80509 LINEAR DIGITAL FILTERING I

PART II: Introductory Filtering Examples

Before concentarting on actual filter synthesis, this part tries to make the reader familar with

1) The needs for filtering.
2) The difference between IIR and linear-phase FIR filters.
3) The roles of poles and zeros in contributing to various filter responses: amplitude, phase, phase delay, group delay, and impulse responses.
4) Characteristics of all-pall transfer functions.

- Questions in the final examination are not coming directly from this part.

INTRODUCTORY FILTERING EXAMPLES

- Purpose of the examples:
  - What is filtering: artifical example
  - Difference between classical infinite impulse response (IIR) filters and linear phase finite impulse response (FIR) filters
  - Roles of the poles and zeros in contributing to the responses of the filter: amplitude, phase, phase delay, group delay, and impulse responses
  - Characteristics of allpass filters
  - Matlab-files for generating the following figures can be found in SUN's in the directory ts/matlab/dsp.
  - The files are firfilt1.m, iirfilt1.m, firfilt2.m, iirfilt2.m, contrir.m, contric.m, and contrial.m.

EXAMPLE ON FILTERING A PERIODIC SIGNAL

- $F_s = 1000$ Hz, $T = 1/F_s = 1$ ms.
- Desired signal in the region $0 \leq t \leq 1$ s, that is, 1000 samples:

$$x_d[n] = \sum_{k=1}^{6} A_k \sin(n(k\omega_0) + \phi_k),$$

where

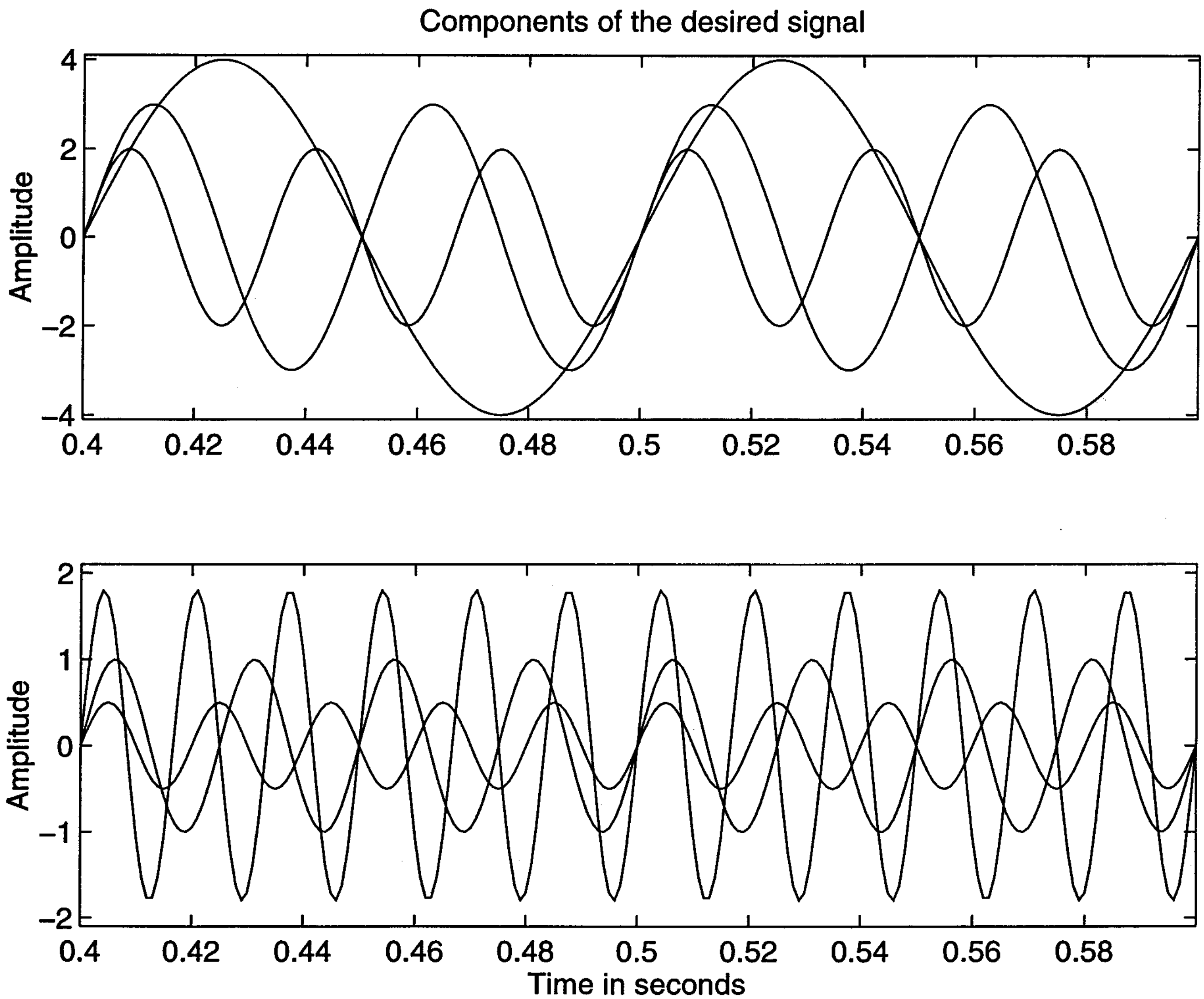
$$\omega_0 = \pi/50,$$

$$A_1 = 4,\ \ A_2 = 3,\ \ A_3 = 2,\ \ A_4 = 1,\ \ A_5 = 0.5,\ \ A_6 = 1.8$$

$$\phi_k = 0,\ \ k = 1, 2, \cdots, 6.$$

- Since $\omega = 2\pi f/F_s$, the oscillation frequencies are 10, 20, 30, 40, 50, and 60 Hz, respectively.
- The components of the desired signal are shown on page 3.

COMPONENTS OF THE DESIRED SIGNAL IN THE RANGE 0.4 s $\leq t \leq$ 0.6 s



NOISY SIGNAL

- The desired signal is contaminated by two noise components.
- The first component contains two sinusoidal signals and is given by

$$x_{noise1}[n] = 10\sin(n(\pi/2) + \pi/3.2) + 15\sin(n(1.2\pi) + \pi/1.6).$$

- For $F_s = 1000$ Hz, the oscillation frequencies are 250 Hz and 600 Hz. 600 Hz ??, this is aliased to 400 Hz.
- The second noise component $x_{noise2}[n]$ is white Gaussian noise with variance 10000. This is filtered with an elliptic highpass filter providing at least a 100-dB attenuation in the band $f \leq 90$ Hz.
- There is no noise in the signal band of interest $0 \leq f \leq 60$ Hz.

- The overall signal is thus

$$x_{ove}[n] = x_d[n] + x_{noise}[n],$$

where

$$x_{noise}[n] = x_{noise1}[n] + x_{noise2}[n].$$

SIGNAL RESPONSES

- Page 8 gives the desired as well as the noisy signal both in the time and frequency domains.

- DFT (Discrete Fourier Transform) is used for the frequency-domain representation:

- z-transform of a sequence $x(n)$, $0 \leq n \leq N-1$:

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}.$$

- Fourier transform or spectrum:

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n) e^{-jn\omega}.$$

- DFT:

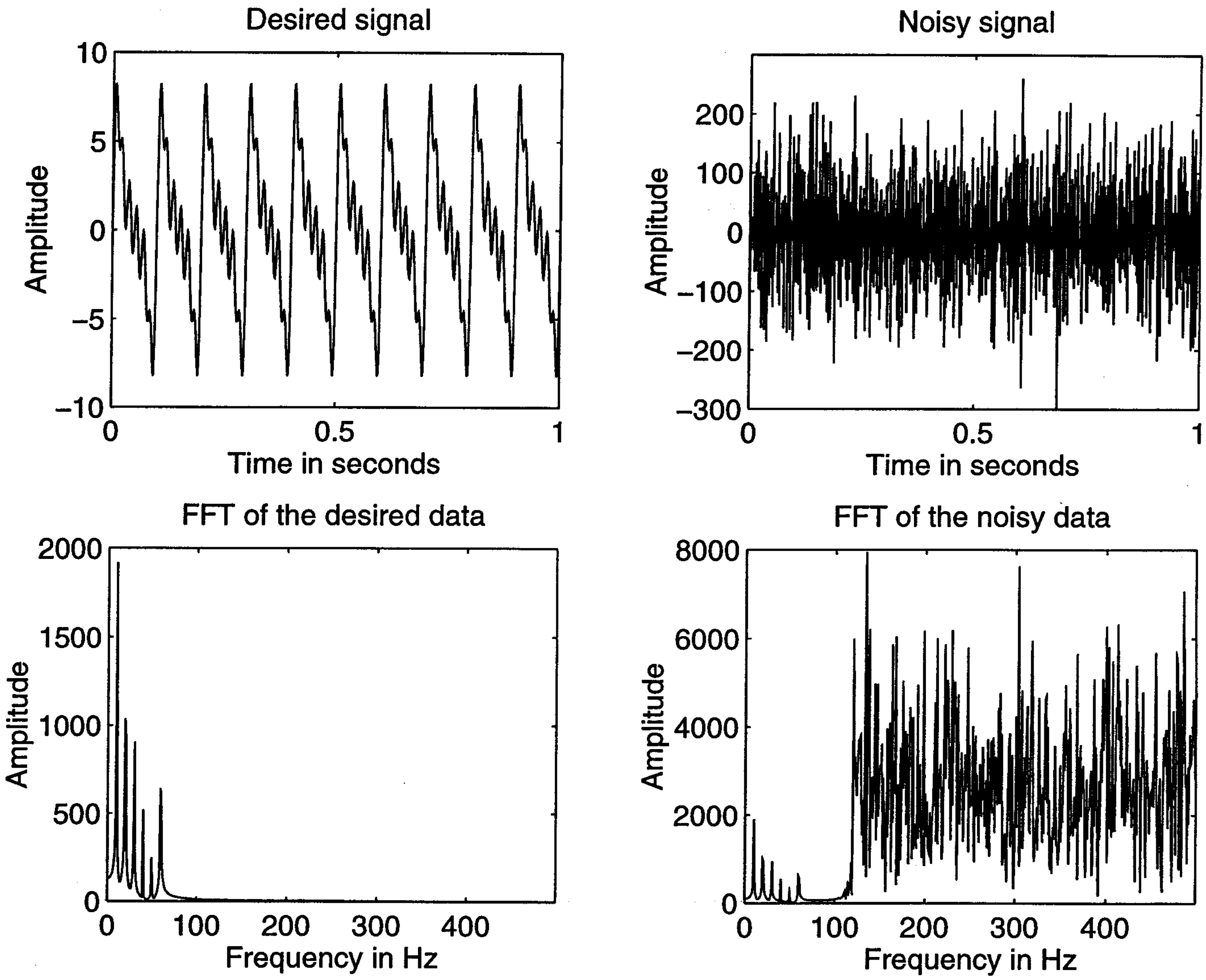
$$X_{DFT}(k) = X(e^{j2\pi k/N}), k = 0, 1, \cdots, N-1.$$

- In the figure of page 8, $N = 1024$ and $|X_{DFT}(k)| = |X(e^{j2\pi k/N})|$ are given for $k = 0, 1, \cdots, 511$.

- The corresponding real frequencies are $f_k = k(F_s/1024)$ $k = 0, 1, \cdots, 511$.

- FFT stands for Fast Fourier Transform, that is, a fast way of evaluating DFT.

SIGNAL RESPONSES IN THE TIME AND FREQUENCY DOMAINS



TWO FILTERING APPROACHES FOR REMOVING THE NOISE

- In both cases, it is desired to preserve the signal in the frequency range 0 Hz $\leq f \leq$ 60 Hz and attenuate the signal in the range 90 Hz $\leq f \leq$ 500 Hz.

- In terms of the angular frequency $\omega = 2\pi f/F_s$, the corresponding ranges are $[0,\ 0.12\pi]$ and $[0.18\pi,\ \pi]$.

I: Elliptic infinite-impulse-response (IIR) filter with the following difference equation and transfer function:

$$y[n] = \sum_{k=1}^{7} a_k y[n-k] + \sum_{k=0}^{7} b_k x[n-k]$$

and

$$H(z) = \sum_{k=0}^{7} b_k z^{-k} / (1 - \sum_{k=1}^{7} a_k z^{-k}),$$

where

$$a_1 = 6.0943, \quad a_2 = -16.1890, \quad a_3 = 24.2687,$$

$$a_4 = -22.1539, \quad a_5 = 12.3073, \quad a_6 = -3.8509,$$

$$a_7 = 0.5234$$

$$b_0 = b_7 = 0.0013947, \quad b_1 = b_6 = -0.005088$$

$$b_2 = b_5 = 0.0076294, \quad b_3 = b_4 = -.0038490.$$

II: Linear-phase finite-impulse-response (FIR) filter with the following difference equation and transfer function:

$$y[n] = \sum_{k=0}^{94} h[k]x[n-k]$$

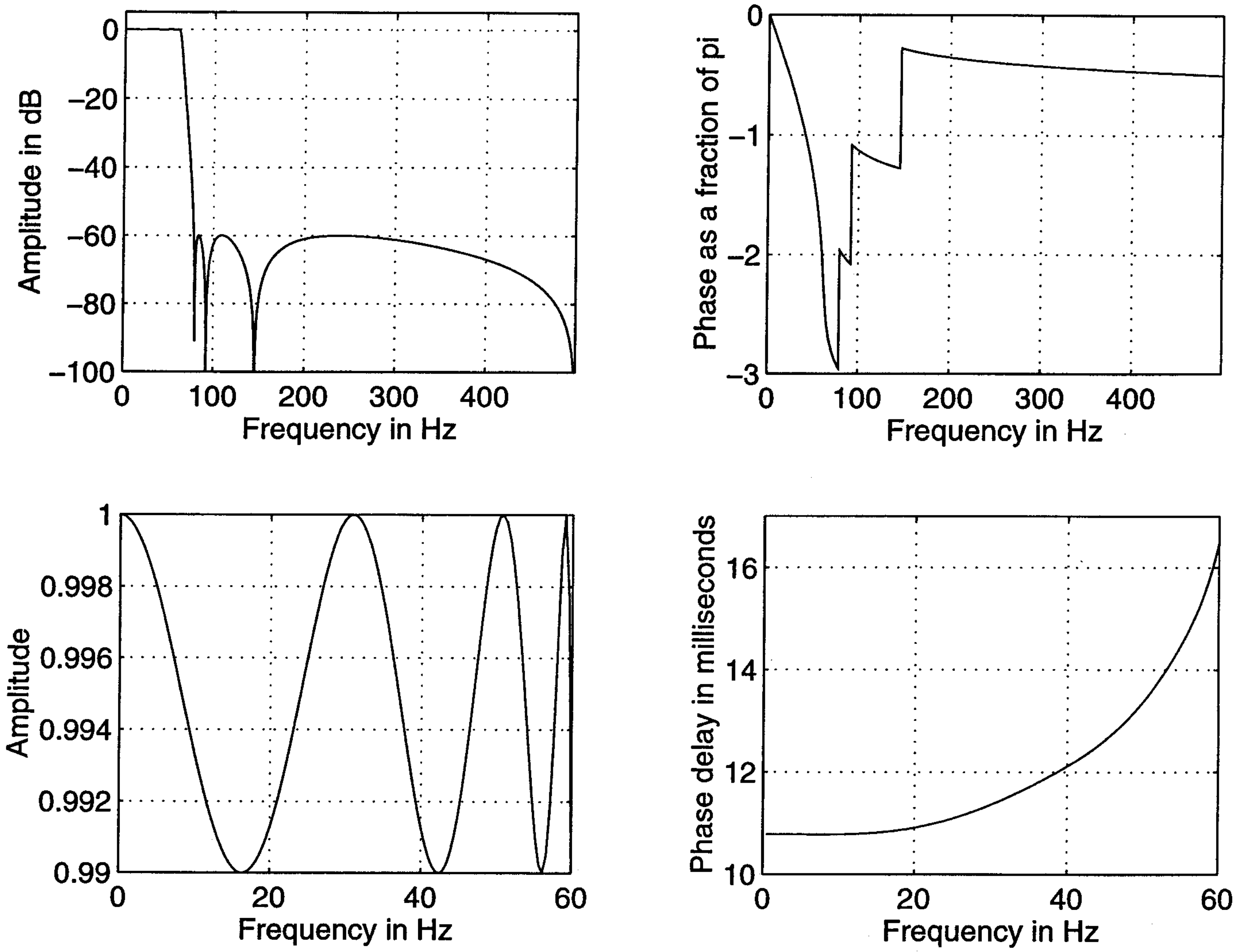
and

$$H(z) = \sum_{k=0}^{94} h(k)z^{-k},$$

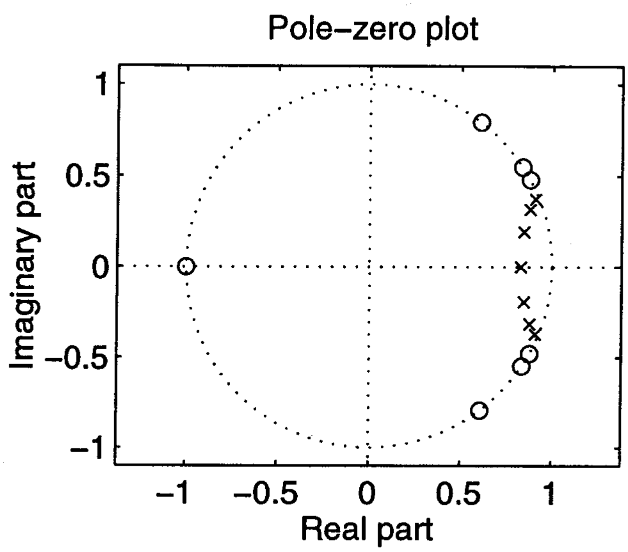
where the $h[k]$'s are the impulse response coefficients of the filter satisfying $h[94-n] = h[n]$.

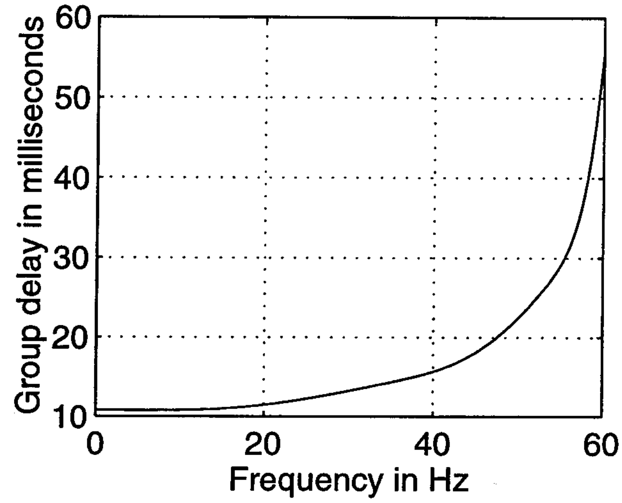
- The following four pages give several characteristics for these filters.

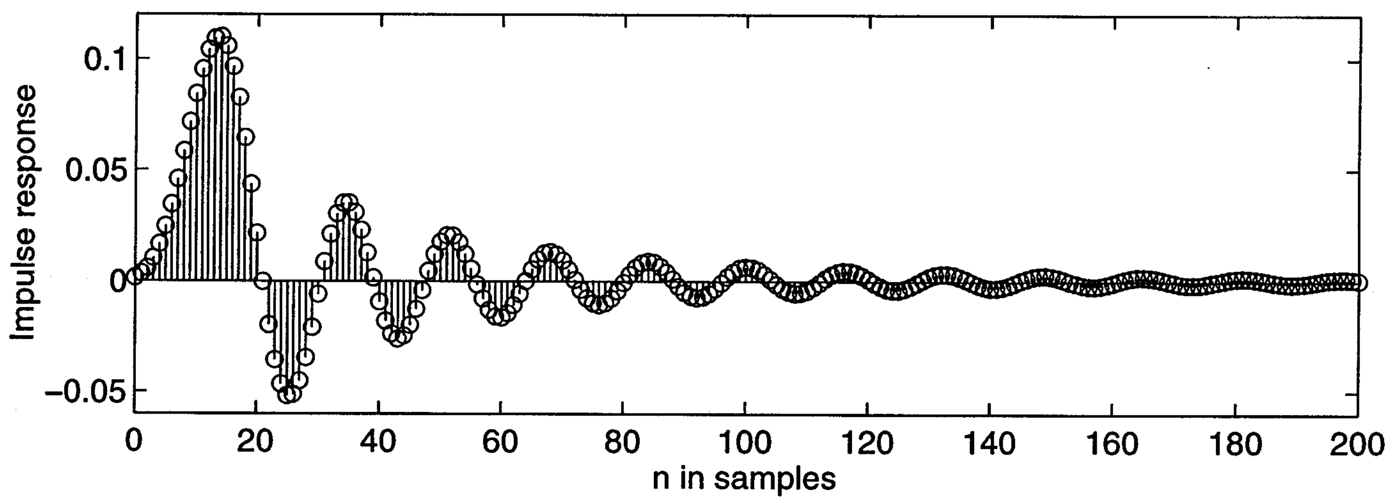
CHARACTERISTICS FOR THE IIR FILTER



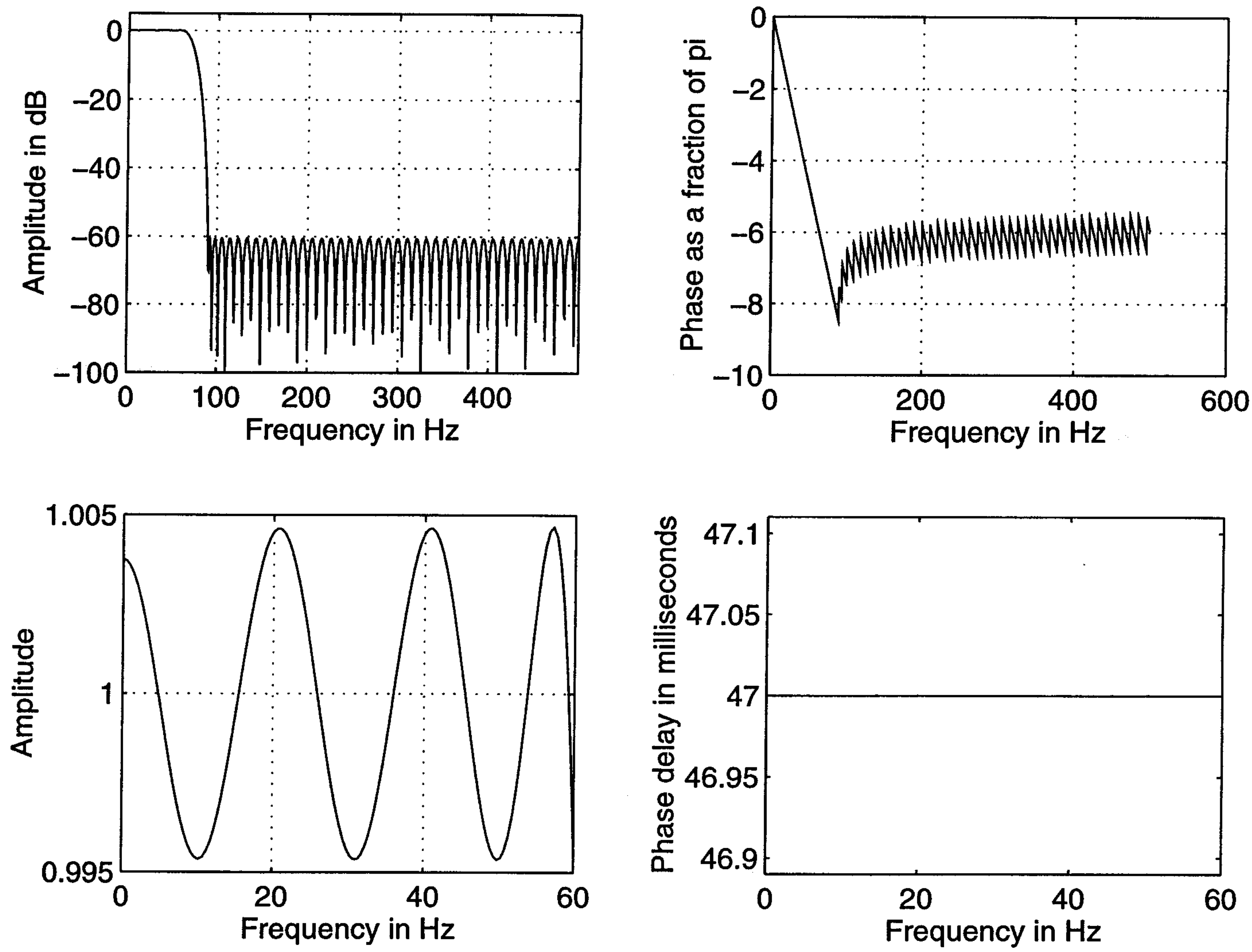
CHARACTERISTICS FOR THE IIR FILTER



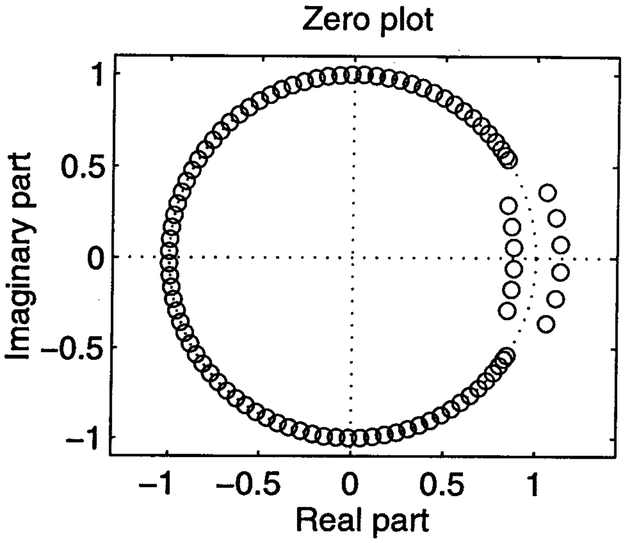


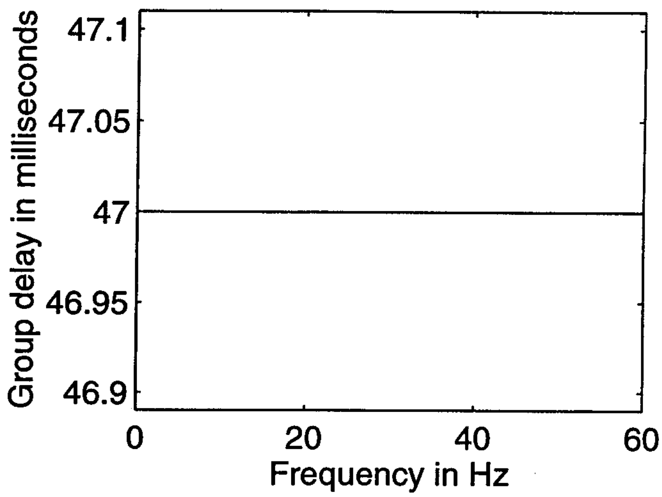


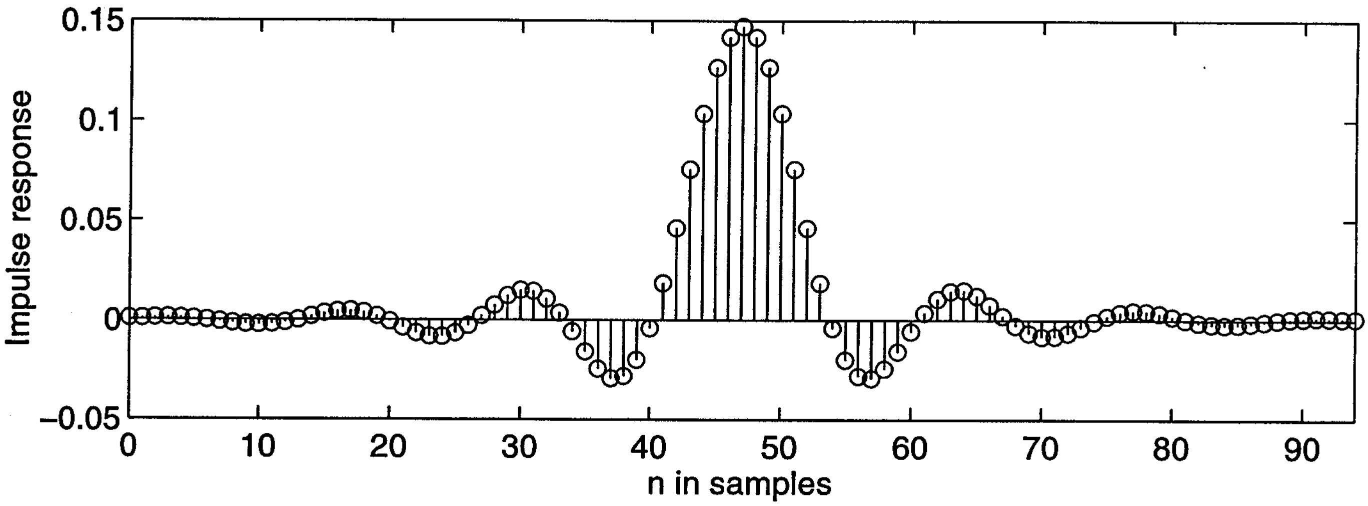
CHARACTERISTICS FOR THE FIR FILTER



CHARACTERISTICS FOR THE FIR FILTER



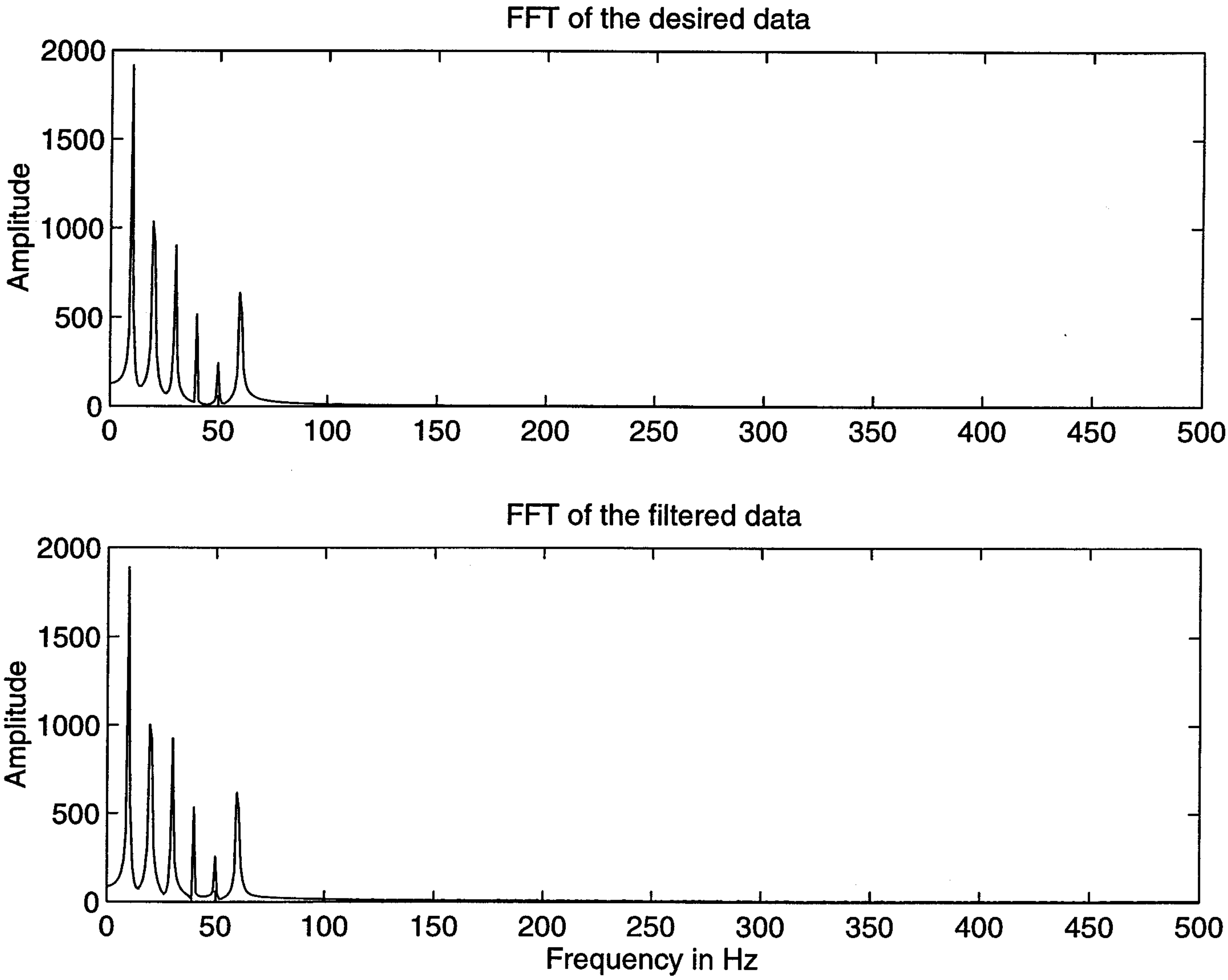




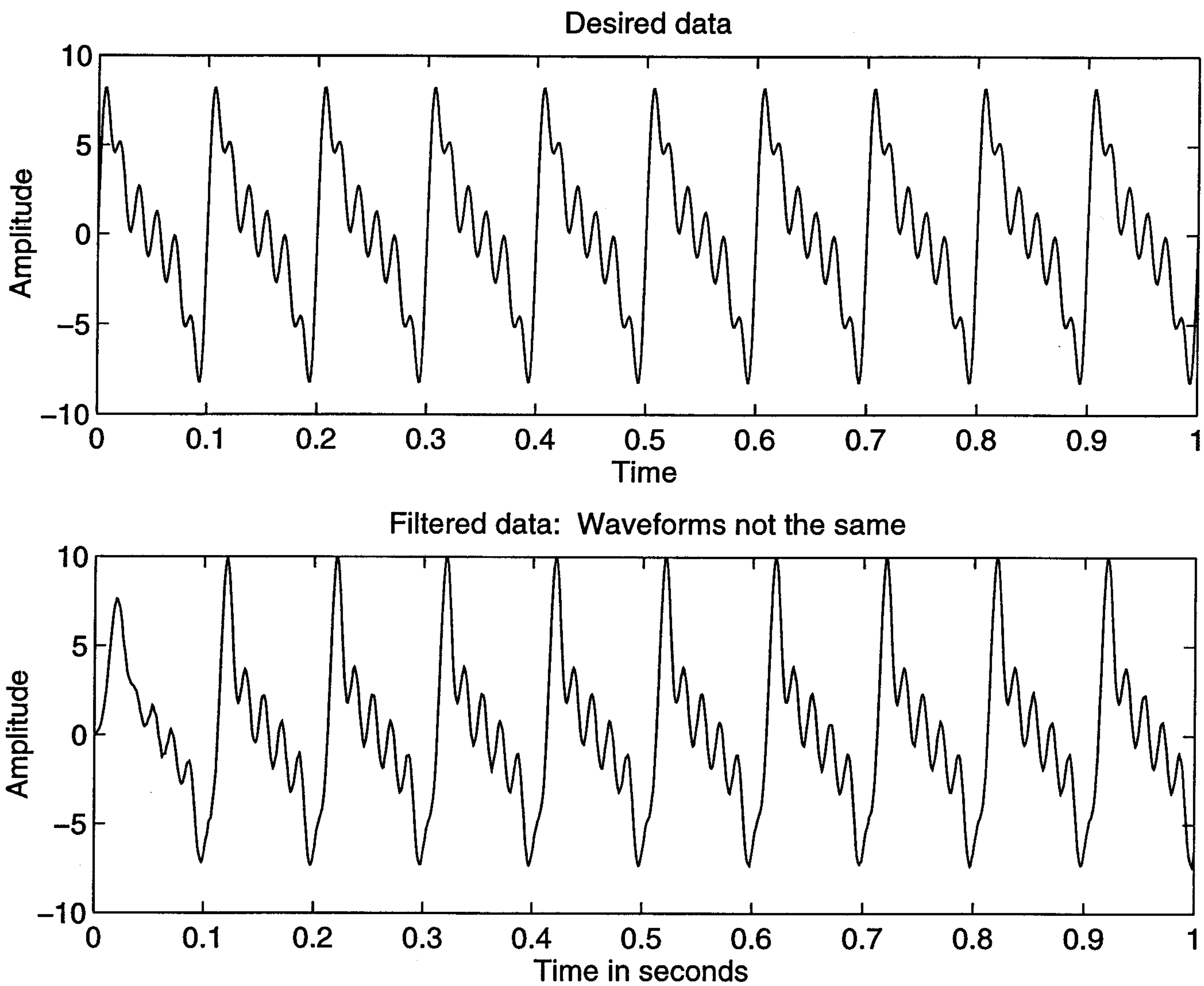
FILTERING RESULTS

- The following four pages compare the original desired signal with the filtered signal.
- After that it shown, how the noise is attenuated very well by both filters.

