# 80509 LINEAR DIGITAL FILTERING I

## PART IV: Design of IIR filters

1) Synthesis of IIR filters by transforming an analog filter to its digital equivalent using the bilinear transformation
2) Basic types of classical analog and digital filters
3) Synthesis of highpass, bandpass, and bandstop digital filters based on transforming a lowpass digital filter into the desired one using a proper transformation.

- What to read for the examination ?:

1) How to use the bilinear transformation for mapping an analog filter into its digital equivalent (the overall synthesis procedure)
2) Basic types of classical filters: design formulas only for Butterworth lowpass analog filters.
3) Basic idea of using digital lowpass-to-lowpass, lowpass-to-highpass, lowpass-to-bandpass, and lowpass-to-band-stop transformations

# DESIGN OF CLASSICAL IIR FILTERS

- Traditionally, the synthesis of infinite impulse response (IIR) lowpass digital filters is accomplished by transforming an analog filter to its digital equivalent.
  - The two basic ways to perform the desired transformation is to use the bilinear transformation or the impulse-invariant technique.
- The design of highpass, bandpass, and bandstop filters is usually performed by applying a proper transformation converting a lowpass filter into the desired one.
- In this cource, we concentrate on
  - Synthesis of lowpass classical digital filters obtainable from an analog prototype filter through the bilinear transformation (the best technique).
  - Syntesis of highpass, bandpass, and bandstop filters based on transforming a lowpass digital filter into the desired one using a proper transformation.
  - It should be pointed out that there exist several more general sophisticated design techniques directly in the $z$-plane (to be considered in the course "Digital Linear Filtering II")

## Organization of This Chapter

**1)** First, the squared-magnitude functions of the four basic classical analog lowpass filter types are considered. For each type, the formulas are given for determining the poles and zeros of the filter as well as the scaling constant.

**2)** Second, it is shown how the analog lowpass filter can be converted to its digital equivalent with the aid of the bilinear transformation. Illustrative examples are included showing how to actually perform the synthesis for given digital filter criteria.

**3)** Third, it is shown how to convert a lowpass filter to another lowpass, highpass, bandpass, or bandstop filter with the aid of transformations.

**4)** Finally, a general-purpose matlab file for designing classical digital filters is introduced in Appendix A.

- The design of digital filters is convenient to perform with the aid of analog filters since there exist rather simple formulas for finding the poles, zeros, and the scaling constant for these filters.

- The purpose is **not** to study analog filters in details.

They are used just as a convenient intermediate tool for finding a proper digital filter. If you are interested in the details, you are encouraged to find a textbook on analog filters.

- The historical reason for this is the fact that when people started designing IIR digital filters, the theory of analog filters was well-known. What was left was to find a transformation to map the analog filter to the corresponding digital filter. The proper transformation is the bilinear transformation.

- It should be pointed out that the similar formulas can be derived directly in the $z$-plane. The next duty for the lecturer.

## Analog Lowpass Filters Under Consideration

- Transfer function of an analog filter is of the form

$$H_a(s) = \frac{C(s)}{D(s)} = \frac{\sum_{k=0}^{M} c_k s^k}{1 + \sum_{k=1}^{N} d_k s^k} = \frac{H_0 \prod_{k=1}^{M}(s - z_k)}{\prod_{k=1}^{N}(s - p_k)}. \quad (1)$$

- In order to guarantee the filter stability, the poles $p_k = \sigma_k + j\Omega_k$ must lie in the left-half $s$-plane, that is, the real parts of the $p_k$'s (the $\sigma_k$') must be less than zero.

- Furthermore, for an analog filter to be realizable, it is required that $M \leq N$ and the coefficients $c_k$ and $d_k$ are real.

- The squared-magnitude function of the filter with transfer function given by the above equation can be expressed in the following forms:

$$|H_a(j\Omega)|^2 = H_a(s)H_a(-s)_{|s\,=\,j\Omega} = H_a(j\Omega)H_a(-j\Omega)$$

$$= \frac{C(j\Omega)C(-j\Omega)}{D(j\Omega)D(-j\Omega)} = \frac{\sum_{k=0}^{M} e_k(\Omega^2)^k}{1 + \sum_{k=1}^{N} f_k(\Omega^2)^k} = \frac{E_M(\Omega^2)}{F_N(\Omega^2)}, \quad (2a)$$

where

$$E_M(\Omega^2) = \sum_{k=0}^{M} e_k(\Omega^2)^k \tag{2b}$$

and

$$F_N(\Omega^2) = 1 + \sum_{k=1}^{N} f_k(\Omega^2)^k. \tag{2c}$$

- Alternatively, if $|H_a(j\Omega)|^2$ is known, then $H_a(s)H_a(-s)$ is obtained from the above squared-magnitude function using the substitution $\Omega = s/j$ giving

$$H_a(s)H_a(-s) = \frac{C(s)C(-s)}{D(s)D(-s)} = \frac{\sum_{k=0}^{M} e_k(-s^2)^k}{1 + \sum_{k=1}^{N} f_k(-s^2)^k} \tag{3}$$

$$= \frac{E_M(-s^2)}{F_N(-s^2)}.$$

- If the **stable** $H_a(s)$ has the poles [roots of $D(s)$] at $p_k = \sigma_k + j\Omega_k$, then the corresponding **unstable** transfer function $H_a(-s)$ has poles [roots of $D(-s)$] at $\widehat{p}_k = -\sigma_k + j\Omega_k$ lying on the right-half $s$-plane.

- Therefore, the poles of $H_a(s)$ can be found by first locating the roots of $F_N(-s^2)$ and then selecting those roots which lie on the left-half $s$-plane.

- For the classical analog filters with $M > 1$, $M$ is even and $E_M(\Omega^2)$ is always factorizable as

$$E_M(\Omega^2) = e_0 \prod_{k=1}^{M/2} [\Omega^2 - \Omega_k^2]^2, \tag{4}$$

so that

$$E_M(-s^2) = e_0 \prod_{k=1}^{M/2} [-s^2 - \Omega_k^2]^2. \tag{5}$$

- This means that both $H_a(s)$ and $H_a(-s)$ contain complex conjugate zero pairs on the imaginary axis at the points $\pm j\Omega_k$ for $k = 1, 2, \ldots, M/2$. At these frequency points, the squared-magnitude function becomes zero.

- Note that for analog filters, the imaginary axis $s = j\Omega$ (analog frequency domain) plays the same role as the unit circle $z = e^{j\omega}$ (digital frequency domain) for digital filters.

# EXAMPLE

- $|H_a(j\Omega)|^2$ is given by
$$|H_a(j\Omega)|^2 = 1/[1+\epsilon^2(\Omega^2)^{14}].$$

- Using the substitution $\Omega = s/j$ gives
$$H_a(s)H_a(-s) = \frac{C(s)C(-s)}{D(s)D(-s)} = 1/[1+\epsilon^2(-s^2)^{14}].$$

- The 28 roots of $1+\epsilon^2(-s^2)^{14} = 0$ are located at
$$p_k = \frac{1}{\epsilon^{1/14}} e^{j\pi[1/2+(2k-1)/(28)]}, \; k = 1, 2, \ldots 14$$
and
$$\widehat{p}_k = \frac{1}{\epsilon^{1/14}} e^{j\pi[-1/2+(2k-1)/(28)]}, \; k = 1, 2, \ldots 14$$

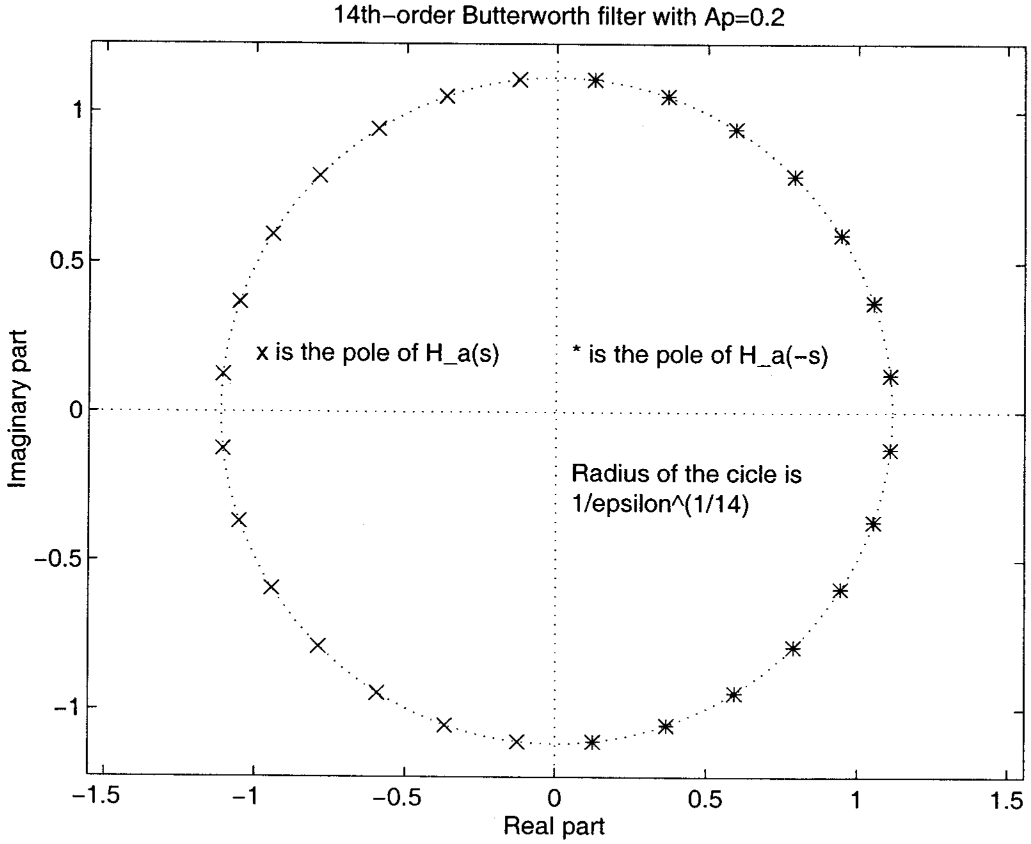
- The roots $p_k$ ($\widehat{p}_k$) for $k = 1, 2, \ldots, 14$ are on the left-half (right-half) $s$-plane.

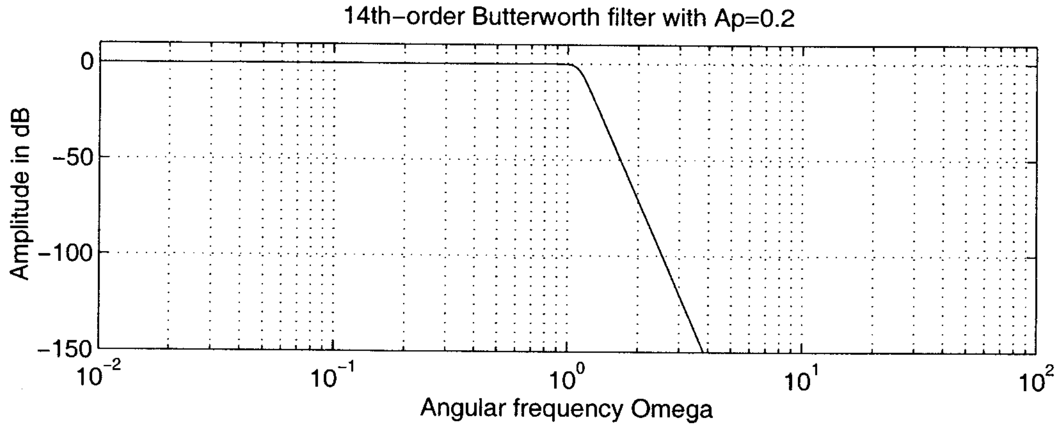
- The stable transfer function is then
$$H_a(s) = \frac{H_0}{\prod_{k=1}^{14}(s-p_k)}, \; H_0 = \prod_{k=1}^{14}(-p_k).$$

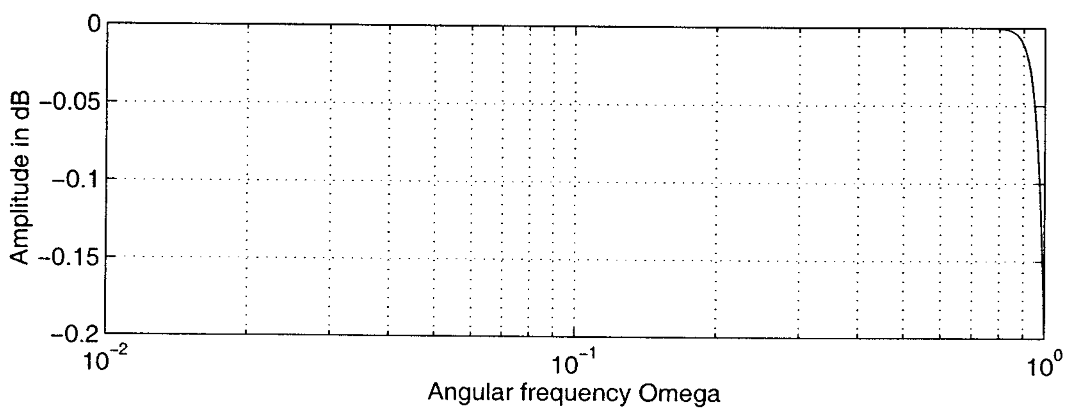
- The above selection of $H_0$ guarantees that $|H_a(j\Omega)|^2$ achieves the value of unity at $\Omega = 0$.

- The next transparency shows the poles of both $H_a(s)$ and $H_a(-s)$ as well as the amplitude response of $H_a(s)$ for $\epsilon^2 = 10^{0.2/10} - 1 = 0.047128548$.

# 14th-Order Butterworth Analog Filter with $\Omega_p = 1$, $\Omega_s = 1.8944272$, $A_p = 0.2$ dB, and $A_s \geq 60$ dB



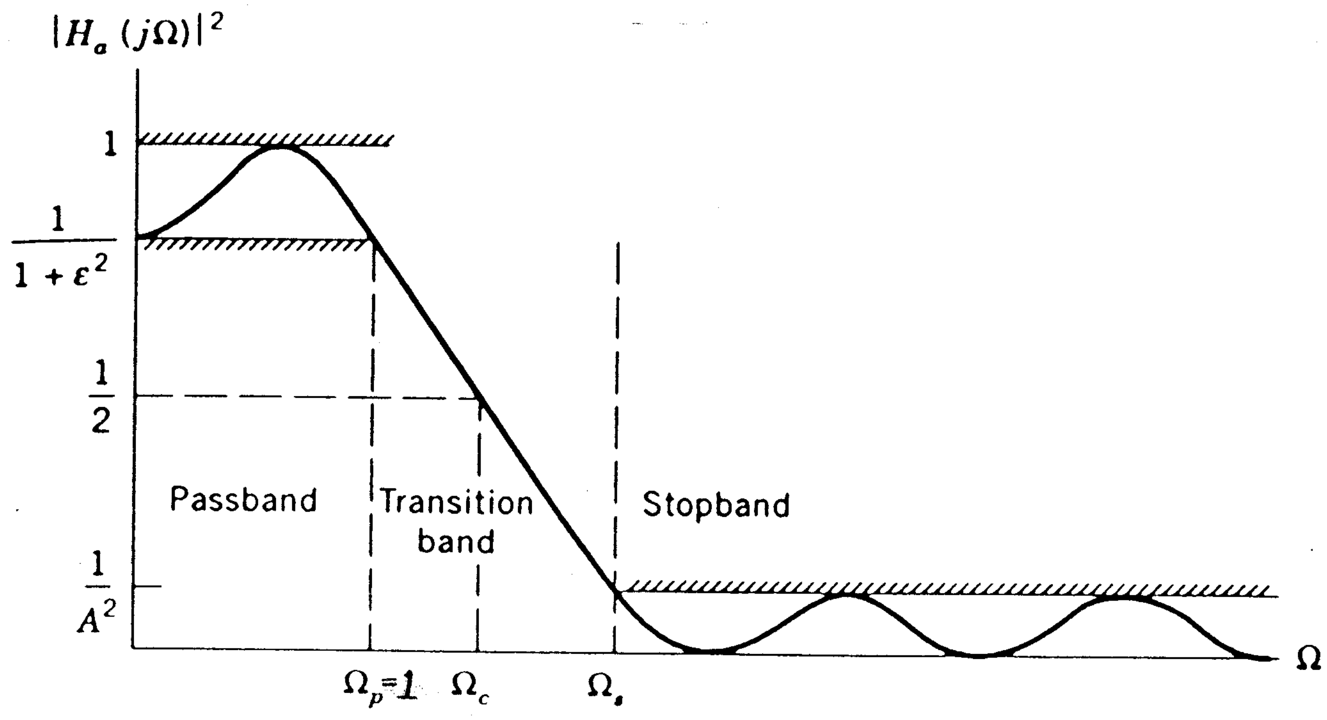



## Normalized Specifications for an Analog Lowpass filter

- The figure below gives tranditional specifications for the squared-magnitude function of an analog lowpass filter in the case where the passband edge is normalized to be located at $\Omega_p = 1$.

- For historical reasons, it is required that in the passband $0 \leq \Omega \leq \Omega_p = 1$ $|H_a(j\Omega)|^2$ stays within 1 and $1/(1+\epsilon^2)$.

- In the stopband $\Omega_s \leq \Omega < \infty$, $|H_a(j\Omega)|^2$ is less than or equal to $1/A^2$.



