing moments.

- This number is N if $H_1(z)$ has N zeros at $z = -1$.
- Polynomial signals of order less than or equal to $N - 1$ are filtered out by the $H_1(z)$'s in the wavelet bank and passed trough the $H_0(z)$'s.

IV. Regularity or smoothness.

- In order to give the definitions, we form the following transfer functions:

$$A^{(i)}(z) = 2^i H_0(z) H_0(z^2) \cdots H_0\left(z^{2^{i-2}}\right) H_0\left(z^{2^{i-1}}\right),$$

$$B^{(i)}(z) = 2^i H_0(z) H_0(z^2) \cdots H_0\left(z^{2^{i-2}}\right) H_1\left(z^{2^{i-1}}\right),$$

$$C^{(i)}(z) = 2^i F_0(z) F_0(z^2) \cdots F_0\left(z^{2^{i-2}}\right) F_0\left(z^{2^{i-1}}\right),$$

and

$$D^{(i)}(z) = 2^i F_0(z) F_0(z^2) \cdots F_0\left(z^{2^{i-2}}\right) F_1\left(z^{2^{i-1}}\right).$$

- Here, it is assumed that $H_0(z)$ and $F_0(z)$ [$H_1(z)$ and $F_1(z)$] have been normalized to achieve the value of unity at $z = -1$ [$z = 1$].
- From the above transfer functions we can generate the following continuous-time analysis wavelet and scaling function:

$$\psi(t) = \lim_{i\to\infty} a^{(i)}(t(2^i - 1))$$

$$\phi(t) = \lim_{i\to\infty} a^{(i)}(t(2^i - 1)) ,$$

where $a^{(i)}(n)$ and $b^{(i)}(n)$ are the impulse-response coefficients of $A^{(i)}(z)$ and $B^{(i)}(z)$, respectively.

- Similarly, the synthesis wavelet and scaling function can be generated as follows:

$$\hat{\psi}(t) = \lim_{i\to\infty} c^{(i)}(t(2^i - 1))$$

$$\hat{\phi}(t) = \lim_{i\to\infty} d^{(i)}(t(2^i - 1)) ,$$

where $c^{(i)}(n)$ and $d^{(i)}(n)$ are the impulse-response coefficients of $C^{(i)}(z)$ and $D^{(i)}(z)$, respectively.

- Regularity is the number of continuous derivatives of the above functions.
- The above impulse responses have been formed in such a way that as i increases $a^{(i)}(t(2^i - 1))$ is all the time nonzero in the same interval and we are getting more points for $\psi(t)$ in this interval without changing its shape. The same is true for the other impulse responses and functions.

- In connection of generating five-level wavelet banks previously, we gave the analysis and synthesis wavelets and scaling functions after five iterations ($i = 5$).
- In the most general case, the regularity of the above functions, denoted by s, is not an integer. Let m be an integer such that $m < s < m + 1$. Then, the function $\psi(t)$ has a regularity of s if the mth derivative of $\psi(t)$ resembles $|t - t_0|^{s-m}$ at each point $t = t_0$ in the interval where $\psi(t)$ is nonzero.

V. Frequency selectivity

- Typically, the selectivity of the filters in the wavelet bank is very poor. This is because wavelet banks are normally used more or less in preserving the waveform of a one- or two-dimensional signal.
- Images are typical cases. We are looking at images and our eyes are the referees of the quality. In audio applications, in turn, the frequency-domain behavior of the filter bank is of great importance as your ears are the referees of the quality.

VI. Number of levels

- This depends on the application. Typically three to five levels is a good selection.

- The above-mentioned measures are very conflicting.
- The selection of a proper wavelet bank depends strongly on the application.
- Hopefully, the MATLAB Wavelet Toolbox manual helps us.

- It should be also pointed out that in the case of biorthogonal wavelets the analysis and synthesis parts may be very different in order to achieve a satisfactory overall performance.
- For biorthogonal wavelets, the selectivity of the analysis part and the smoothness of the synthesis part are of great importance.
- There are also available very useful pseudo wavelets. If you are interested in them, contact the lecturer, e-mail: ts@cs.tut.fi.
- There are also several MATLAB files (all the designs and plots lecture notes have been generated by own files).