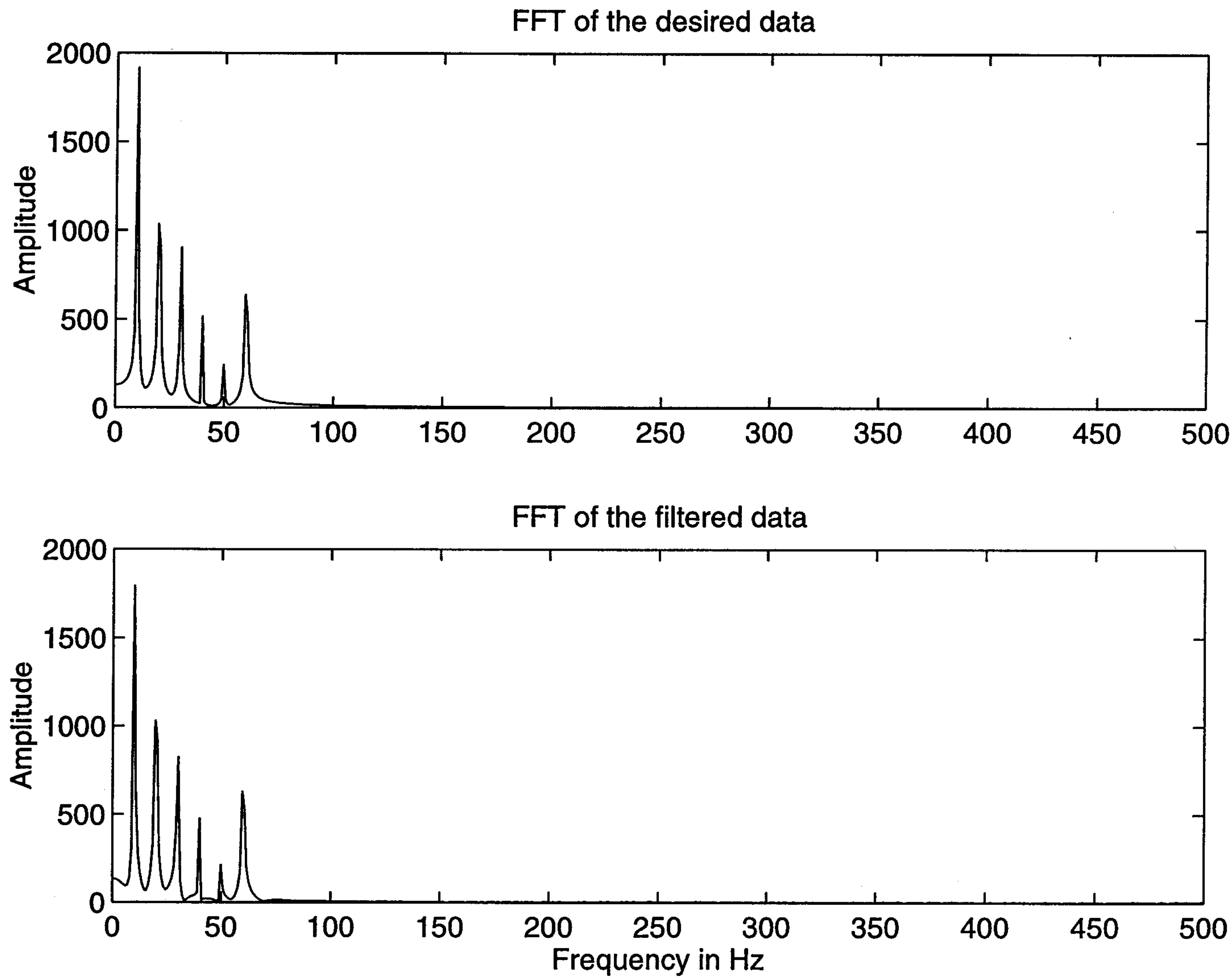
COMPARISON BETWEEN THE DESIRED SIGNAL AND THE NOISY SIGNAL FILTERED WITH THE AID OF AN IIR FILTER



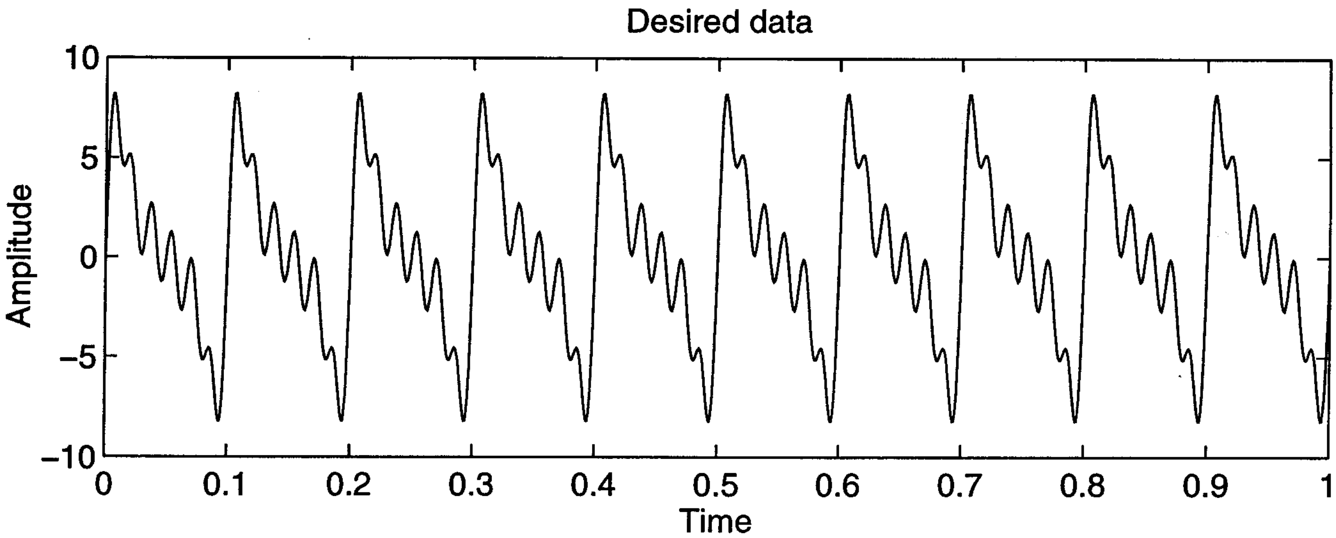
COMPARISON BETWEEN THE DESIRED SIGNAL AND THE NOISY SIGNAL FILTERED WITH THE AID OF AN IIR FILTER

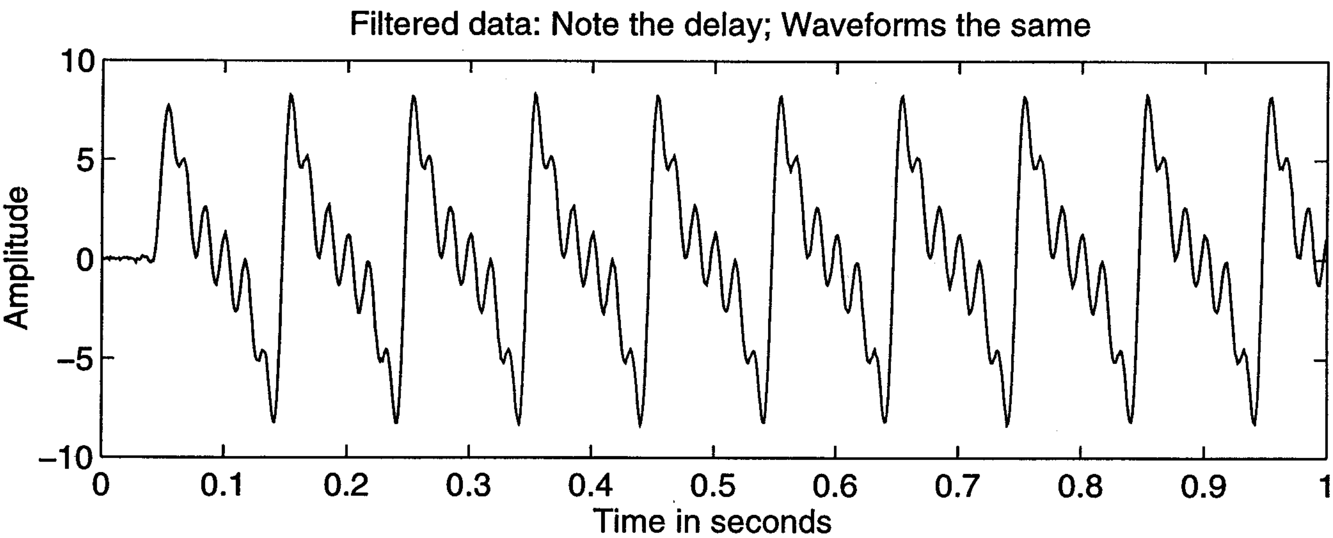


COMPARISON BETWEEN THE DESIRED SIGNAL AND THE NOISY SIGNAL FILTERED WITH THE AID OF AN FIR FILTER

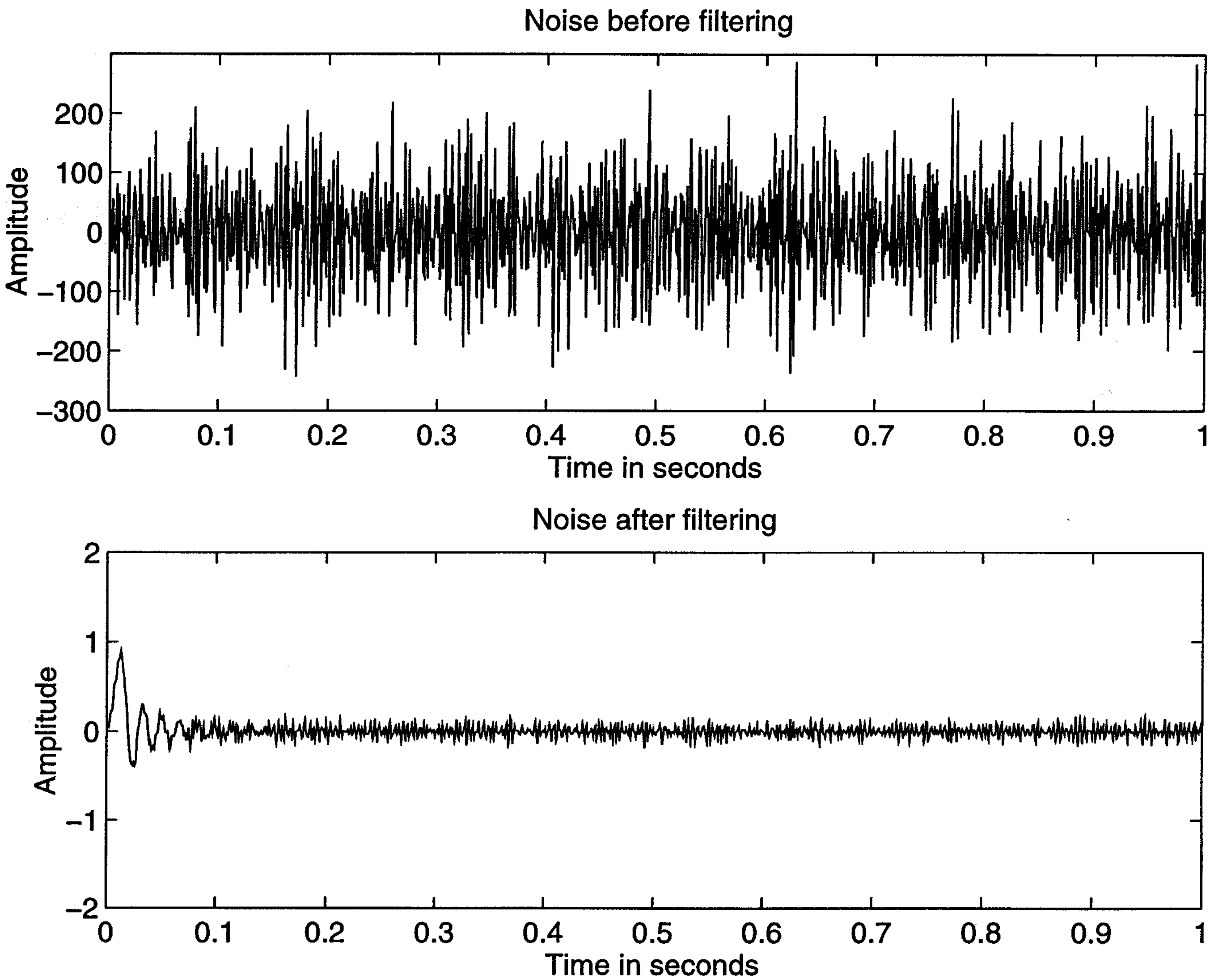


COMPARISON BETWEEN THE DESIRED SIGNAL AND THE NOISY SIGNAL FILTERED WITH THE AID OF AN FIR FILTER

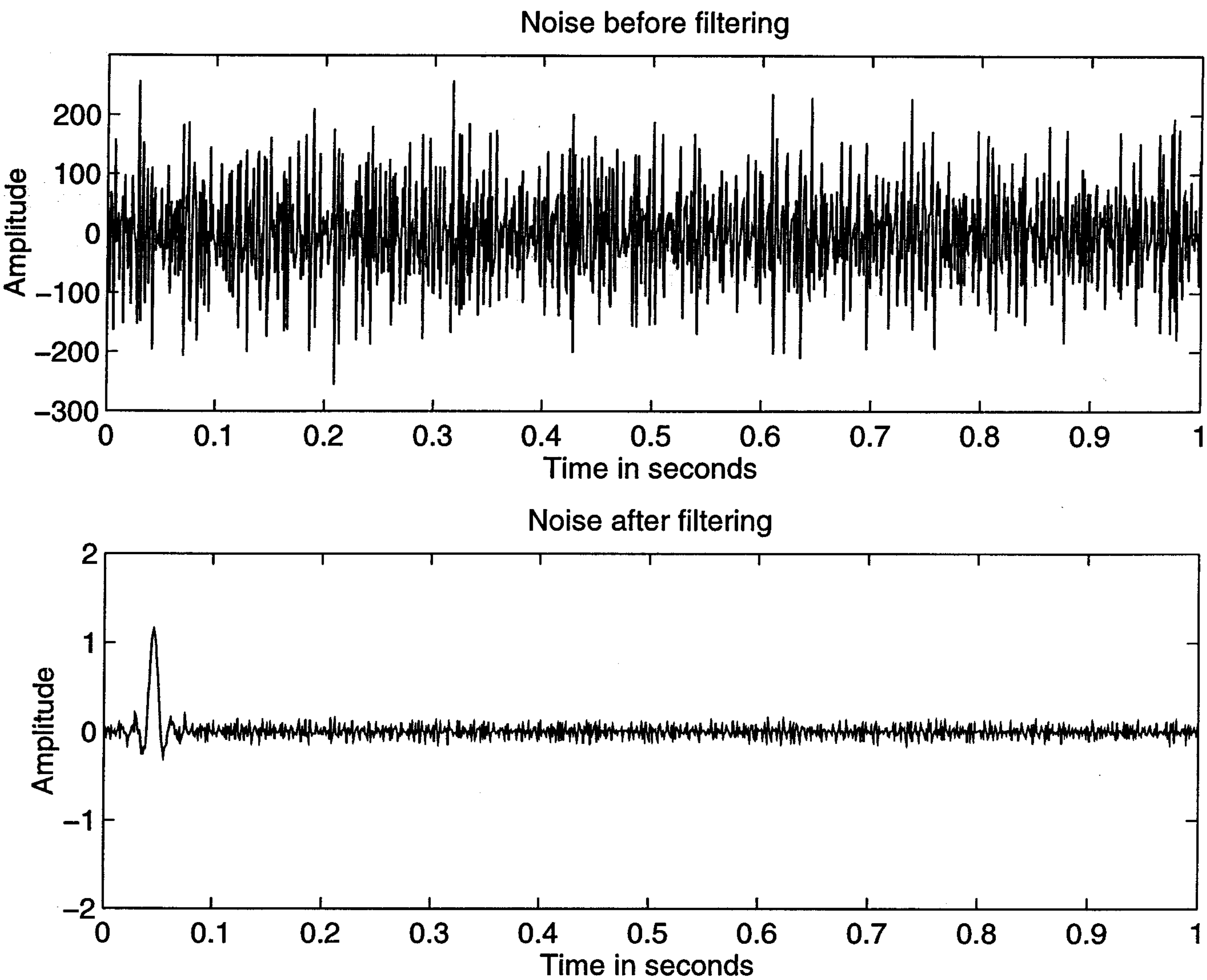




NOISE ATTENUATION WITH THE AID OF AN IIR FILTER



NOISE ATTENUATION WITH THE AID OF AN FIR FILTER



COMPARISON BETWEEN THE FILTERED SIGNALS

- In both cases, the filtered signal can be expressed after the transient in the form

$$
\begin{aligned}
x_f(n) &= \sum_{k=1}^{6} |H(e^{jk\omega_0})| A_k \sin(n(k\omega_0) + \arg H(e^{jk\omega_0})) \\
&= \sum_{k=1}^{6} |H(e^{jk\omega_0})| A_k \sin(n[k - \tau_p(k\omega_0)]\omega_0),
\end{aligned}
$$

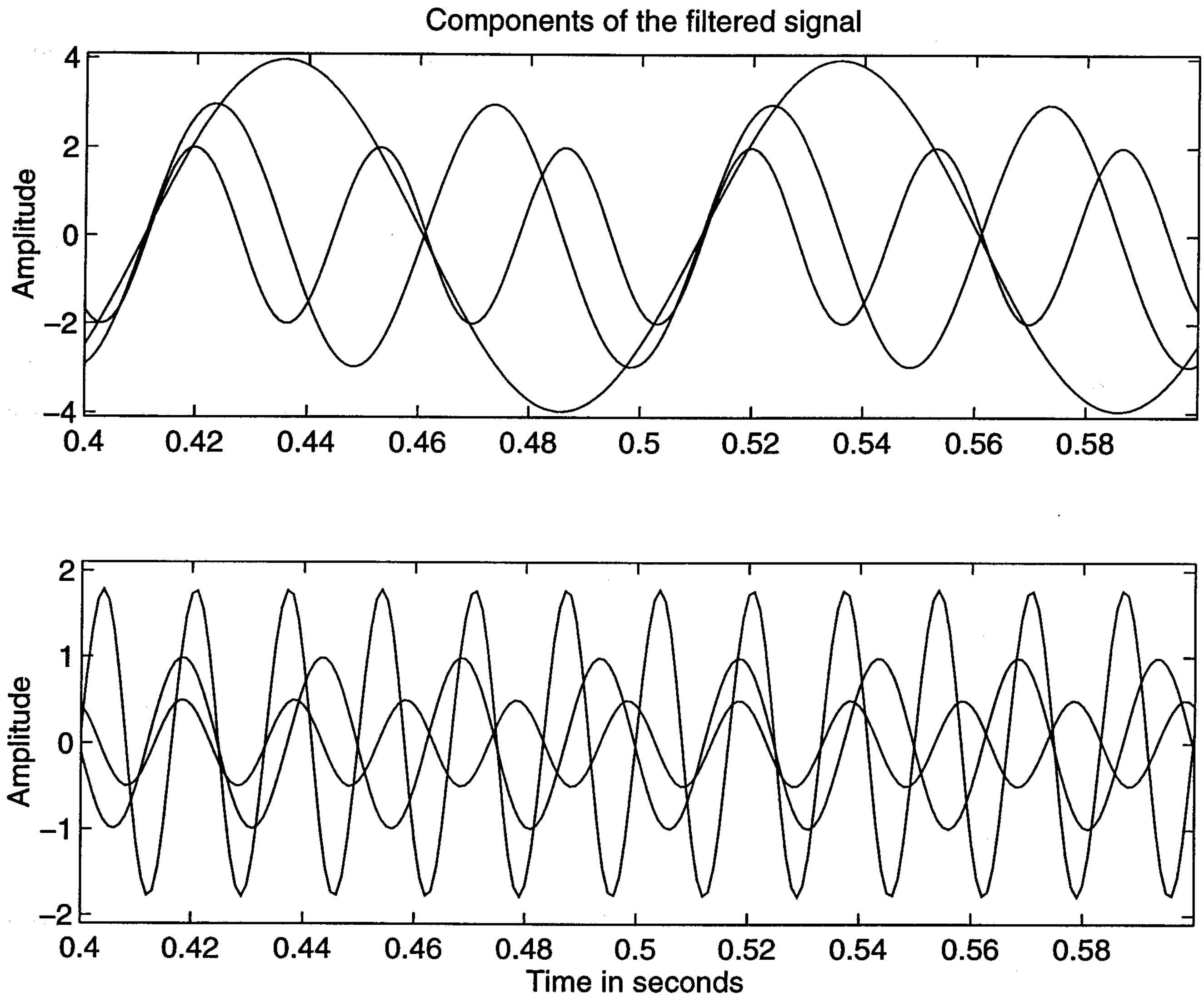
where

$$\tau_p(\omega) = -\frac{\arg H(e^{j\omega})}{\omega}.$$

- Here, $|H(e^{j\omega})|$, $\arg H(e^{j\omega})$, and $\tau_p(\omega)$ are the amplitude, the phase, and the phase delay responses of the filter, respectively.

- For the IIR filter, the $|H(e^{jk\omega_0})|$'s have the values 0.9934, 0.9913, 0.9999, 0.9913, 0.9997, and 0.9915, respectively. Thus, the oscillation amplitudes of the filtered components of the desired signal, shown on the next page, are practically preserved.

COMPONENTS OF THE SIGNAL FILTERED WITH THE AID OF AN IIR FILTER



- For the IIR filter, $\tau_p(k\omega_0)$'s have the values 10.7828, 10.9115, 11.3625, 12.1078, 13.3674, 16.4348. These are in samples. Since the sampling period is 1 ms, these are directly the delays in milliseconds.

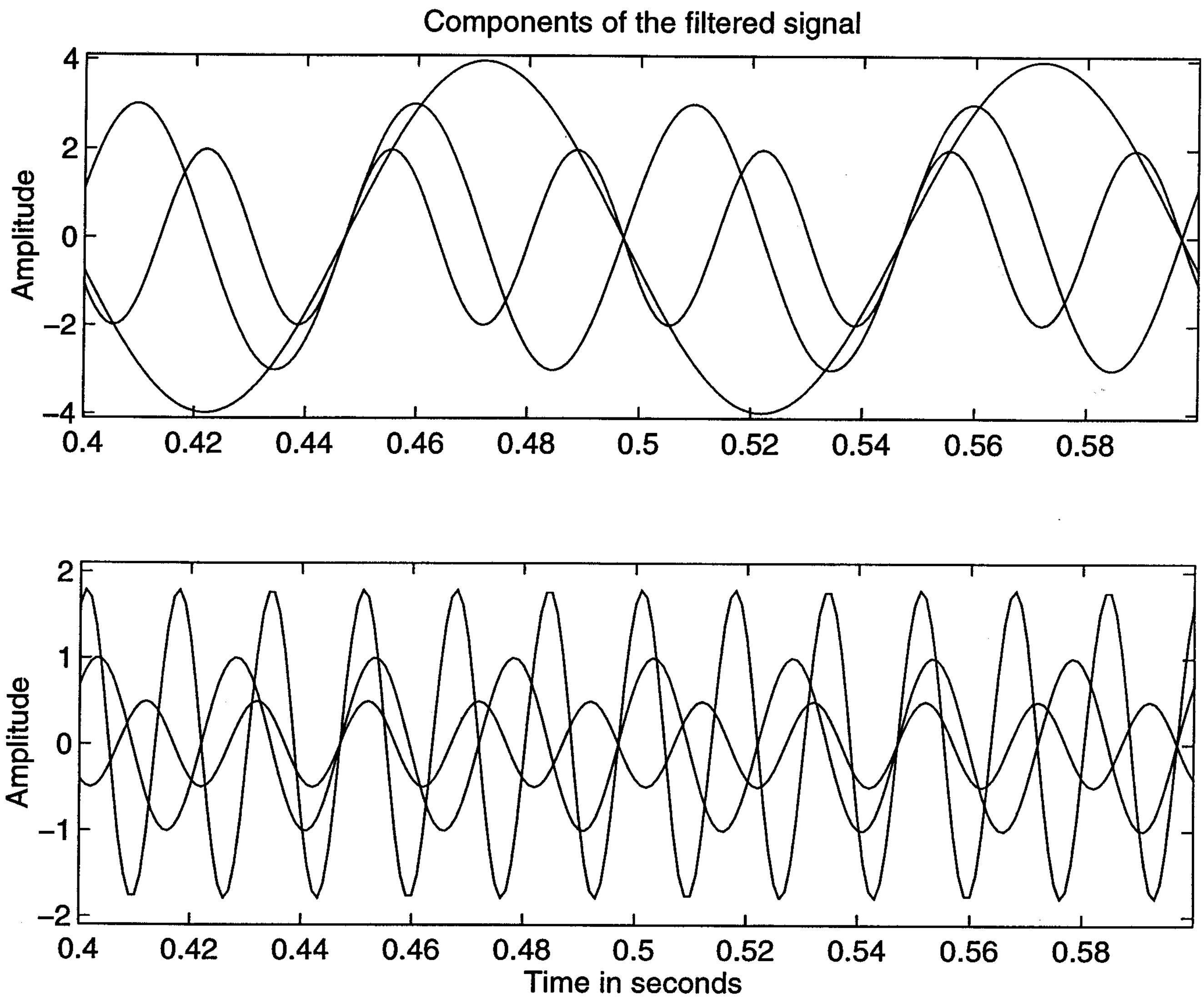
- Since the delays for the sinusoidals are different (compare figures on pages 23 and 3), the signal shape is not preserved (see page 17).

- For the FIR filter, all the $\tau_p(k\omega_0)$'s are 47 samples, and the $|H(e^{jk\omega_0})|$'s are 0.9954, 1.0045, 0.9956, 1.0045, 0.9954, and 0.9958, respectively.

- For the FIR filter, the waveform remains thus practically the same at the expense of a longer delay (see page 19.)

- The filtered components are shown on the next page. Notice the delay of 0.047 s compared to the original signals of page 3.

COMPONENTS OF THE SIGNAL FILTERED WITH THE AID OF AN FIR FILTER



PARTIAL RESPONSES OF THE IIR FILTER

- The transfer function can be expressed as

$$H(z) = kN(z)/D(z),$$

where

$$k = 0.0013947$$

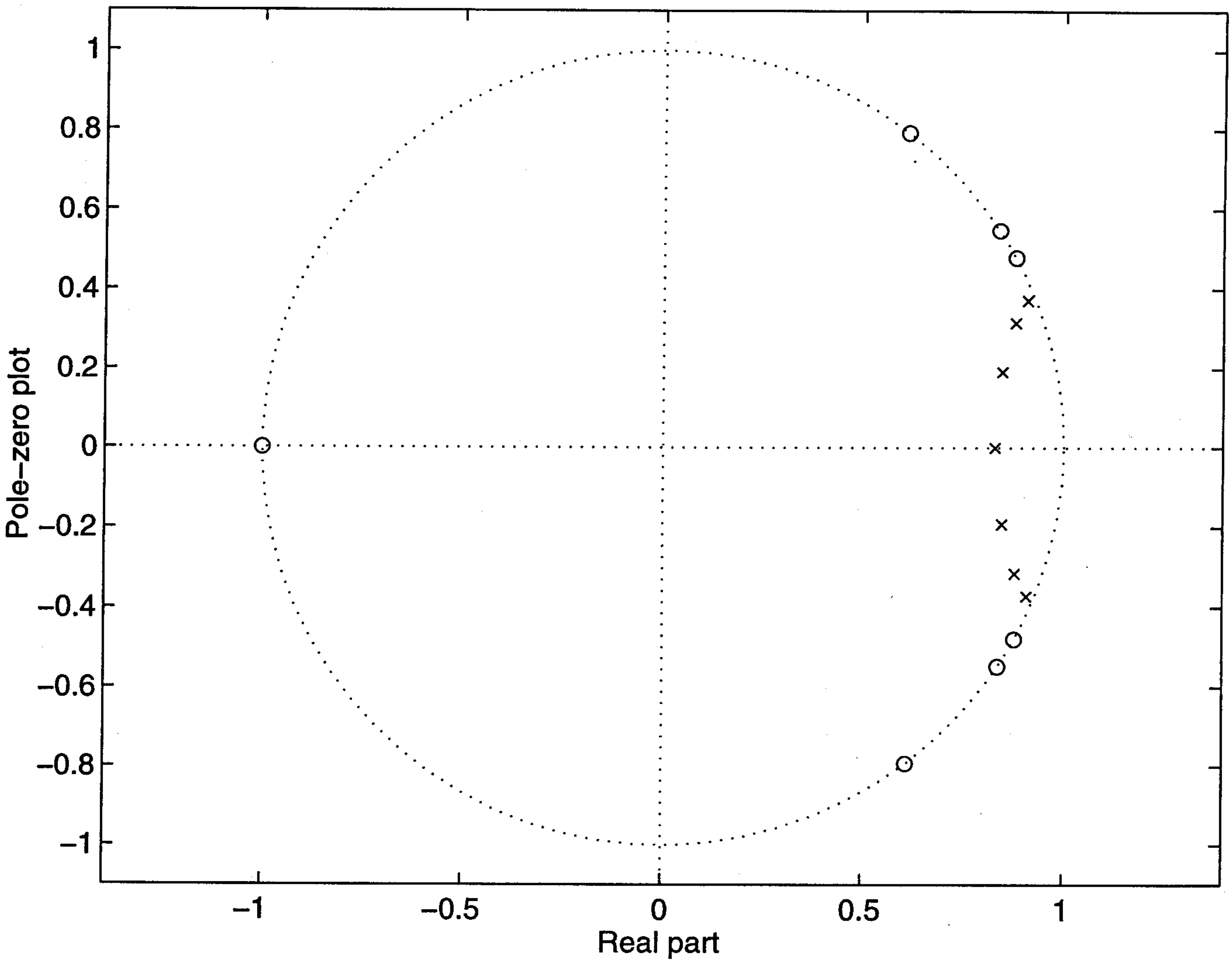
$$D(z) = (1 - 0.8286z^{-1})(1 - 1.8163z^{-1} + 0.9628z^{-2})\times$$
$$(1-1.7573z^{-1}+0.8715z^{-2})(1-1.6922z^{-1}+0.7528z^{-2})$$

$$N(z) = (1 + z^{-1})(1 - 1.2196z^{-1} + z^{-2})\times$$
$$(1 - 1.6731z^{-1} + z^{-2})(1 - 1.7554z^{-1} + z^{-2}).$$

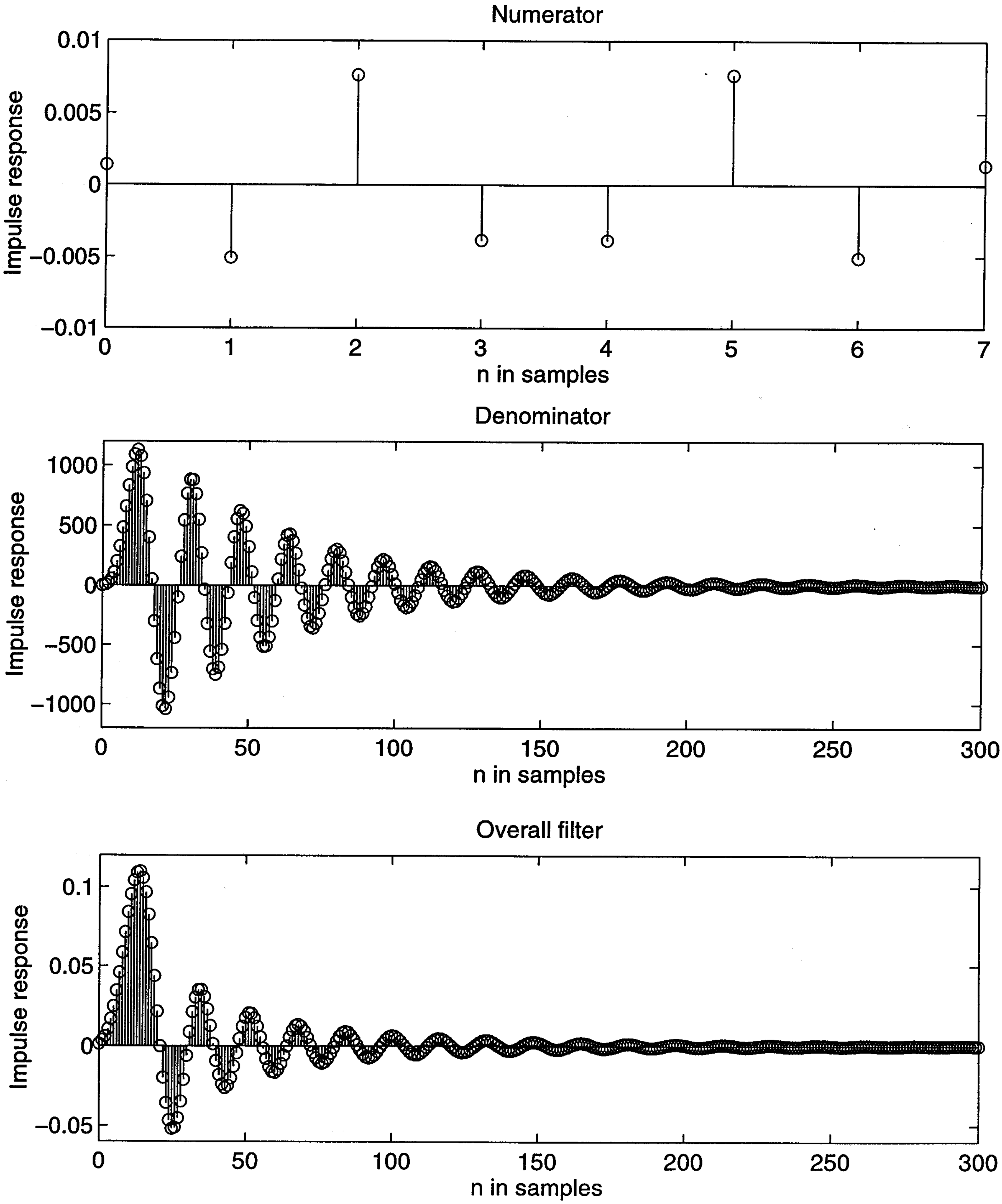
- The poles are thus located at $z = 0.8286$, $z = 0.9812\exp(\pm j0.12364\pi)$, $z = 0.9335\exp(\pm j0.10969\pi)$, and $z = 0.8677\exp(\pm j0.07115\pi)$.

- One zero is at $z = -1$, and there are three pairs of zeros on the unit circle at frequencies $\omega = \pm 0.1591\pi$, $\omega = \pm 0.1846\pi$, and $\omega = \pm 0.2913\pi$.