## DETERMINATION OF $\epsilon^2$ AND $A^2$ FROM SPECIFICATIONS IN DESIBELS

- Usually, the criteria are given as follows:

$$-A_p \leq 10\log_{10}|H_a(j\Omega)|^2 \leq 0 \text{ for } 0 \leq \Omega \leq 1 \quad (6a)$$

$$10\log_{10}|H_a(j\Omega)|^2 \leq -A_s \text{ for } \Omega_s \leq \Omega < \infty. \quad (6b)$$

  - Here, $A_p$ and $A_s$ are the maximum passband variation in dB and the minimum stopband attenuation in dB, respectively.
  - Both of these quantities are positive.
- After some manipulations, we get

$$\epsilon^2 = 10^{A_p/10} - 1 \quad (7a)$$

and

$$A^2 = 10^{A_s/10}. \quad (7b)$$

# CLASSICAL FILTERS

1) Butterworth filters

   - Maximally flat both in the passband and stopband

2) Chebyshev Type I filters or simply Chebyshev filters

   - Equiripple in the passband and maximally flat in the stopband

3) Chebyshev Type II filters or inverse Chebyshev filters

   - Maximally flat in the passband and equiripple in the stopband

4) Elliptic (Cauer) filters

   - Equiripple both in the passband and stopband

# BUTTERWORTH FILTERS

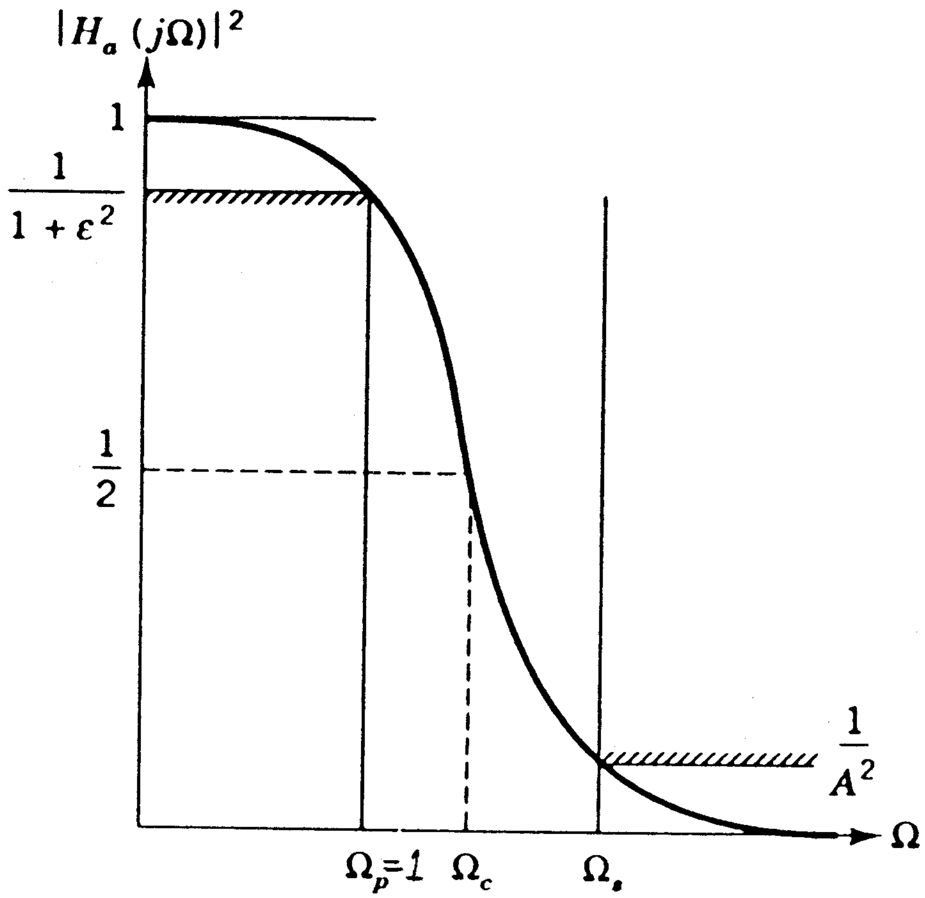
- For the $N$th-order Butterworth filter the squared-magnitude function is given by

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 \Omega^{2N}}. \tag{8}$$

- This function achieves the value of unity at $\Omega = 0$ with the first $2N - 1$ derivatives being zero at this point (**maximally flat passband**).

- At infinity, the value is zero and the first $2N-1$ derivatives are zero (**maximally flat stopband**).

  - The following figure gives the normalized specifications, where $\Omega_p = 1$, $|H_a(j\Omega_p)|^2 = 1/(1+\epsilon^2)$, and it is required that $|H_a(j\Omega_s)|^2 \leq 1/A^2$.



# TRANSFER FUNCTION

- In order to meet the criteria given in the figure of the previous transparency, the filter order $N$ must be selected such that
$$\frac{1}{1+\epsilon^2\Omega_s^{2N}} \leq \frac{1}{A^2}. \tag{9}$$

- Alternatively, this condition can be expressed as
$$N \geq \frac{\log_{10}[(A^2-1)/\epsilon^2]}{2\cdot\log_{10}\Omega_s} \tag{10}$$

- By analytic continuation ($j\Omega = s$), equation (8) can be extended to the complex $s$-domain, giving
$$H_a(s)H_a(-s) = \frac{1}{1+\epsilon^2(-s^2)^N}. \tag{11}$$

- The poles of the **stable** $H_a(s)$ are the left-half plane roots of $1+\epsilon^2(-s^2)^N = 0$, whereas the poles of the **unstable** $H_a(-s)$ are the right-half plane roots of $1+\epsilon^2(-s^2)^N = 0$.

- The $N$ poles of the **stable** $H_a(s)$ are given by
$$p_k = \frac{1}{\epsilon^{1/N}}e^{j\pi[1/2+(2k-1)/(2N)]},\ \ k=1,2,\ldots N. \tag{12}$$
or
$$p_k = \sigma_k + j\Omega_k,\ \ k=1,2,\ldots N, \tag{13a}$$
where
$$\sigma_k = \frac{1}{\epsilon^{1/N}}\cos\Big(\frac{\pi}{2}+\frac{\pi(2k-1)}{2N}\Big) \tag{13b}$$
$$\Omega_k = \frac{1}{\epsilon^{1/N}}\sin\Big(\frac{\pi}{2}+\frac{\pi(2k-1)}{2N}\Big). \tag{13c}$$

- For above design, the passband criteria are just met, whereas

$$\widehat{A}^2 = 1 + \epsilon^2 \Omega_s^{2N} \tag{14a}$$

or

$$\widehat{A}_s = 10 \log_{10}(1 + \epsilon^2 \Omega_s^{2N}). \tag{14b}$$

- These values are in most cases larger than the specified values $A^2$ and $A_s$.

- All the poles of $H_a(s)$ lie on a circle of radius $1/\epsilon^{1/N}$ centered at the origin of the $s$-plane. The pole with largest imaginary part appears at angle $\pi/2 + \pi/(2N)$ relative to the positive real axis. The other poles appear at angular increments of $\pi/N$. This is exemplified in transparency 8.

- The transfer function for which $|H_a(j0)|^2 = 1$ or $H_a(0) = 1$ is then

$$H_a(s) = \frac{H_0}{\prod_{k=1}^{N}(s - p_k)}, \tag{15a}$$

where

$$H_0 = \prod_{k=1}^{N}(-p_k). \tag{15b}$$

- All the $N$ zeros are lying at infinity.

  - For the very original Butterworth filters, $\epsilon^2 = 1$ so that $|H_a(j\Omega_p)|^2 = 1/2$ and $A_p = 3$ dB.

## EXAMPLE

- Later on, when designing digital filters with the aid of analog classical filters, we end up with the following specifications:

$$A_p = 0.2 \text{ dB}, \quad A_s \geq 60 \text{ dB}, \quad \Omega_s = 1.8944272.$$

- Using the formulas described above, we obtain

$$\epsilon^2 = 10^{A_p/10} - 1 = 10^{0.2/10} - 1 = 0.047238748,$$
$$A^2 = 10^{A_s/10} = 10^{60/10} = 10^6,$$

and

$$N \geq \log_{10}[(A^2 - 1)/\epsilon^2]/[2 \cdot \log_{10} \Omega_s]$$
$$= 13.202340 \Rightarrow N = 14.$$

- The fourteen poles are located at

$$-0.12487140 \pm j1.10826429,$$
$$-0.36835261 \pm j1.05269129,$$
$$-0.59336309 \pm j0.94433195,$$
$$-0.78861987 \pm j0.78861987,$$
$$-0.94433195 \pm j0.59336309,$$
$$-1.05269129 \pm j0.36835261,$$

and

$$-1.10826429 \pm j0.12487140,$$

whereas

$$H_0 = 4.60636100,$$

and

$$\widehat{A}_s = 10 \log_{10}(1 + \epsilon^2 \Omega_s^{2N}) = 64.4267$$

- The pole-plot for this filter (those given by x) as well as the amplitude response are shown in transparency 8 (all the zeros are lying at the infinity (not visible)).

# CHEBYSHEV FILTERS OR CHEBYSHEV TYPE I FILTERS

- The squared-magnitude function of this filter of order $N$ is given by

$$|H_a(j\Omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega)}, \qquad (16a)$$

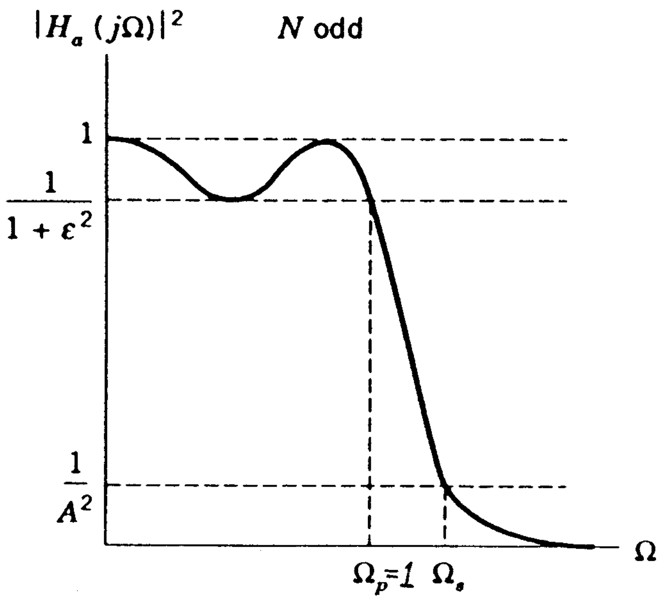
where

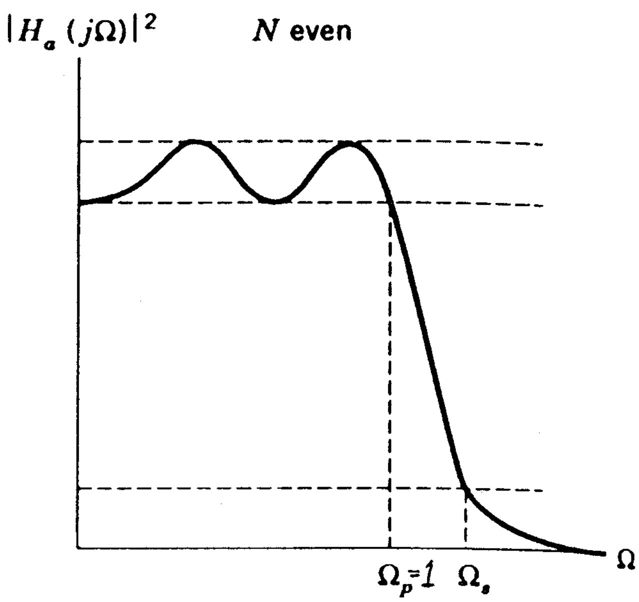
$$T_N(\Omega) = \begin{cases} \cos(N\cos^{-1}\Omega), & |\Omega| \le 1 \\ \cosh(N\cosh^{-1}\Omega) & |\Omega| > 1 \end{cases} \qquad (16b)$$

is the $N$th-degree Chebyshev polynomial.

- In the normalized passband $0 \le \Omega \le \Omega_p = 1$, this function alternatingly achieves the values of 1 and $1/(1+\epsilon^2)$ at $N+1$ points such that $|H_a(j\Omega_p)|^2 = 1/(1+\epsilon^2)$. For $N$ even, $|H_a(j0)|^2 = 1/(1+\epsilon^2)$ and for $N$ odd, $|H_a(j0)|^2 = 1$ (**equiripple passband**).

- At infinity, the value of $|H_a(j\Omega)|^2$ is zero and the first $2N-1$ derivatives are zero (**maximally flat stopband**) (see the figure shown below).



# TRANSFER FUNCTION

- In order to meet the criteria of the previous transparency, it is required that

$$\frac{1}{1+\epsilon^2 T_N^2(\Omega_s)} \leq \frac{1}{A^2} \Rightarrow N \geq \frac{\cosh^{-1}\{\sqrt{[(A^2-1)/\epsilon^2]}\}}{\cosh^{-1}(\Omega_s)}, \tag{16a}$$

where $\cosh^{-1} x$ can be evaluated from

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}). \tag{16b}$$

- Like for Butterworth filters, the transfer function is of the form

$$H_a(s) = \frac{H_0}{\prod_{k=1}^{N}(s - p_k)}, \tag{17a}$$

where

$$H_0 = \begin{cases} \prod_{k=1}^{N}(-p_k), & N \text{ odd} \\ \sqrt{1/(1+\epsilon^2)}\prod_{k=1}^{N}(-p_k), & N \text{ even.} \end{cases} \tag{17b}$$

- It can be shown (for details see a textbook on analog filters) that the poles $H_a(s)$ are located at

$$p_k = \sigma_k + j\Omega_k, \quad k = 1, 2, \ldots N, \tag{18a}$$

where

$$\sigma_k = -\frac{\gamma - \gamma^{-1}}{2}\sin[\frac{(2k-1)\pi}{2N}] \tag{18b}$$

$$\Omega_k = \frac{\gamma + \gamma^{-1}}{2} \cos[\frac{(2k-1)\pi}{2N}] \qquad (18c)$$

and

$$\gamma = (\frac{1 + \sqrt{1+\epsilon^2}}{\epsilon})^{1/N} \quad . \qquad (18d)$$

- These poles are on an ellipse centered about the origin in the $s$-plane. The ellipse has minor-axis length $\frac{\gamma-\gamma^{-1}}{2}$ and major-axis length $\frac{\gamma+\gamma^{-1}}{2}$ and is given by the equation

$$\frac{4\sigma_k^2}{(\gamma-\gamma^{-1})^2} + \frac{4\Omega_k^2}{(\gamma+\gamma^{-1})^2} = 1. \qquad (19)$$

- For details see transparency 21, which gives the poles of $H_a(s)$ by $x$ and the poles of $H_a(-s)$ by astrisk.

- Like for Butterworth filters, all the zeros are located at infinity.

- Futhermore, the squared-magnitude function of the above filter just meets the passband criteria, whereas

$$\widehat{A}^2 = 1 + \epsilon^2 T_N^2(\Omega_s) \qquad (20a)$$

or

$$\widehat{A}_s = 10 \log_{10}[1 + \epsilon^2 T_N^2(\Omega_s)], \qquad (20b)$$

which are in most cases larger than the specified values of $A^2$ and $A_s$.

## EXAMPLE

- We again consider the specifications:

$$A_p = 0.2 \text{ dB}, \quad A_s \geq 60 \text{ dB}, \quad \Omega_s = 1.8944272.$$

- Like for Butterworth filters, $\epsilon^2 = 0.047238748$ and $A^2 = 10^6$, whereas

$$N \geq \cosh^{-1}\{\sqrt{[(A^2 - 1)/\epsilon^2]}\}/\cosh^{-1}(\Omega_s)$$
$$= 7.2808916 \;\Rightarrow\; N = 8.$$

- The eight poles are located at

$$-0.05514327 \pm j1.01921190,$$
$$-0.15703476 \pm j0.86404612,$$
$$-0.23501912 \pm j0.57733716,$$

and

$$-0.27722396 \pm j0.20273385,$$

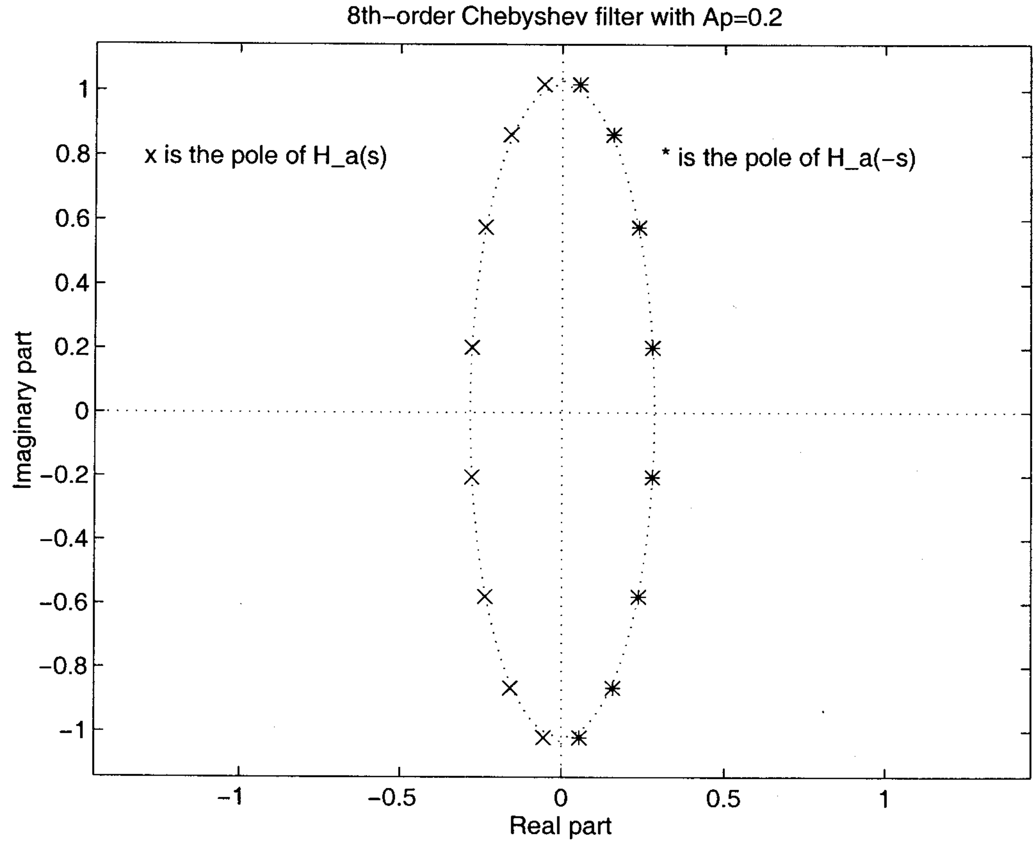
whereas

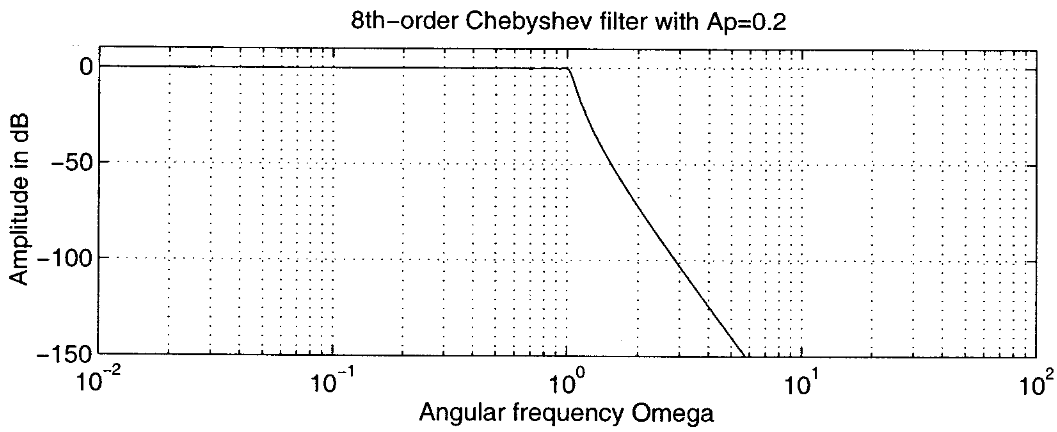
$$H_0 = 0.035987195.$$

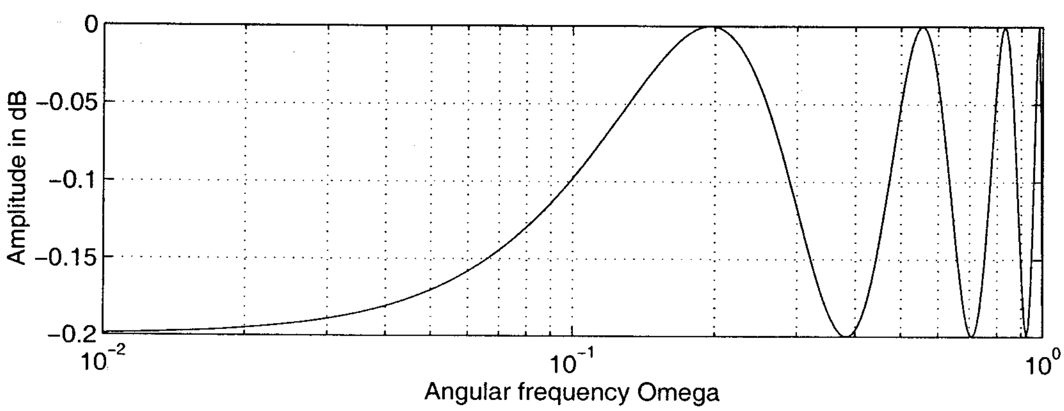
- The pole-plot for this filter (those given by x) as well as the amplitude response are shown in the next transparency (all the zeros are lying at the infinity (not visible)).