Some Comments

- In the above, we considered only some starting-point two-channel filter banks for generating multilevel wavelet banks.
- Furthermore, we concentrated only on FIR wavelet banks, although there are also IIR wavelet banks.
- We generated our banks by further processing the lowpass filtered and decimated signal.
- In the most general case, some of the highpass filtered and decimated signals are processed by the basic building-block two-channel filter bank, yielding the so-called wavelet packet.
- The extreme case is the tree-structured filter bank generated by using the same building-block two-channel filter bank.

Part V.F: Octave Filter Banks

- The multilevel wavelet banks generated in Part V.E are examples of octave filter banks, although their frequency selectivity is very poor.
- However, we can mimic the same procedure with the exception that now also IIR two-channel filter banks are under consideration.
- Hence, given a two-channel filter bank with analysis filter transfer functions $H_0(z)$ and $H_1(z)$ and the synthesis filter transfer functions $F_0(z)$ and $F_1(z)$, the analysis and synthesis filter can be generated in the five-level case as shown in Figures 1 and 2.
- Figure 3 shows the overall filter bank.
- If for the building-block two-channel filter bank , the input-output transfer function is $T(z)$ an allpass filter, like in the case of two-channel IIR filters built using half-band IIR filters (Part V.B), then for the overall system, the input-output transfer function becomes in the case of Figure 3

$$T_{ove}(z) = \prod_{l=0}^{5} T\left(z^{2l}\right)$$

if $C_6(z) = C_5(z) = 1$, $C_4(z) = T(z)$, $C_3(z) = T(z)T(z^2)$, $C_2(z) = T(z)T(z^2)T(z^4)$, and $C_1(z) = \Pi_{l=0}^{4} T\left(z^{2l}\right)$.

- In the case of a perfect-reconstruction two-channel filter bank with $T(z) = z^{-K}$, $T_{ove}(z) = z^{-31K}$, $C_6(z) = C_5(z) = 1$, $C_4(z) = z^{-K}$, $C_3(z) = z^{-3K}$, $C_2(z) = z^{-7K}$, and $C_1(z) = z^{-15K}$.

Figure 1. Five-Level Octave Filter Bank: Analysis Part.

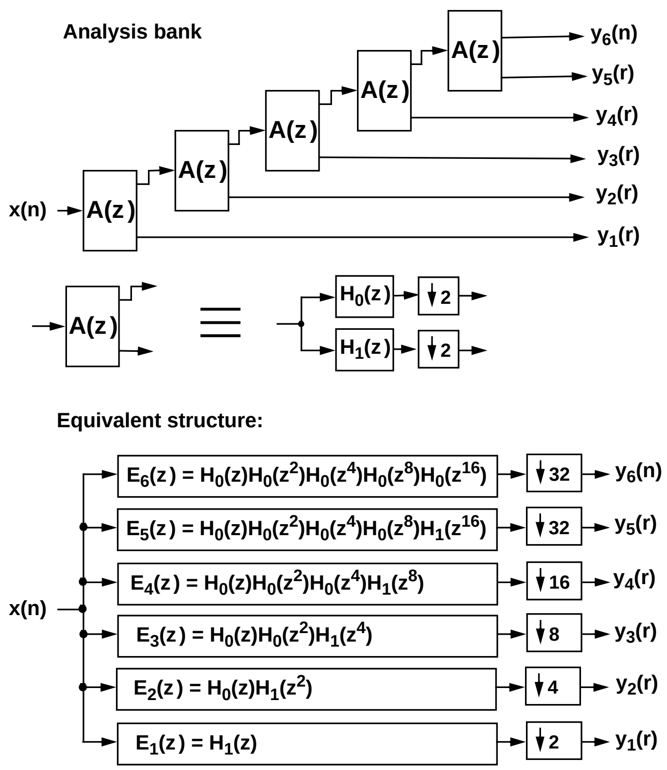


Figure 2. Five-Level Octave Filter Bank: Synthesis Part.