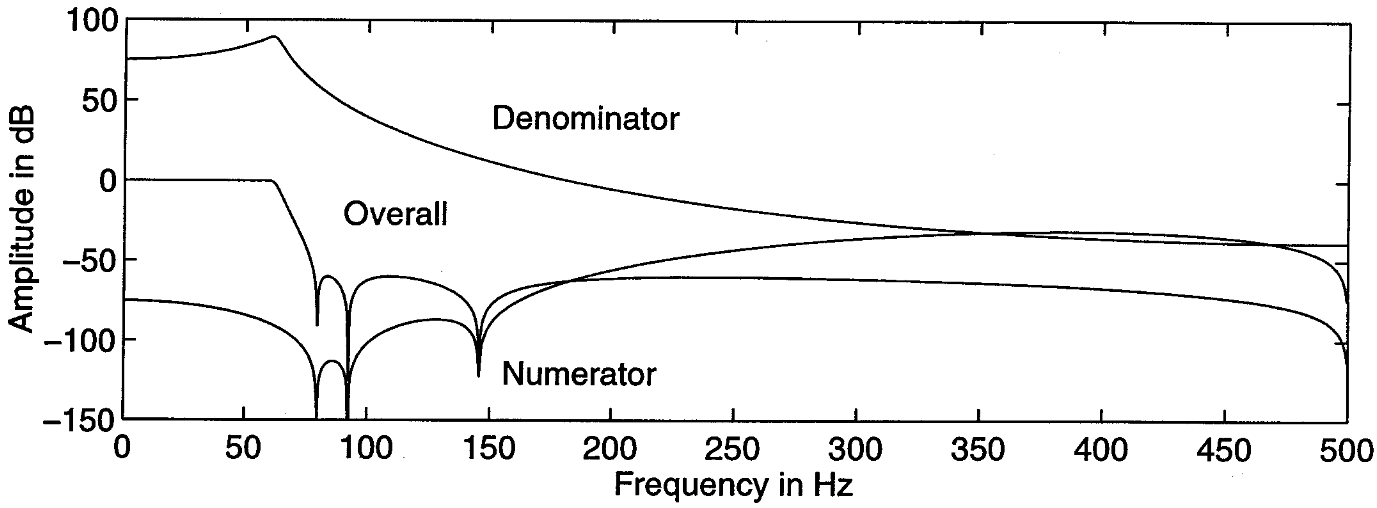
POLE-ZERO PLOT

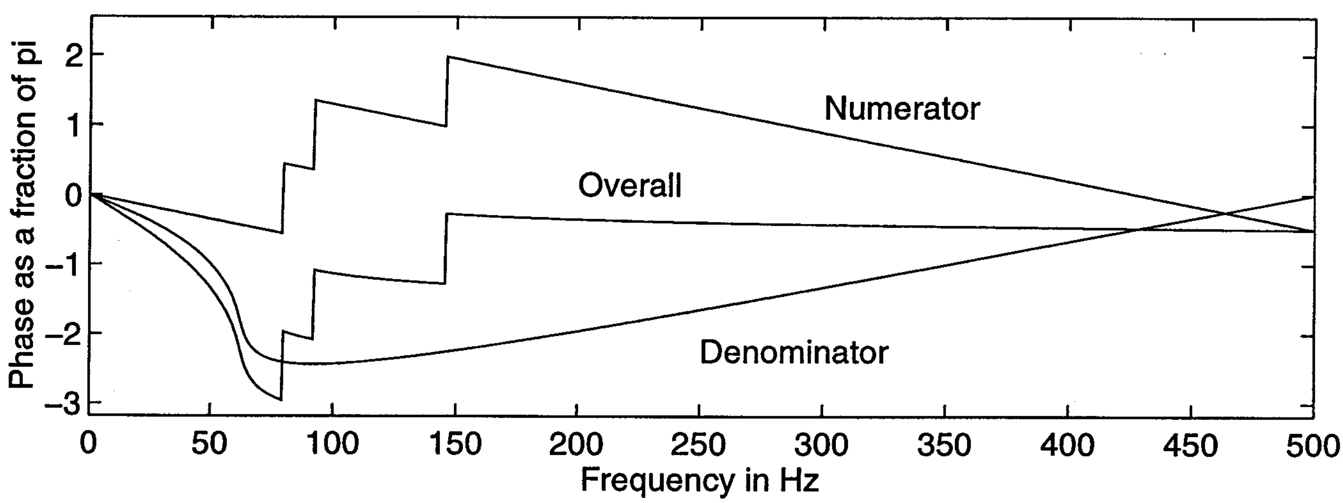


IMPULSE RESPONSES



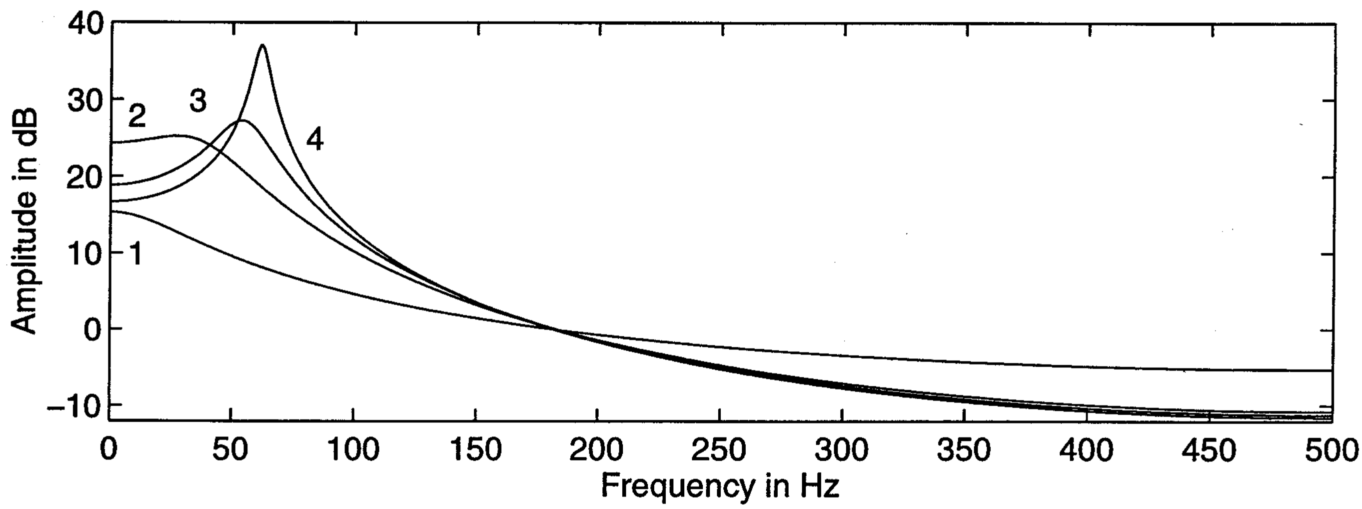
AMPLITUDE AND PHASE RESPONSES

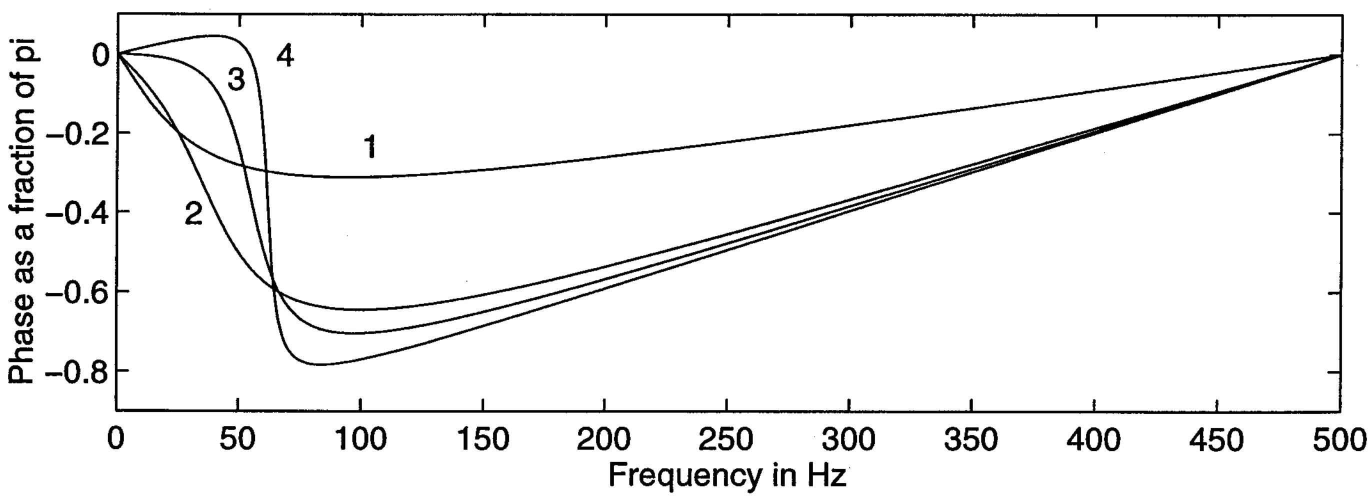




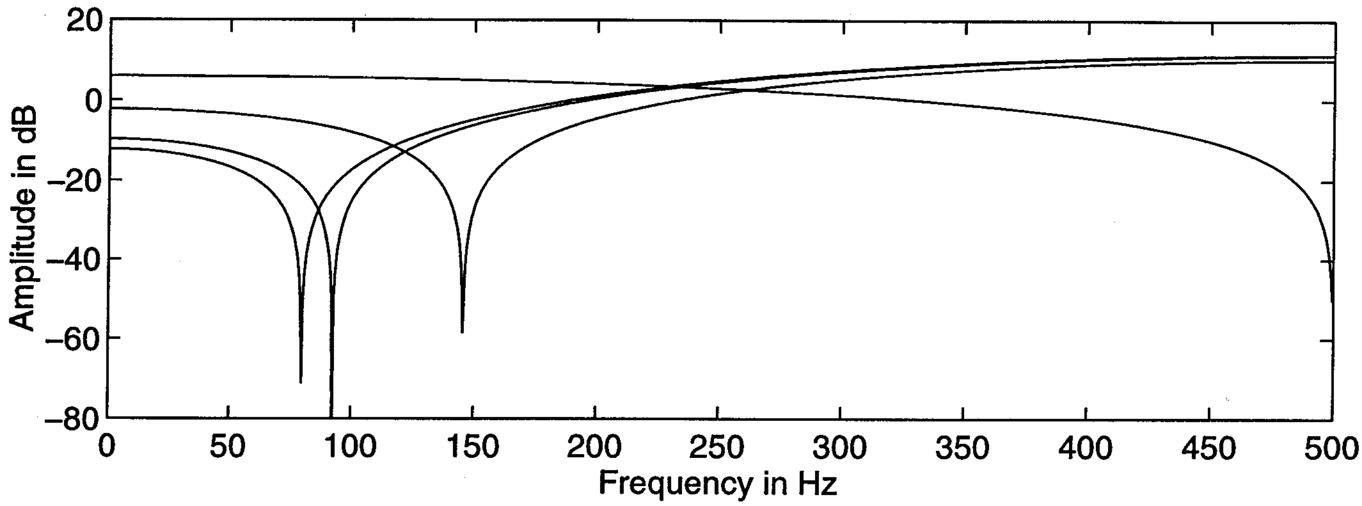
CONTRIBUTION OF THE POLES TO THE AMPLITUDE AND PHASE RESPONSES

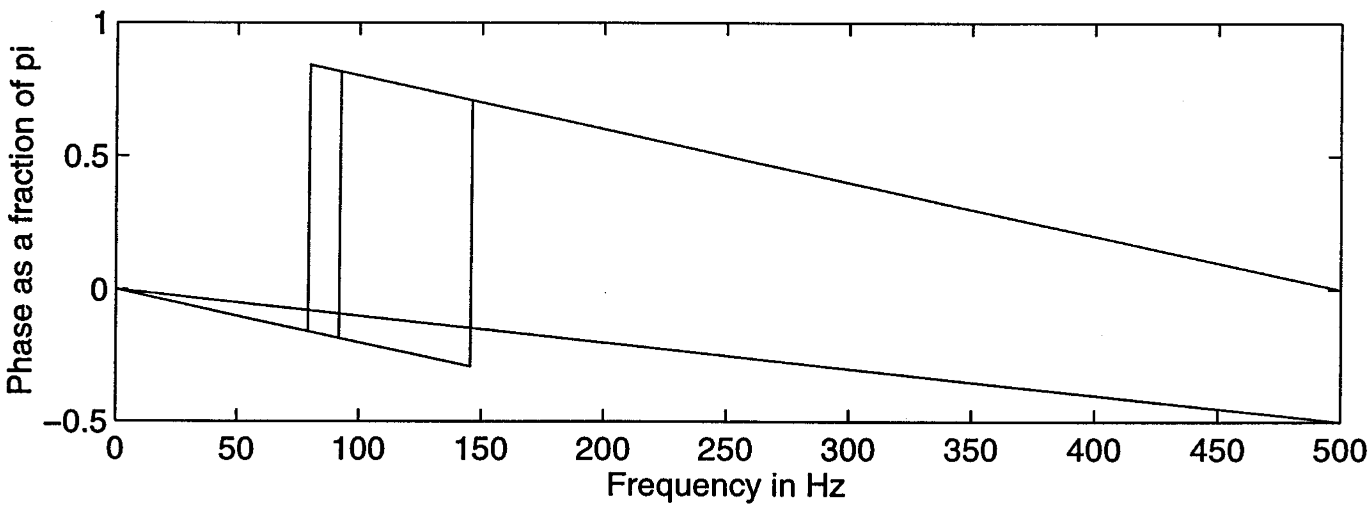
- 1 for real pole, 2 for innermost, 3 for second innermost, 4 for outermost.



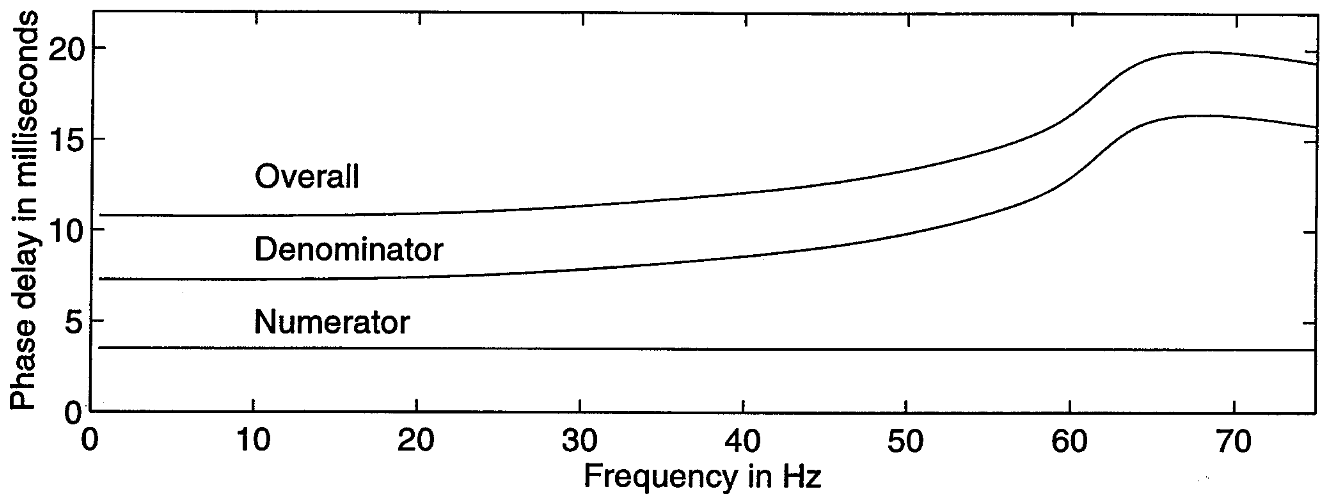


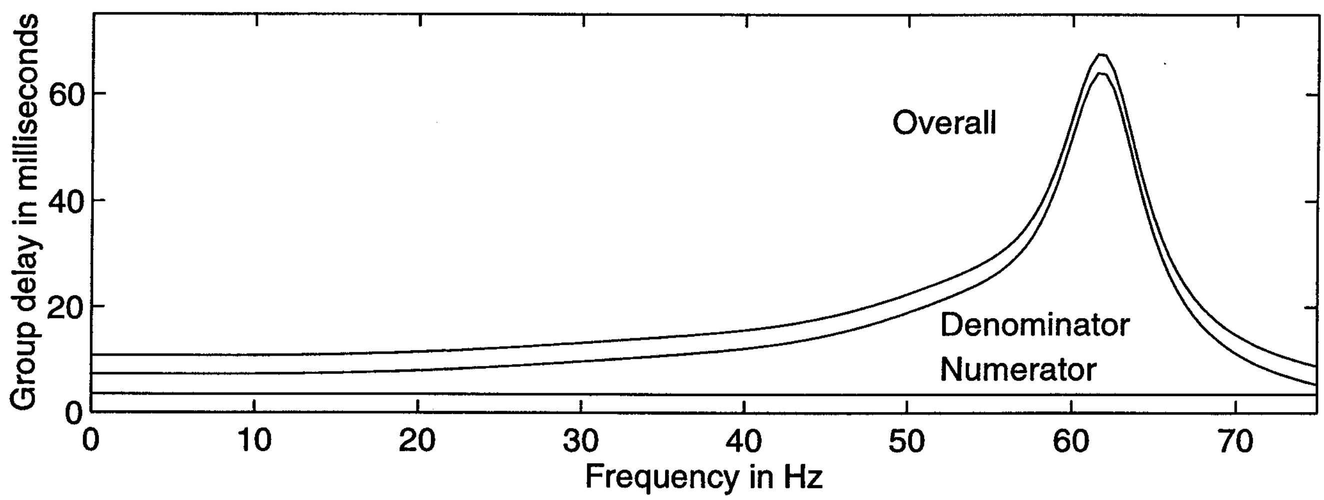
CONTRIBUTION OF THE ZEROS TO THE AMPLITUDE AND PHASE RESPONSES





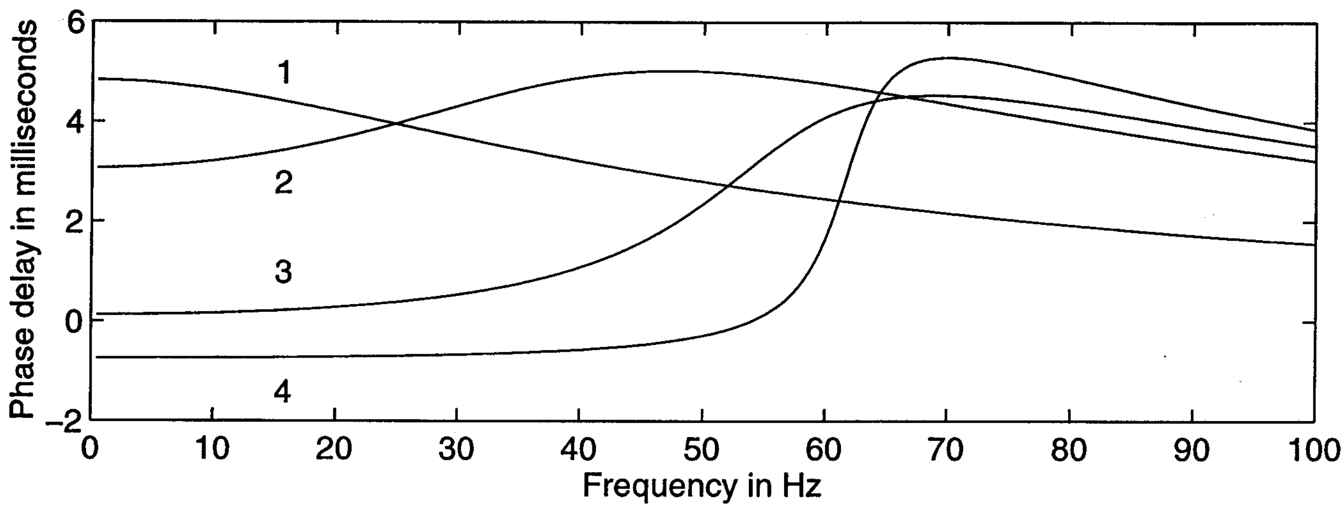
PHASE AND GROUP DELAY RESPONSES

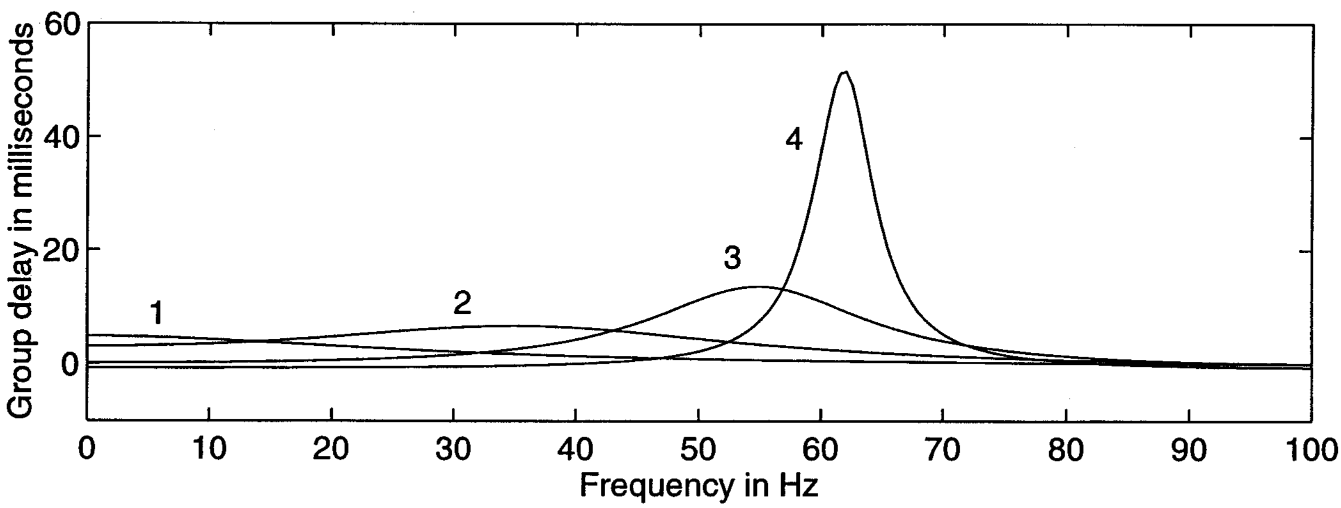




CONTRIBUTION OF THE POLES TO THE PHASE AND GROUP DELAYS

- 1 for real pole, 2 for innermost, 3 for second innermost, 4 for outermost.





COMMENTS ON THE IIR FILTER

- As seen from the first figure of page 29, the filter denominator, that is, the poles provide most of the attenuation of the filter at the expense of a large deviation of the passband phase response (the second figure of page 29) from a straight line of the form $-\alpha\omega$ (linear phase).

- The numerator, that is, the zeros provide a linear phase in the passband and help in achieving the desired overall attenuation in the beginning of the stopband region.

CHARACTERISTICS OF AN IIR FILTER OBTAINED BY DIVIDING THE PASSBAND AND STOPBAND EDGES BY TWO

- Pages 37, 38, and 39 give responses of an IIR filter obtained from the previous filter (see pages 11, 12, and 29) by dividing the passband and stopband edges by two.

- The resulting filter is characterized by the following properties:

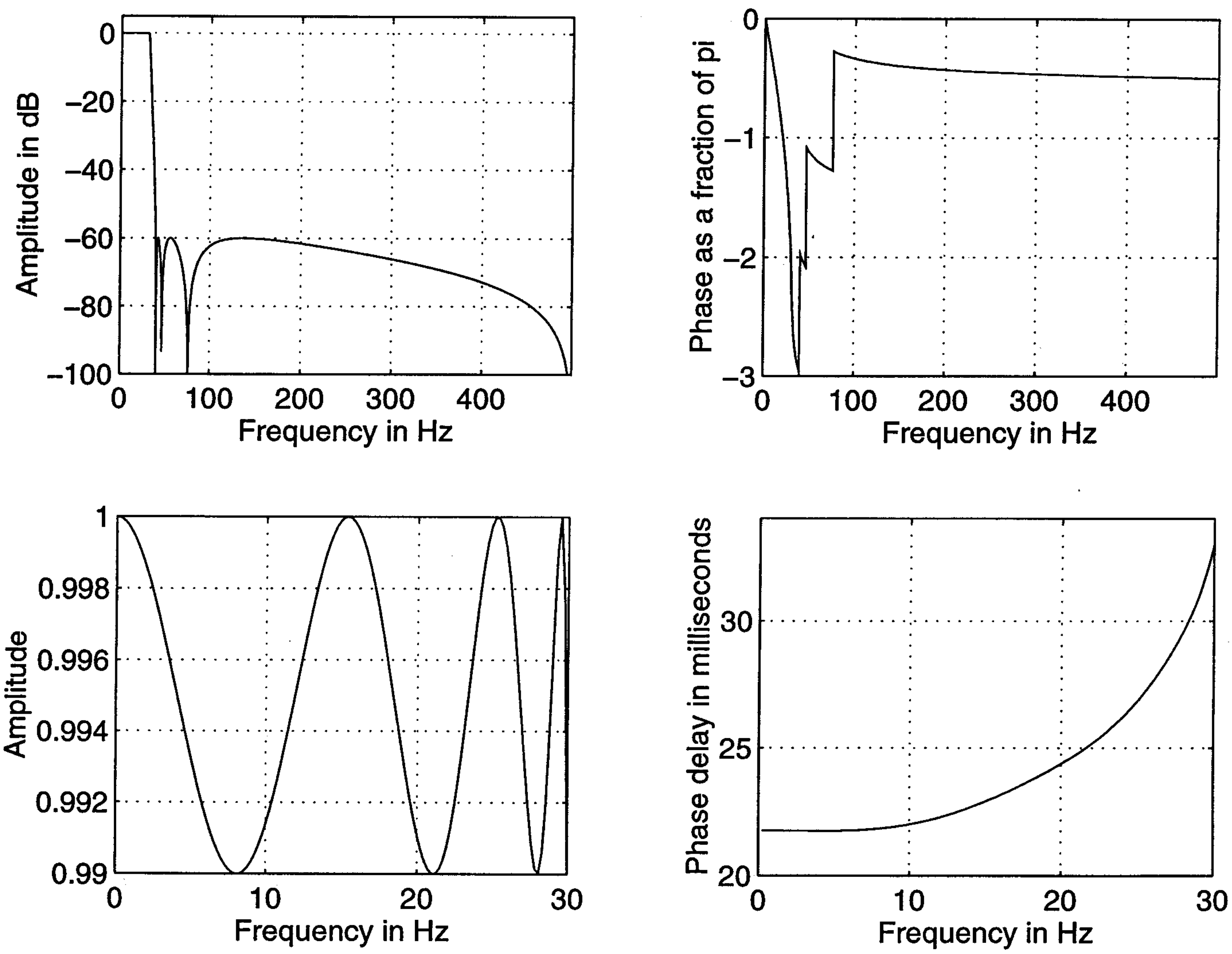
- The amplitude and phase responses are roughly obtained at low frequencies by shrinking the frequency axis by two.

- Since the phase response of the new filter changes two times faster, the values of the phase and group delay responses are roughly two times those of the wider filter.

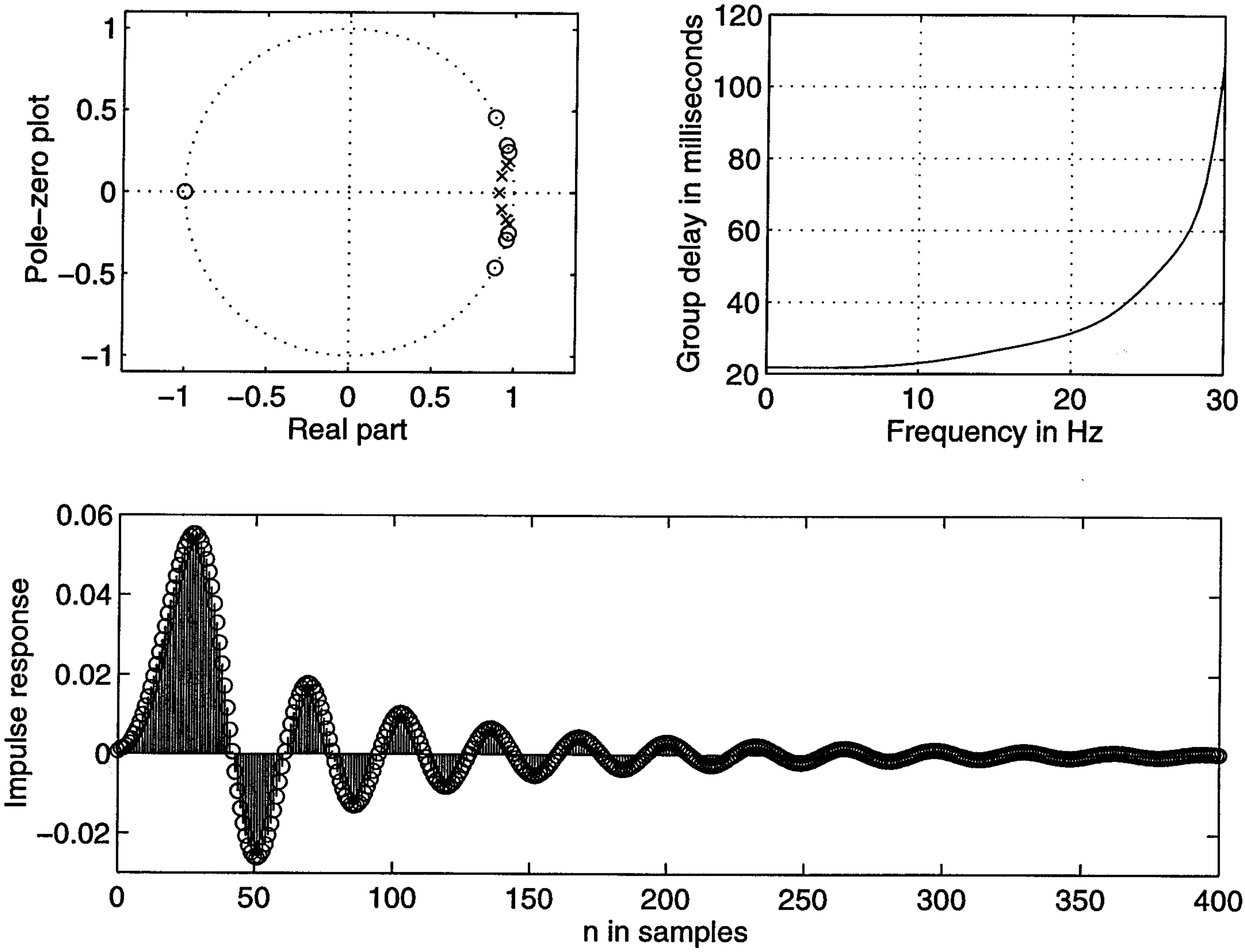
- The angles of the complex conjugate pole and zero pairs are roughly related to those of the old wider filter via $\theta_{new} = \theta_{old}/2$, whereas the radii of the poles satisfy roughly $r_{new} = \sqrt{r_{old}}$.

- The impulse response of the narrower filter is roughly obtained from that of the wider one by interpolating one more sample point between the existing samples and by dividing the resulting sample values by two.

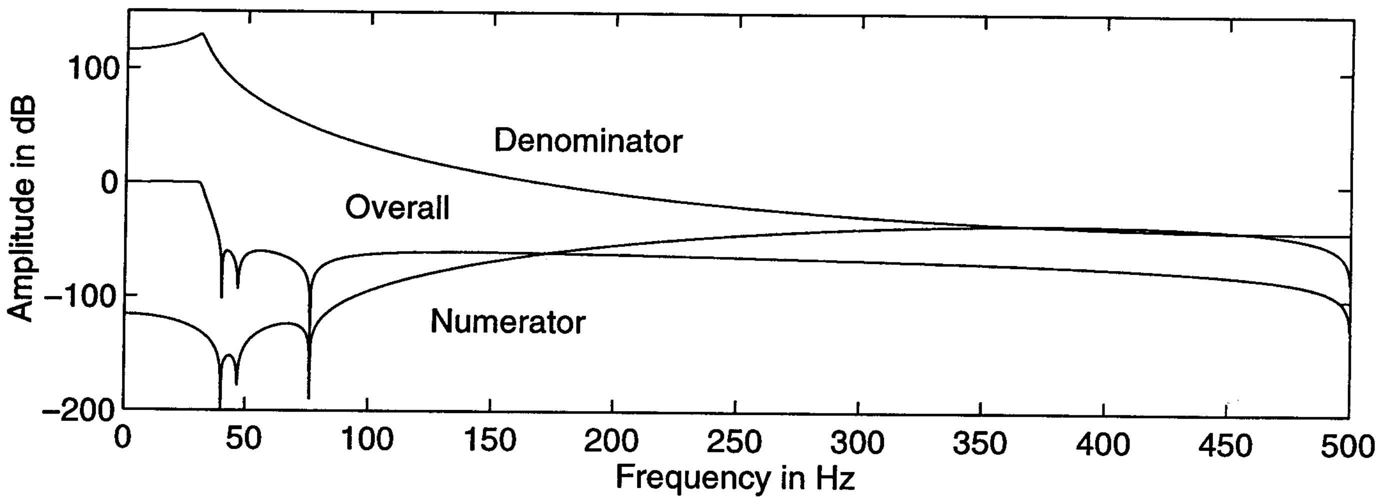
CHARACTERISTICS FOR THE NARROW-BAND IIR FILTER

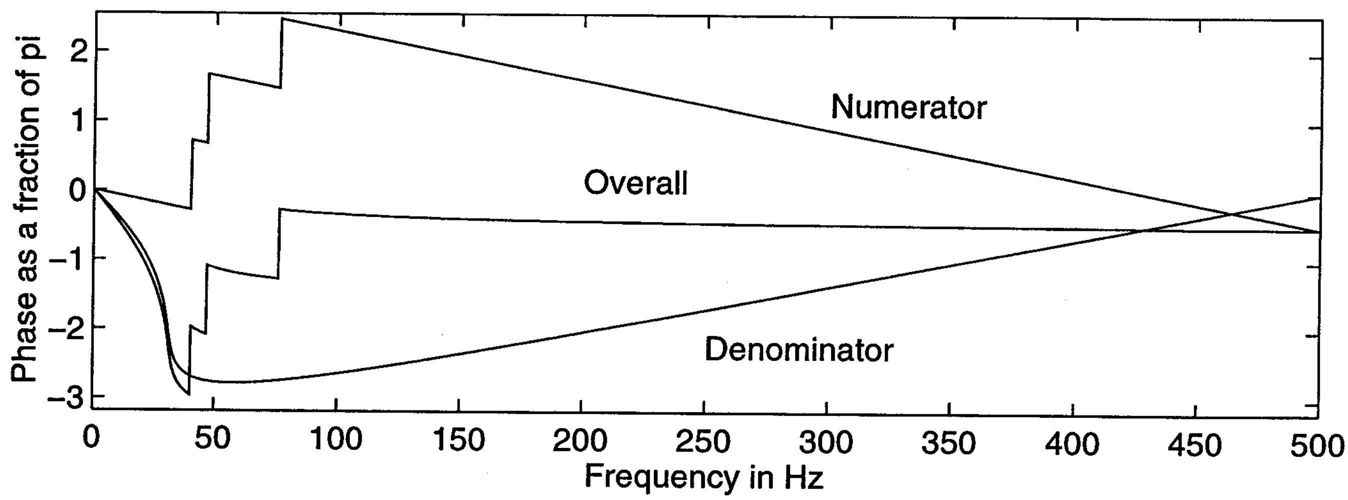


CHARACTERISTICS FOR THE NARROW-BAND IIR FILTER



CHARACTERISTICS FOR THE NARROW-BAND IIR FILTER





PARTIAL RESPONSES FOR THE FIR FILTER

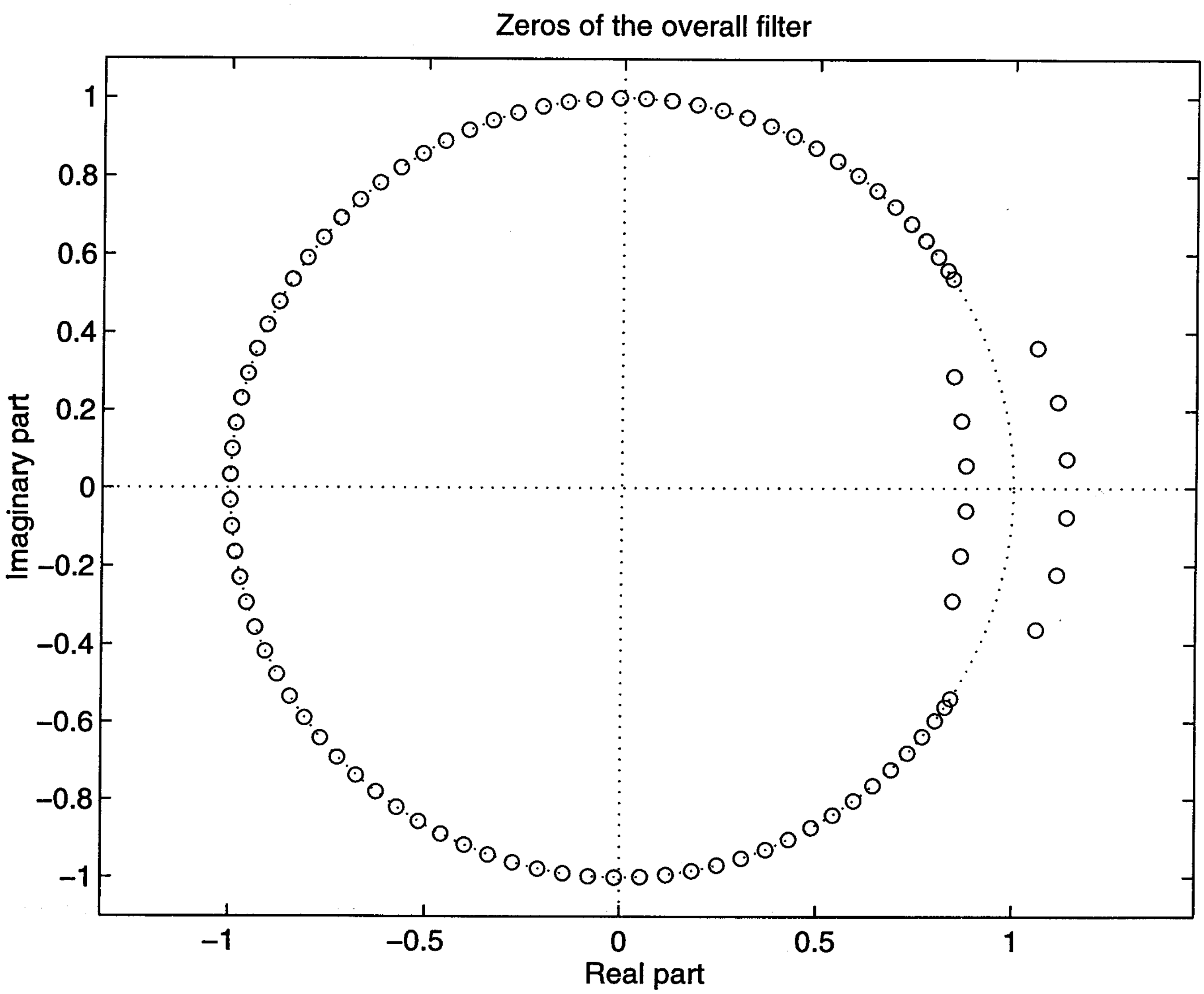
- The linear-phase property is based on the facts:

1. Each zero pair on the unit circle gives a linear-phase contribution, also zero at $z = 1 = \exp(\pi)$.

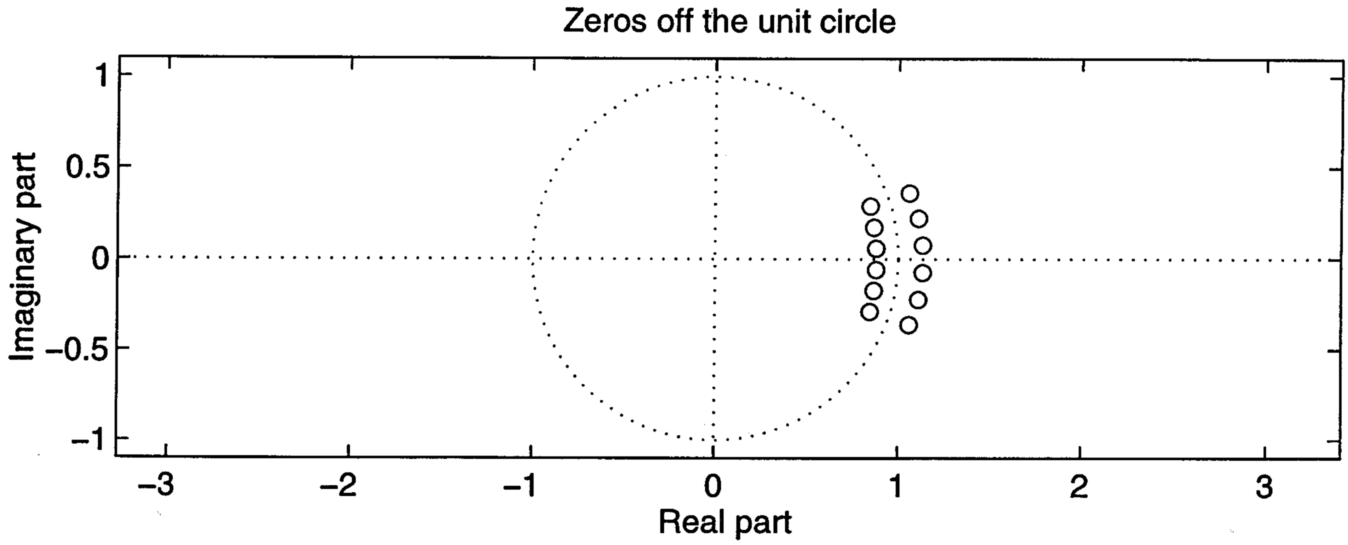
2. Zeros off the unit circle occur in quadruplets, that is, there are four zeros located at $z = r\exp(\pm j\theta)$, $z = (1/r)\exp(\pm j\theta)$, or in reciprocal pairs on the unit circle, that is, there are two zeros located at $z = r$, $z = (1/r)$. Each of these zero groups gives a linear-phase contribution.