- For this filter, $A_s = 67.8310$ dB.

# Eighth-Order Chebyshev Analog Filter with $\Omega_p = 1$, $\Omega_s = 1.8944272$, $A_p = 0.2$ dB, and $A_s \geq 60$ dB



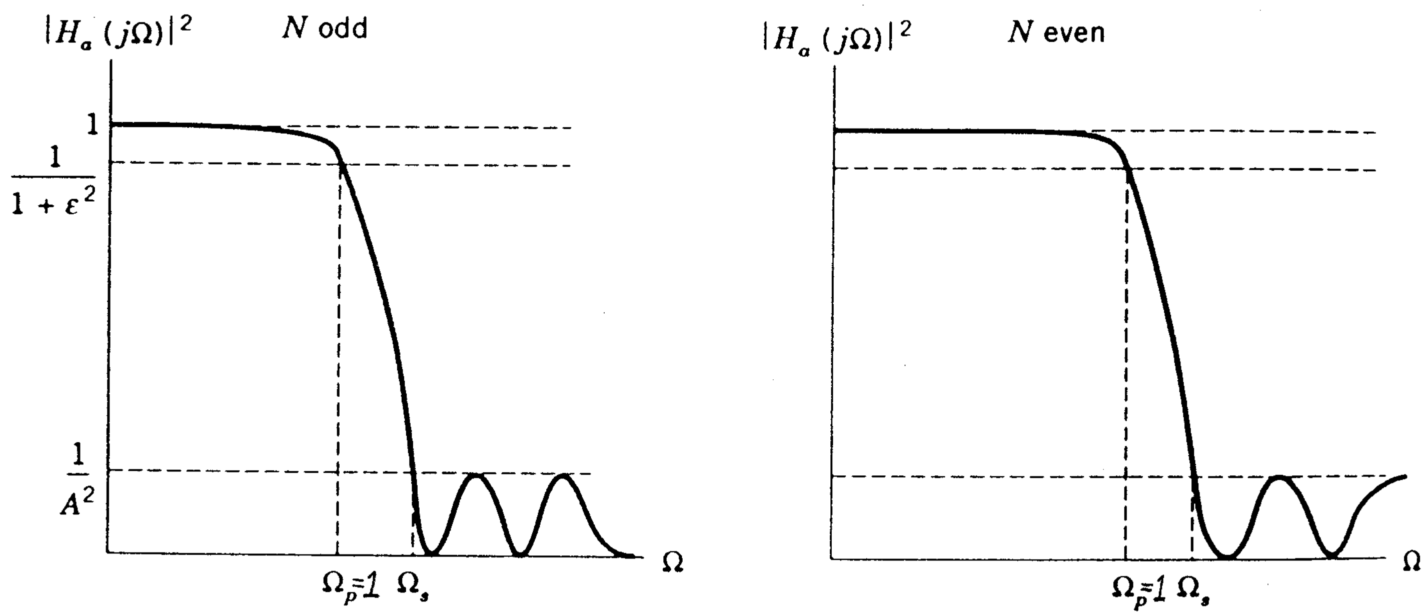
# INVERSE CHEBYSHEV FILTERS OR CHEBY-SHEV TYPE II FILTERS

- The squared-magnitude function of this filter of order $N$ is given by

$$|H_a(j\Omega)|^2 = \frac{1}{1 + (A^2 - 1)/[T_N(\Omega_s/\Omega)]^2}. \qquad (21)$$

- Like for Butterworth filters, this function achieves the value of unity at $\Omega = 0$ with the first $2N - 1$ derivatives being zero at this point (**maximally flat passband**).

- In the stopband $\Omega_s \leq \Omega \leq \infty$, $|H_a(j\Omega)|^2$ alternatingly achieves the values of $1/A^2$ and zero at $N + 1$ points such that $|H_a(j\Omega_s)|^2 = 1/A^2$. For $N$ even, $|H_a(j\infty)|^2 = 1/A^2$ and for $N$ odd, $|H_a(j\infty)|^2 = 0$ (**equiripple stopband**) (see the figure below).



# TRANSFER FUNCTION

- Like for Chebyshev filters, in order to satisfy $1/[1 + (A^2-1)/[T_N(\Omega_s/\Omega_p)]^2] \geq 1/(1+\epsilon^2)$, it is required that

$$N \geq \frac{\cosh^{-1}\{\sqrt{[(A^2-1)/\epsilon^2]}\}}{\cosh^{-1}(\Omega_s)}. \tag{22}$$

- The transfer function is of the form

$$H_a(s) = \begin{cases} H_0 \prod_{k=1}^{N}(s-z_k) / \prod_{k=1}^{N}(s-p_k), & N \text{ even} \\ H_0 \prod_{k=1}^{(N-1)}(s-z_k) / \prod_{k=1}^{N}(s-p_k), & N \text{ odd} \end{cases} \tag{23a}$$

where

$$H_0 = \begin{cases} \prod_{k=1}^{N}(-p_k) / \prod_{k=1}^{N}(-z_k), & N \text{ even} \\ \prod_{k=1}^{N)}(-p_k) / \prod_{k=1}^{(N-1)}(-z_k), & N \text{ odd.} \end{cases} \tag{23b}$$

- The zeros are located at

$$z_k = j\frac{\Omega_s}{\cos\{[(2k-1)/(2N)]\pi\}}, \quad k = 1, 2, \cdots, N. \tag{24}$$

- Note that if $N$ is odd, then for $k = (N+1)/2$, the zero lies at the infinity (numerator order is $N-1$).

- The poles are located at

$$p_k = \sigma_k + j\Omega_k, \quad k = 1, 2, \ldots, N, \qquad (25a)$$

where

$$\sigma_k = \frac{\Omega_s \alpha_k}{\alpha_k^2 + \beta_k^2} \qquad (25b)$$

$$\Omega_k = \frac{-\Omega_s \beta_k}{\alpha_k^2 + \beta_k^2} \qquad (25c)$$

with

$$\alpha_k = -\frac{\gamma - \gamma^{-1}}{2} \sin[\frac{(2k-1)\pi}{2N}] \qquad (25d)$$

$$\beta_k = \frac{\gamma + \gamma^{-1}}{2} \cos[\frac{(2k-1)\pi}{2N}] \qquad (25e)$$

and

$$\gamma = (A + \sqrt{A^2 - 1})^{1/N}. \qquad (25f)$$

- In this case, the poles do not lie on a simple geometric figure as they do for Butterworth and Chebyshev filters.

- Note that the above formulas for Chebyshev and inverse Chebyshev filters have been constructed such that the Chebyshev filter just meets the passband criteria, whereas the inverse Chebyshev filter just meets the stopband criteria.

- If it is desired that the inverse Chebyshev filter just meets the given passband criteria, then then the desired result is achieved by evaluating $A^2$ according to equation (20a).

# EXAMPLE

- We again consider the specifications:

$$A_p = 0.2 \text{ dB}, \quad A_s \geq 60 \text{ dB}, \quad \Omega_s = 1.8944272.$$

- Like for Chebyshev filters, $\epsilon^2 = 0.047238748$ and $A^2 = 10^6$, whereas

$$N \geq \cosh^{-1}\{\sqrt{[(A^2-1)/\epsilon^2]}\}/\cosh^{-1}(\Omega_s)$$
$$= 7.2808916 \;\Rightarrow\; N = 8.$$

- Knowing the fact that the Chebyshev and inverse Chebyshev filter meet the same amplitude criteria, our filter can be forced to have $A_p = 0.2$ dB by selecting $A_s = 67.8310$ dB, giving $A^2 = 6068718.6$.

- The eight poles of the resulting filter are located at

$$-0.18212766 \pm j1.16381690,$$

$$-0.57926246 \pm j1.10192829,$$

$$-1.03855485 \pm j0.88204869,$$

and

$$-1.42446611 \pm j0.36015085.$$

- The eight zeros are located at

$$\pm j1.93154121,$$

$$\pm j2.27840821,$$

$$\pm j3.40987886,$$

and

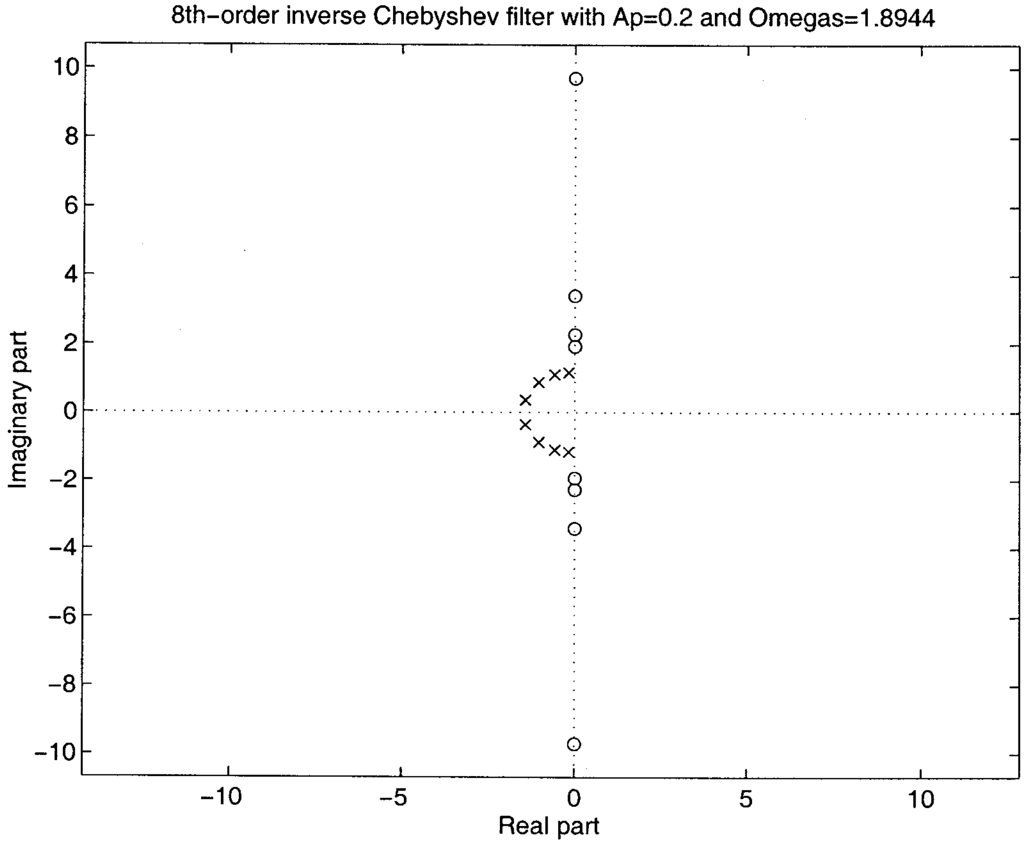
$$\pm j9.71051342,$$

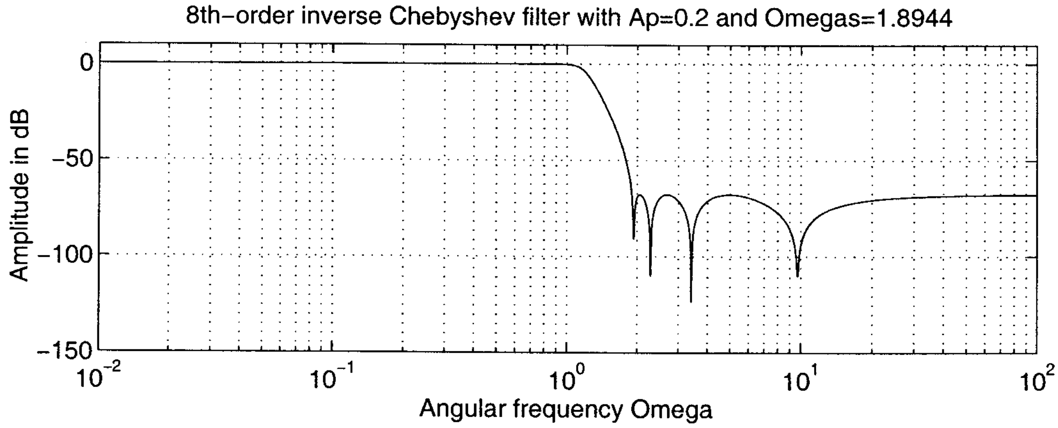
whereas

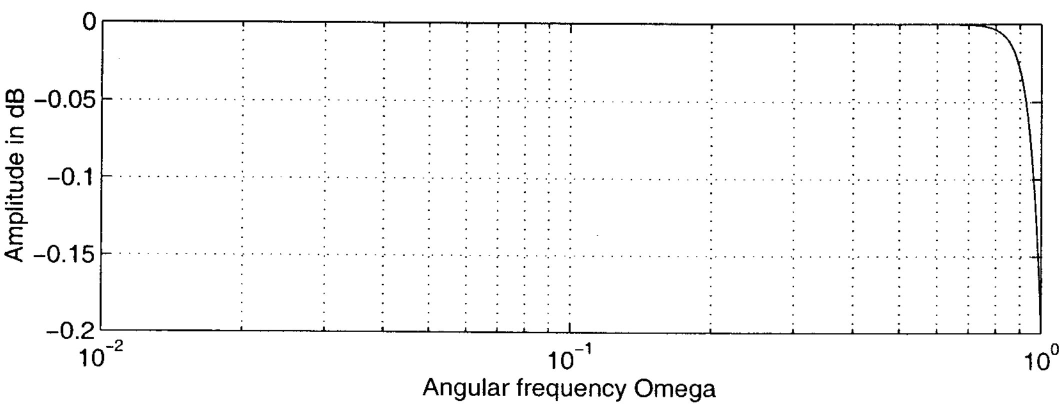
$$H_0 = 0.00036795086.$$

- The following transparency gives the pole-zero plot as well as the amplitude response for this filter.

# Eighth-Order Inverse Chebyshev Analog Filter with $\Omega_p = 1$, $\Omega_s = 1.8944272$, $A_p = 0.2$ dB, and $A_s \geq 60$ dB

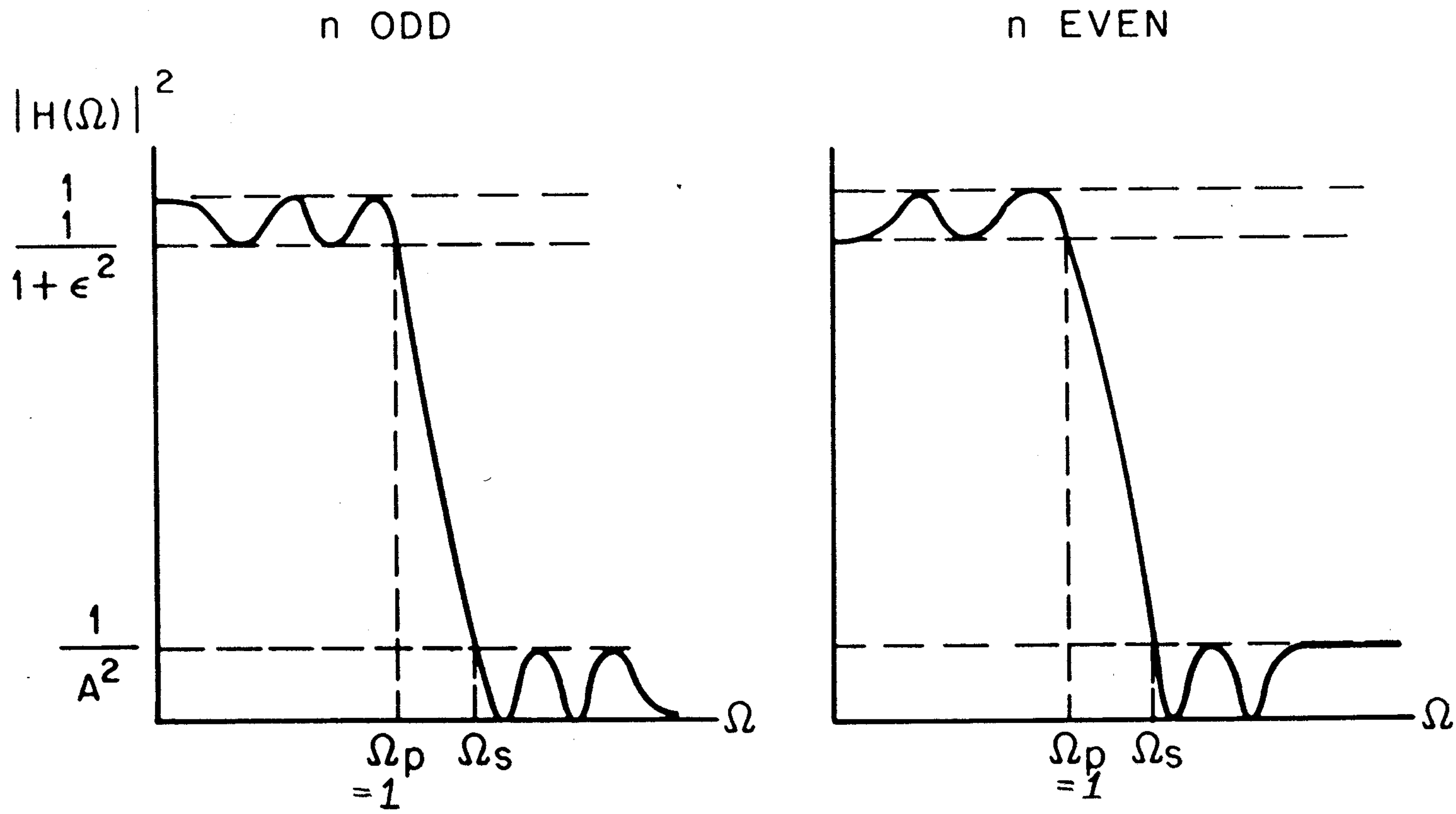


# ELLIPTIC (CAUER) FILTERS

- Like for Chebyshev filters, in the normalized passband $0 \leq \Omega \leq \Omega_p = 1$, the squared-magnitude function of a filter of order $N$ alternatingly achieves the values of 1 and $1/(1+\epsilon^2)$ at $N+1$ points such that $|H_a(j\Omega_p)|^2 = 1/(1+\epsilon^2)$. For $N$ even, $|H_a(j0)|^2 = 1/(1+\epsilon^2)$ and for $N$ odd, $|H_a(j0)|^2 = 1$ (**equiripple passband**).
- Like for inverse Chebyshev filters, in the stopband $\Omega_s \leq \Omega \leq \infty$, $|H_a(j\Omega)|^2$ alternatingly achieves the values of $1/A^2$ and zero at $N+1$ points such that $|H_a(j\Omega_s)|^2 = 1/A^2$. For $N$ even, $|H_a(j\infty)|^2 = 1/A^2$ and for $N$ odd, $|H_a(j\infty)|^2 = 0$ (**equiripple stopband**).