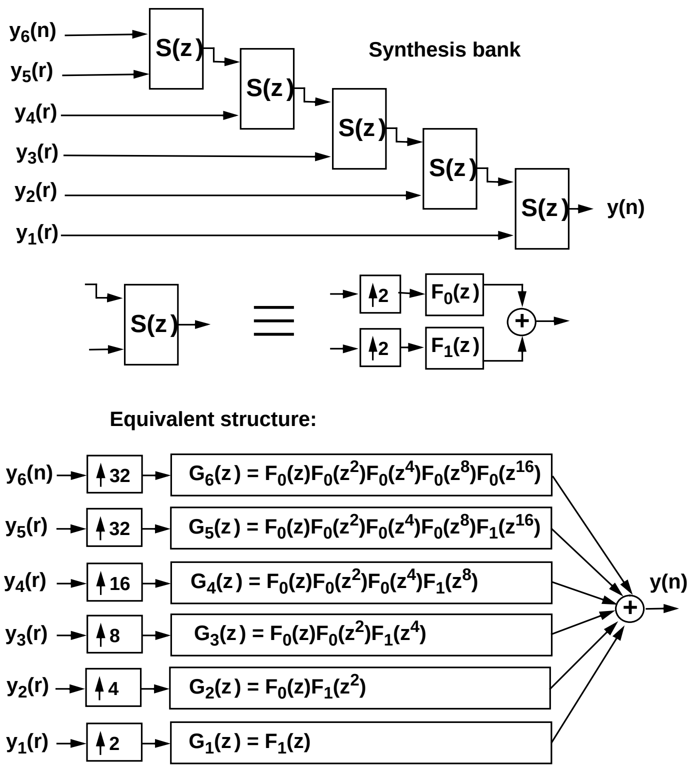
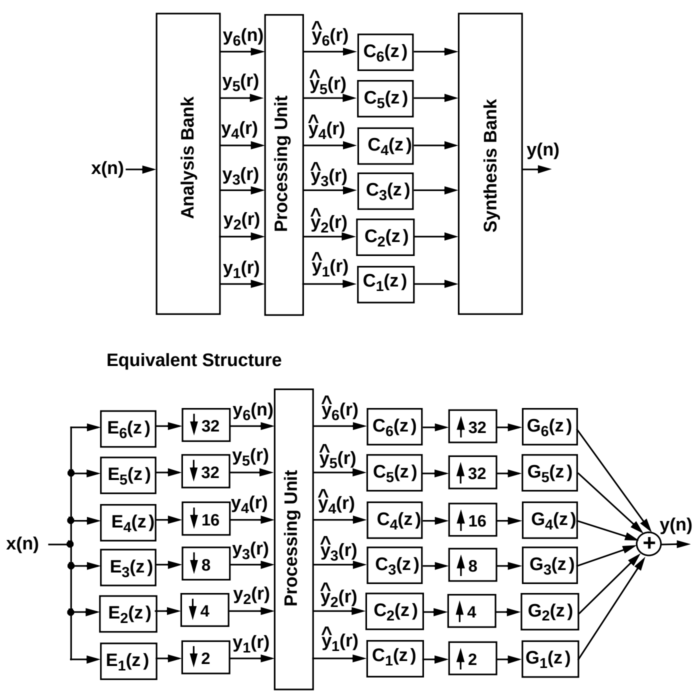


Figure 3. Overall Five-Level Octave Filter Bank.



Example 1: Five-Level FIR Octave Filter Bank

- It is desired to build a five-level FIR filter bank using the minimax orthogonal perfect-reconstruction two-channel filter bank considered on Pages 51 and 52 in this part of lecture notes.
- Figure 4 shows the amplitude responses for the $E_k(z)$'s and $G_k(z)$'s (see Figures 1,2,and 3).
- The amplitude responses for the $G_k(z)$'s have been normalized such that their maximum value is equal to unity .The actual responses are obtained by multiplying the amplitude responses by the corresponding interpolation factor.
- Figures 5 and 6 show the corresponding impulse responses for the $E_k(z)$'s and $G_k(z)$'s. In this case, no normalization has been performed.

Figure 4. Amplitude Responses for the Analysis and Synthesis Filters in an Example Five-Level Octave FIR Filter Bank.