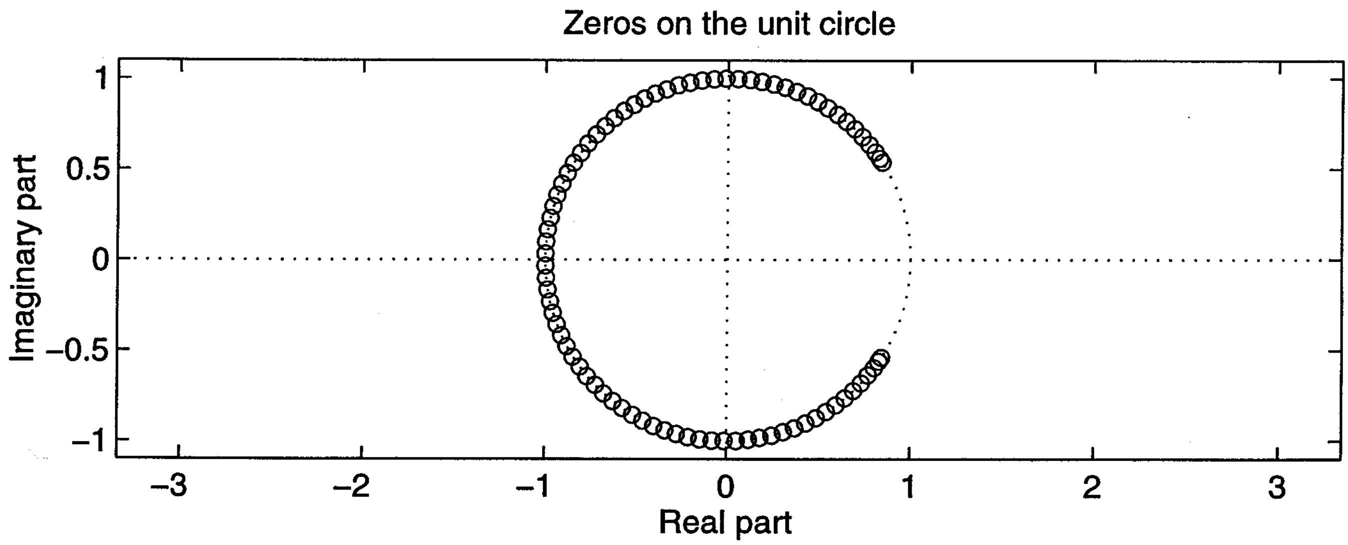
- In the following, we consider separately the zeros on the unit circle, which concentrate on the stopband shaping, and zeros off the unit circle, which concentrate on the passband shaping.

OVERALL ZERO PLOT

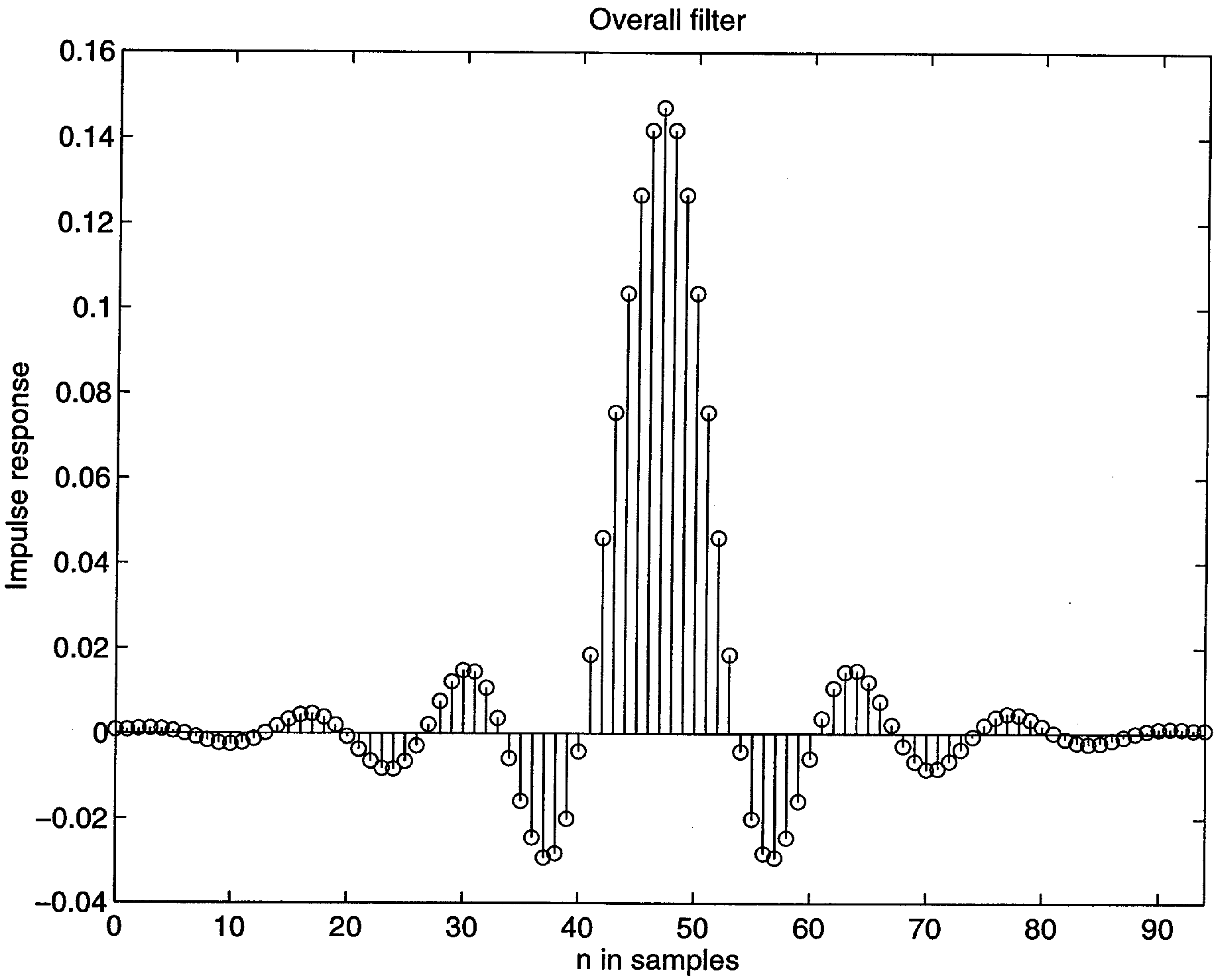


PARTIAL ZERO PLOTS

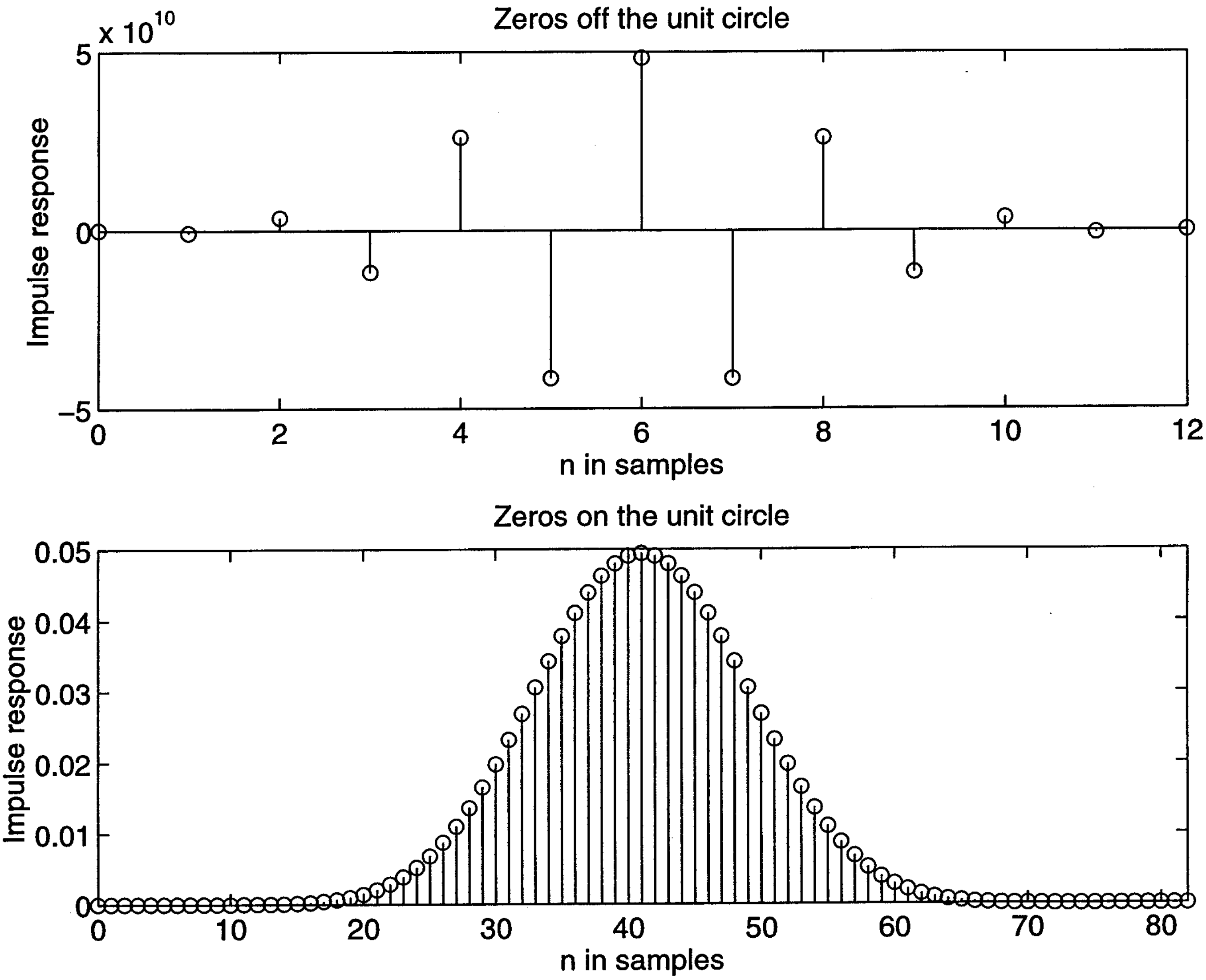




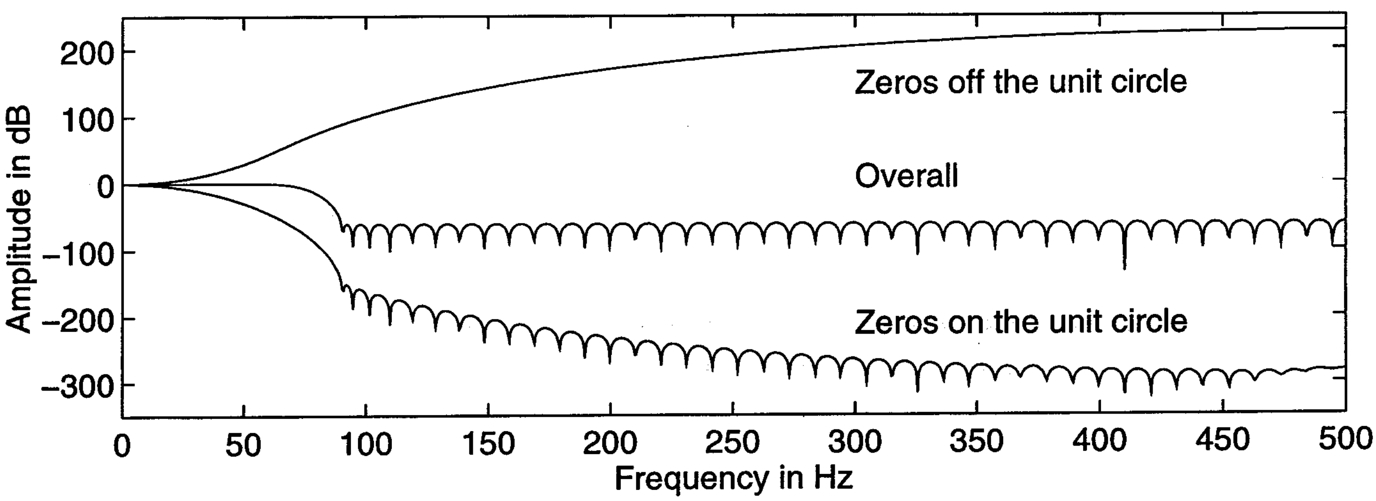
OVERALL IMPULSE RESPONSE

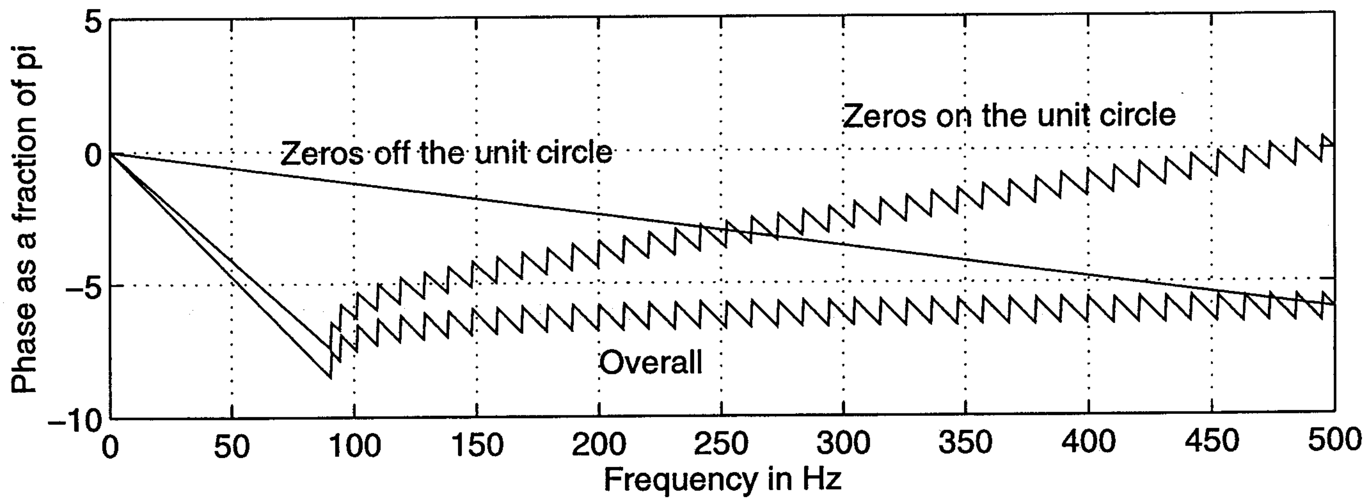


PARTIAL IMPULSE RESPONSES

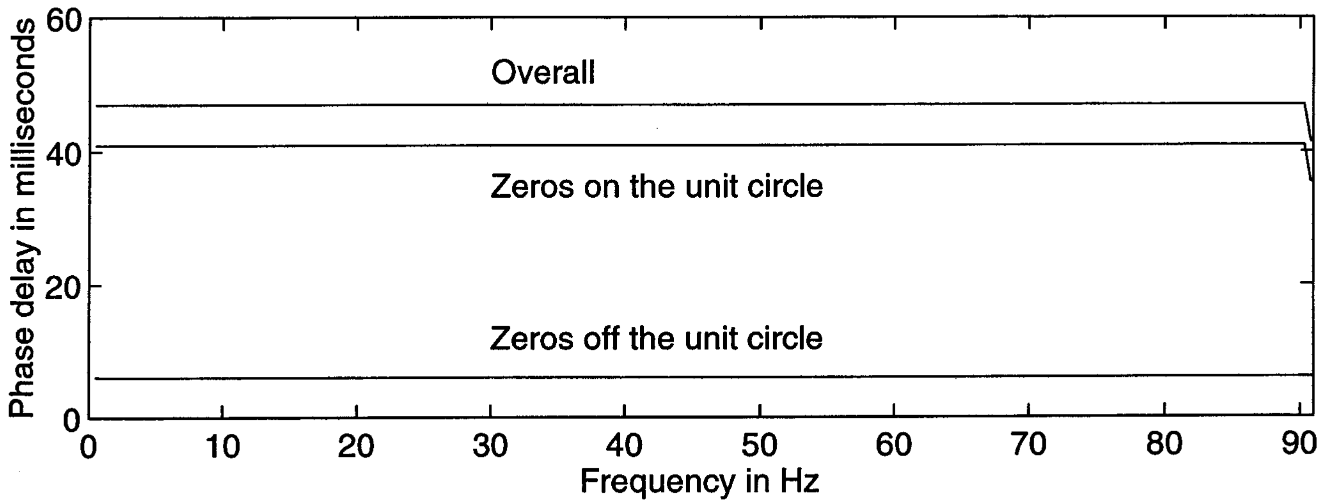


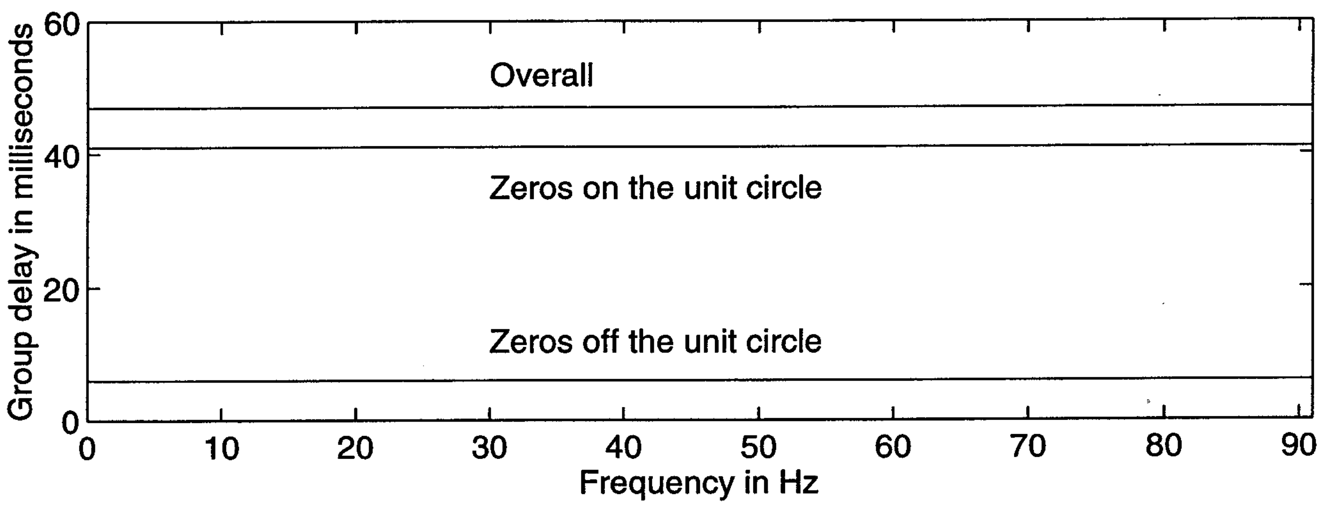
AMPLITUDE AND PHASE RESPONSES





PHASE AND GROUP DELAY RESPONSES





COMMENTS ON THE FIR FILTERS

- As seen from the first figure of page 45, the zeros off the unit circle concentrating in making the overall passband response flat, compensating the effect of the off-the-unit-circle zeros.

- Simultaneously, instead of proving attenuation in the stopband region, they amplify frequencies in the stopband region.

- Therefore, several zeros on the unit circle are needed to give the desired attenuation for the overall filter.

- This explains why there are many more zeros on the unit circle compared to IIR filters.

- The overall passband group delay and phase delay is equal to $N/2$, where $n = N$ is the location of the last impulse response sample.

CHARACTERISTICS OF AN FIR FILTER OBTAINED BY DIVIDING THE PASSBAND AND STOPBAND EDGES BY TWO

- Pages 50, 51, and 52 give responses of an FIR filter obtained from the previous FIR filter (see pages 13, 14, and 44) by dividing the passband and stopband edges by two.

- The resulting filter is characterized by the following properties:

- The amplitude and phase responses are roughly obtained at low frequencies by shrinking the frequency axis by two.

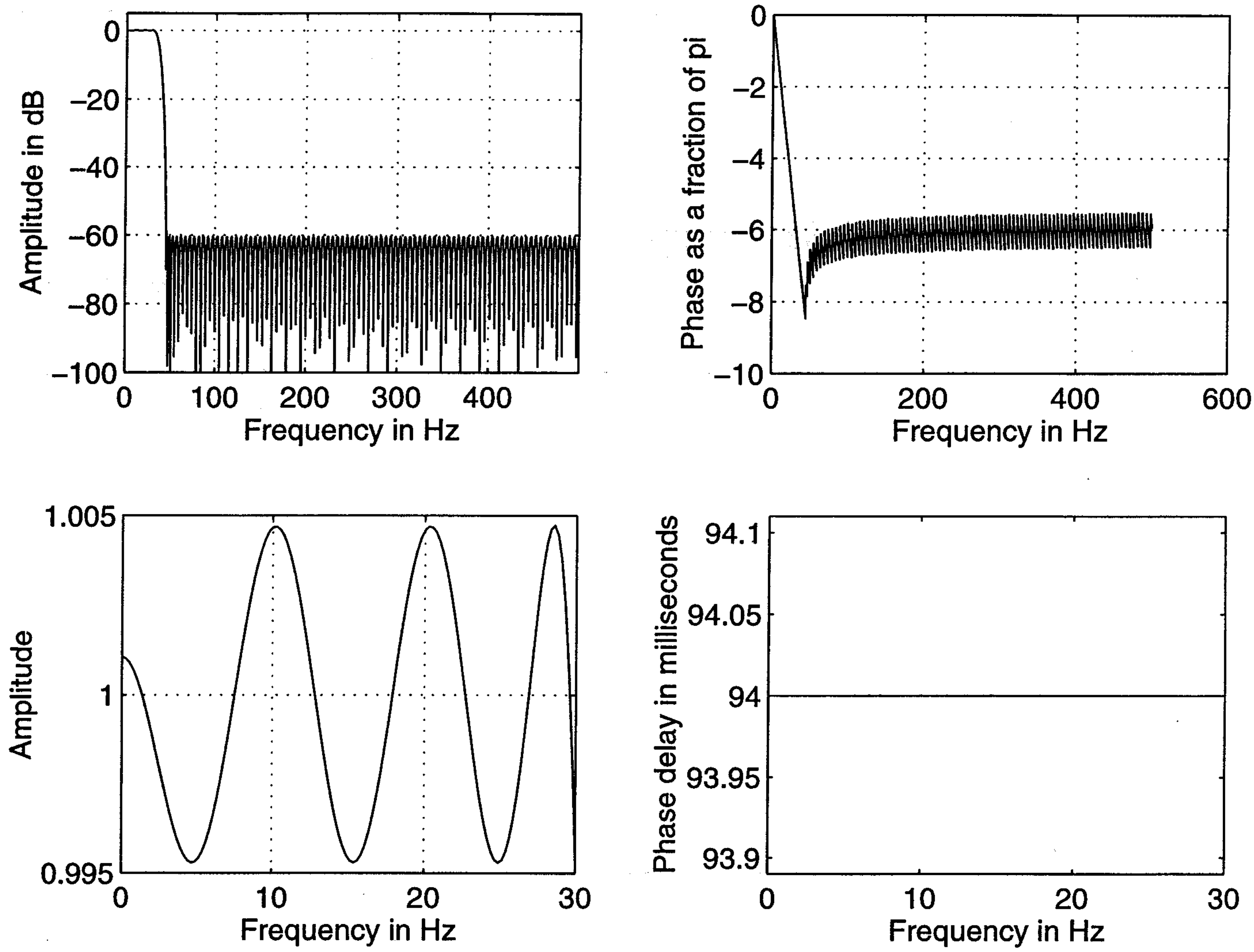
- The unit circle region $[\pi/2,\ \pi]$ is filled roughly by N more zeros, making the number of impulse response samples equal to $2N$, that is, the number of samples is increased by a factor of two.

- Since the number of impulse response values are incresed by a factor of two, also the delays are increased by the same factor.

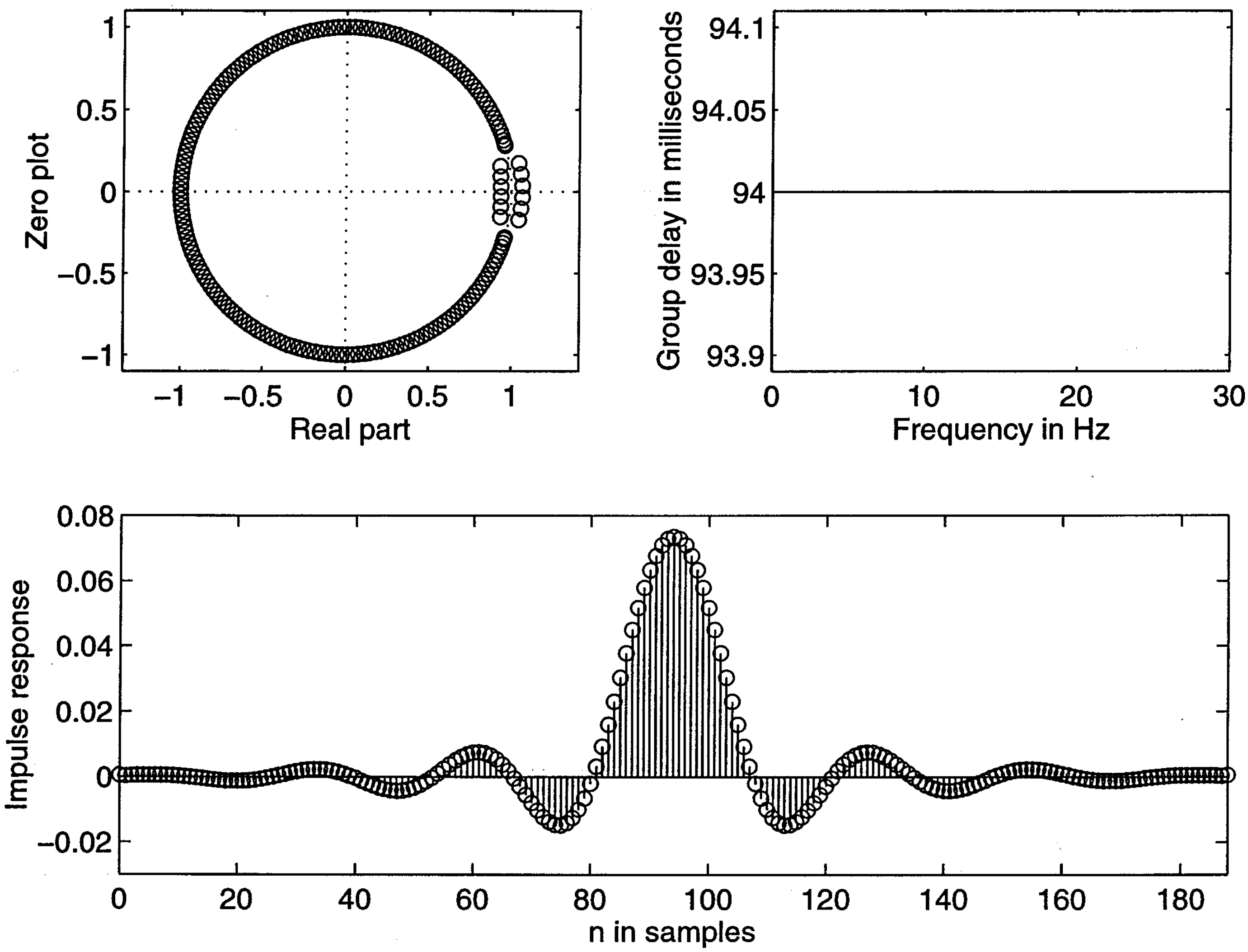
- The number of off-the-unit circle zeros remains the same and the angles of these zeros are roughly related to those of the old wider filter via $\theta_{new} = \theta_{old}/2$, whereas the radii of satisfy roughly $r_{new} = \sqrt{r_{old}}$.

- The impulse response of the narrower filter is roughly obtained from that of the wider one by interpolating one more sample point between the existing samples and by dividing the resulting sample values by two.

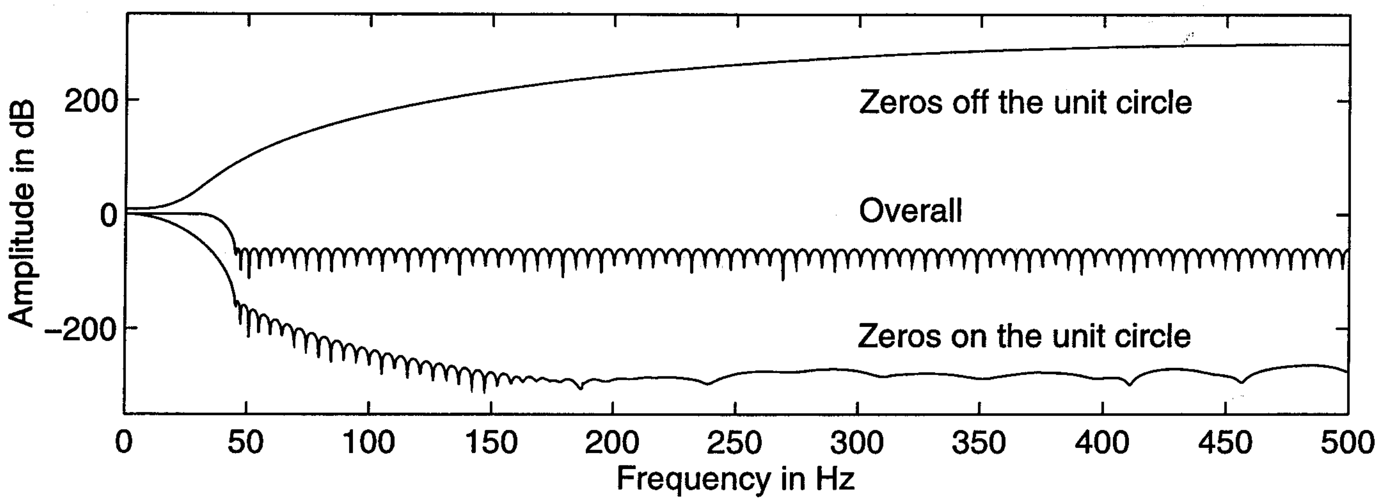
CHARACTERISTICS FOR THE NARROW-BAND FIR FILTER

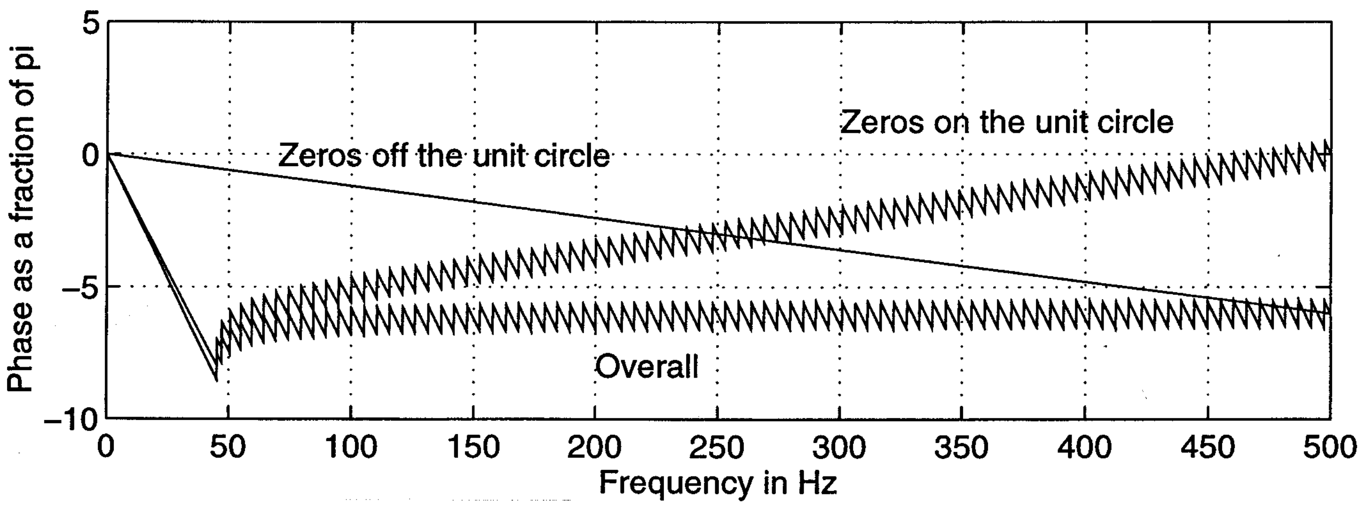


CHARACTERISTICS FOR THE NARROW-BAND FIR FILTER



CHARACTERISTICS FOR THE NARROW-BAND FIR FILTER





CONTRIBUTION OF THE POLES AND ZEROS TO THE FILTER RESPONSES

- Consider a digital filter characterized by the difference equation

$$y[n] = \sum_{k=0}^{M} a_k x[n-k] + \sum_{k=1}^{N} b_k y[n-k].$$

- The corresponding transfer function can be written as

$$H(z) = \frac{\sum_{k=0}^{M} a_k z^{-k}}{1 - \sum_{k=1}^{N} b_k z^{-k}}.$$

- This transfer function can be factorized as

$$H(z) = a_0 \frac{[\prod_{k=1}^{M_r} A_k^{(r)}(z)][\prod_{k=1}^{M_c} A_k^{(c)}(z)]}{[\prod_{k=1}^{N_r} B_k^{(r)}(z)][\prod_{k=1}^{N_c} B_k^{(c)}(z)]}.$$

- Here, $M = M_r + 2M_c$ and

$$A_k^{(r)}(z) = 1 - r_k^{(zr)} z^{-1}$$

and

$$\begin{aligned} A_k^{(c)}(z) =& (1 - r_k^{(zc)} e^{j\theta_k^{(zc)}} z^{-1})(1 - r_k^{(zc)} e^{-j\theta_k^{(zc)}} z^{-1}) \\ =& 1 - 2r_k^{(zc)} \cos(\theta_k^{(zc)}) z^{-1} + (r_k^{(zc)})^2 z^{-2} \end{aligned}$$

contain the kth real zero located at $z = r_k^{(zr)}$ and the kth complex-conjugate zero pair located at $z = r_k^{(zc)} e^{\pm j\theta_k^{(zc)}}$.