## TRANSFER FUNCTION

- For the elliptic filter, the squared-magnitude function has a very complicated form (see a textbook on analog filters).

- The transfer function is of the form

$$H_a(s) = \frac{H_0}{D_0(s)} \prod_{i=1}^{r} \frac{s^2 + A_{0i}}{s^2 + B_{1i}s + B_{0i}}, \qquad (26a)$$

where

$$r = \begin{cases} (N-1)/2, & N \text{ odd} \\ N/2, & N \text{ even} \end{cases} \qquad (26b)$$

and

$$D_0(s) = \begin{cases} s + \sigma_0, & N \text{ odd} \\ 1, & N \text{ even} \end{cases} \qquad (26c)$$

- The transfer-function coefficients and the multiplier constant can be computed using the following formulas in sequence:

$$k = 1/\Omega_s \qquad (27a)$$

$$k' = \sqrt{1 - k^2} \qquad (27b)$$

$$q_0 = (1/2)(1 - \sqrt{k'})/(1 + \sqrt{k'}) \qquad (27c)$$

$$q = q_0 + 2q_0^5 + 15q_0^9 + 150q_0^{13} \qquad (27d)$$

$$D = (A^2 - 1)/\epsilon^2 \qquad (27e)$$

$$N \geq \log_{10}(16D)/\log_{10}(1/q) \qquad (27f)$$

$$\Delta = \frac{1}{2N} \ln\left[\frac{\sqrt{1+\epsilon^2}+1}{\sqrt{1+\epsilon^2}-1}\right] \qquad (27g)$$

$$\widehat{\sigma}_0 = \left|\frac{2q^{1/4}\sum_{m=0}^{\infty}(-1)^m q^{m(m+1)}\sinh[(2m+1)\Delta]}{1+2\sum_{m=1}^{\infty}(-1)^m q^{m^2}\cosh[2m\Delta]}\right| \tag{27h}$$

$$W = \sqrt{(1+k\widehat{\sigma}_0^2)(1+\widehat{\sigma}_0^2/k)} \tag{27i}$$

$$\Omega_i = \frac{2q^{1/4}\sum_{m=0}^{\infty}(-1)^m q^{m(m+1)}\sin[(2m+1)]\pi\mu/N]}{1+2\sum_{m=1}^{\infty}(-1)^m q^{m^2}\cos[2m\pi\mu/N]}, \tag{27j}$$

where

$$\mu = \begin{cases} i, & N \text{ odd} \\ i-1/2, & N \text{ even} \end{cases}, \quad i = 1, 2\cdots, r \tag{27k}$$

$$\sigma_0 = \widehat{\sigma}_0/\sqrt{k} \tag{27l}$$

$$V_i = \sqrt{(1-k\Omega_i^2)(1-\Omega_i^2/k)} \tag{27m}$$

$$A_{0i} = \frac{1}{k\Omega_i^2} \tag{27n}$$

$$B_{0i} = \frac{(\widehat{\sigma}_0 V_i)^2 + (\Omega_i W)^2}{k(1+\widehat{\sigma}_0^2\Omega_i^2)^2} \tag{270}$$

$$B_{1i} = \frac{2\widehat{\sigma}_0 V_i}{\sqrt{k}(1+\widehat{\sigma}_0^2\Omega_i^2)^2} \tag{27p}$$

$$H_0 = \begin{cases} \sigma_0\prod_{i=1}^{r}[B_{0i}/A_{0i}], & \text{for } N \text{ odd} \\ \frac{1}{\sqrt{1+\epsilon^2}}\prod_{i=1}^{r}[B_{0i}/A_{0i}], & \text{for } N \text{ even.} \end{cases} \tag{27q}$$

- In practice, three or four terms in the series of equations (27h) and (27j) are sufficient.

- The passband criteria are just met, whereas the resulting minimum stopband attenuation is

$$A_s = 10\log_{10}[\epsilon^2/(16q^N)+1]. \tag{27r}$$

## EXAMPLE

- We again consider the specifications:

$$A_p = 0.2 \text{ dB}, \quad A_s \geq 60 \text{ dB}, \quad \Omega_s = 1.8944272.$$

- $\epsilon^2 = 0.047238748$ and $A^2 = 10^6$, whereas

$$k = 1/\Omega_s = 0.527864, \ k' = \sqrt{1-k^2} = 0.8493289$$
$$q_0 = (1/2)(1-\sqrt{k'})/(1+\sqrt{k'}) = 0.0204022$$
$$q = q_0 + 2q_0^5 + 15q_0^9 + 150q_0^{13} = 0.0204022$$
$$D = (A^2-1)/\epsilon^2 = 21218562$$
$$N \geq \log_{10}(16D)/\log_{10}(1/q)$$
$$= 5.046816 \ \Rightarrow \ N = 6.$$

- The six poles of our filter are located at

$$-0.08205619 \pm j1.03019607,$$

$$-0.25402886 \pm j0.79507992,$$

and

$$-0.39500663 \pm j0.30821324.$$

- The six zeros are located at

$$\pm j1.95117116,$$

$$\pm j2.57623214,$$

and

$$\pm j6.79458015,$$

whereas

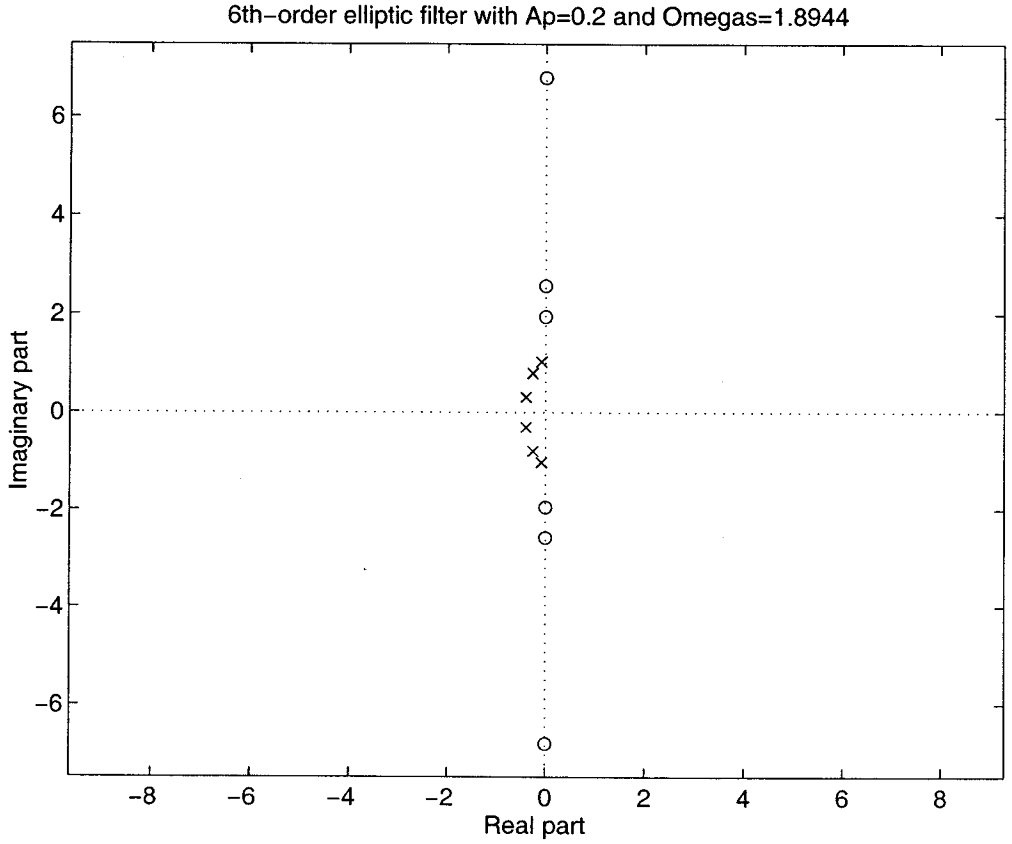
$$H_0 = 0.00015647808$$

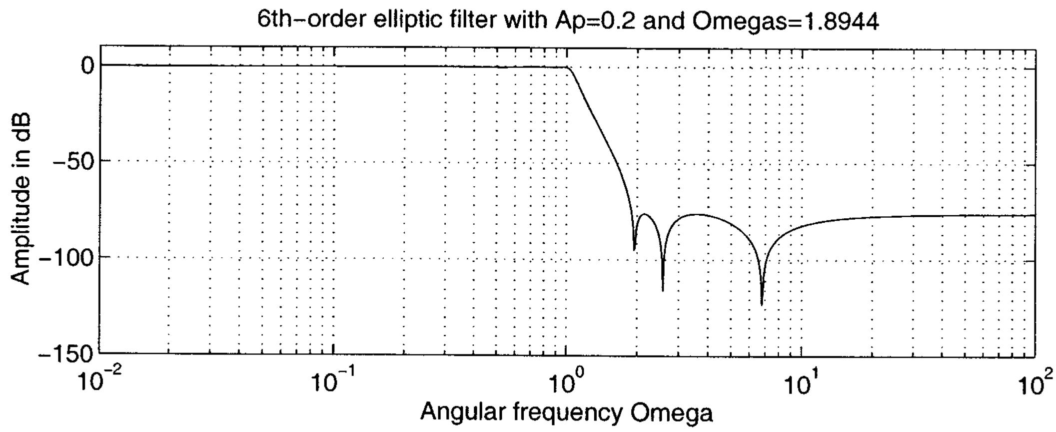
and

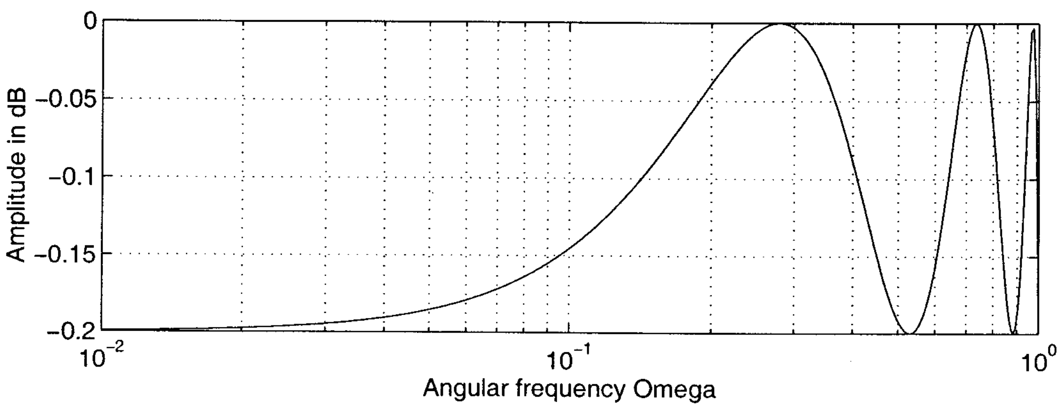
$$A_s = 76.1109.$$

- The next transparency shows the pole-zero plot and the amplitude response for this filter.

# Sixth-Order Elliptic Analog Filter with $\Omega_p = 1$, $\Omega_s = 1.8944272$, $A_p = 0.2$ dB, and $A_s \geq 60$ dB



## BILINEAR TRANSFORMATION

- The most efficient and popular analog-to-digial transformation is the bilinear transformation.
- It transforms $H_a(s)$ to its digital equivalent $H(z)$ via the relation

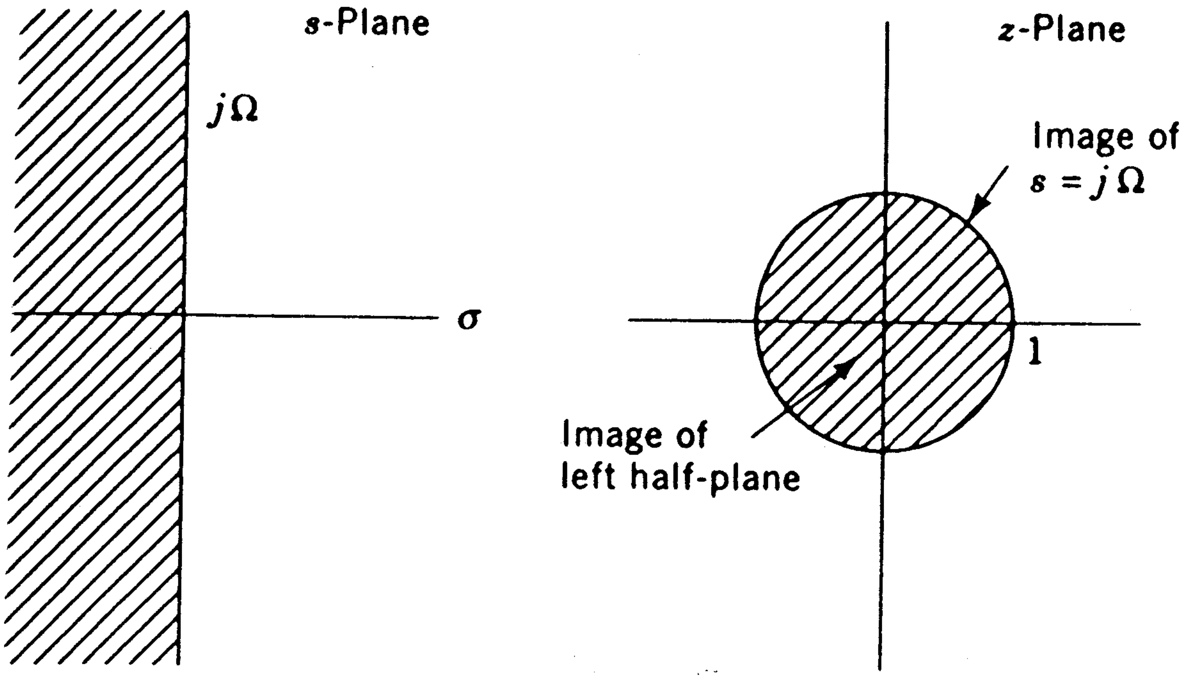
$$H(z) = H_a(s)|_{s = c(z-1)/(z+1)} \cdot \tag{23}$$

  - Here, $c$ can be selected arbitrarily.
- Alternatively, $H_a(s)$ can be obtained from $H(z)$ via the relation

$$H_a(z) = H(z)|_{z = (1+s/c)/(1-s/c)} \cdot \tag{24}$$

- The transformation $s = c(z-1)/(z+1)$ is a one-to-one mapping between the $s$-plane and the $z$-plane and has the following desired properties:
  - The left-half $s$-plane is mapped to the interior of the unit circle $\Rightarrow$ A stable $H_a(s)$ is mapped into a stable $H(z)$.
  - The right-half $s$-plane (unstable region for poles) is mapped to the exterior of the unit circle (unstable region for poles).
  - The imaginary axis $s = j\Omega$ is mapped to the $z$-plane unit circle $z = e^{j\omega}$ $\Rightarrow$ The analog frequency domain (imaginary axis) maps onto the digital frequency domain (unit circle), albeit, as we shall see, nonlinearly.

# RELATIONS BETWEEN THE $s$- AND $z$-PLANES




- Substituting $z = e^{j\omega}$ into equation (23) results in, after some manipulations,

$$H(e^{j\omega}) = H_a(jc\tan(\omega/2)). \qquad (25)$$

- Alternatively substituting $z = e^{j\omega}$ and $s = j\Omega$ into $s = c(z-1)/(z+1)$ we end up with the realtions

$$\Omega = c\tan(\omega/2) \qquad (26)$$

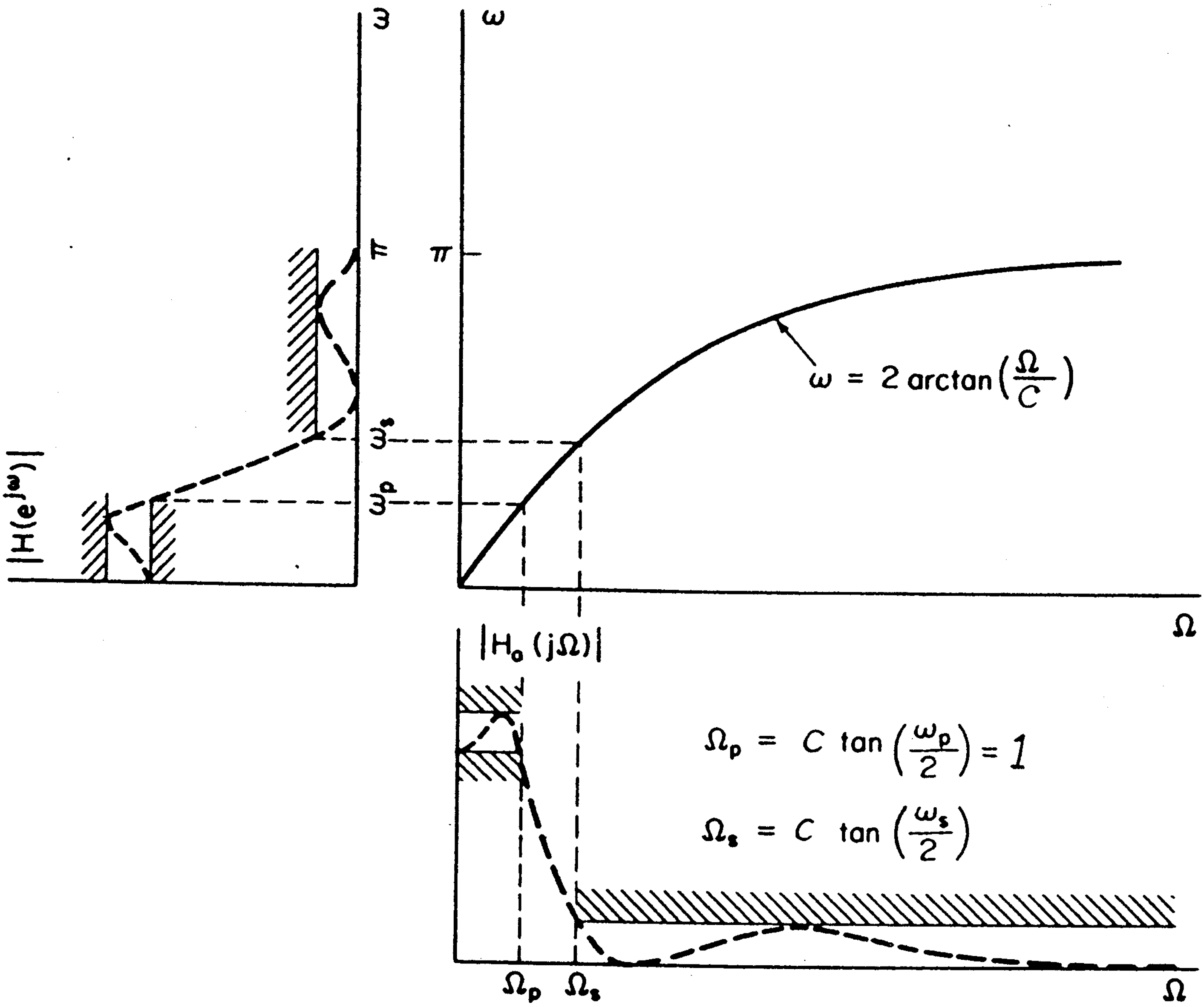
or

$$\omega = 2\arctan(\Omega/c). \qquad (27)$$

- $s = 0$ and $s = \infty$ are mapped to $z = 1$ and $z = -1$, respectively.