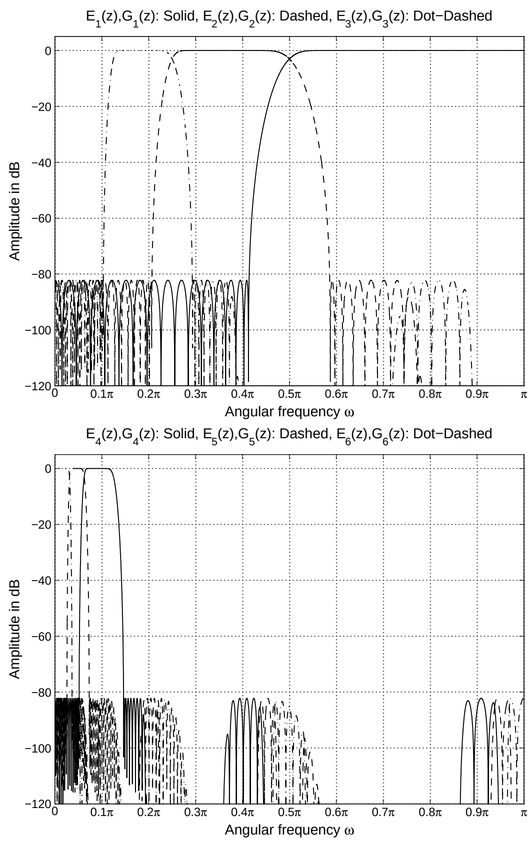


Figure 5. Impulse Responses for the Analysis Filters in an Example Five-Level Octave FIR Filter Bank.

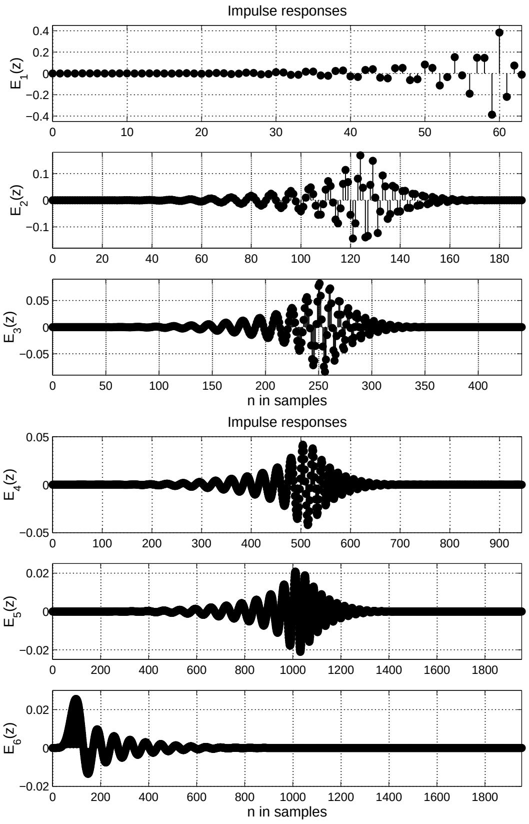
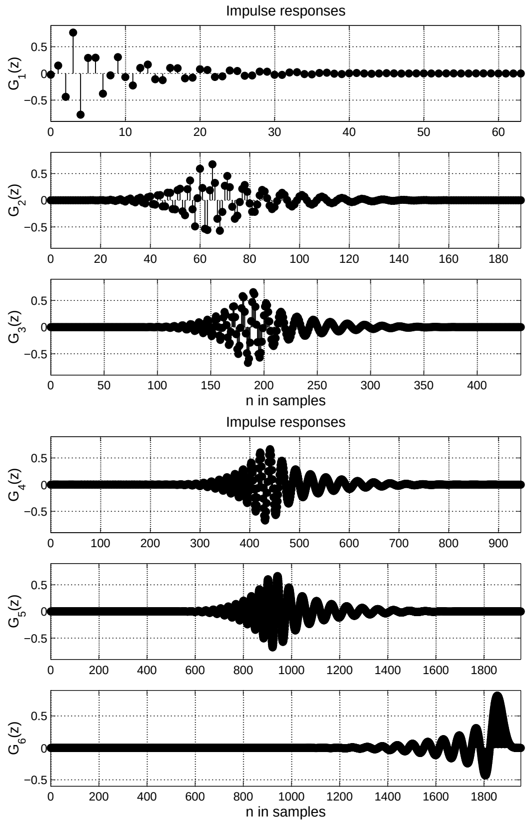


Figure 6. Impulse Responses for the Synthesis Filters in an Example Five-Level Octave FIR Filter Bank.



Example 2: Five-Level IIR Octave Filter Bank

- It is desired to build a five-level IIR filter bank using the two-channel IIR filter bank considered on Pages 91–94 in this part of lecture notes.
- Figure 7 shows the amplitude responses for the $E_k(z)$'s and $G_k(z)$'s (see Figures 1,2,and 3).
- The amplitude responses for the $G_k(z)$'s have again been normalized such that their maximum value is equal to unity.
- In this case, the amplitude response for the input-output transfer function is equal to unity at all frequencies, but there is a phase distortion.

Figure 7. Amplitude Responses for the Analysis and Synthesis Filters in an Example Five-Level Octave IIR Filter Bank.