- Similarly, $N = N_r + 2N_c$ and

$$B_k^{(r)}(z) = 1 - r_k^{(pr)} z^{-1}$$

and

$$\begin{aligned} B_k^{(c)}(z) =& (1 - r_k^{(pc)} e^{j\theta_k^{(pc)}} z^{-1})(1 - r_k^{(pc)} e^{-j\theta_k^{(pc)}} z^{-1}) \\ &= 1 - 2r_k^{(pc)} \cos(\theta_k^{(pc)}) z^{-1} + (r_k^{(pc)})^2 z^{-2} \end{aligned}$$

contain the kth real pole located at $z = r_k^{(pr)}$ and the kth complex-conjugate pole pair located at $z = r_k^{(pc)} e^{\pm j\theta_k^{(pc)}}$.

- The overall amplitude response linearly and logarithmically can be expressed as

$$|H(e^{j\omega})| = |a_0| \frac{[\prod_{k=1}^{M_r} |A_k^{(r)}(e^{j\omega})|][\prod_{k=1}^{M_c} |A_k^{(c)}(e^{j\omega})|]}{[\prod_{k=1}^{N_r} |B_k^{(r)}(e^{j\omega})|][\prod_{k=1}^{n_c} |B_k^{(c)}(e^{j\omega})|]}$$

- and

$$\begin{aligned}20\cdot\log_{10}(|H(e^{j\omega})|) &= 20\cdot\log_{10}(|a_0|)\\&+\sum_{k=1}^{M_r}20\cdot\log_{10}(|A_k^{(r)}(e^{j\omega})|)+\sum_{k=1}^{M_c}20\cdot\log_{10}(|A_k^{(c)}(e^{j\omega})|)\\&-\sum_{k=1}^{N_r}20\cdot\log_{10}(|B_k^{(r)}(e^{j\omega})|)-\sum_{k=1}^{N_c}20\cdot\log_{10}(|B_k^{(c)}(e^{j\omega})|).\end{aligned}$$

- The overall phase response is expressible as

$$\begin{aligned}\arg\ H(e^{j\omega}) =&\alpha+\sum_{k=1}^{M_r}\arg\ A_k^{(r)}(e^{j\omega})+\sum_{k=1}^{M_c}\arg\ A_k^{(c)}(e^{j\omega})\\&-\sum_{k=1}^{N_r}\arg\ B_k^{(r)}(e^{j\omega})-\sum_{k=1}^{N_c}\arg\ B_k^{(c)}(e^{j\omega}),\end{aligned}$$

where $\alpha=0$ if $a_0>0$ and $\alpha=\pi$ if $a_0<0$.

- The phase and group delays are given by

$$\begin{aligned}\tau_p(\omega) =&\alpha/\omega+\sum_{k=1}^{M_r}\tau_p^{(zr)}(\omega)+\sum_{k=1}^{M_c}\tau_p^{(zc)}(\omega)\\&+\sum_{k=1}^{N_r}\tau_p^{(pr)}(\omega)+\sum_{k=1}^{N_c}\tau_p^{(pc)}(\omega)\end{aligned}$$

and

$$\begin{aligned}\tau_g(\omega) =&\sum_{k=1}^{M_r}\tau_g^{(zr)}(\omega)+\sum_{k=1}^{M_c}\tau_g^{(zc)}(\omega)\\&+\sum_{k=1}^{N_r}\tau_g^{(pr)}(\omega)+\sum_{k=1}^{N_c}\tau_g^{(pc)}(\omega).\end{aligned}$$

- After some manipulations,

$$|A_k^{(r)}(e^{j\omega})| = \sqrt{1 + (r_k^{(zr)})^2 - 2r_k^{(zr)}\cos(\omega)}$$

$$|A_k^{(c)}(e^{j\omega})| = \sqrt{1 + (r_k^{(zr)})^2 - 2r_k^{(zc)}\cos(\omega - \theta_k^{(zc)})} \times \sqrt{1 + (r_k^{(zc)})^2 - 2r_k^{(zc)}\cos(\omega + \theta_k^{(zc)})}$$

$$|B_k^{(r)}(e^{j\omega})| = \sqrt{1 + (r_k^{(pr)})^2 - 2r_k^{(pr)}\cos(\omega)}$$

$$|B_k^{(c)}(e^{j\omega})| = \sqrt{1 + (r_k^{(pc)})^2 - 2r_k^{(pr)}\cos(\omega - \theta_k^{(pc)})} \times \sqrt{1 + (r_k^{(pc)})^2 - 2r_k^{(pc)}\cos(\omega + \theta_k^{(pc)})}$$

$$\arg\ A_k^{(r)}(e^{j\omega}) = \tan^{-1}\left[\frac{r_k^{(zr)}\sin(\omega)}{1 - r_k^{(zr)}\cos(\omega)}\right]$$

$$\arg\ A_k^{(c)}(e^{j\omega}) = \tan^{-1}\left[\frac{r_k^{(zc)}\sin(\omega - \theta_k^{(zc)})}{1 - r_k^{(zc)}\cos(\omega - \theta_k^{(zc)})}\right] + \tan^{-1}\left[\frac{r_k^{(zc)}\sin(\omega + \theta_k^{(zc)})}{1 - r_k^{(zc)}\cos(\omega + \theta_k^{(zc)})}\right]$$

$$\arg\ B_k^{(r)}(e^{j\omega}) = \tan^{-1}\left[\frac{r_k^{(pr)}\sin(\omega)}{1 - r_k^{(pr)}\cos(\omega)}\right]$$

$$\arg\ B_k^{(c)}(e^{j\omega}) = \tan^{-1}\left[\frac{r_k^{(pc)}\sin(\omega - \theta_k^{(pc)})}{1 - r_k^{(pc)}\cos(\omega - \theta_k^{(pc)})}\right] + \tan^{-1}\left[\frac{r_k^{(pc)}\sin(\omega + \theta_k^{(pc)})}{1 - r_k^{(pc)}\cos(\omega + \theta_k^{(pc)})}\right].$$

- Furthermore,

$$\tau_g^{(zr)}(\omega) = \frac{(r_k^{(zr)})^2 - r_k^{(zr)}\cos(\omega)}{1 + (r_k^{(zr)})^2 - 2r_k^{(zr)}\cos(\omega)}$$

$$\begin{aligned}\tau_g^{(zc)}(\omega) =& \frac{(r_k^{(zc)})^2 - r_k^{(zc)}\cos(\omega - \theta_k^{(zc)})}{1 + (r_k^{(zc)})^2 - 2r_k^{(zc)}\cos(\omega - \theta_k^{(zc)})} \\ &+ \frac{(r_k^{(zc)})^2 - r_k^{(zc)}\cos(\omega + \theta_k^{(zc)})}{1 + (r_k^{(zc)})^2 - 2r_k^{(zc)}\cos(\omega + \theta_k^{(zc)})}\end{aligned}$$

$$\tau_g^{(pr)}(\omega) = -\frac{(r_k^{(pr)})^2 - r_k^{(pr)}\cos(\omega)}{1 + (r_k^{(pr)})^2 - 2r_k^{(pr)}\cos(\omega)}$$

$$\begin{aligned}\tau_g^{(pc)}(\omega) =& -\frac{(r_k^{(pc)})^2 - r_k^{(pc)}\cos(\omega - \theta_k^{(pc)})}{1 + (r_k^{(pc)})^2 - 2r_k^{(pc)}\cos(\omega - \theta_k^{(pc)})} \\ &- \frac{(r_k^{(pc)})^2 - r_k^{(pc)}\cos(\omega + \theta_k^{(pc)})}{1 + (r_k^{(pc)})^2 - 2r_k^{(pc)}\cos(\omega + \theta_k^{(pc)})}\end{aligned}$$

$$\tau_p^{(zr)}(\omega) = -\frac{\arg\ A_k^{(r)}(e^{j\omega})}{\omega}$$

$$\tau_p^{(zc)}(\omega) = -\frac{\arg\ A_k^{(c)}(e^{j\omega})}{\omega}$$

$$\tau_p^{(pr)}(\omega) = \frac{\arg\ B_k^{(r)}(e^{j\omega})}{\omega}$$

$$\tau_p^{(pc)}(\omega) = \frac{\arg\ B_k^{(c)}(e^{j\omega})}{\omega}.$$

RESPONSES FOR POLE AND ZERO GROUPS

- The next 24 pages give amplitude, phase, phase delay, and group delay responses for the following six cases:

1. $H(z) = 1/[1 - 2r\cos(\theta)z^{-1} + r^2z^{-2}]$, that is, a pole pair at $z = re^{\pm j\theta}$.

2. $H(z) = 1 - 2r\cos(\theta)z^{-1} + r^2z^{-2}$, that is, a zero pair at $z = re^{\pm j\theta}$.

3.

$$\begin{aligned} H(z) =& (1 + z^{-4}) - [2(r + \frac{1}{r})\cos\theta](z^{-1} + z^{-3}) \\ &+ [r^2 + \frac{1}{r^2} + 4\cos^2\theta]z^{-2}, \end{aligned}$$

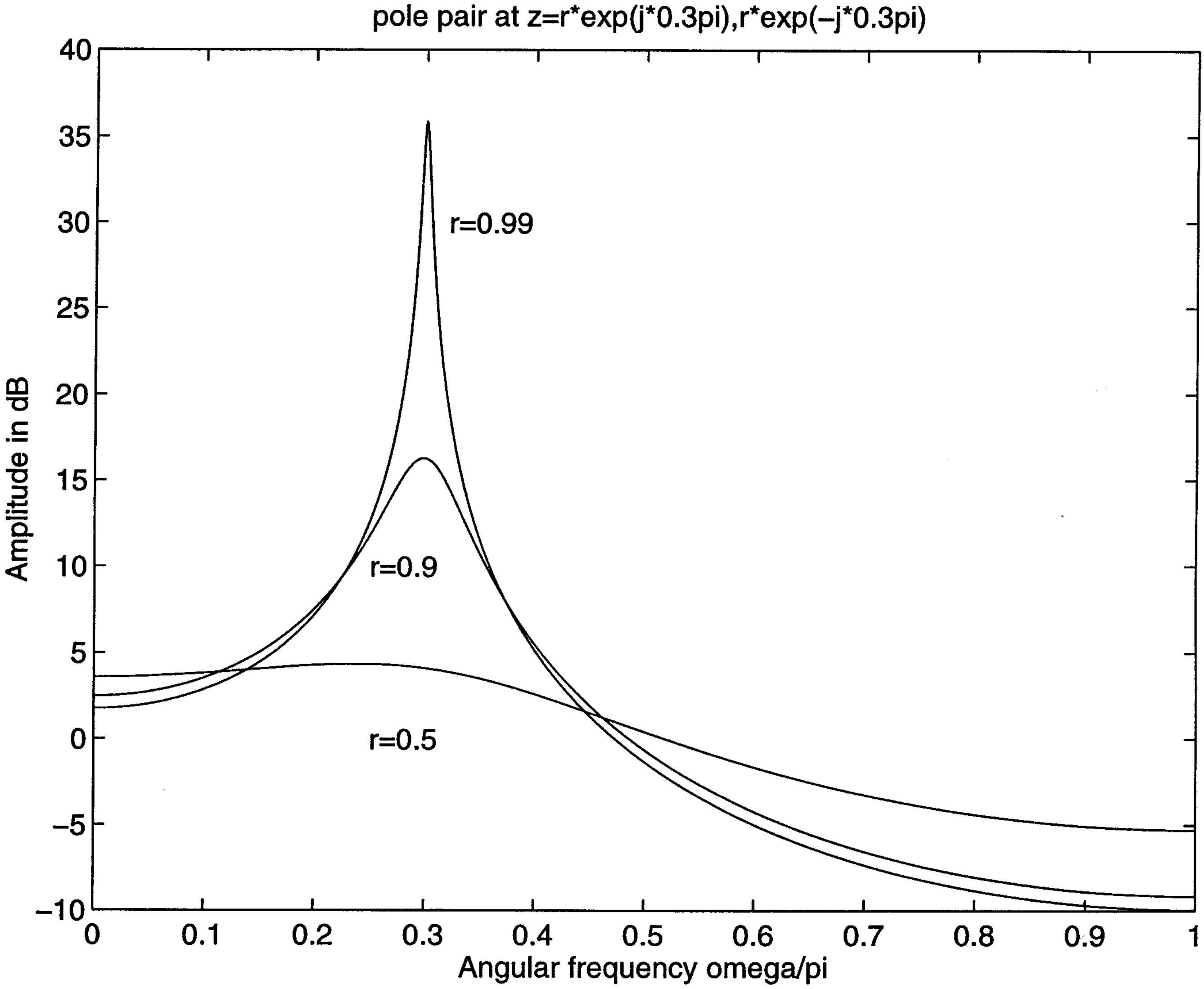
that is, a zero quadruplet at $z = re^{\pm j\theta}, (1/r)e^{\pm j\theta}$.

4. $H(z) = 1/[1 - rz^{-1}]$, that is, a pole at $z = r$.

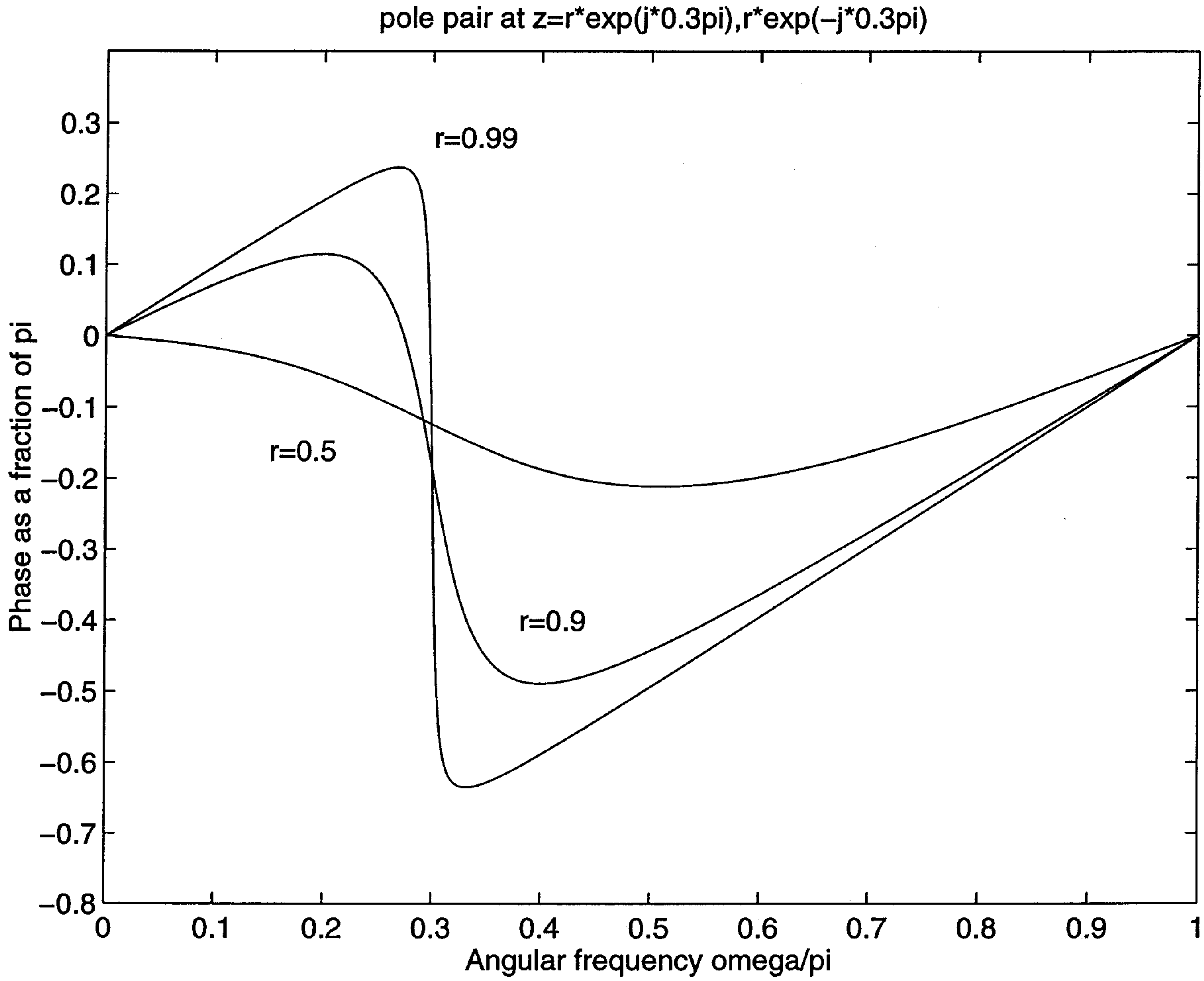
5. $H(z) = 1 - rz^{-1}$, that is, a zero at $z = r$.

6. $H(z) = (r + 1/r)z^{-1} + (1 + z^{-2})$, that is, a reciprocal zero pair at $z = r, (1/r)$.

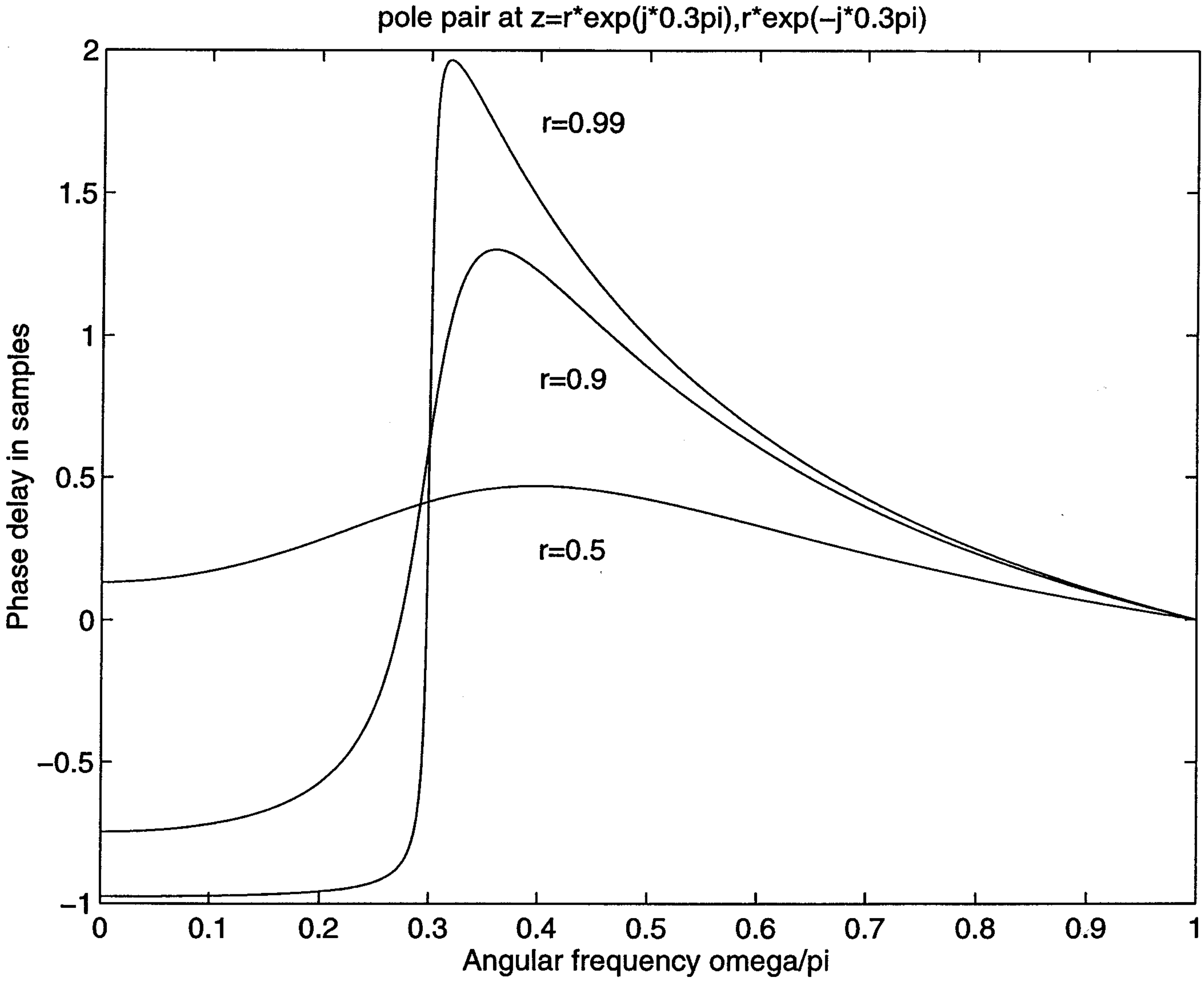
RESPONSES FOR THE COMPLEX-CONJUGATE POLE PAIR



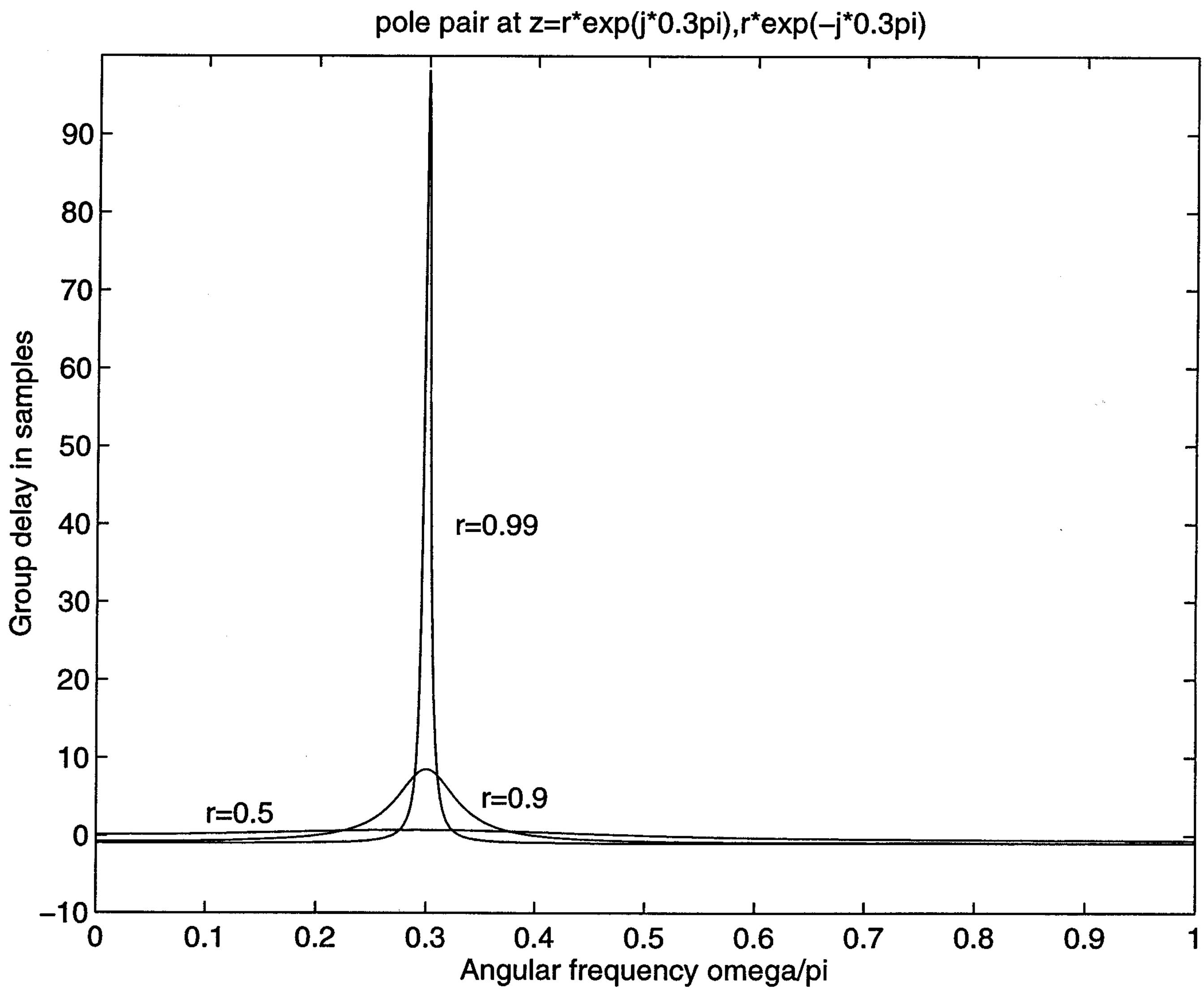
RESPONSES FOR THE COMPLEX-CONJUGATE POLE PAIR



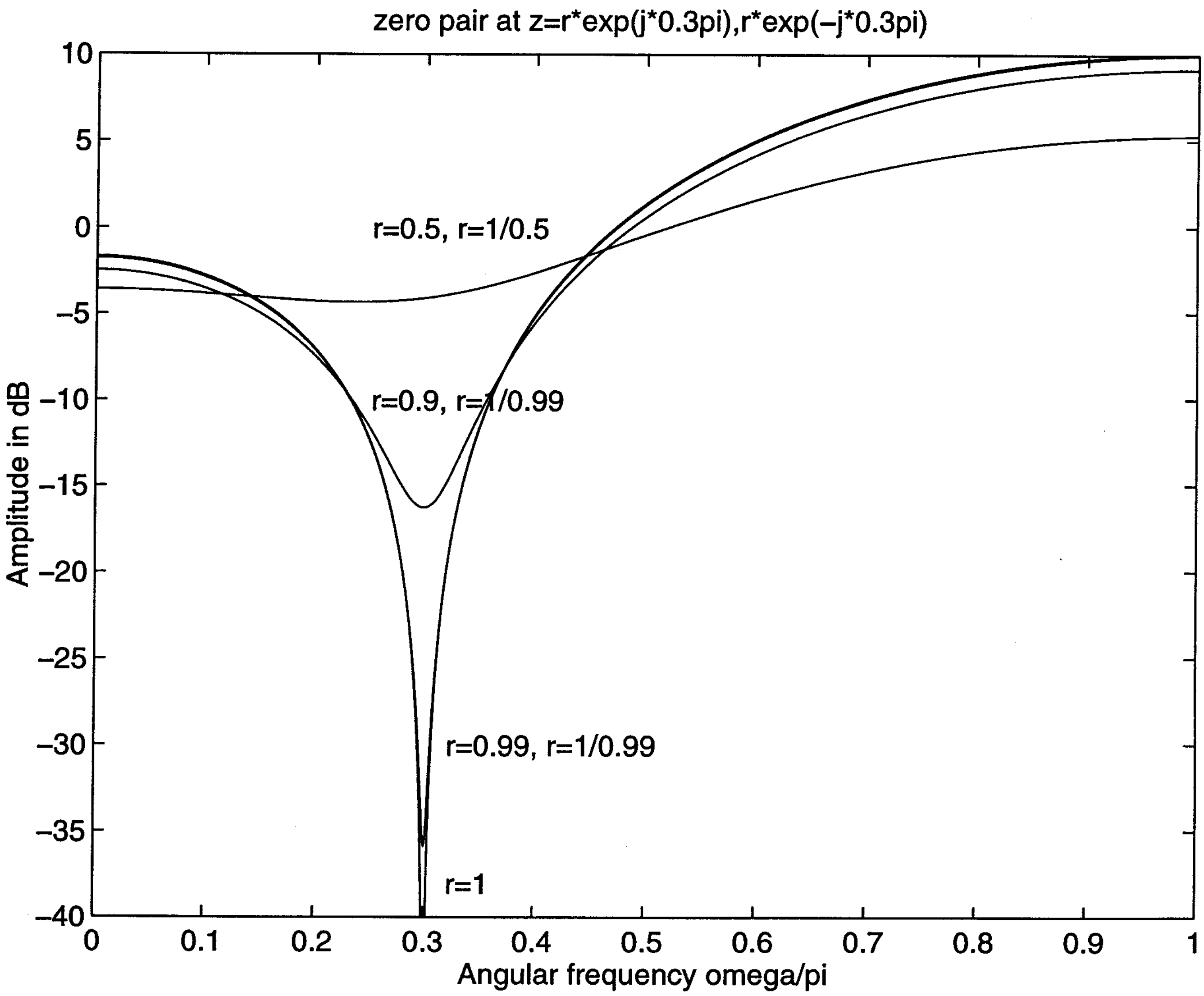
RESPONSES FOR THE COMPLEX-CONJUGATE POLE PAIR



RESPONSES FOR THE COMPLEX-CONJUGATE POLE PAIR

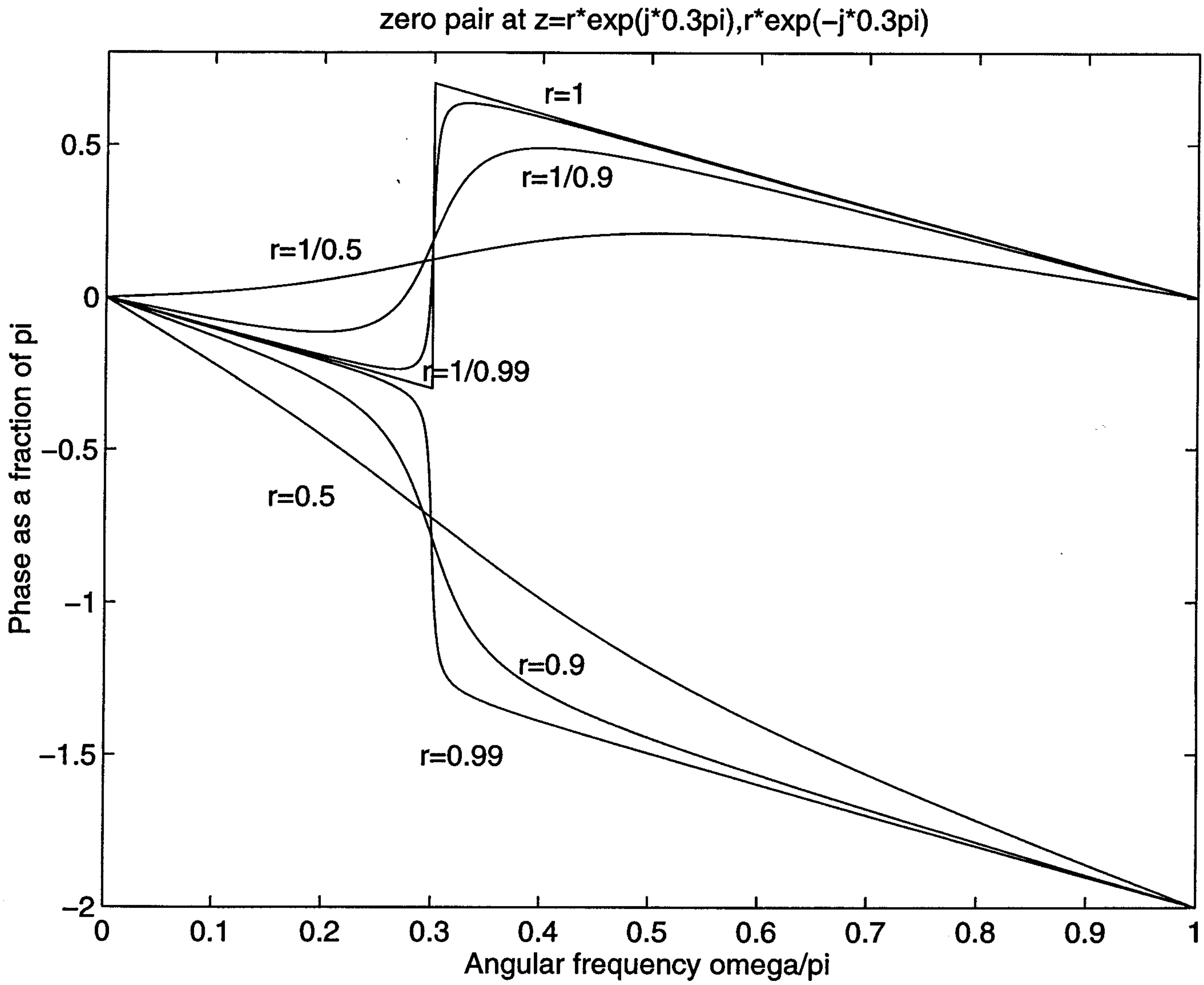


RESPONSES FOR THE COMPLEX-CONJUGATE ZERO PAIR

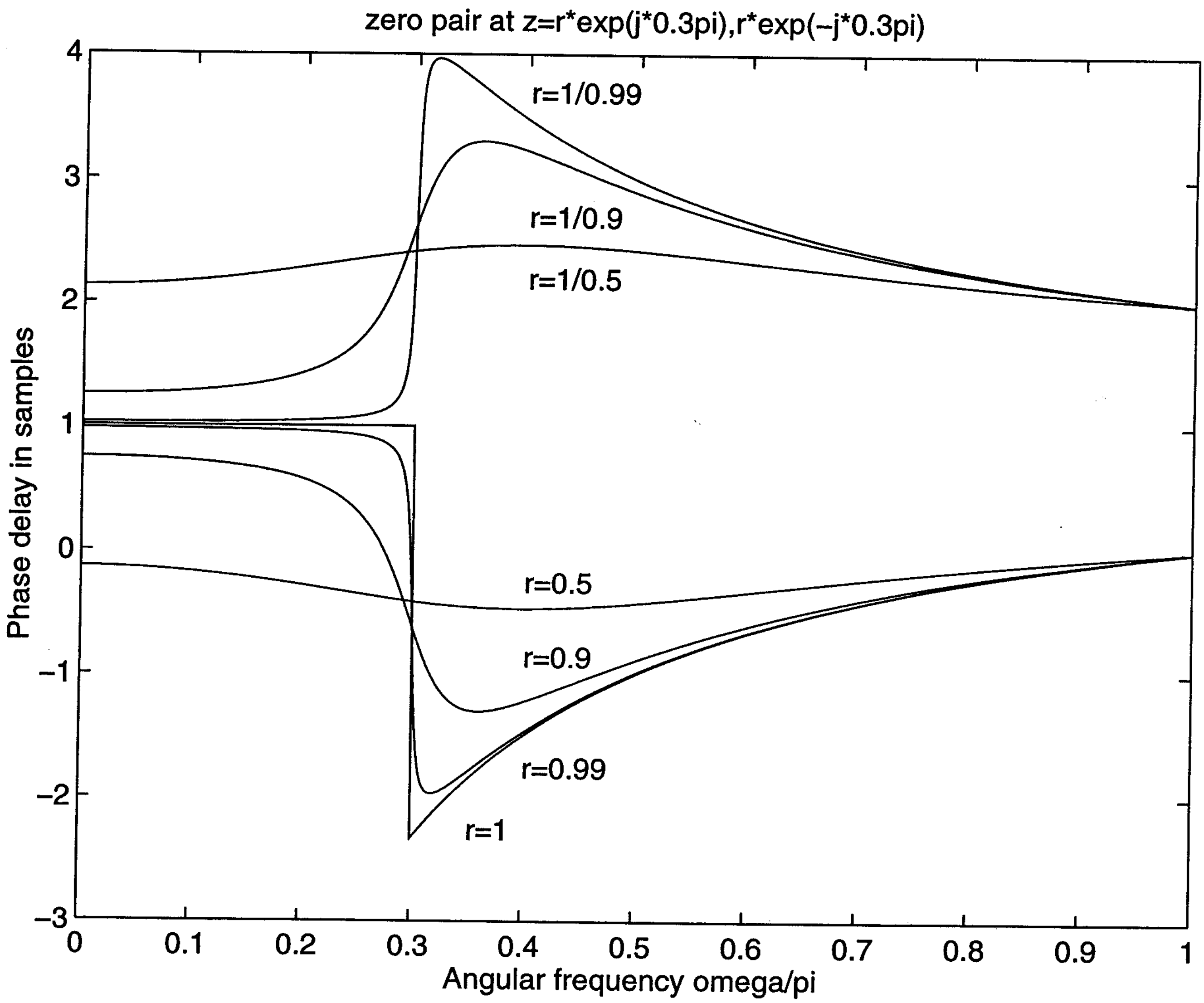


- For $r > 1$, the actual amplitude response is obtained from that of the figure by multiplying it by $(1 - 2r\cos(\theta) + r^2)/(1 - 2(1/r)\cos(\theta) + (1/r)^2)$