# DIGITAL FILTER DESIGN USING THE BILINEAR TRANSFORMATION

- The following figure exemplifies the design process, where the bandedges of the digital filter, $\omega_p$ and $\omega_s$, as well as allowable passband and stopband variations are specified.
- Here, the design of the digital filter is converted to that of the analog filter for which the required passband and stopband variations are the same.



## SYNTHESIS PROCEDURE

**Step 1** Determine $c$ such that $\omega_p$ is mapped to $\Omega_p = 1$. The condition for $c$ is $\Omega_p = 1 = c\tan(\omega_p/2)$, giving

$$c = \cot(\omega_p/2). \tag{28}$$

**Step 2** Determine $\Omega_s = c\tan(\omega_s/2)$.

**Step 3** Determine $\epsilon^2$ and $A^2$ from the passband and stopband ripples $A_p$ and $A_s$ using equation (7).

**Step 4** Select the analog filter type (Butterworth, Chebyshev, inverse Chebyshev, or elliptic) and synthesize the minimum-order filter transfer function $H_a(s)$ whose squared-magnitude resonse stays within the limits unity and $1/(1+\epsilon^2)$ in the passband $0 \leq \Omega \leq 1$ the limits zero and $1/A^2$ in the stopband $\Omega_s \leq \Omega \leq \infty$.

**Step 5** The desired digital filter is then

$$H(z) = H_a(s)_{|s} = c(z-1)/(z+1)\cdot \tag{29}$$

- In practice, the desired $H(z)$ can be conveniently generated by first determining the poles and zeros of the digital filter using the relation $z = (1 + s/c)/(1 - s/c)$ to each pole and zero of the analog filter. This gives the poles and zeros of the digital filter, denoted by $\beta_k$ and $\alpha_k$ for $k = 1, 2, \cdots, N$, respectively.

- The resulting $H(z)$ is then

$$H(z) = k_0 \prod_{k=1}^{N} (1 - \alpha_k z^{-1}) / \prod_{k=1}^{N} (1 - \beta_k z^{-1}). \qquad (30)$$

- The constamt $k_0$ can be then determined from the condition that $H(1) = 1$ ($|H(e^{j\omega})| = 1$ at $\omega = 0$) for Butterworth and inverse Chebyshev filters as well as for Chebyshev filters and elliptic filters for $N$ even. For $N$ odd, $H(1) = 1/\sqrt{1 + \epsilon^2}$ for Chebyshev filters and elliptic filters.

- Note that the zero at infinity ($N$ zeros for Butterworth and Chebyshev filters and one zero for inverse Chebyshev and elliptic filters for $N$ odd) are mapped to $z = -1$.

# ILLUSTRATIVE EXAMPLE

- It is desired to synthesize a Butterworth, Chebyshev, inverse Chebyshev, and an elliptic filter such that the sampling frequency is $F_s = 10$ kHz, the minimum passband ripple $A_p = 0.2$ dB in the passband $0 \leq f \leq 2$ kHz and the minimum stopband attenuation is $A_s = 60$ dB in the stopband 3kHz $\leq f \leq 5$ kHz$= F_s/2$.

**Step 1** $\omega$ and the 'real' frequency are related via $\omega = 2\pi f/F_s$ so that in terms of $\omega$ the passband and stopband edges are $\omega_p = 0.4\pi$ and $\omega_s = 0.6\pi$, respectively.

**Step 2** $\omega_p = 0.4\pi$ is mapped to $\Omega_p = 1$ in the bilinear transformation by selecting $c = \cot(\omega_p/2) = 1.3763819$. Then, $\Omega_s = c\tan(\omega_s/2) = 1.8944272$.

**Step 3** Applying equations (7a) and (7b) gives $\epsilon^2 = 10^{A_p/10} - 1 = 0.0471285$ and $A^2 = 10^{A_s/10} = 10^6$

**Step 4** These are the same criteria we considered previously for all the four classical analog filter types. All what is left is to apply the bilinear transformation.

## BUTTERWORTH FILTER

- By applying the substitution $z = (1 + s/c)/(1 - s/c)$ with $c = 1.3763819$ to the poles of the anaolg filter given in transparency 15, we end up with the following fourteen $z$-plane pole locations:

$$0.89585800 exp(\pm j0.43312181\pi)$$

$$0.71526001 exp(\pm j0.42970219\pi)$$

$$0.56158624 exp(\pm j0.42193975\pi)$$

$$0.42686164 exp(\pm j0.40731799\pi)$$

$$0.30642203 exp(\pm j0.37934682\pi)$$

$$0.19959206 exp(\pm j0.31841870\pi)$$

$$0.11888906 exp(\pm j0.15472269\pi)$$

- All the fourteen zeros of the analog filter lying at infinity are mapped to $z = -1$.

- This filter can be implemented in following the cascade form (see the next transparency)

$$H(z) = k_0 \prod_{i=1}^{R} \left[\frac{1 + a_{0i}z^{-1} + a_{1i}z^{-2}}{1 - b_{0i}z^{-1} - b_{1i}z^{-2}}\right] \qquad (31)$$

- Combining complex-conjugate pole pairs, seven ($R = 7$) second-order denominator sections become

$$1 - 0.21023698z^{-1} + 0.01413460z^{-2}$$

$$1 - 0.21556526z^{-1} + 0.03983699z^{-2}$$

$$1 - 0.22677174z^{-1} + 0.09389446z^{-2}$$
$$1 - 0.24508032z^{-1} + 0.18221086z^{-2}$$
$$1 - 0.27268700z^{-1} + 0.31537911z^{-2}$$
$$1 - 0.31336428z^{-1} + 0.51159689z^{-2}$$
$$1 - 0.37368323z^{-1} + 0.80256154z^{-2}$$

- Seven second-order numerator sections are

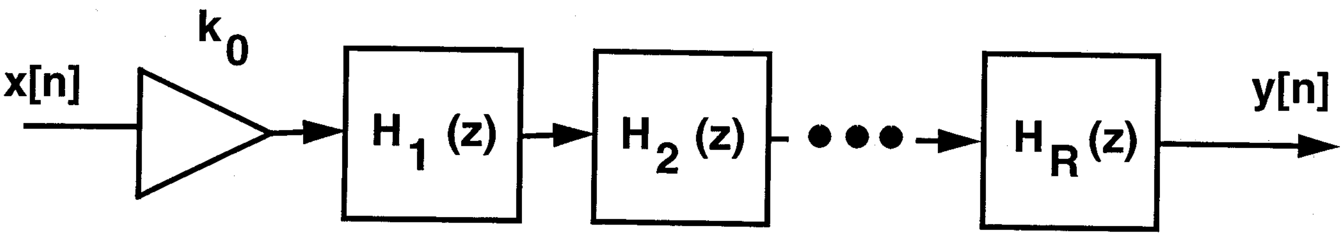
$$1 + 2z^{-1} + z^{-2}.$$

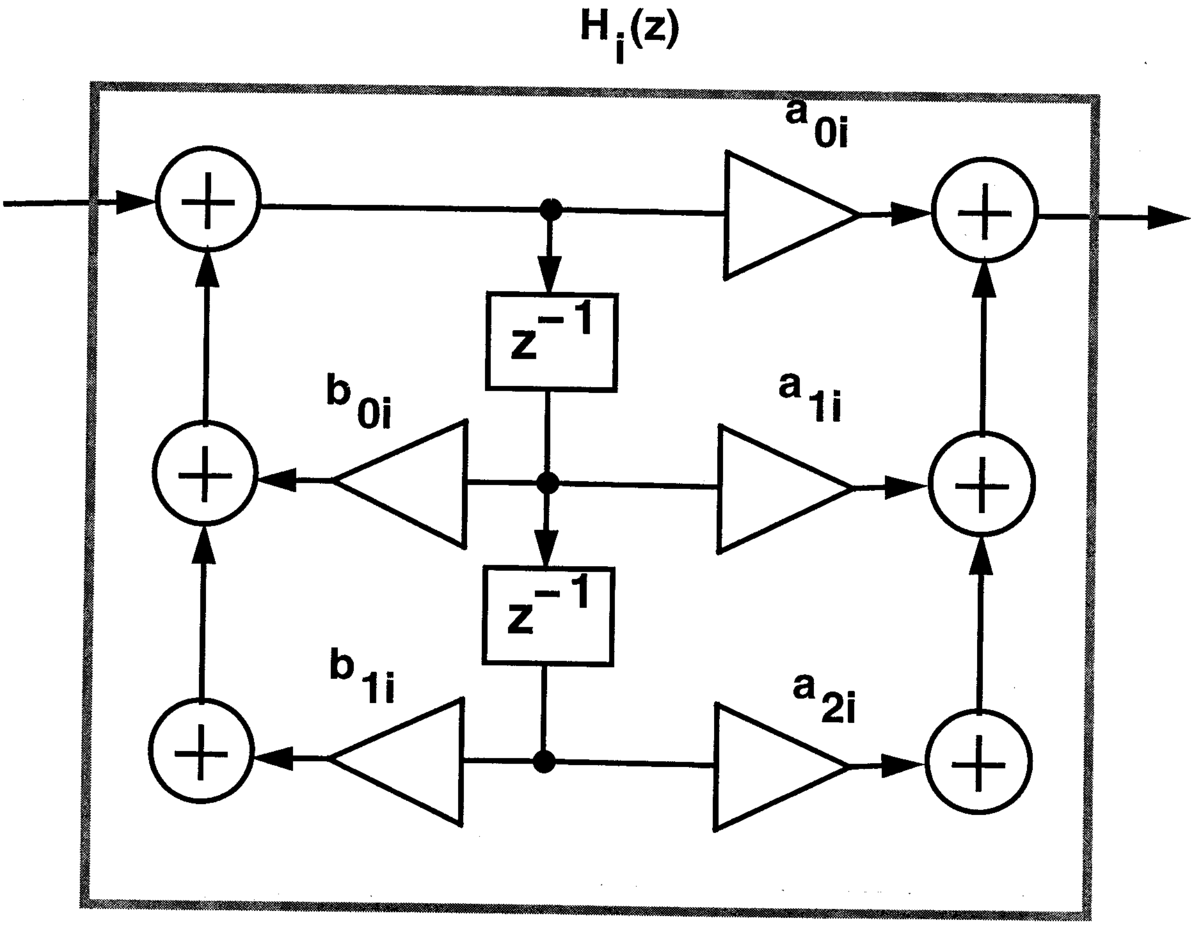
- $k_0$ can be determined from the condition $H(1)$=1, giving $k_0 = 5.8671114210 \cdot 10^{-5}$.

- $A_s = 64.42665$dB

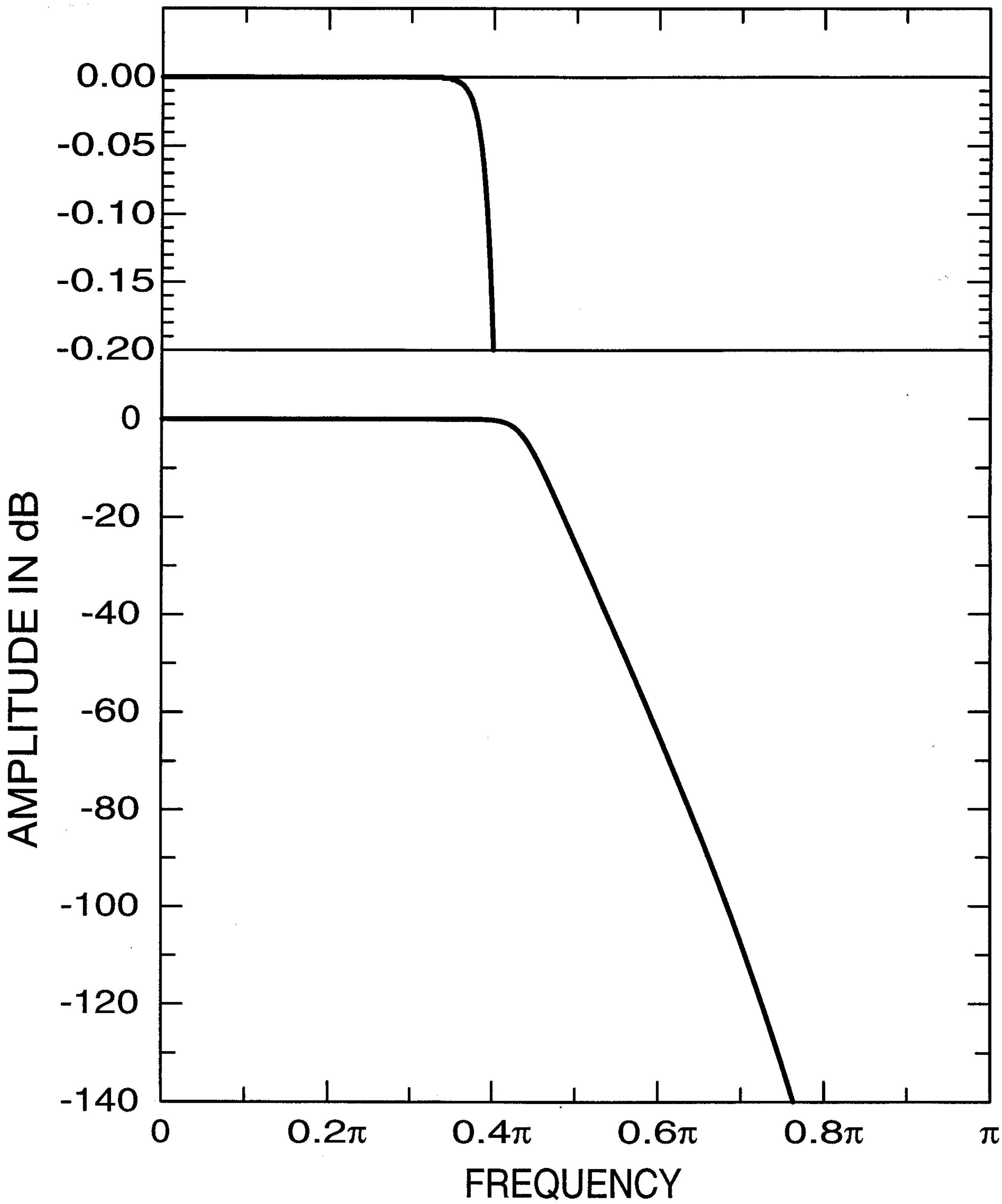
## CASCADE-FORM REALIZATION

- The following figure shows the corresponding realization. For the unscaled case, $a_{0i} = a_{2i} = 1$ for $i = 1, 2, \ldots, R$. For scaling this realization, see Part ~~IV: Infinite~~ V: Finite Wordlength Effects.



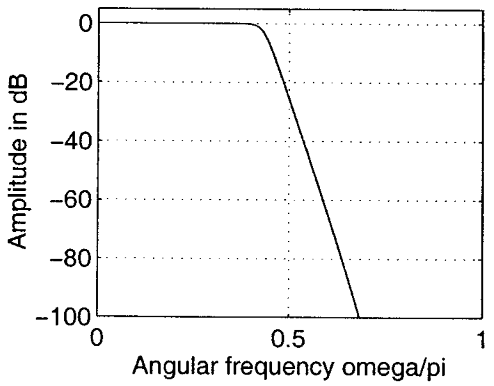



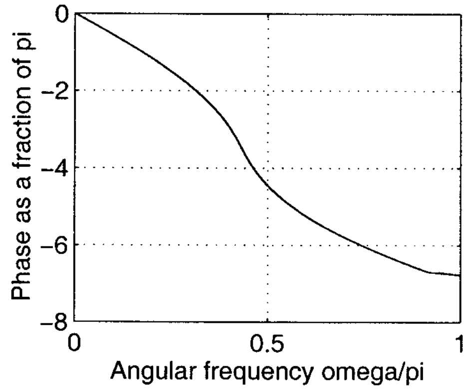
# RESPONSE

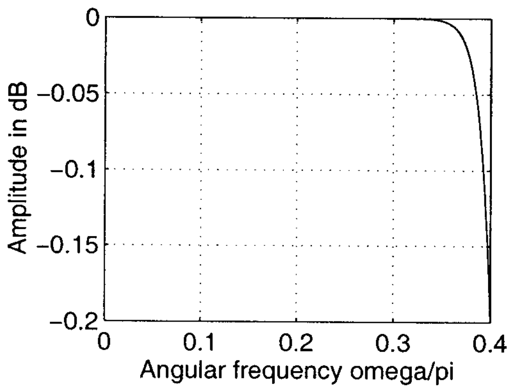


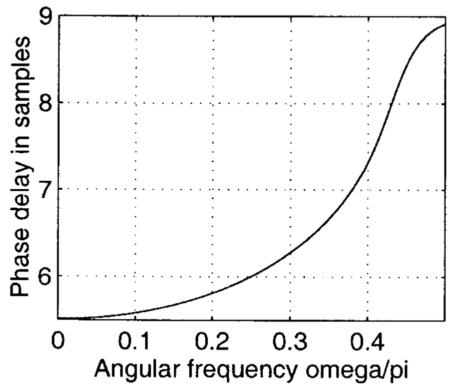
0.00
-0.05
-0.10
-0.15
-0.20
0
-20
-40
-60
-80
-100
-120
-140
AMPLITUDE IN dB
0
0.2π
0.4π
0.6π
0.8π
π
FREQUENCY

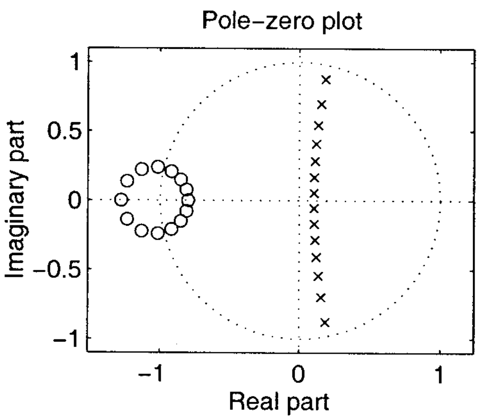
# MORE DETAILS

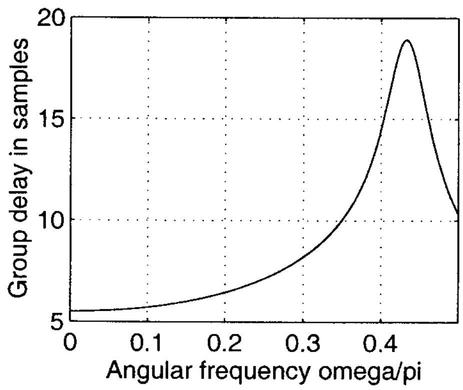


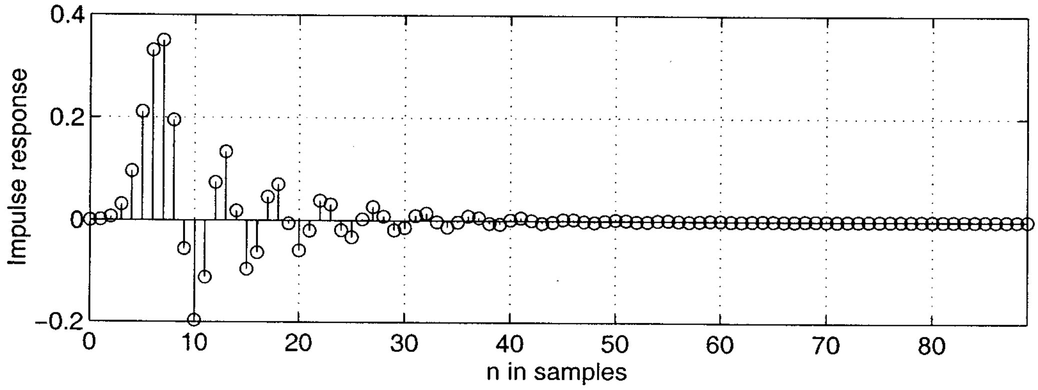



## CHEYSHEV FILTER

- By applying the substitution $z = (1 + s/c)/(1 - s/c)$ with $c = 1.3763819$ to the poles of the anaolg filter given in transparency 20, we end up with the following eight $z$-plane pole locations:

$$0.94957258 \exp(\pm j0.40609325\pi)$$
$$0.84907285 \exp(\pm j0.35956778\pi)$$
$$0.74725104 \exp(\pm j0.25857469\pi)$$
$$0.67089242 \exp(\pm j0.09688941\pi)$$

- All eight zeros of the digital filter are located at $z = -1$.