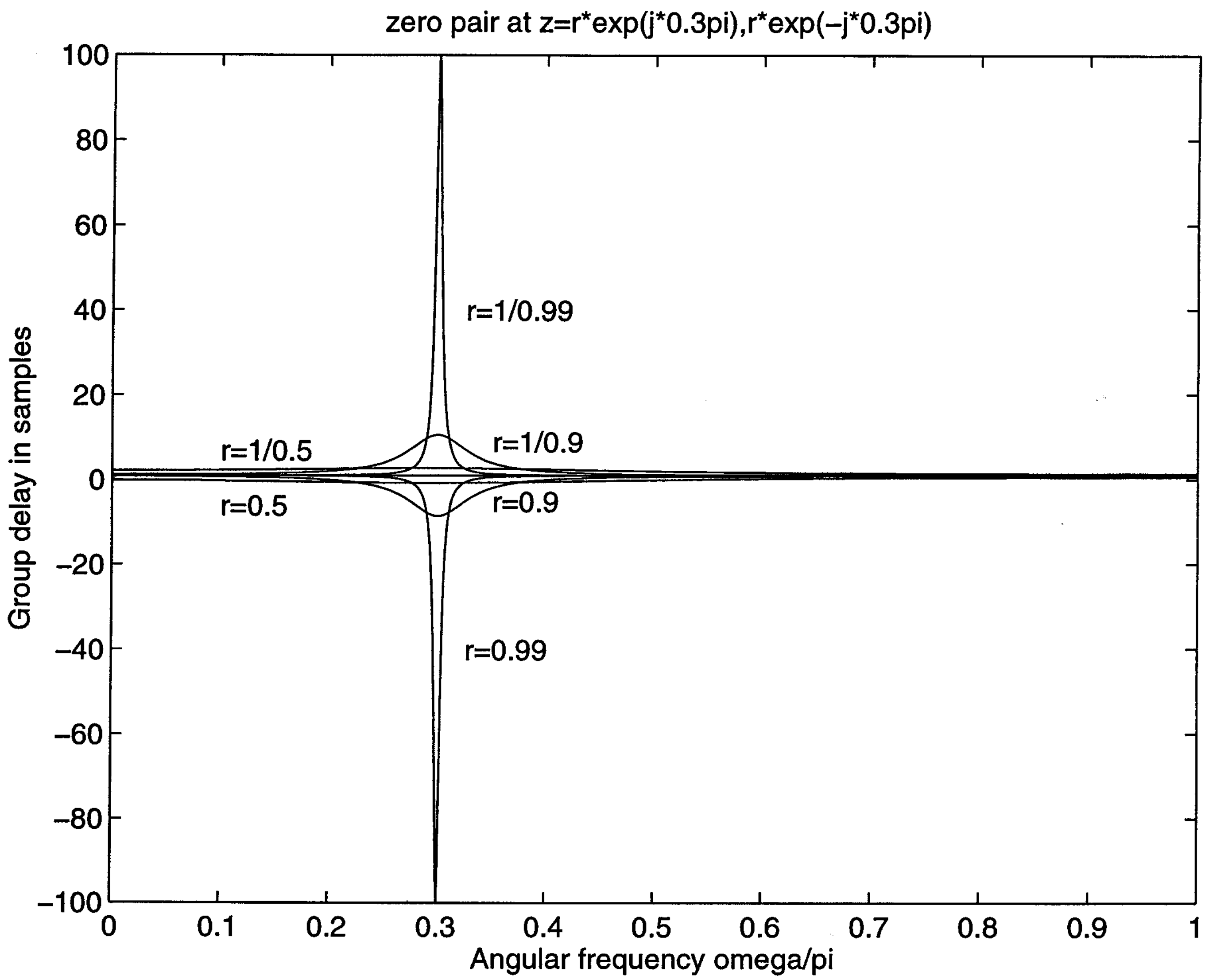
RESPONSES FOR THE COMPLEX-CONJUGATE ZERO PAIR



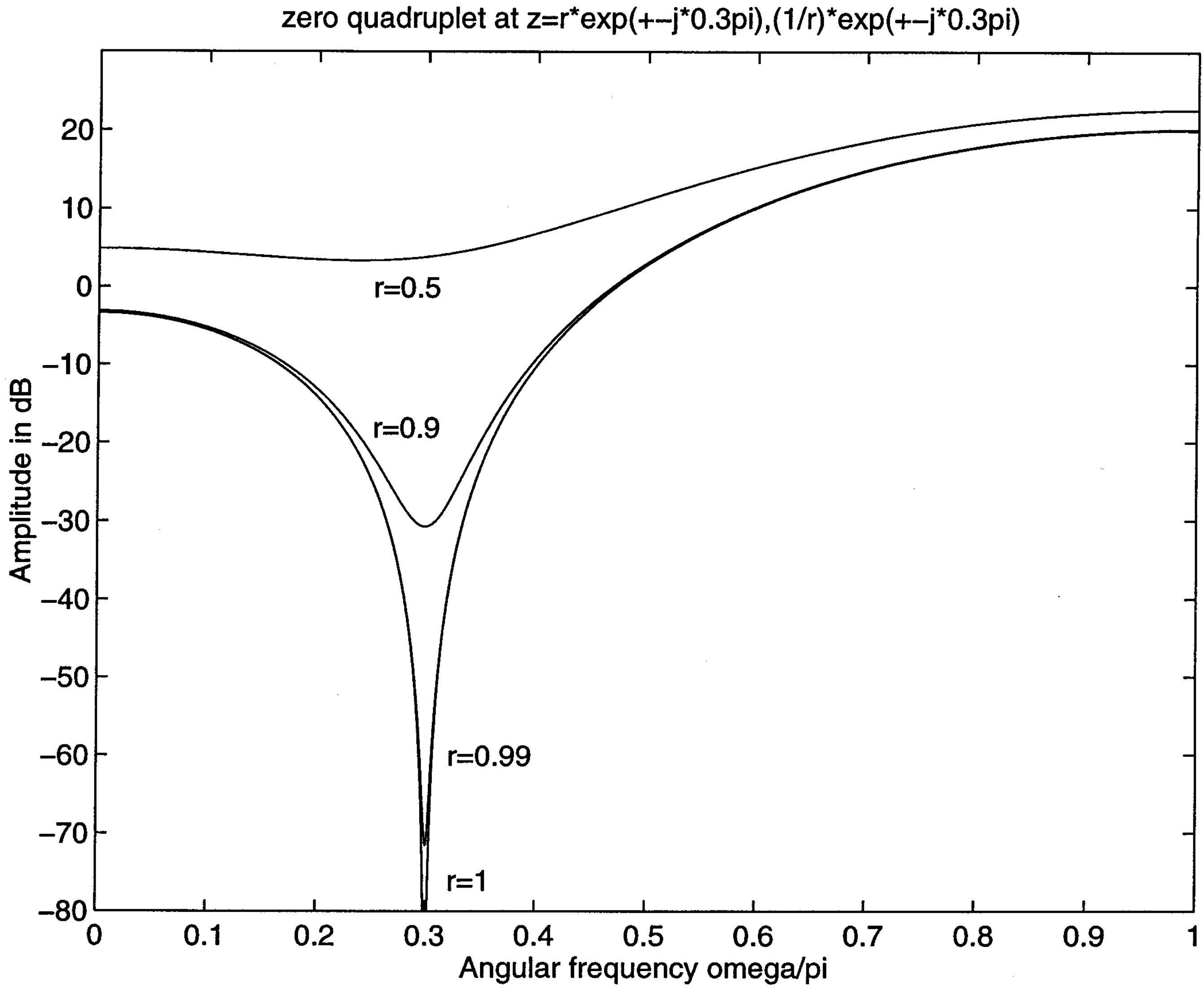
RESPONSES FOR THE COMPLEX-CONJUGATE ZERO PAIR



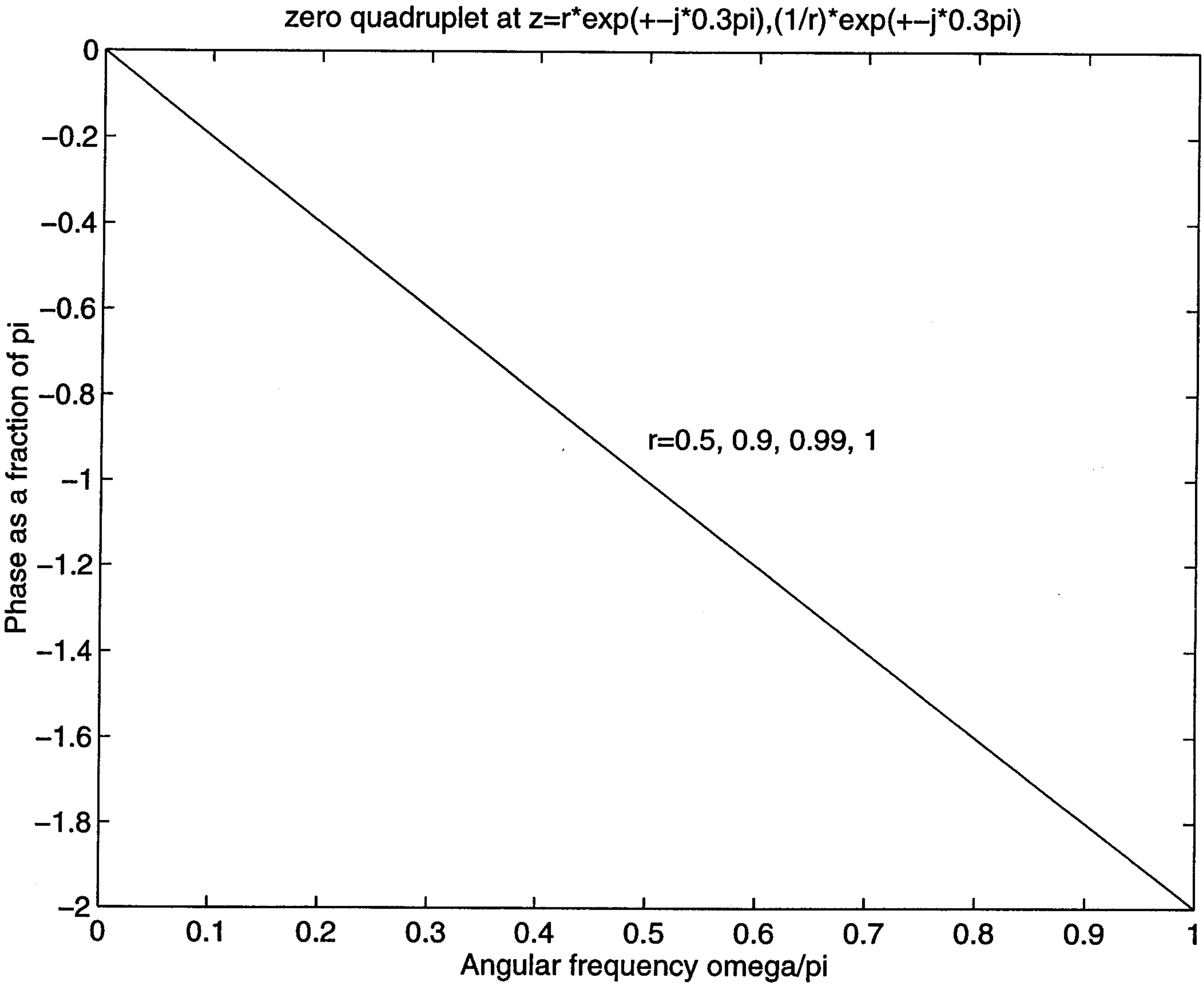
RESPONSES FOR THE COMPLEX-CONJUGATE ZERO PAIR