- This filter can be implemented in the form of equation (31) with $R = 4$.

- Four second-order denominator sections are:

$$1 - 1.28010410z^{-1} + 0.45009663z^{-2}$$
$$1 - 1.02792505z^{-1} + 0.55838412z^{-2}$$
$$1 - 0.72512101z^{-1} + 0.72092470z^{-2}$$
$$1 - 0.55218764z^{-1} + 0.90168809z^{-2}$$

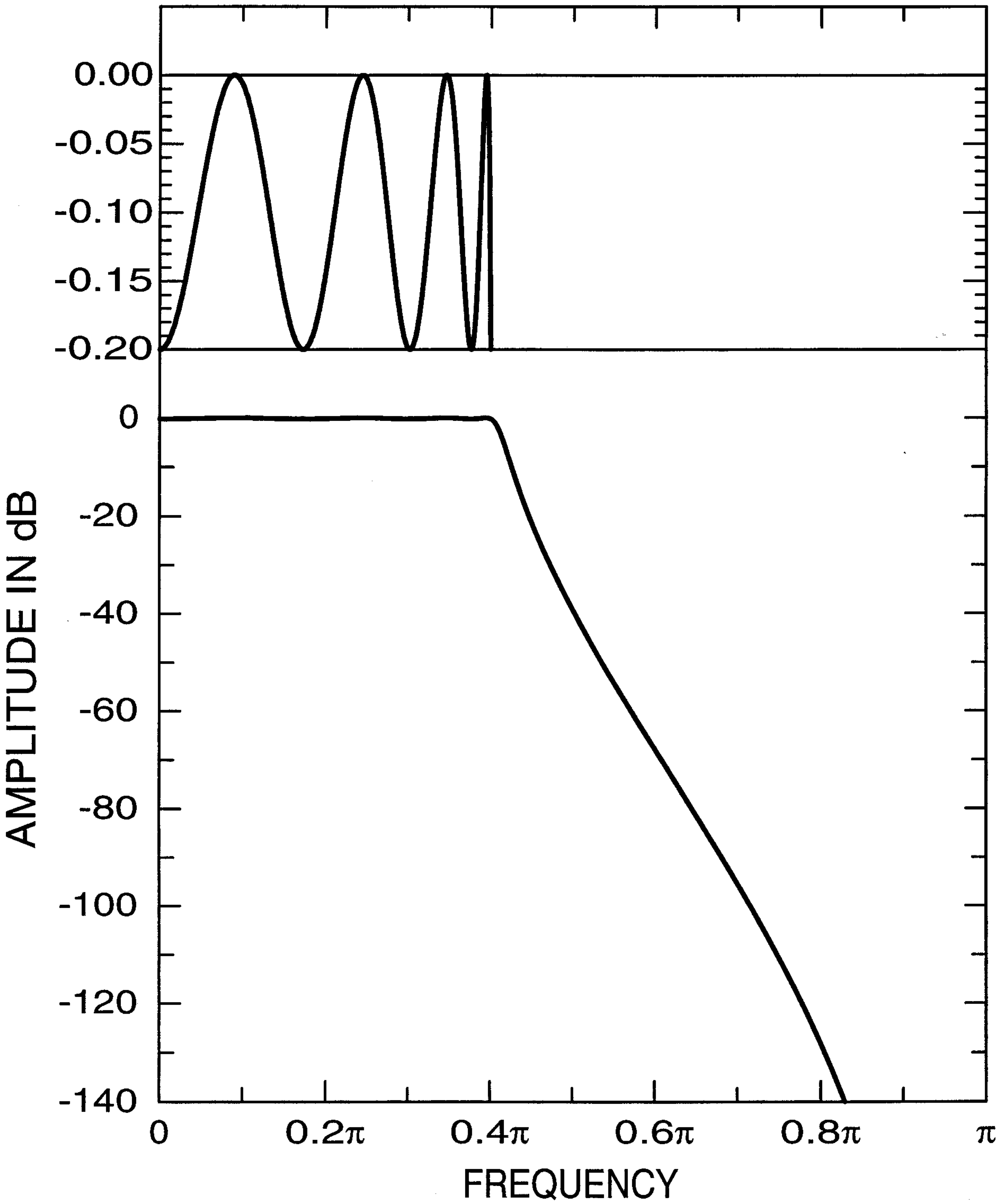
- Four second-order numerator sections are:

$$1 + 2z^{-1} + z^{-2}$$

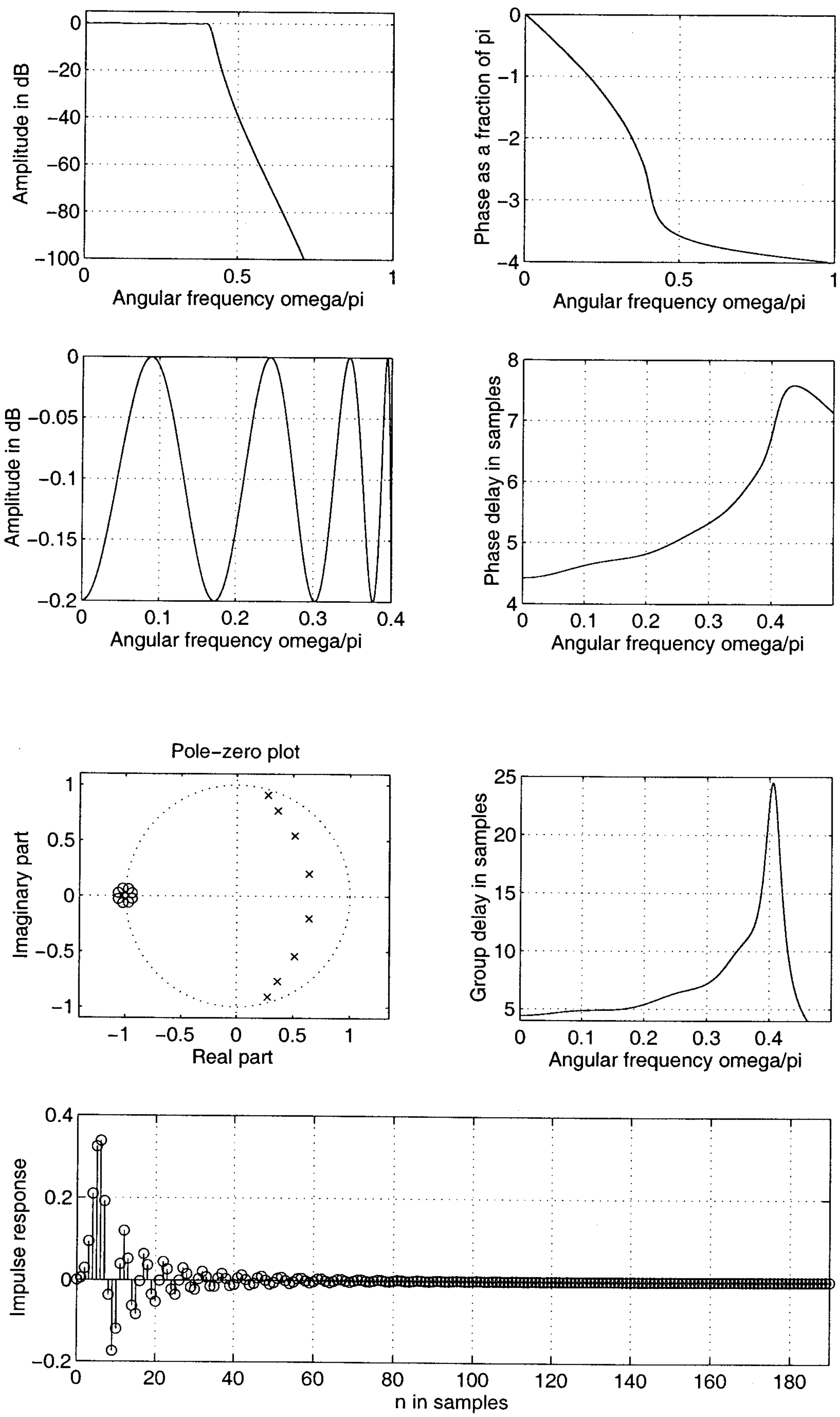
- $H(1) = 1/\sqrt{1 + \epsilon^2} \Rightarrow k_0 = 4.6258177 \cdot 10^{-4}$

- $A_s = 67.83097$ dB

# RESPONSE


0.00
-0.05
-0.10
-0.15
-0.20
0
-20
-40
-60
-80
-100
-120
-140
AMPLITUDE IN dB
0
0.2π
0.4π
0.6π
0.8π
π
FREQUENCY

# MORE DETAILS



# INVERSE CHEBYSHEV FILTER

- By applying the substitution $z = (1 + s/c)/(1 - s/c)$ with $c = 1.3763819$ to the poles of the anaolg filter given in transparency 25, we end up with the following eight $z$-plane pole locations:

$$0.85730567 \exp(\pm j0.45006120\pi)$$

$$0.60587226 \exp(\pm j0.46398940\pi)$$

$$0.36738157 \exp(\pm j0.49504221\pi)$$

$$0.12866797 \exp(\pm j0.58295504\pi)$$

- Eight zeros of the digital filter are located on the unit circle at

$$\exp(\pm j0.60585559\pi)$$

$$\exp(\pm j0.65404342\pi)$$

$$\exp(\pm j0.75576400\pi)$$

$$\exp(\pm j0.91036173\pi)$$

- This filter can be implemented in the form of equation (31) with $R = 4$.

- Four second-order denominator sections are:

$$1 + 0.06630799z^{-1} + 0.01655545z^{-2}$$

$$1 - 0.01144373z^{-1} + 0.13496922z^{-2}$$

$$1 - 0.13679322z^{-1} + 0.36708120z^{-2}$$

$$1 - 0.26789871z^{-1} + 0.73497301z^{-2}$$

- Four second-order numerator sections:

$$1 + 1.92122022z^{-1} + z^{-2}$$

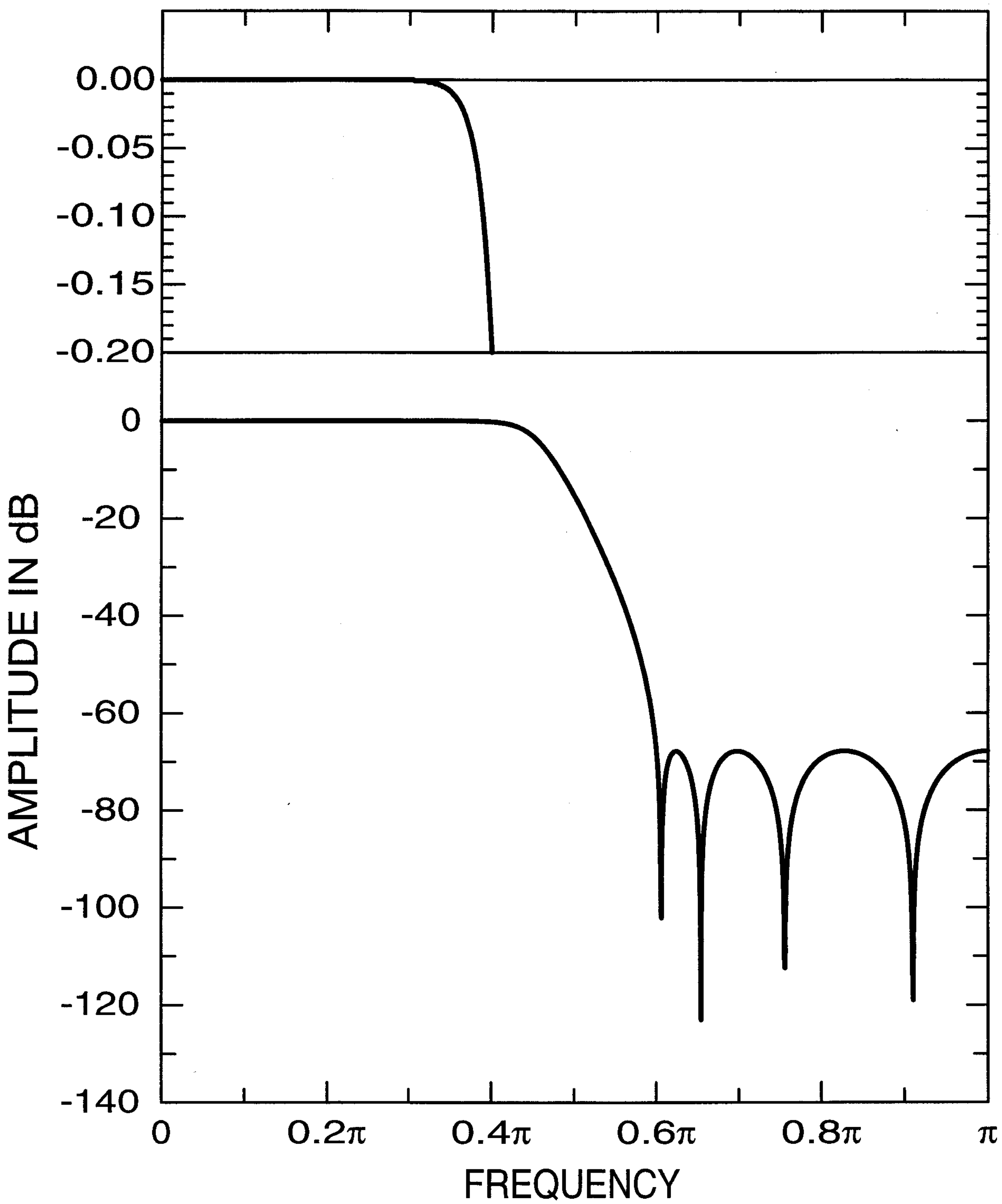
$$1 + 1.43958909z^{-1} + z^{-2}$$

$$1 + 0.93054369z^{-1} + z^{-2}$$

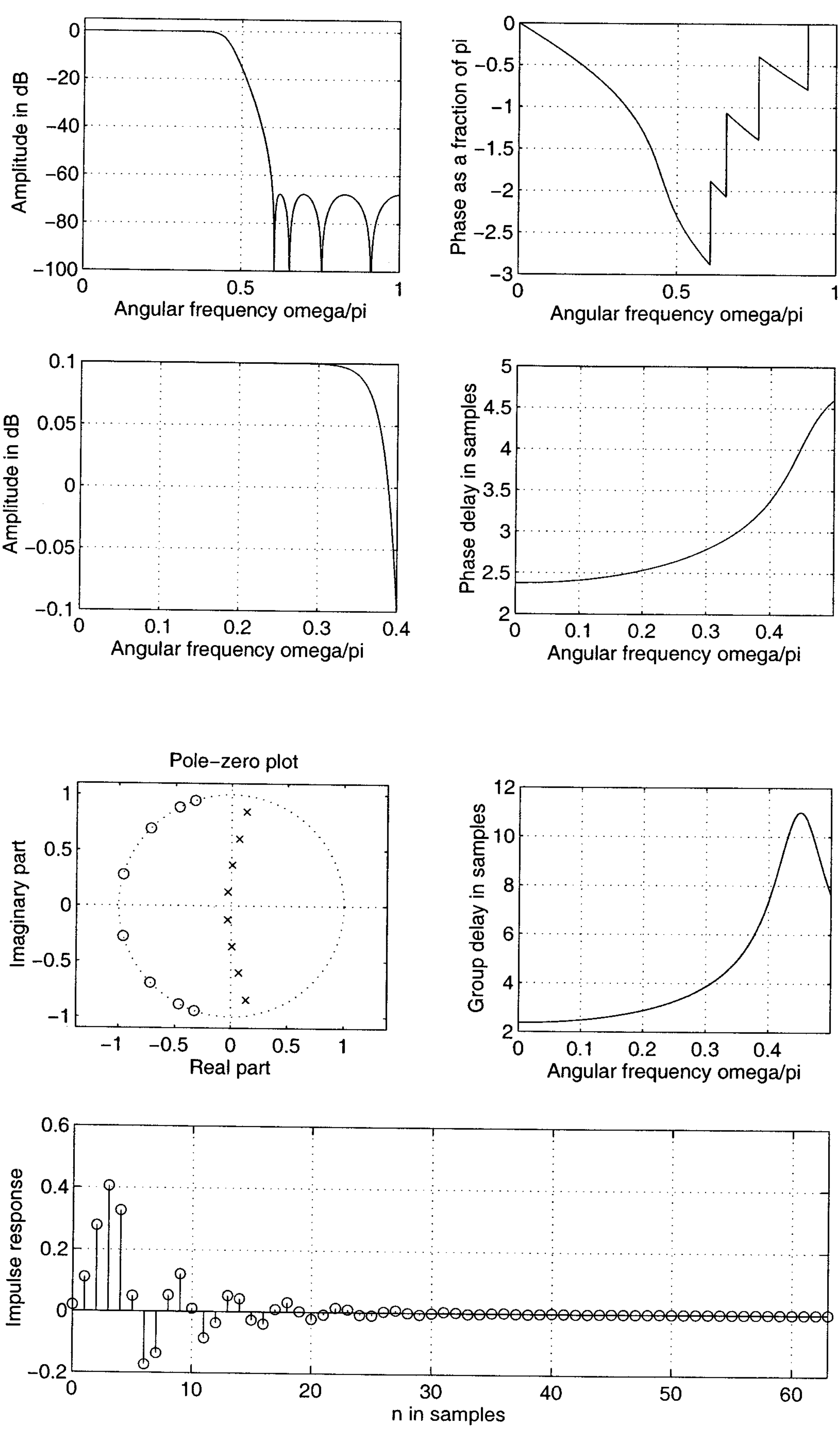
$$1 + 0.65291851z^{-1} + z^{-2}$$

- $H(1) = 1 \Rightarrow k_0 = 2.0941877 \cdot 10^{-2}$
- $A_s = 67.8310$ dB

# RESPONSE


0.00
-0.05
-0.10
-0.15
-0.20
0
-20
-40
-60
-80
-100
-120
-140
AMPLITUDE IN dB
0
0.2π
0.4π
0.6π
0.8π
π
FREQUENCY

# MORE DETAILS



## ELLIPTIC FILTER

- By applying the substitution $z = (1 + s/c)/(1 - s/c)$ with $c = 1.3763819$ to the poles of the anaolg filter given in transparency 31, we end up with the following six $z$-plane pole locations:

$$0.92644921 \exp(\pm j0.40974245\pi)$$

$$0.75825817 \exp(\pm j0.34061328\pi)$$

$$0.57209956 \exp(\pm j0.15169962\pi)$$

- The six poles of the digital filter are located at

$$\exp(\pm j0.60889279\pi)$$

$$\exp(\pm j0.68762357\pi)$$

$$\exp(\pm j0.87276133\pi)$$

- This filter can be implemented in the form of equation (31) with $R = 3$.

- Three second-order denominator sections:

$$1 - 1.01670072z^{-1} + 0.32729791z^{-2}$$

$$1 - 0.72802553z^{-1} + 0.57495546z^{-2}$$

$$1 - 0.51838171z^{-1} + 0.85830814z^{-2}$$

- Three second-order numerator sections:

$$1 + 1.84233061z^{-1} + z^{-2}$$

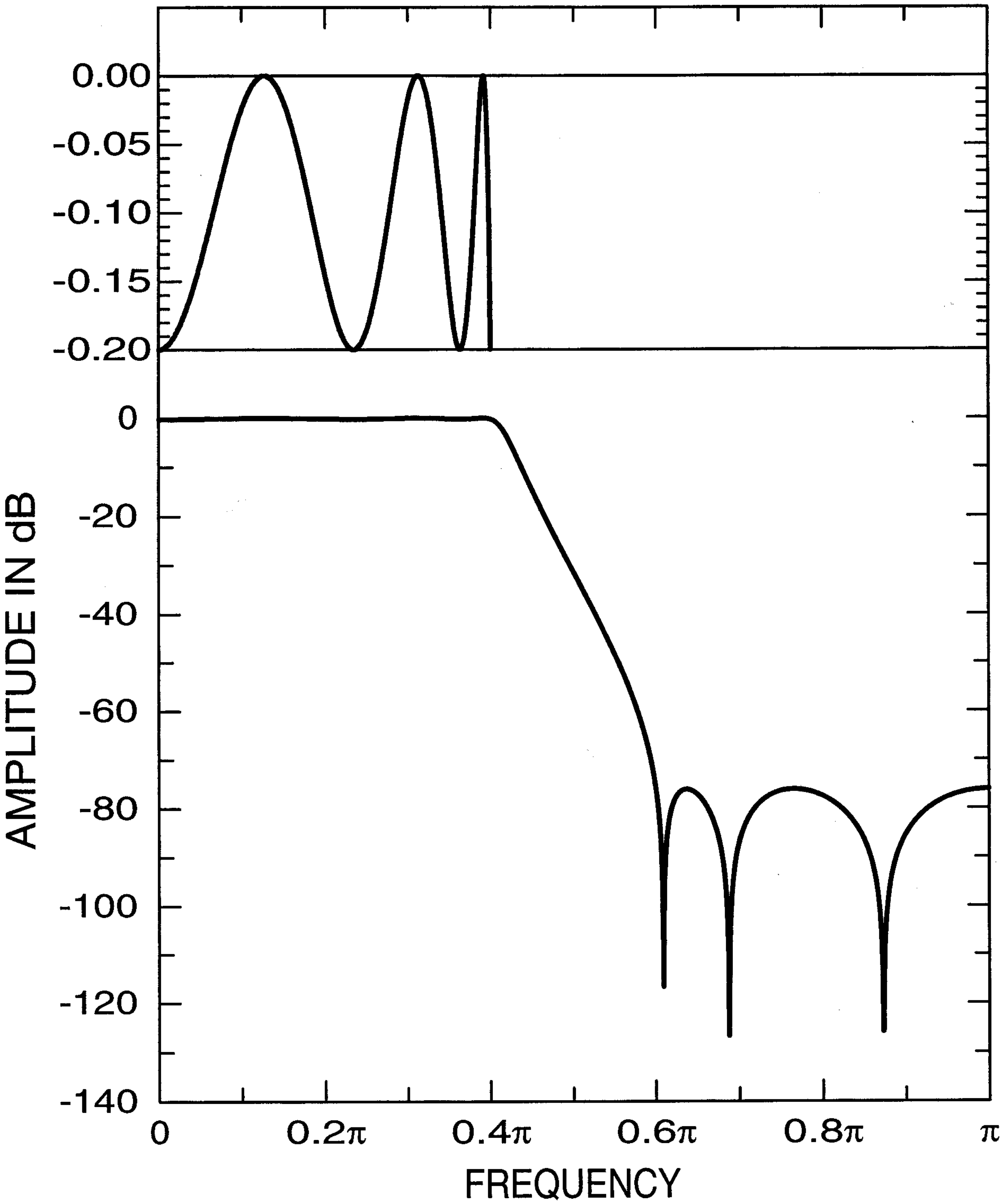
$$1 + 1.11178594z^{-1} + z^{-2}$$

$$1 + 0.67092626z^{-1} + z^{-2}$$

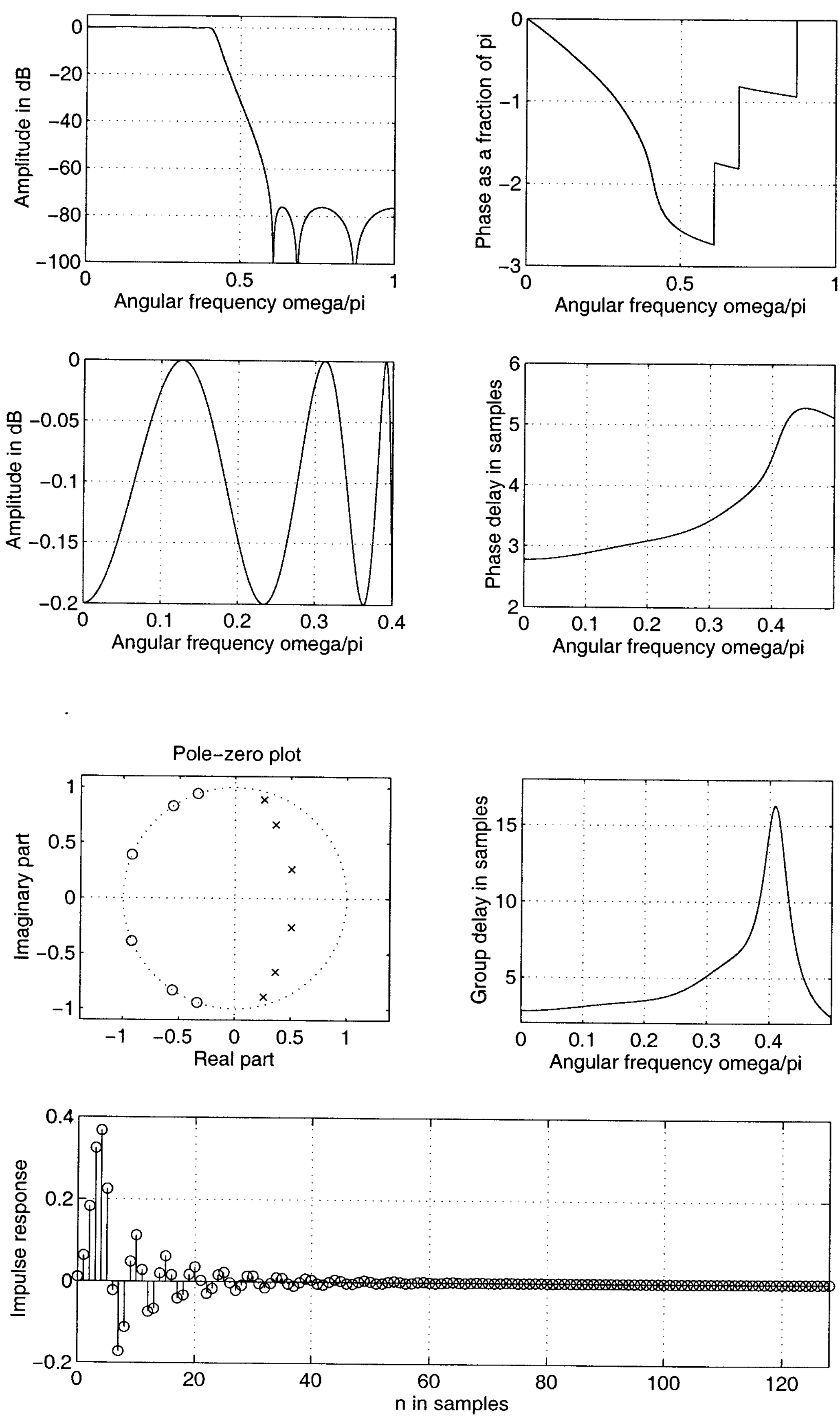
- $H(1) = 1/\sqrt{1+\epsilon^2} \Rightarrow k_0 = 1.078595980 \cdot 10^{-2}$

- $A_s = 76.11092$ dB.

# RESPONSE


0.00
-0.05
-0.10
-0.15
-0.20
0
-20
-40
-60
-80
-100
-120
-140
AMPLITUDE IN dB
0
0.2π
0.4π
0.6π
0.8π
π
FREQUENCY

# MORE DETAILS



# FREQUENCY TRANSFORMATIONS

- Up to now, we have considered only lowpass filters.
- Given a prototype lowpass filter transfer function $H_{lp}(Z)$, there exist transformations converting this prototype filter into another lowpass filter, a highpass filter, a bandpass filter, or a bandstop filter.
- In all these cases, the desired transfer function is constructed as

$$H(z) = H_{lp}(Z)\Big|_{Z^{-1}=G(z^{-1})} \qquad (32)$$

- It is required that
  - $G(z^{-1})$ must be a rational function of $z^{-1}$
  - The inside of the unit circle of the $Z$-plane must map to the inside the unit circle of the $z$-plane.
  - The unit circle of the $Z$-plane must map onto the unit circle of the $z$-plane.
- The most general transformation is of the following allpass type

$$Z^{-1} = G(z^{-1}) = \pm \prod_{k=1}^{N} \frac{z^{-1} - \alpha_k}{1 - \alpha_k z^{-1}}. \qquad (33)$$

# LOWPASS-TO-LOWPASS TRANFORMATION

- The desired transformation is

$$Z^{-1} = G(z^{-1}) = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}. \tag{34}$$

- Substituting $Z = e^{j\theta}$ and $z = e^{j\omega}$, we obtain, after some manipulations,

$$\omega = \text{arctan2}((1 - \alpha^2)\sin\theta, 2\alpha + (1 + \alpha^2)\cos\theta) \tag{35}$$

or

$$\theta = 2\text{arctan2}((1-\alpha^2)\sin\omega, -2\alpha+(1+\alpha^2)\cos\omega), \tag{36}$$

where

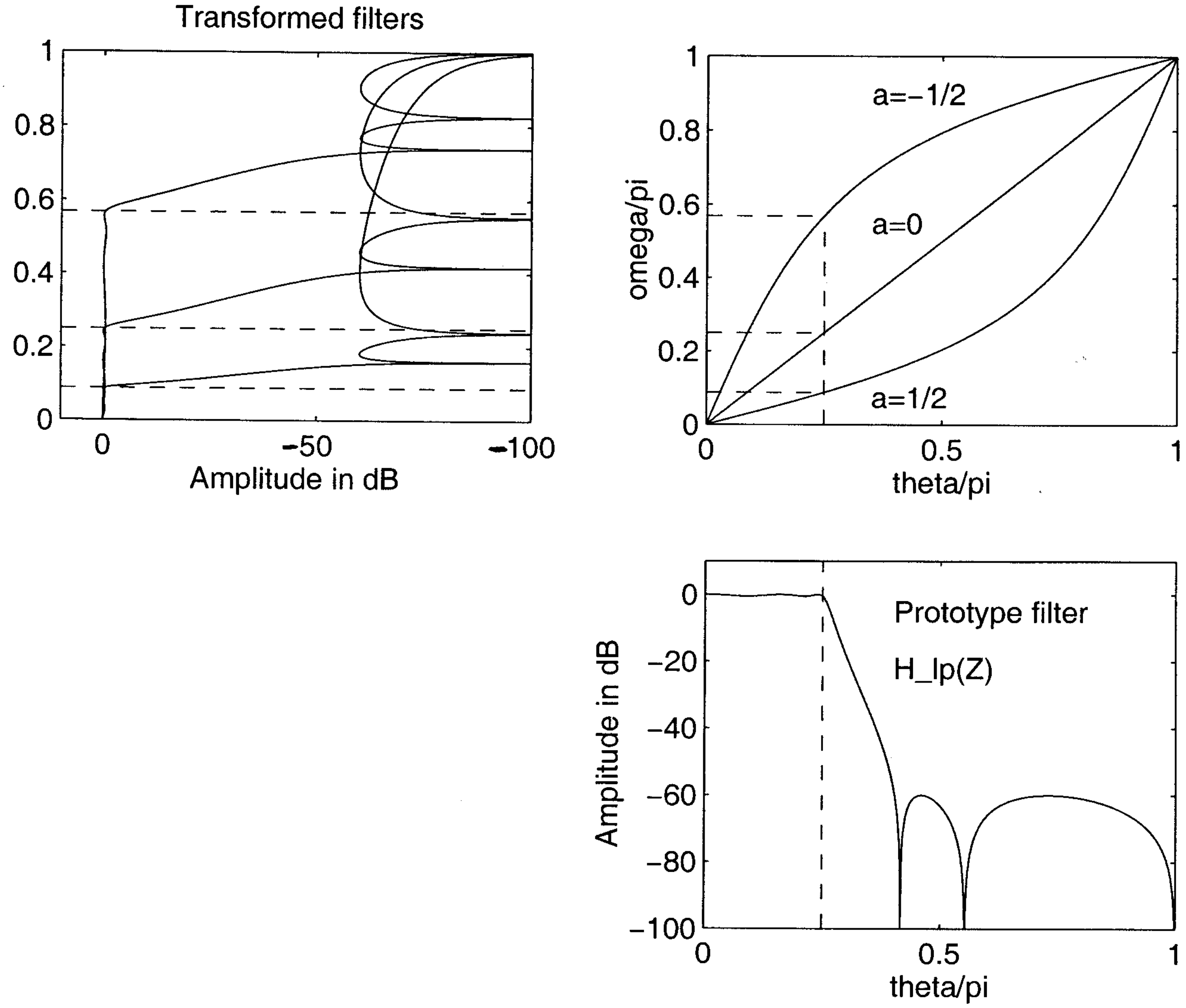
$$\text{arctan2}(y, x) = \begin{cases} \arctan(y/x), & x \geq 0 \\ \arctan(y/x) + \pi, & x < 0 \text{ and } y \geq 0 \\ \arctan(y/x) - \pi, & x < 0 \text{ and } y \leq 0. \end{cases} \tag{37}$$

- If the passband edge of the prototype is $\theta_p$ and the edge of the desired filter is $\omega_p$, then

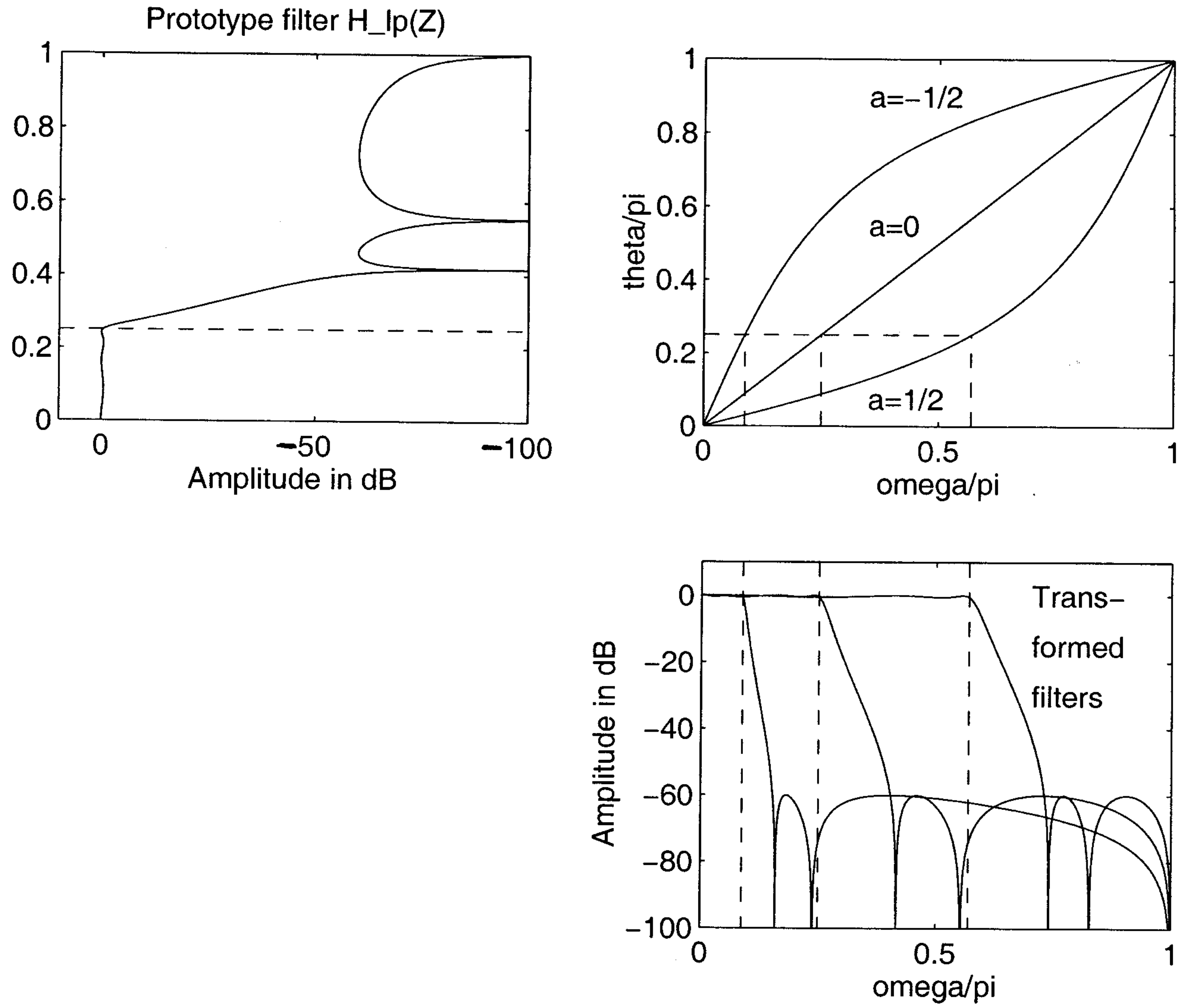
$$\alpha = \frac{\sin[(\theta_p - \omega_p)/2]}{\sin[(\theta_p + \omega_p)/2]}. \tag{38}$$

- The following two transparencies exemplify these relationships for $\theta_p = 0.25\pi$. For $\alpha = 1/2$, $\alpha = 0$, and $\alpha = -1/2$, $\omega_p$ achieves the values of $0.0876\pi$, $0.25\pi$, and $0.5686\pi$, respectively.