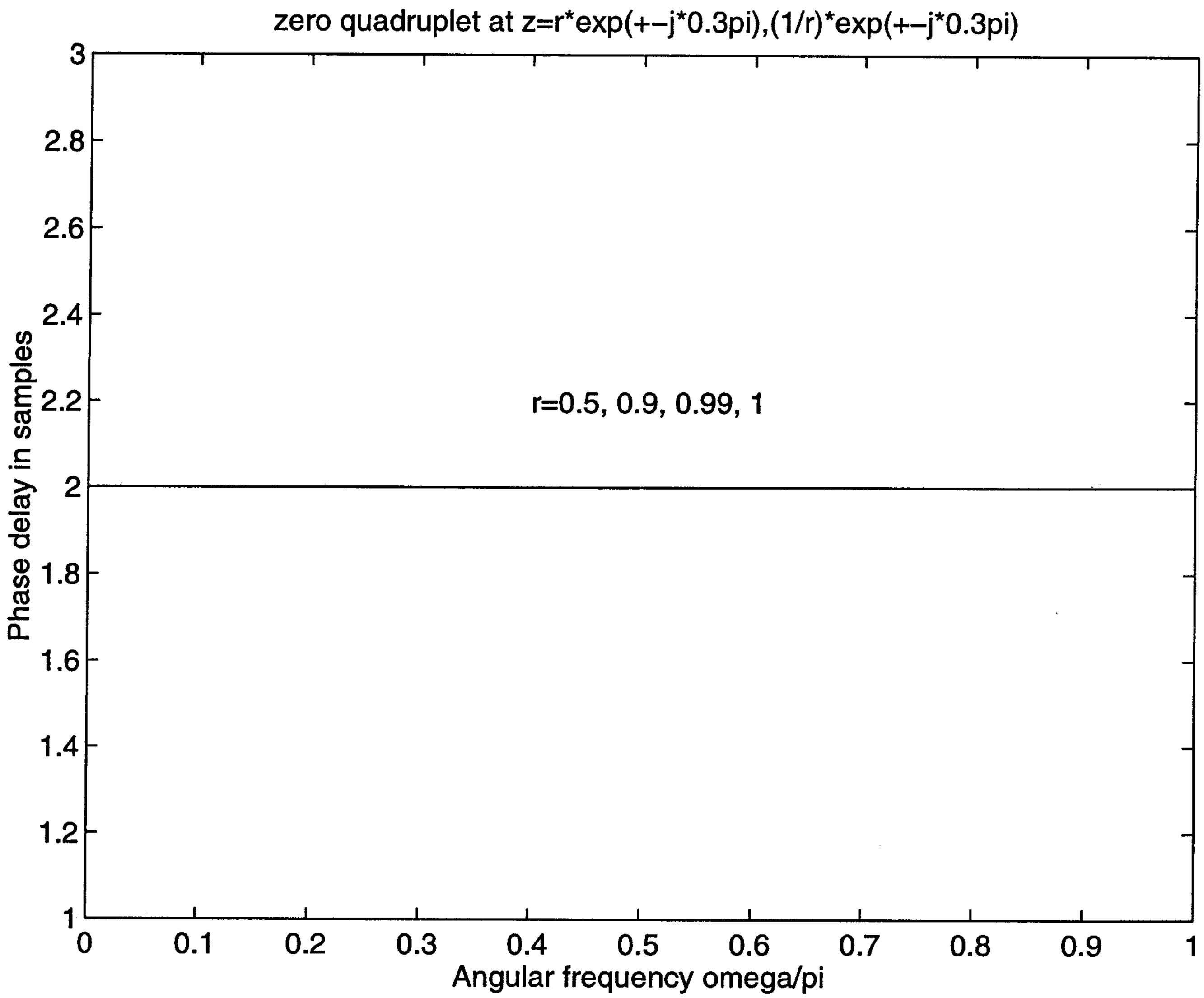
RESPONSES FOR THE ZERO QUADRUPLET



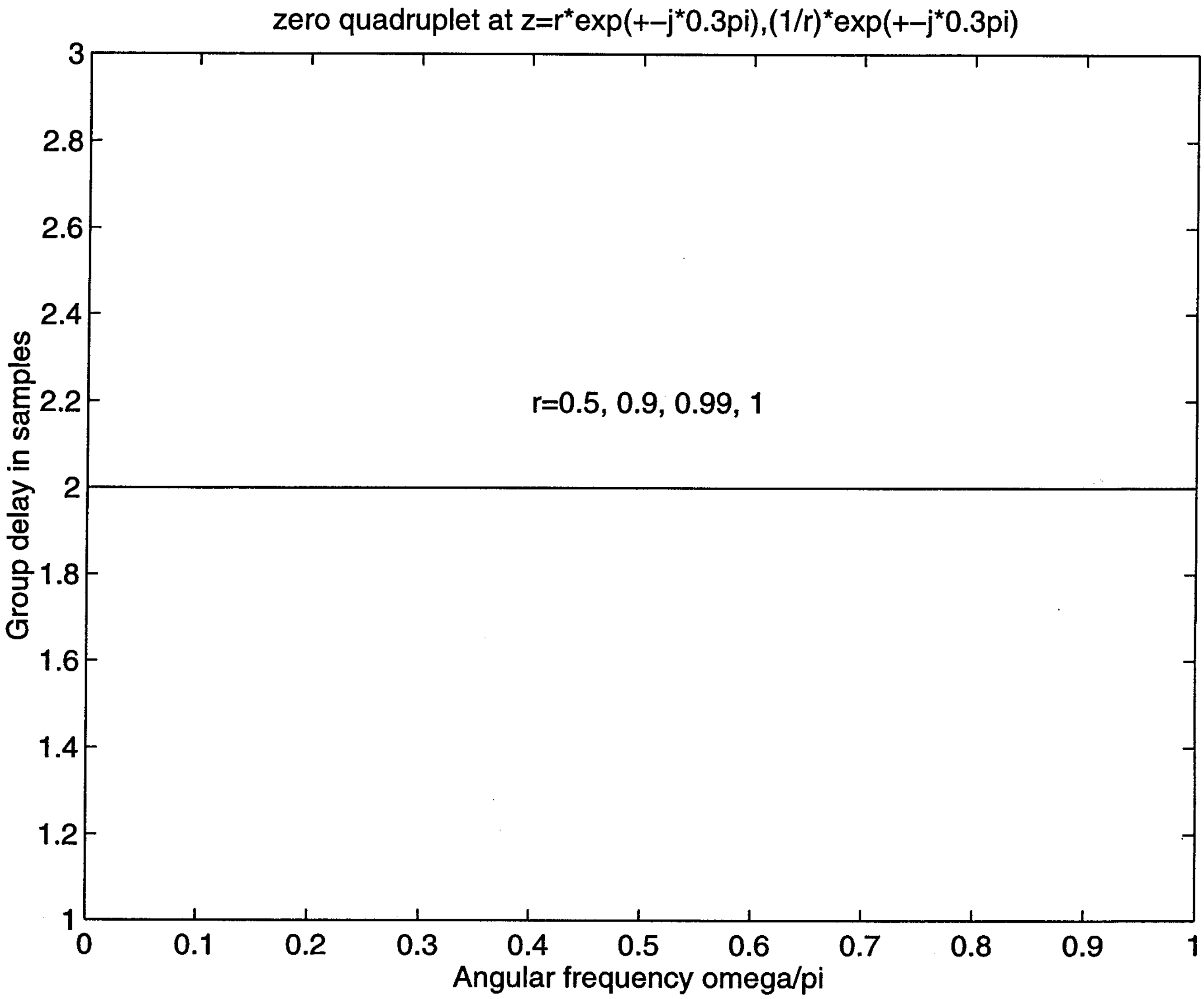
RESPONSES FOR THE ZERO QUADRUPLET



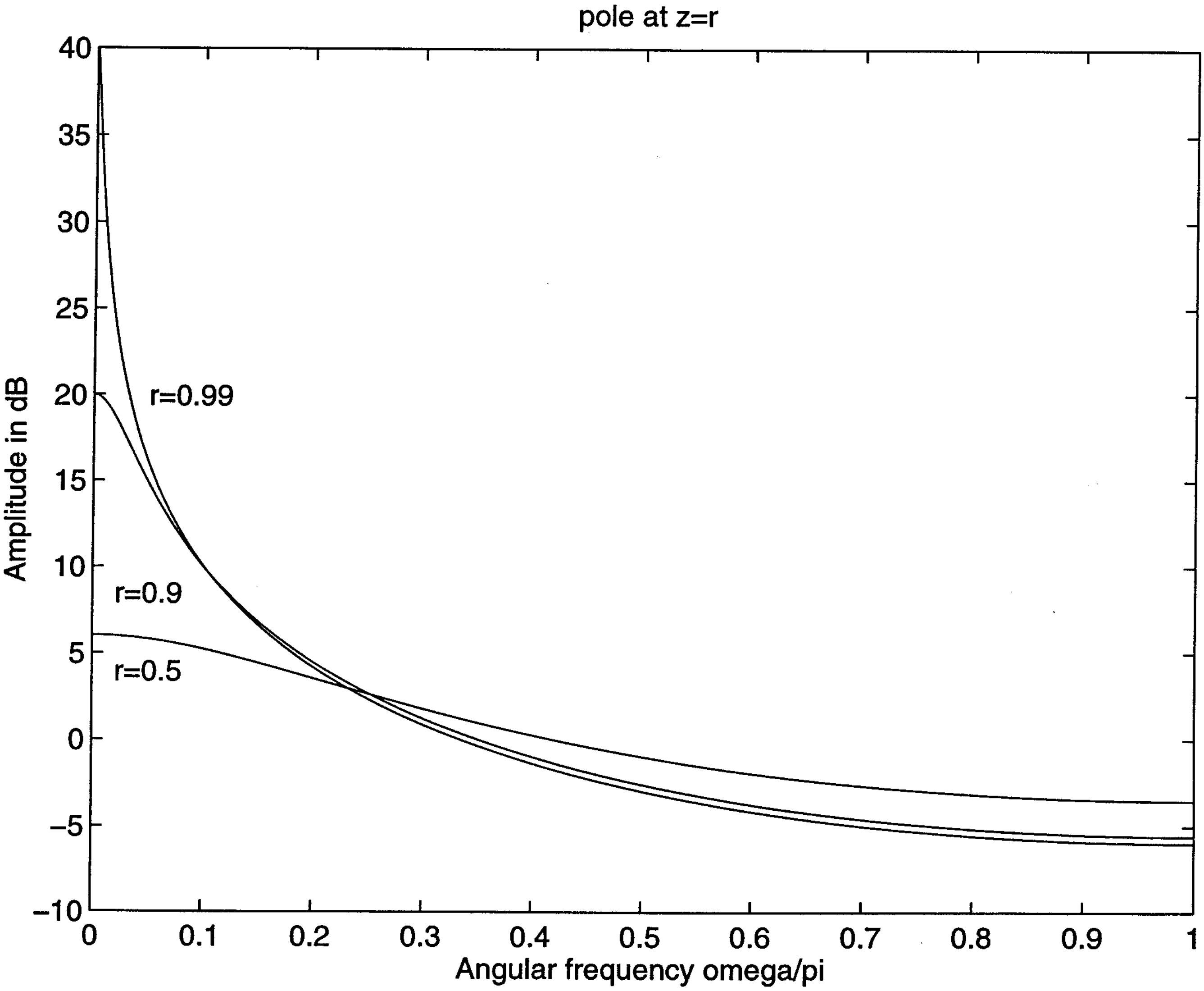
RESPONSES FOR THE ZERO QUADRUPLET



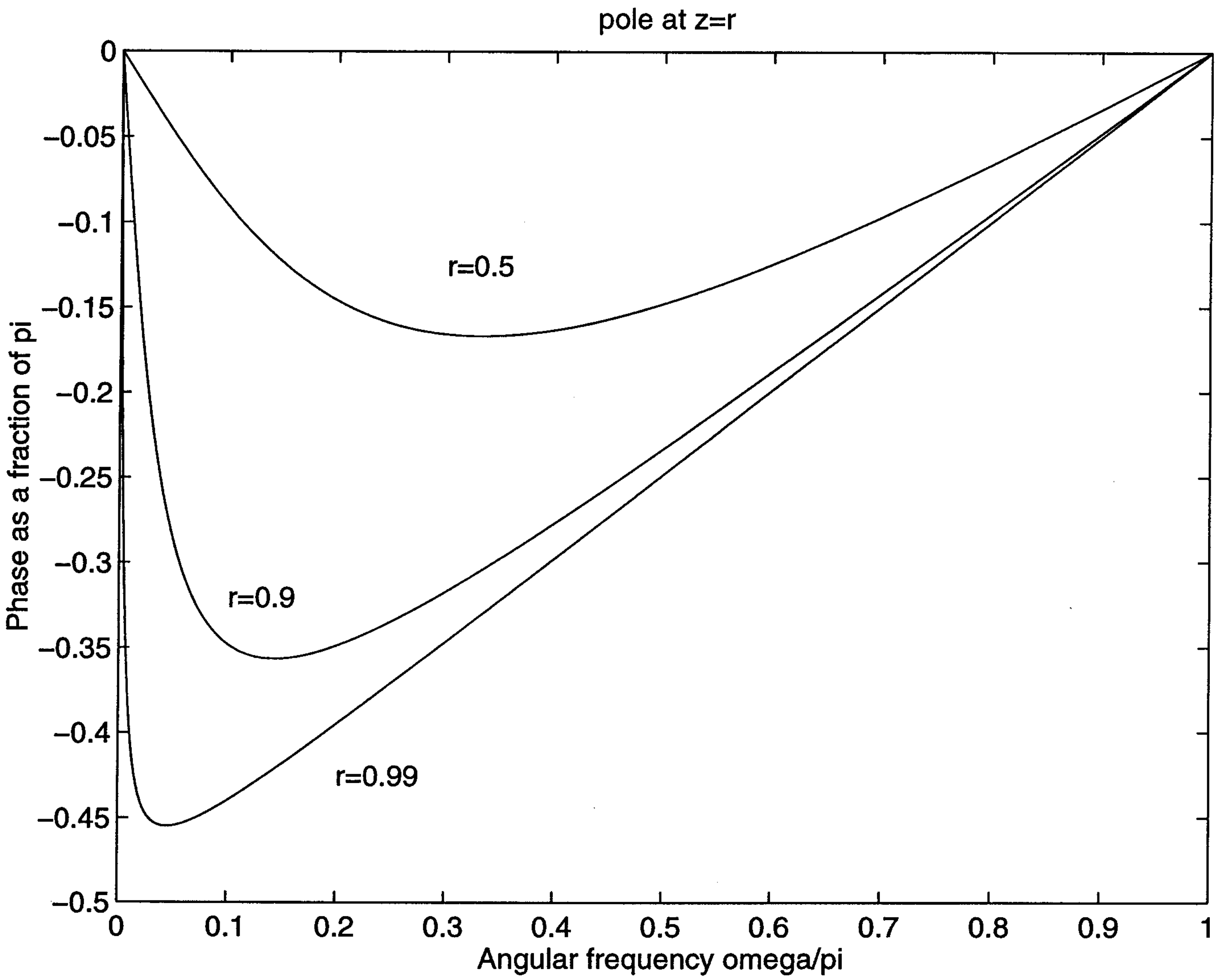
RESPONSES FOR THE ZERO QUADRUPLET



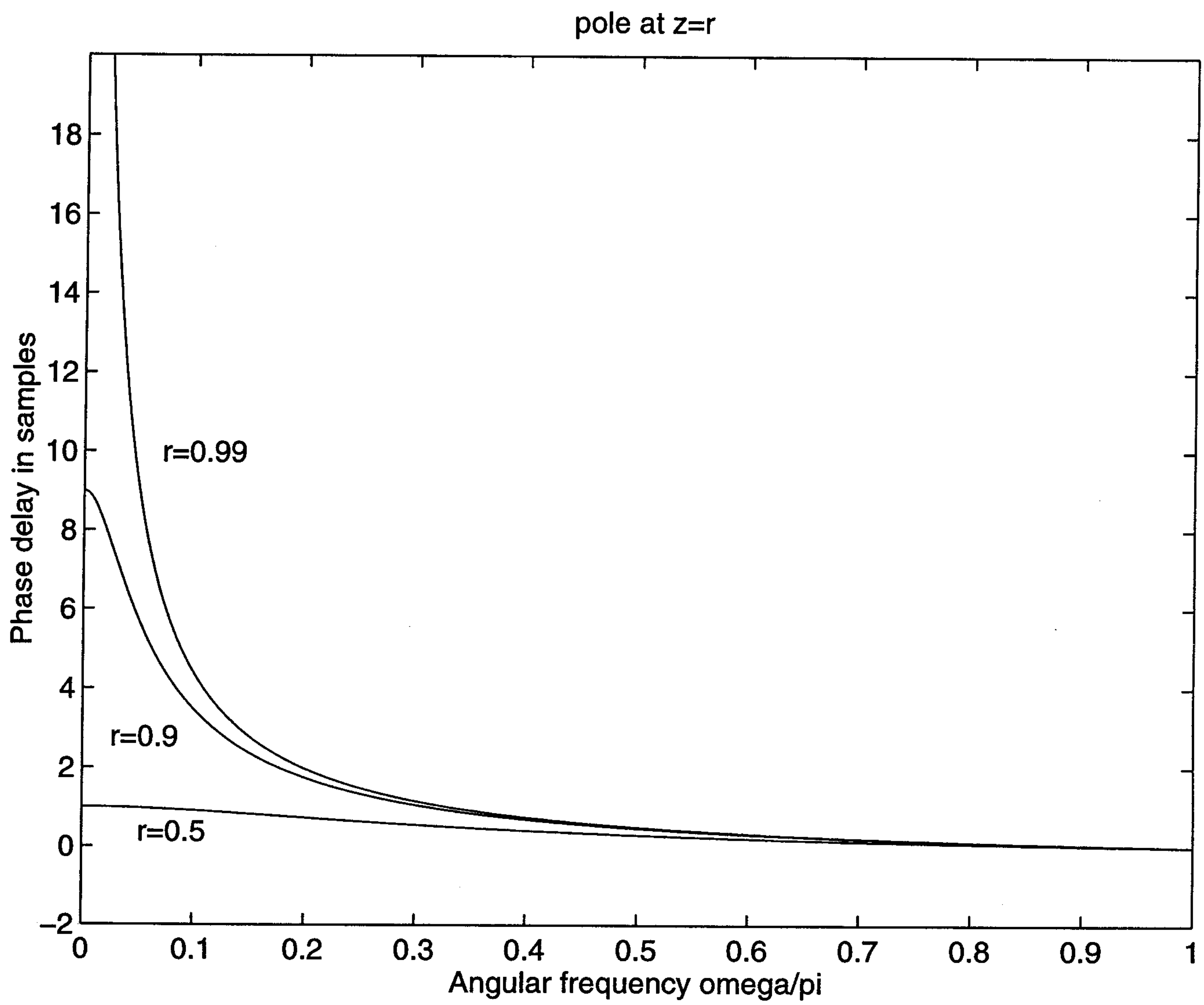
RESPONSES FOR THE REAL POLE



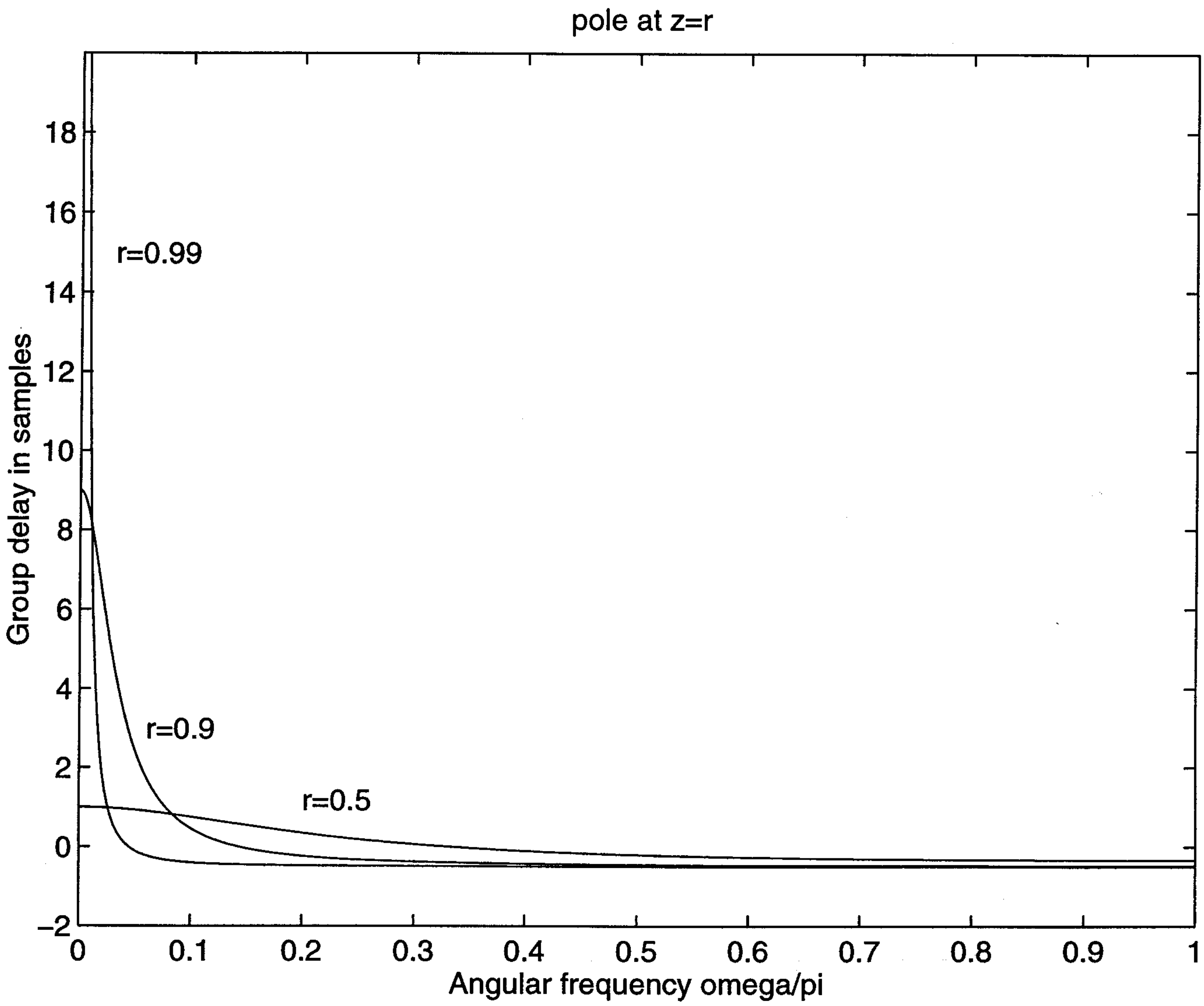
RESPONSES FOR THE REAL POLE



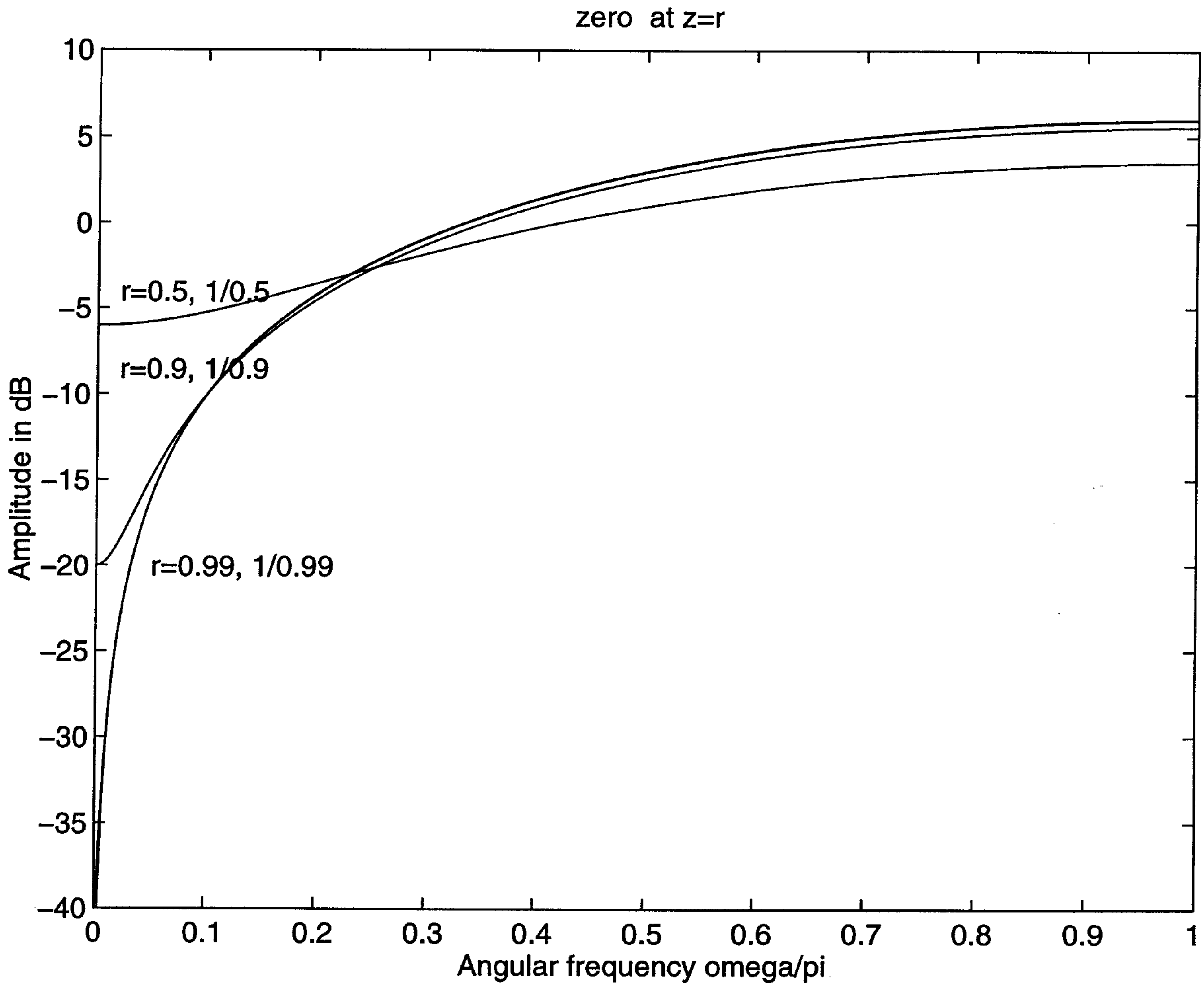
RESPONSES FOR THE REAL POLE



RESPONSES FOR THE REAL POLE

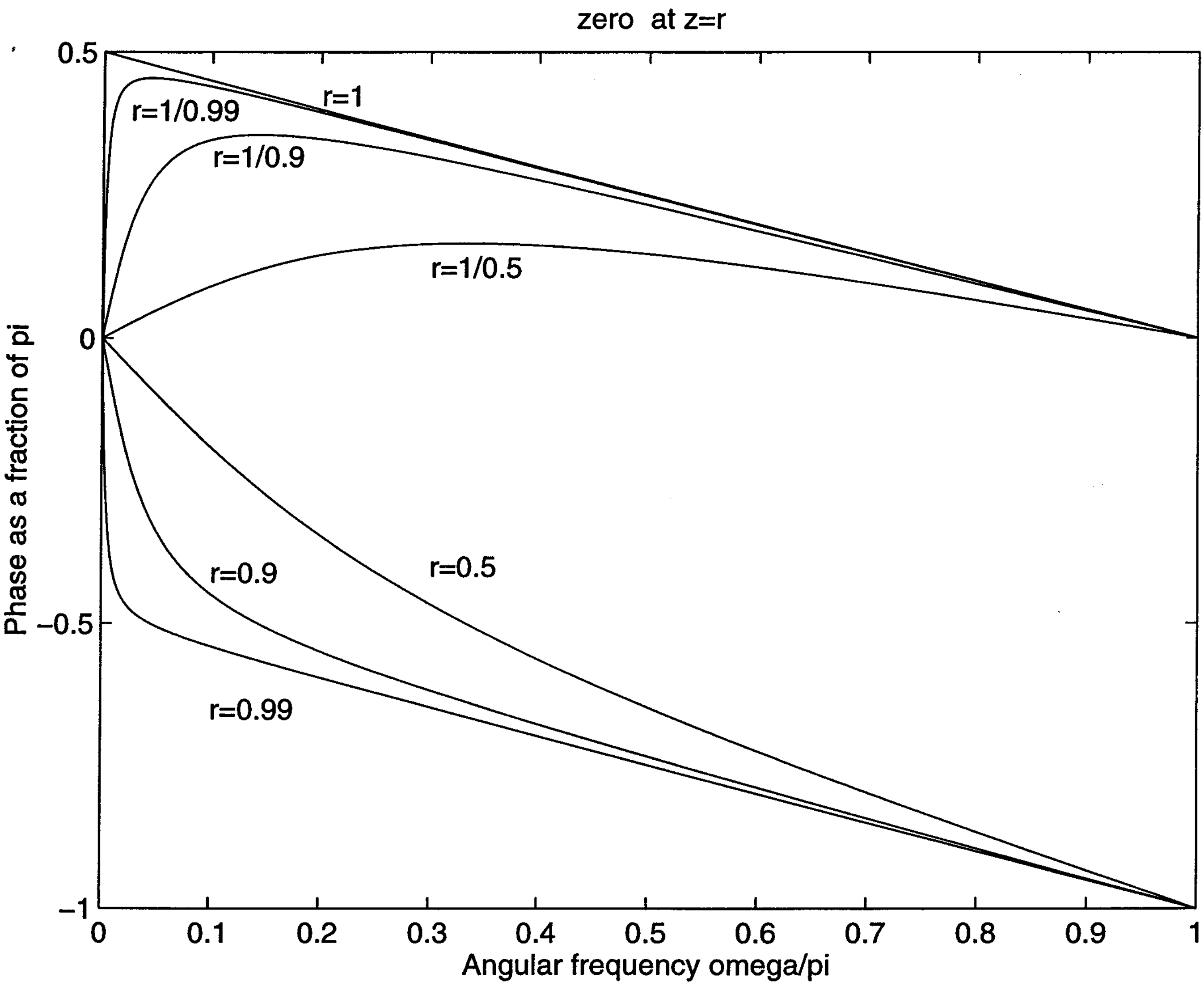


RESPONSES FOR THE REAL ZERO



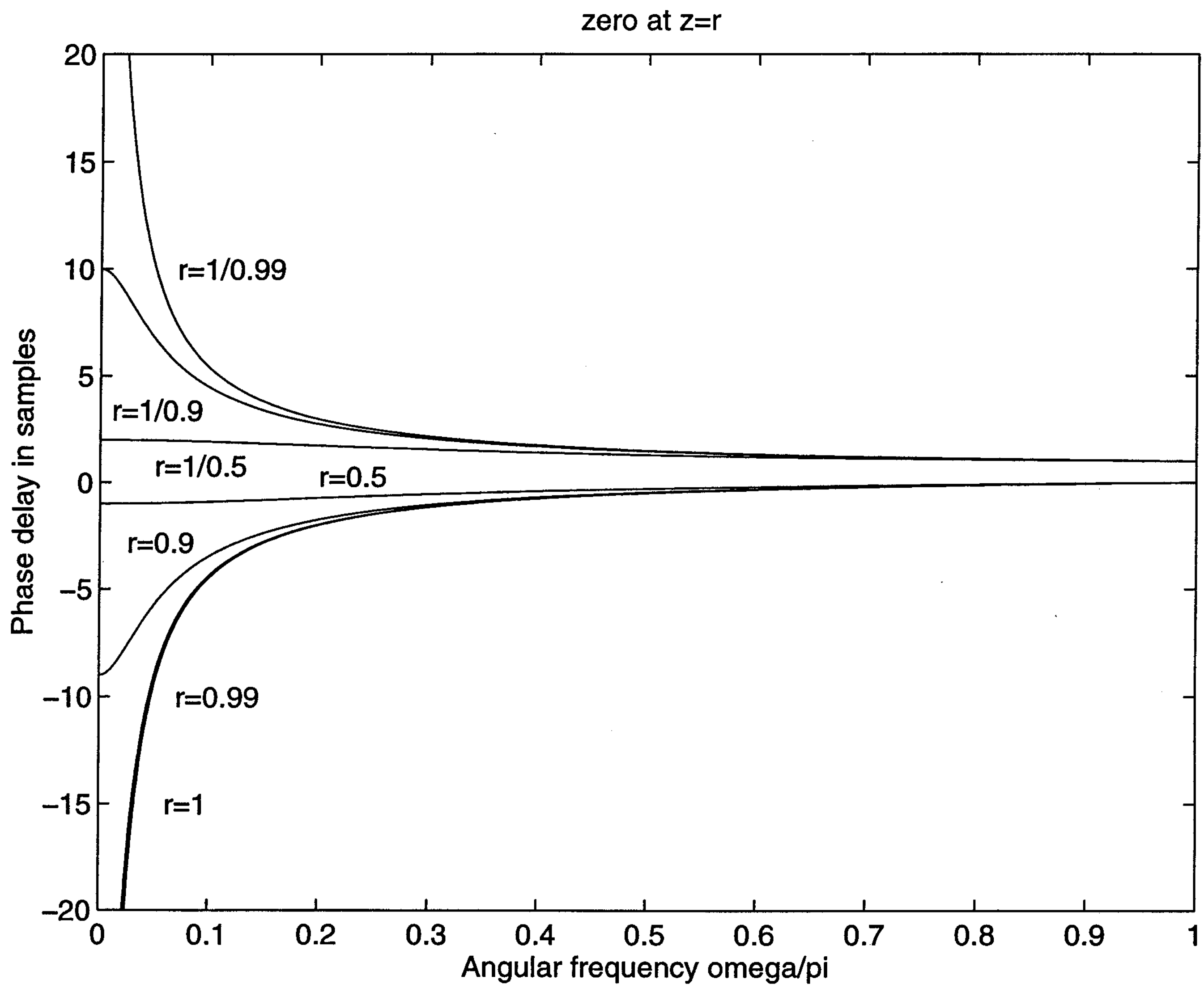
- For $r > 1$, the actual amplitude response is obtained from that of the figure by multiplying it by $(1 - r)/(1 - (1/r))$

RESPONSES FOR THE REAL ZERO

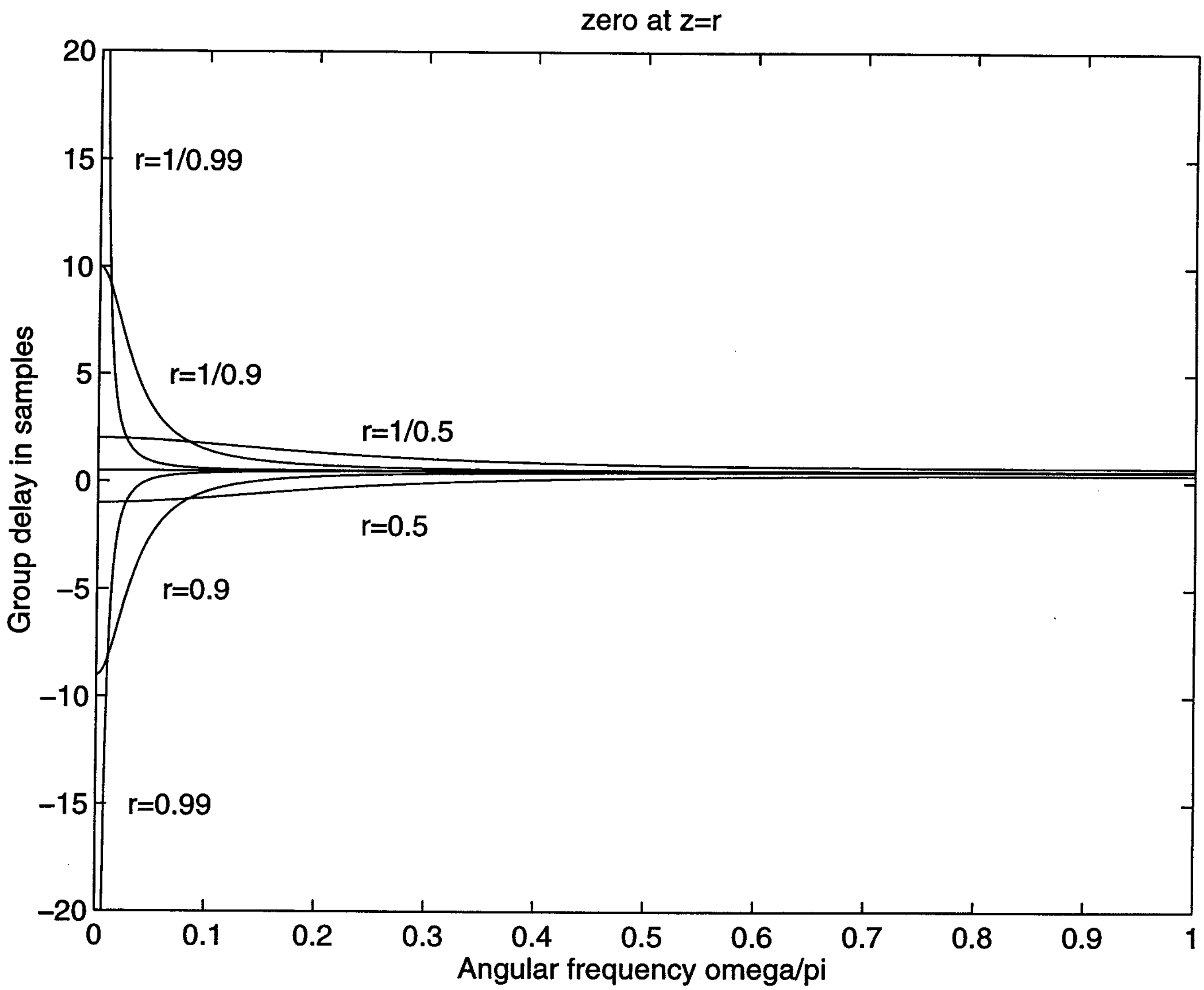


- For $r > 1$, the response is given for $-1 + rz^{-1} = -(1 - rz^{-1})$ to avoid a phase shift equal to π.

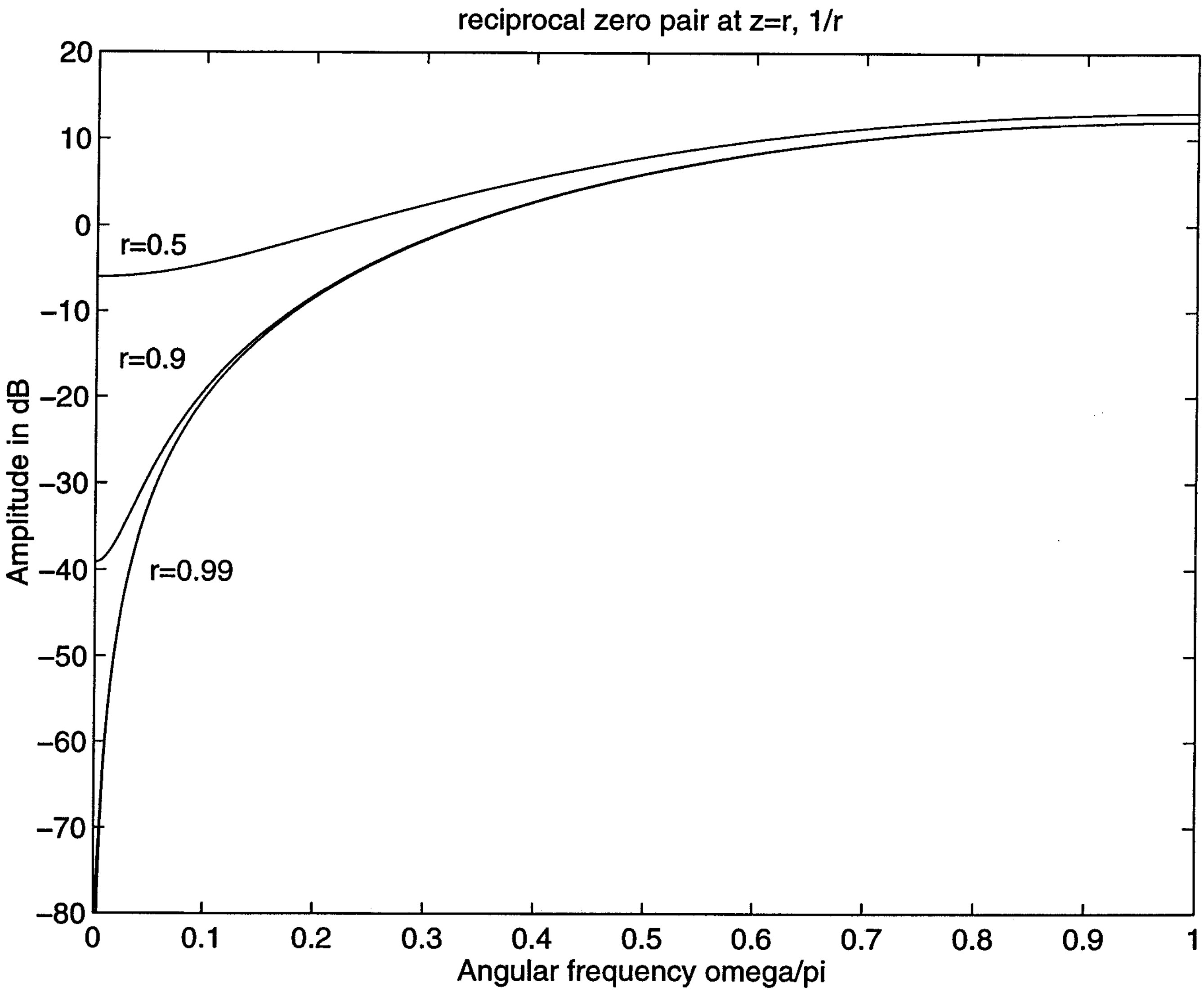
RESPONSES FOR THE REAL ZERO



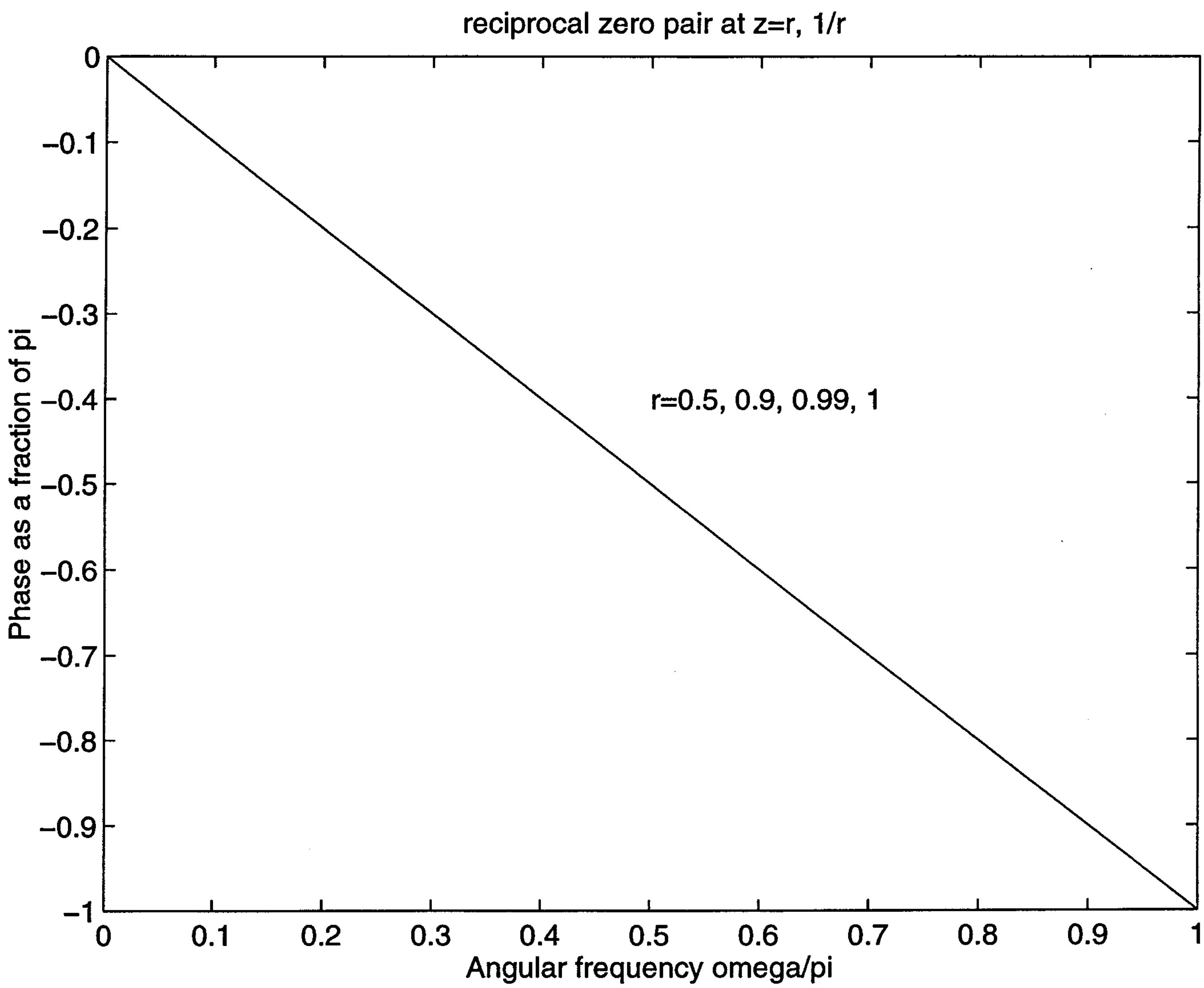
RESPONSES FOR THE REAL ZERO



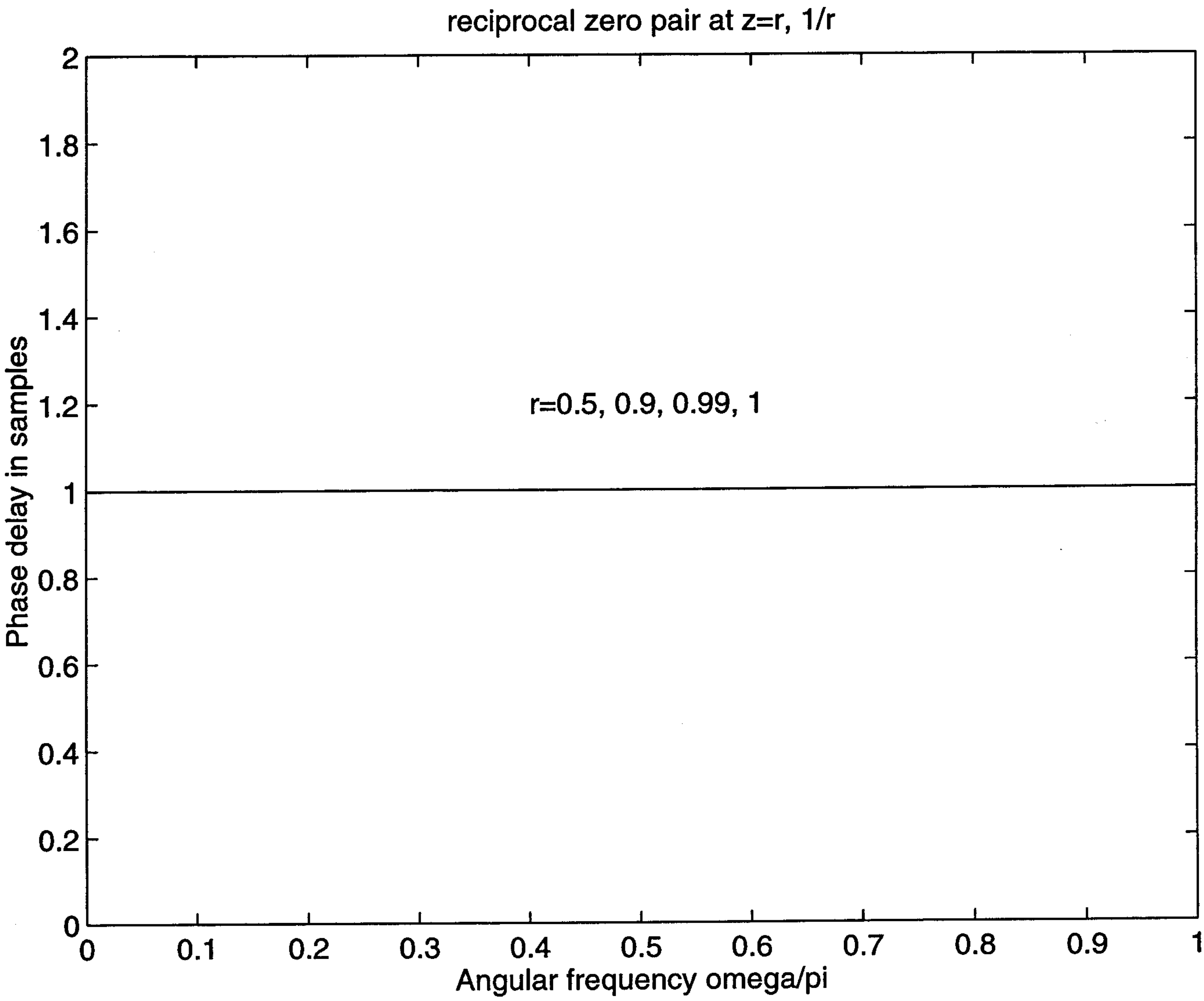
RESPONSES FOR THE RECIPROCAL ZERO PAIR



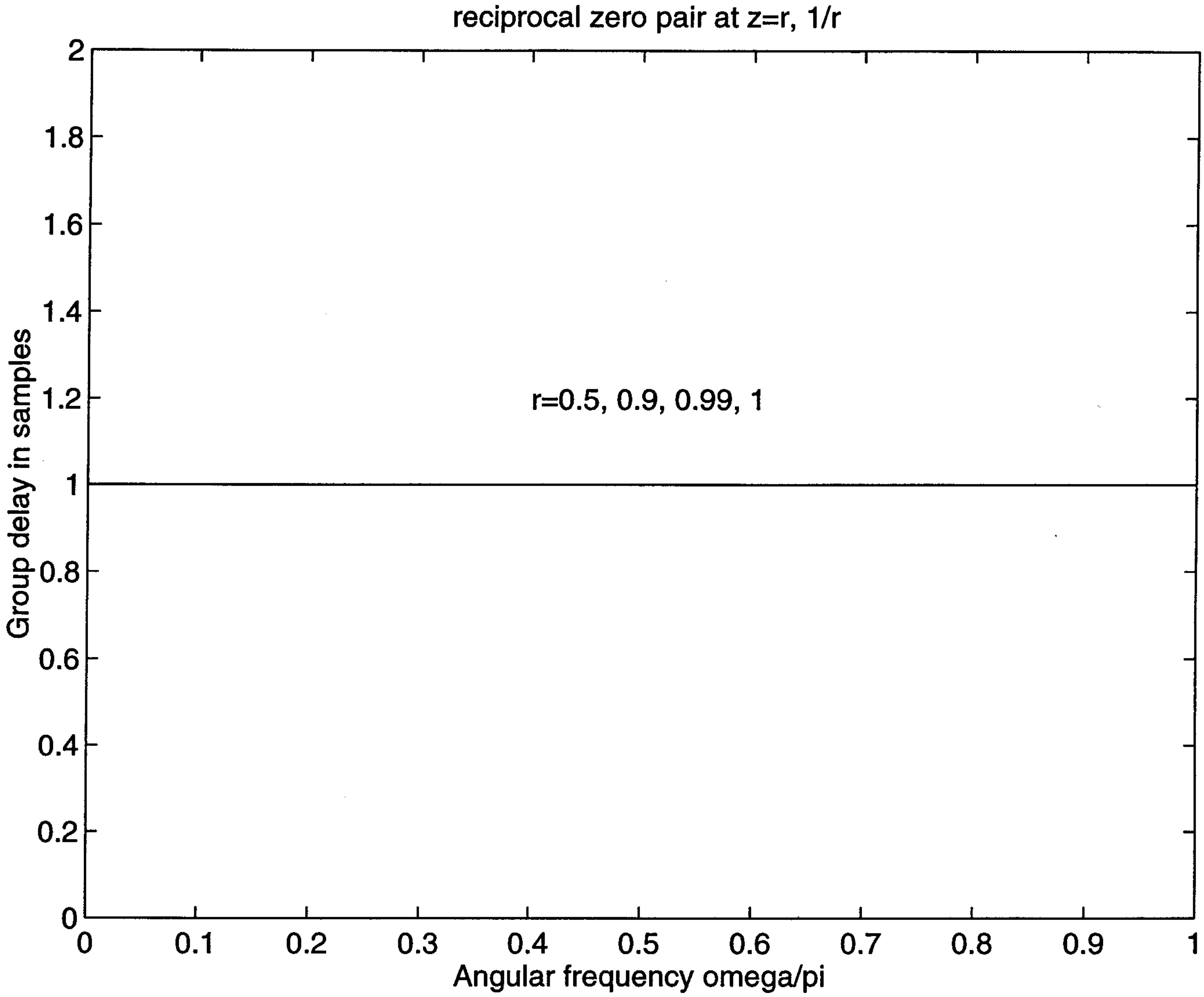
RESPONSES FOR THE RECIPROCAL ZERO PAIR



RESPONSES FOR THE RECIPROCAL ZERO PAIR



RESPONSES FOR THE RECIPROCAL ZERO PAIR



ALLPASS FILTERS

- The transfer function of an allpass filter order N is given by

$$H(z) = \frac{z^{-N}D_N(z^{-1})}{D_N(z)}$$
$$= \frac{a_N + a_{N-1}z^{-1} \cdots a_1 z^{-(N-1)} + z^{-N}}{1 + a_1 z^{-1} + \cdots + a_{N-1}z^{-(N-1)} + a_N z^{-N}}.$$

- For the stable filter, the poles [roots of $D_N(z)$] are inside the unit circle.

- These filters are characterized by the fact that if the filter has a real pole at $z = r_k^{(rp)}$, then it has a zero at $z = 1/r_k^{(rp)}$. If it has a complex conjugate pole pair at $z = r_k^{(cp)} e^{\pm j\theta_k^{(pc)}}$, then it has a zero pair at $z = (1/r_k^{(cp)})e^{\pm j\theta_k^{(pc)}}$.

- Hence, the filter is completely characterized by the pole locations.

- The special feature of these filters is that the amplitude response is identically equal to unity for all values of ω.

- Therefore, the frequency response of these filters is expressible as

$$H(e^{j\omega}) = e^{j\arg H(e^{j\omega})}.$$

- Allpass filters are very important building blocks in synthesizing various kinds of filters, such as conventional filters, filters for sampling rate alteration, Hilbert transformers, fractional delay filters, phase equalizers and so on.

- Assuming that $H(z)$ possesses N_r real poles at $z = r_k^{(rp)}$ for $k = 1, 2, \cdots, N_r$ and N_c $(N = N_r + 2N_c)$ complex conjugate pole pairs at $z = r_k^{(cp)} e^{\pm j\theta_k^{(pc)}}$, the overall transfer function is factorizable as

$$H(z) = [\prod_{k=1}^{N_r} H_k^{(r)}(z)][\prod_{k=1}^{N_c} H_k^{(c)}(z),$$

where

$$H_k^{(r)}(z) = \frac{r_k^{(pr)} - z^{-1}}{1 - r_k^{(pr)} z^{-1}}$$

and

$$H_k^{(c)}(z) = \frac{(r_k^{(pc)})^2 - 2r_k^{(pc)} \cos(\theta_k^{(pc)}) z^{-1} + z^{-2}}{1 - 2r_k^{(pc)} \cos(\theta_k^{(pc)}) z^{-1} + (r_k^{(pc)})^2 z^{-2}}.$$

- The overall phase response of the above filter is expressible as

$$\arg\ H(e^{j\omega}) = \sum_{k=1}^{N_r} \arg\ H_k^{(r)}(e^{j\omega}) + \sum_{k=1}^{N_c} \arg\ H_k^{(c)}(e^{j\omega}),$$

where

$$\arg\ H_k^{(r)}(e^{j\omega}) = -\omega - 2\tan^{-1}\left[\frac{r_k^{(pr)}\sin(\omega)}{1 - r_k^{(pr)}\cos(\omega)}\right]$$

$$\arg\ H_k^{(c)}(e^{j\omega}) = -\ 2\omega - 2\tan^{-1}\left[\frac{r_k^{(pc)}\sin(\omega - \theta_k^{(pc)})}{1 - r_k^{(pc)}\cos(\omega - \theta_k^{(pc)})}\right]$$
$$-\ 2\tan^{-1}\left[\frac{r_k^{(pc)}\sin(\omega 1s + \theta_k^{(pc)})}{1 - r_k^{(pc)}\cos(\omega + \theta_k^{(pc)})}\right].$$

- The phase delay response can be written as

$$\tau_p(\omega) = \sum_{k=1}^{N_r} \tau_p^{(rk)}(\omega) + \sum_{k=1}^{N_c} \tau_p^{(ck)}(\omega),$$

where

$$\tau_p^{(rk)}(\omega) = \frac{-\arg\ H_k^{(r)}(e^{j\omega})}{\omega}$$

and

$$\tau_p^{(ck)}(\omega) = \frac{-\arg\ H_k^{(c)}(e^{j\omega})}{\omega}.$$

- The group delay response is expressible as

$$\tau_p(\omega) = \sum_{k=1}^{N_r} \tau_p^{(rk)}(\omega) + \sum_{k=1}^{N_c} \tau_p^{(ck)}(\omega),$$

where

$$\tau_p^{(rk)}(\omega) = 1 - 2\frac{(r_k^{(pr)})^2 - r_k^{(pr)}\cos(\omega)}{1 + (r_k^{(pr)})^2 - 2r_k^{(pr)}\cos(\omega)}$$

and

$$\begin{aligned}\tau_p^{(ck)}(\omega) =& 2 - 2\frac{(r_k^{(pc)})^2 - r_k^{(pc)}\cos(\omega - \theta_k^{(pc)})}{1 + (r_k^{(pc)})^2 - 2r_k^{(pc)}\cos(\omega - \theta_k^{(pc)})} \\ & - 2\frac{(r_k^{(pc)})^2 - r_k^{(pc)}\cos(\omega + \theta_k^{(pc)})}{1 + (r_k^{(pc)})^2 - 2r_k^{(pc)}\cos(\omega + \theta_k^{(pc)})}\end{aligned}$$

EXAMPLE

- As an example, we consider the allpass filter discussed on pages 54–61 in Part II of these lecture notes.

- This filter is of order 8 and has the following coefficient values for the denominator $a_1 = -4.0861$, $a_2 = 8.3380$, $a_3 = -11.0126$, $a_4 = 10.2425$, $a_5 = -6.8497$, $a_6 = 3.2152$, $a_7 = -0.9706$, and $a_8 = 0.1450$.

- This filter has four pole pairs located at

$$z = 0.778277exp(\pm j0.061161\pi),$$

$$z = 0.778650exp(\pm j0.183884\pi),$$

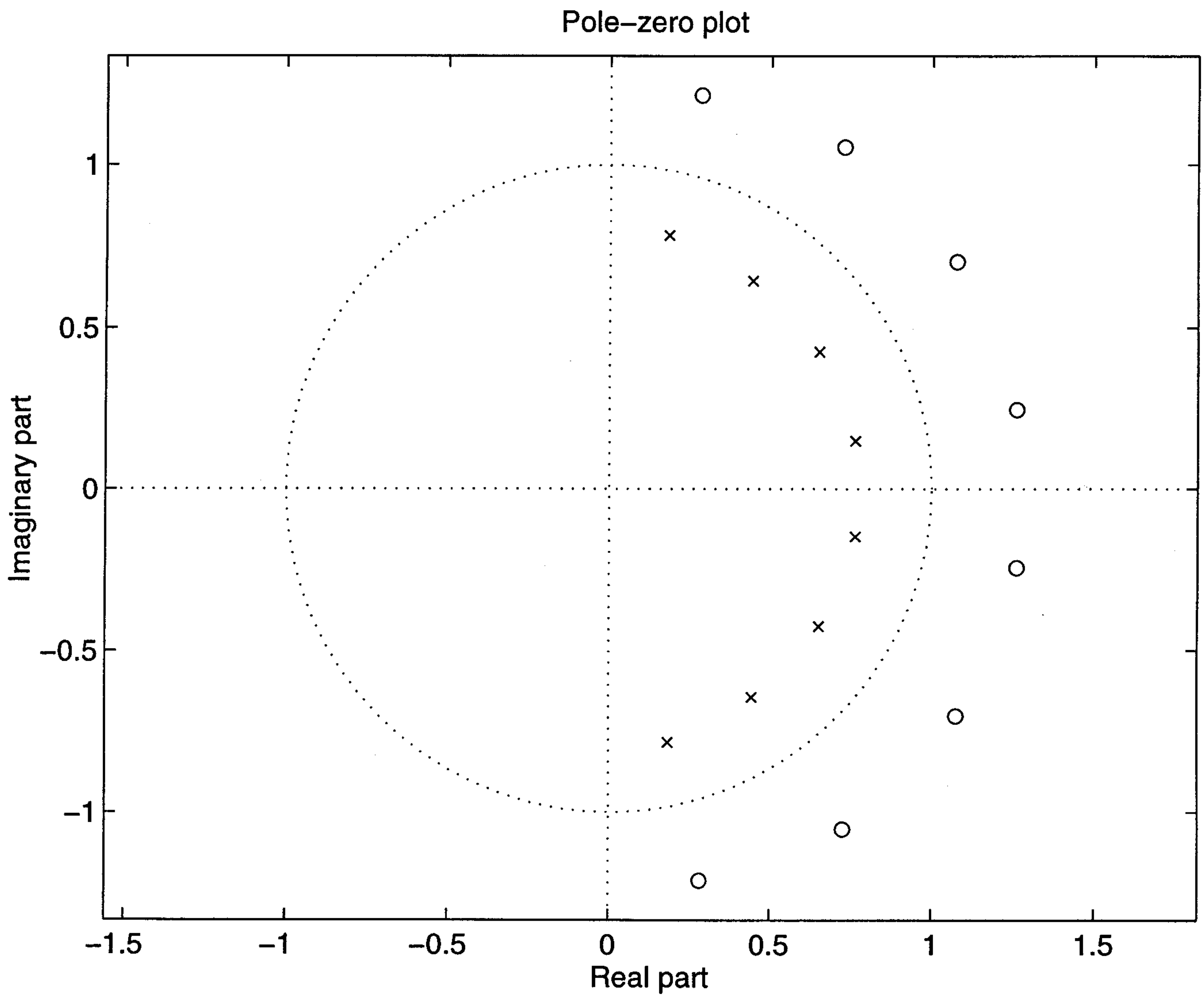
$$z = 0.781718exp(\pm j0.307657\pi),$$

$$z = 0.803864exp(\pm j0.427027\pi).$$

- The following four transparencies show some characteristics of this allpass filter.

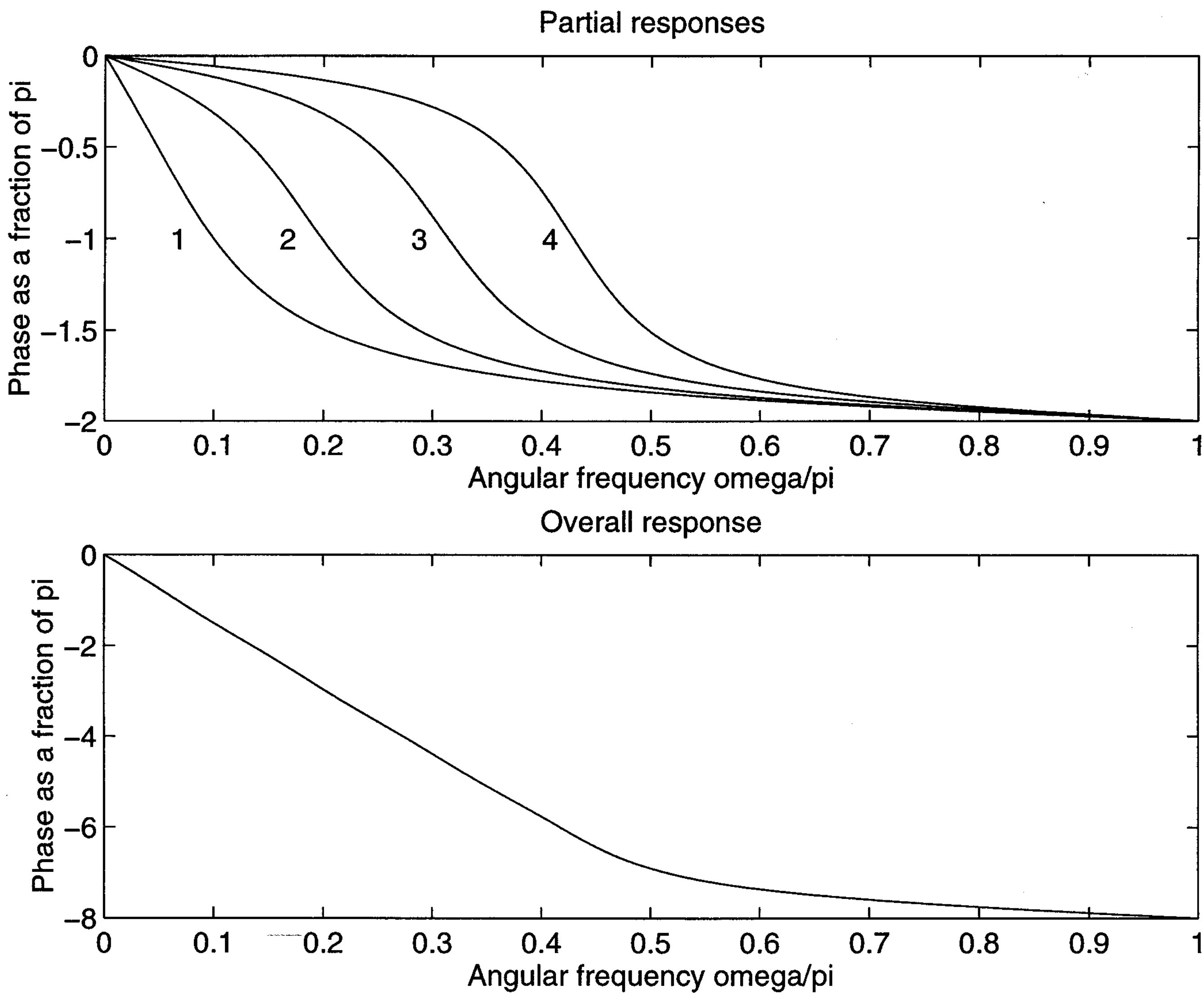
- Note that the phase response of our allpass filter decreases monotonically from zero to -8π as ω varies from zero to π. In general, the phase response decreases from zero to $-N\pi$, where N is the order of the filter.

Example allpass filter: Pole-zero plot



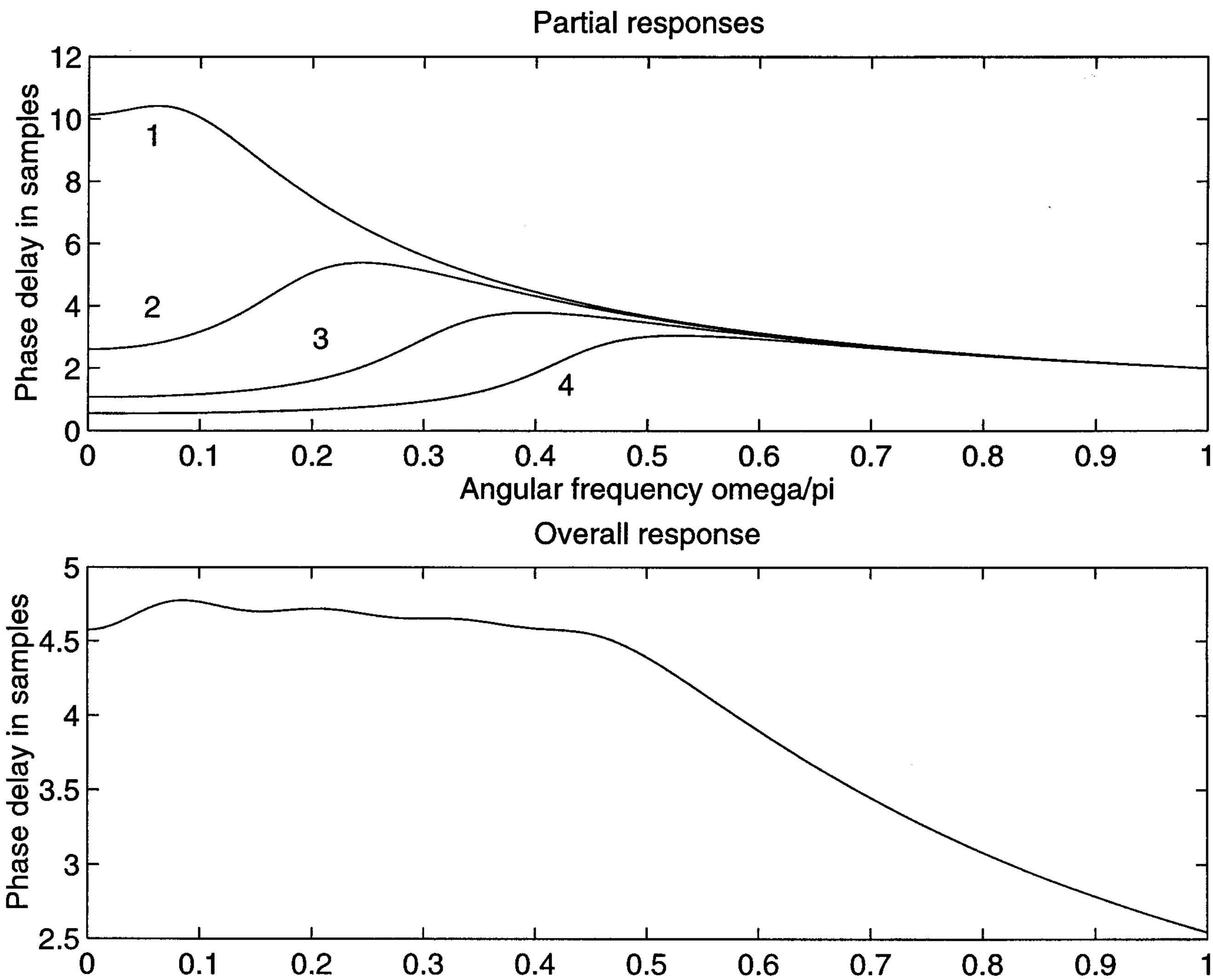
Example allpass filter: Partial and overall phase responses

- 1 for the pole with the smallest angle, 2 for the pole with the second smallest angle, 3 for the pole with the second largest angle, 4 for the pole with the largest angle.



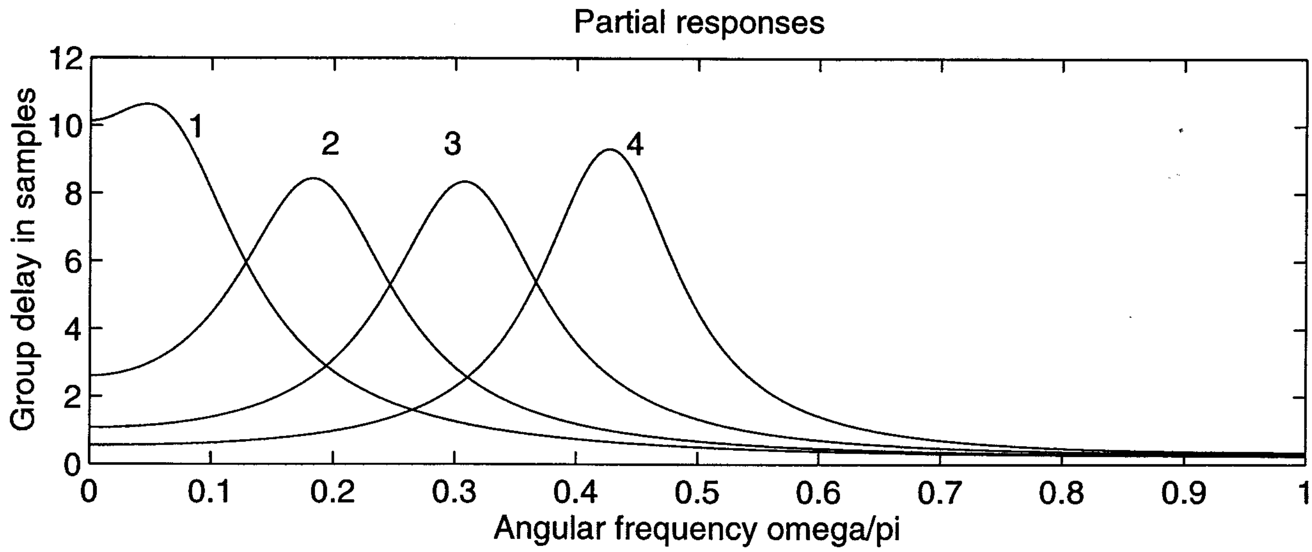
Example allpass filter: Partial and overall phase delay responses

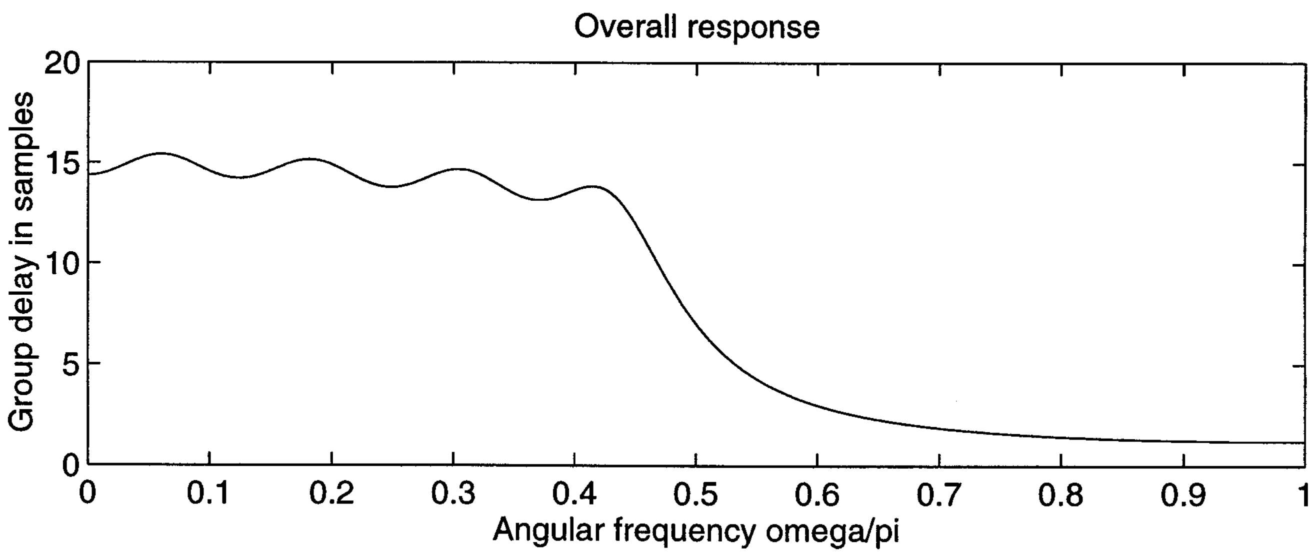
- 1 for the pole with the smallest angle, 2 for the pole with the second smallest angle, 3 for the pole with the second largest angle, 4 for the pole with the largest angle.



Example allpass filter: Partial and overall group delay responses

- 1 for the pole with the smallest angle, 2 for the pole with the second smallest angle, 3 fosecond largest angle, 4 for the pole with the largest angle.

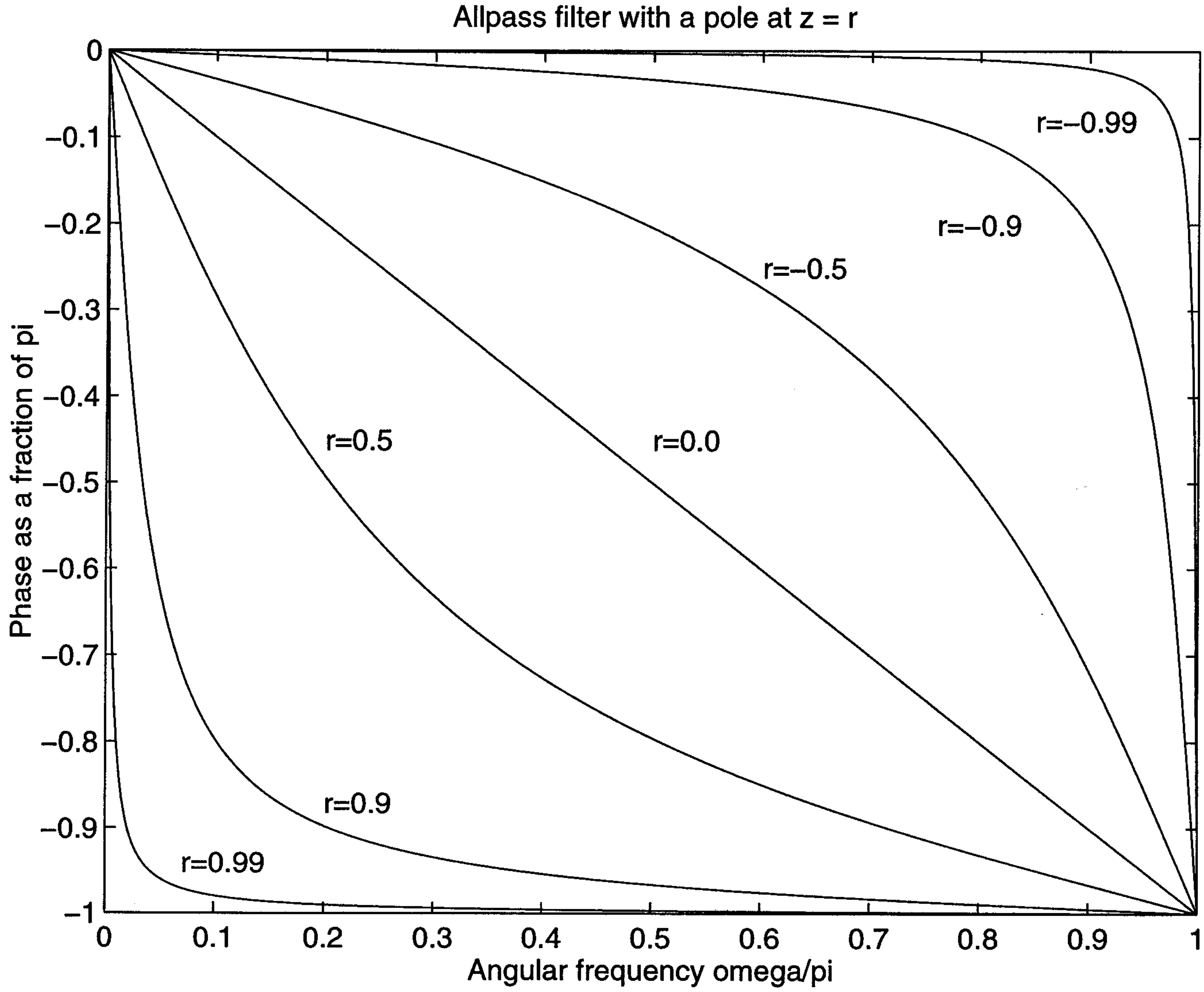




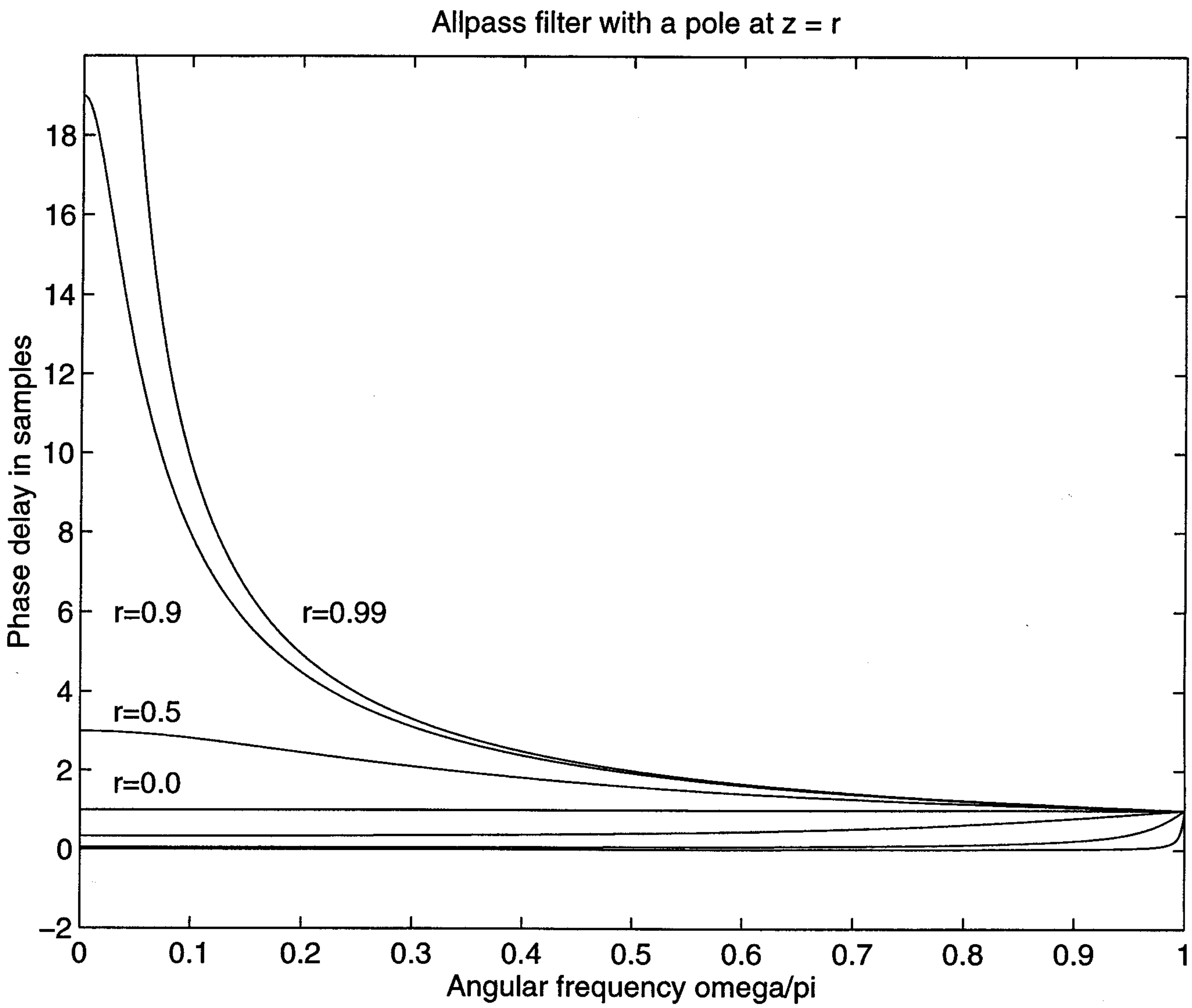
Responses for first-order and second-order all-pass filters

- The next six transparencies show the phase, phase delay, and group delay responses for first-order and second-order allpass filters.

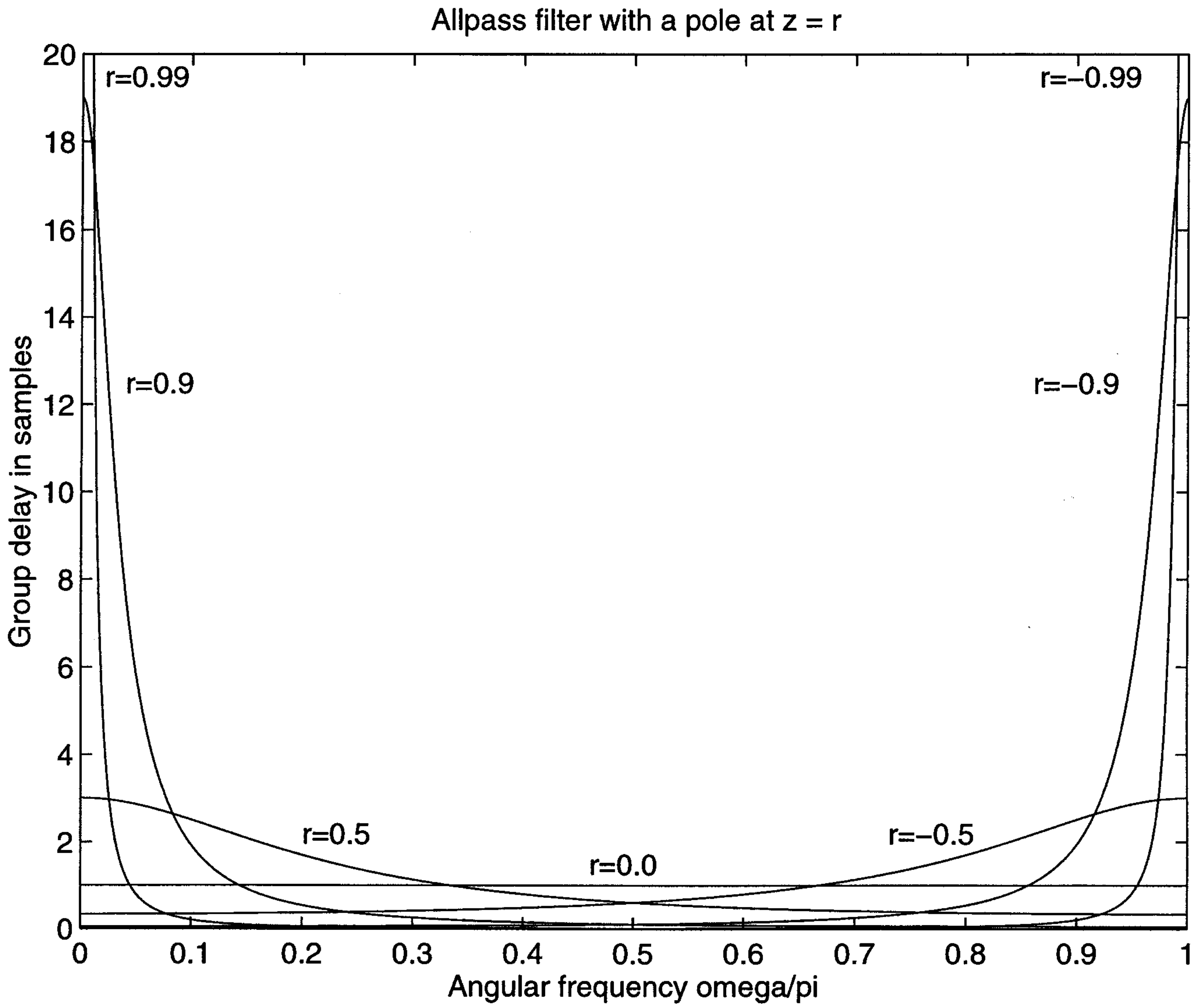
Responses for first-order allpass filters



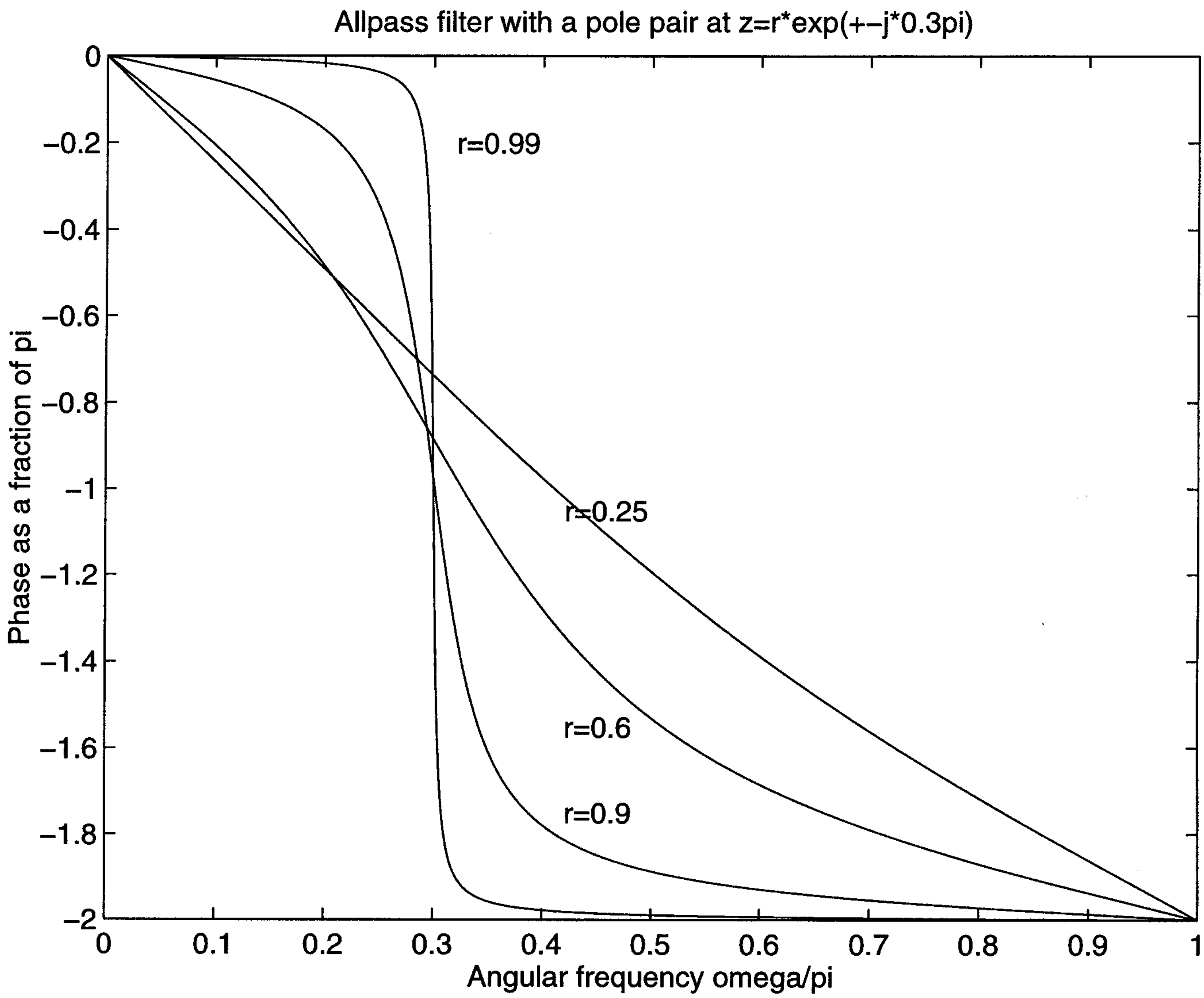
Responses for first-order allpass filters



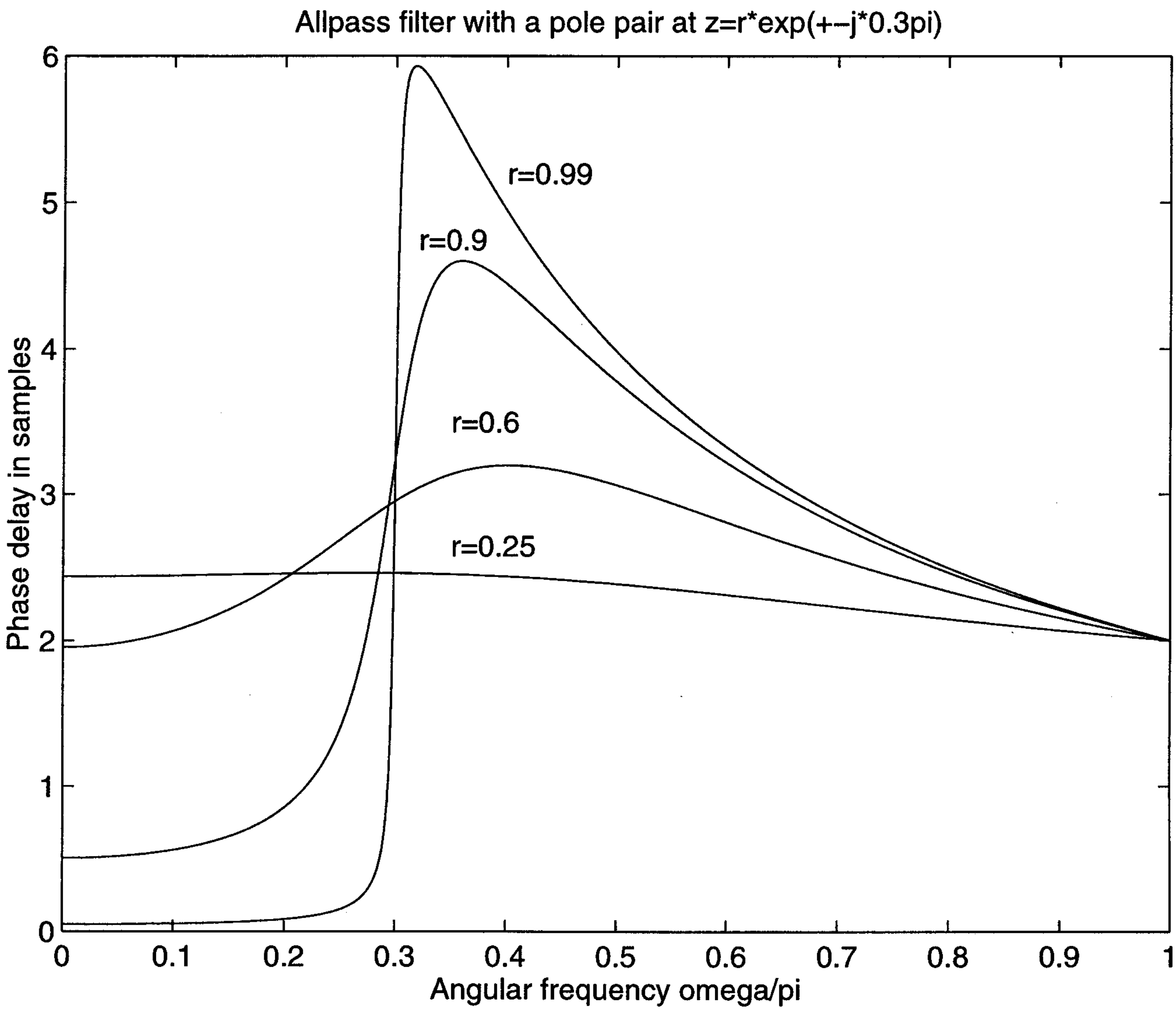
Responses for first-order allpass filters



Responses for second-order allpass filters



Responses for second-order allpass filters



Responses for second-order allpass filters