# Relations Between the Prototype Filter and the Resulting Filter in the Lowpass-to-Lowpass Transformation (instead of $\alpha$, a is used)



# Relations Between the Prototype Filter and the Resulting Filter in the Lowpass-to-Lowpass Transformation (instead of $\alpha$, a is used)



**Example : Convert the elliptic design of pages 52–55 ($\theta_p = 0.4\pi$) to a filter with $\omega_p = 0.1\pi$**

- In this case, $\alpha = 0.64203952$.
- We end up with a filter with the following pole locations

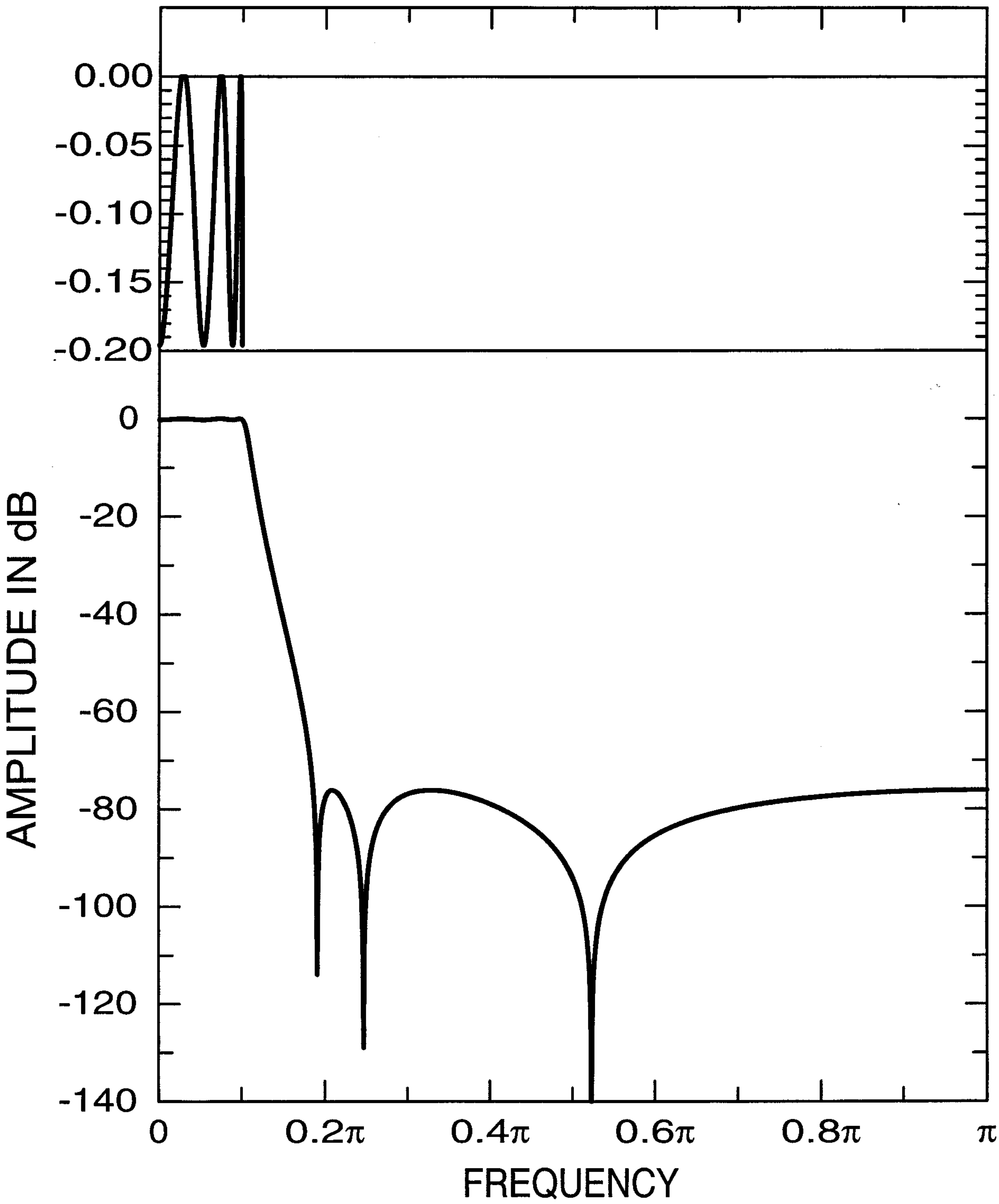
$$0.97499633 \exp(\pm j0.10298475\pi)$$
$$0.92380565 \exp(\pm j0.07987339\pi)$$
$$0.88250858 \exp(\pm j0.10298475\pi).$$

- The zero locations are:

$$\exp(\pm j0.19083677\pi)$$
$$\exp(\pm j0.24666981\pi)$$
$$\exp(\pm j0.52339441\pi).$$

- $\omega_s = 0.1856\pi$, $k_0 = 3.2987935 \cdot 10^{-4}$.

# RESPONSE


0.00
-0.05
-0.10
-0.15
-0.20
0
-20
-40
-60
-80
-100
-120
-140
AMPLITUDE IN dB
0
0.2π
0.4π
0.6π
0.8π
π
FREQUENCY

# LOWPASS-TO-HIGHPASS TRANFORMATION

- The transformation

$$Z^{-1} = G(z^{-1}) = -\frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}, \qquad (39a)$$

where

$$\alpha = -\frac{\cos[(\theta_p + \omega_p)/2]}{\cos[(\theta_p - \omega_p)/2]} \qquad (39b)$$

converts a prototype filter with passband edge at $\theta_p$ to a highpass filter with passband edge at $\omega_p$.

- Substituting $Z = e^{j\theta}$ and $z = e^{j\omega}$, we obtain, after some manipulations,

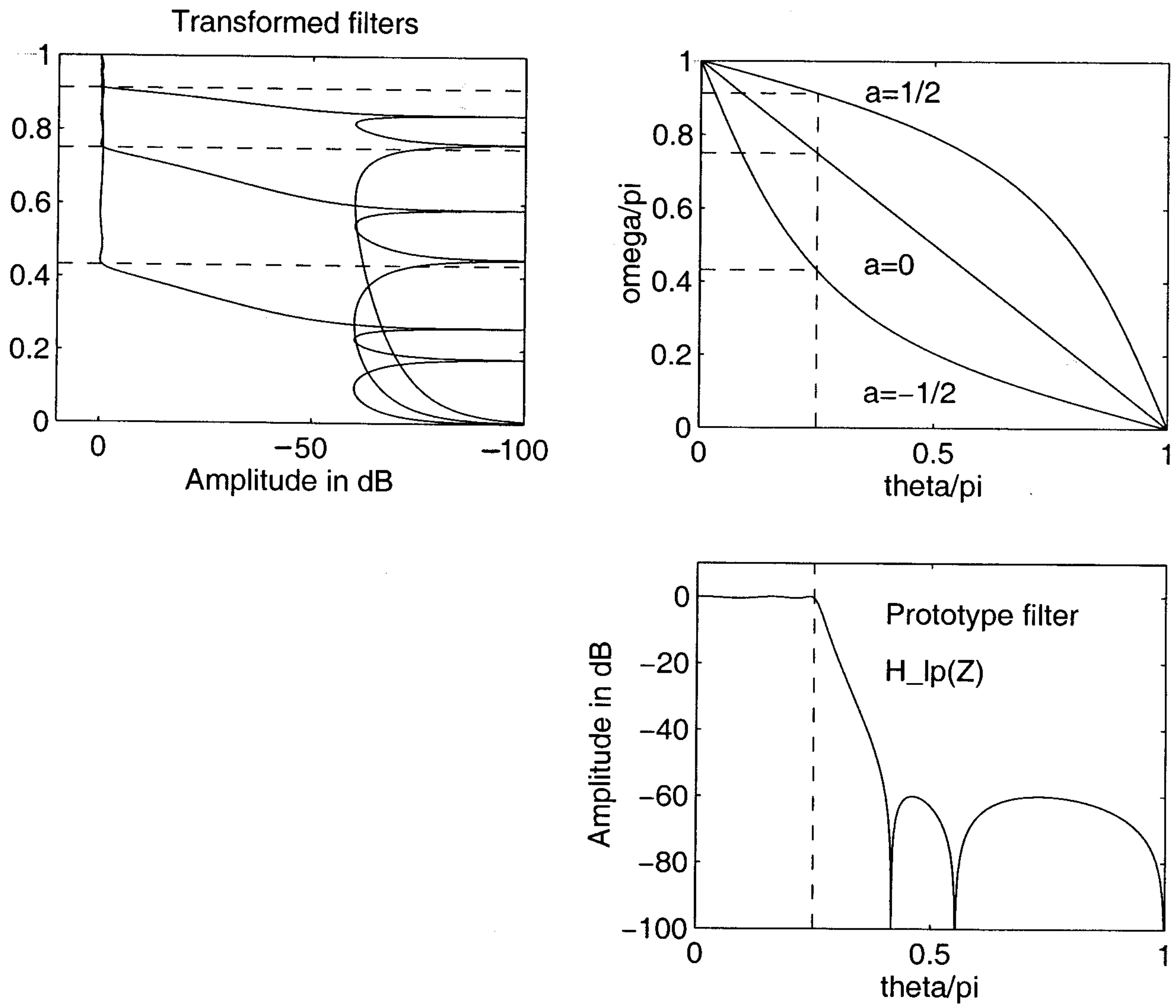
$$\omega = \pi - \text{arctan2}((1-\alpha^2)\sin\theta, 2\alpha + (1+\alpha^2)\cos\theta) \quad (40)$$

or

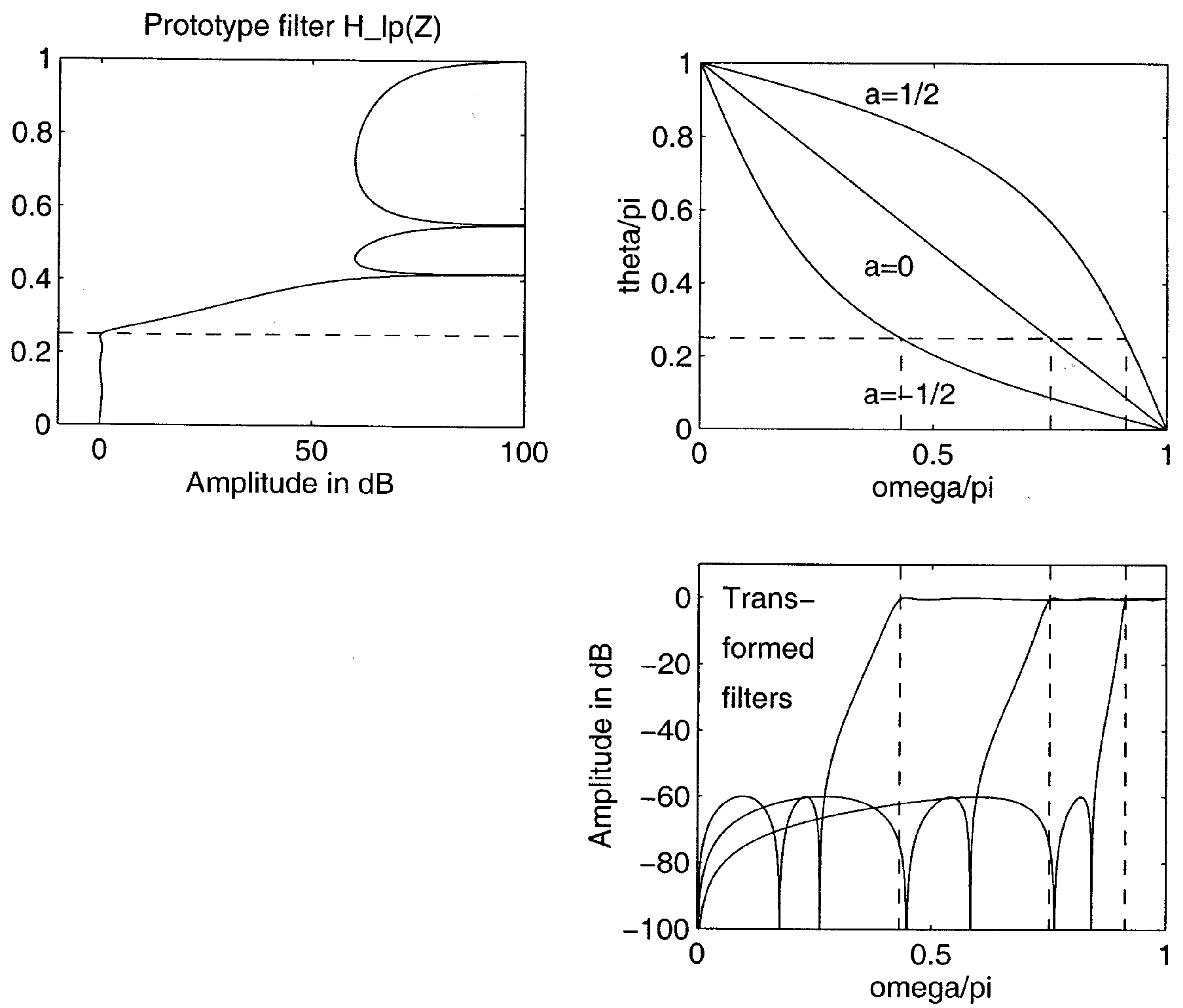
$$\theta = pi - \text{arctan2}((1-\alpha^2)\sin\omega, -2\alpha + (1+\alpha^2)\cos\omega), \qquad (41)$$

- The following two transparencies exemplify these relationships for $\theta_p = 0.25\pi$. For $\alpha = 1/2$, $\alpha = 0$, and $\alpha = -1/2$, $\omega_p$ achieves the values of $0.9124\pi$, $0.75\pi$, and $0.4314\pi$, respectively.

# Relations Between the Prototype Filter and the Resulting Filter in the Lowpass-to-Highpass Transformation (instead of $\alpha$, a is used)



# Relations Between the Prototype Filter and the Resulting Filter in the Lowpass-to-Highpass Transformation (instead of $\alpha$, a is used)



## EXAMPLE

- If the elliptic filter of pages 60 and 61 ($\theta_p = 0.1\pi$) is used as a prototype filter and it is desired to design a highpass filter with passband edge at $\omega_p = 0.9\pi$, then $\alpha = 0$ and the transformation is

$$Z^{-1} = -z^{-1}.$$

- The poles and zeros of the resulting filter are obtained by changing the angles by $\pi - \phi$, where $\phi$ is the angle of the prototype pole or zero.

- $\omega_s = \pi - 0.1856\pi = 0.8144\pi$.

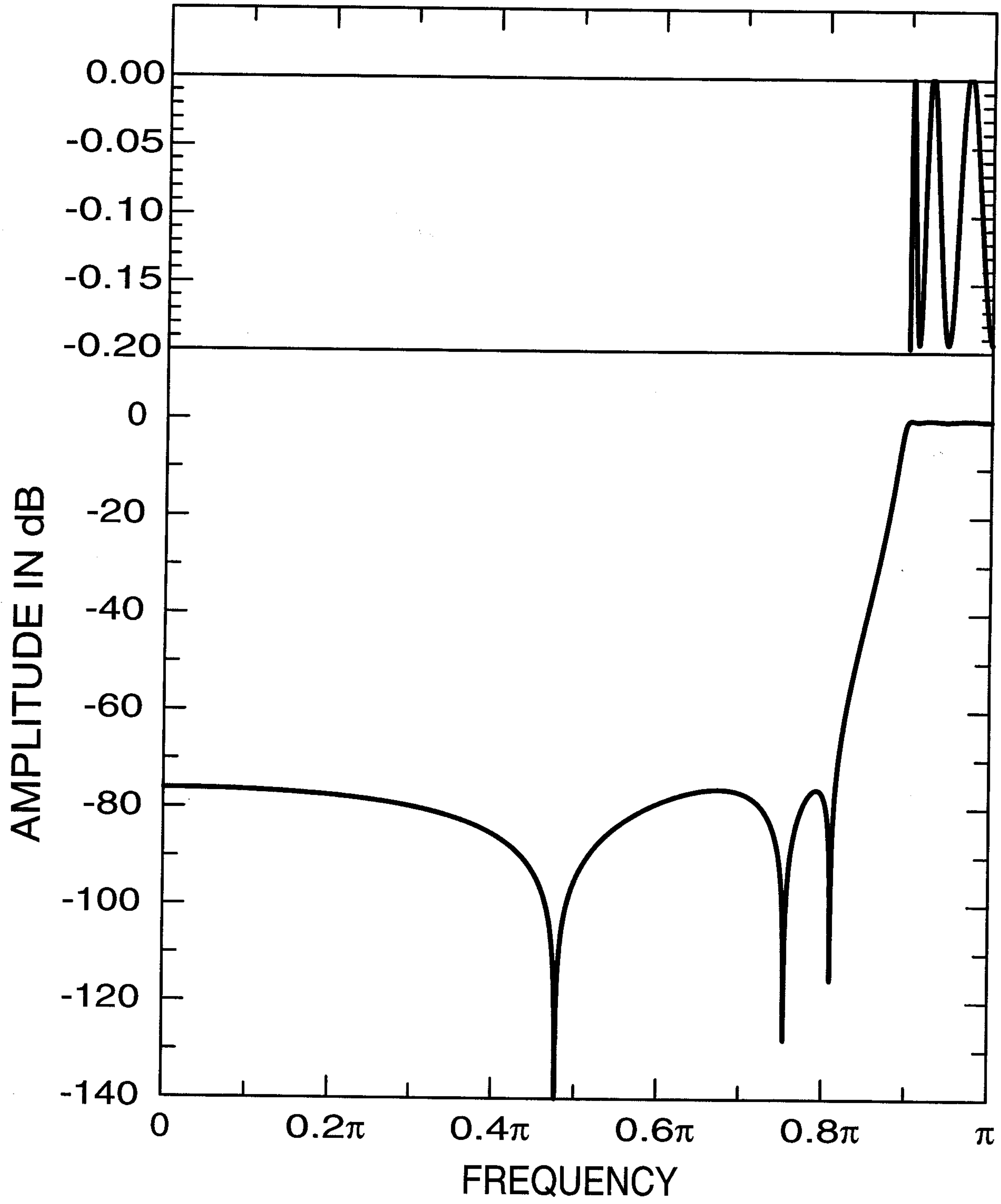
- Note that, in general, if it is desired to design a highpass filter with passband and stopband edges at $\omega_p$ and $\omega_s$ the design can be performed as follows:

**Step 1:** Design a prototype lowpass filter with the same passband and stopband requirements and having the passband and stopband edges at

$$\theta_p = \pi - \omega_p, \quad \theta_s = \pi - \omega_s.$$

**Step 2:** Convert this filter to the desired highpass filter using the substitution $Z^{-1} = -z^{-1}$.

# RESPONSE



0.00
-0.05
-0.10
-0.15
-0.20
0
-20
-40
-60
-80
-100
-120
-140
AMPLITUDE IN dB
0
0.2π
0.4π
0.6π
0.8π
π
FREQUENCY

# LOWPASS-TO-BANDPASS AND LOWPASS-TO-BANDSTOP TRANFORMATIONS

- The transformation

$$Z^{-1} = -\frac{z^{-2} - [2\alpha k/(k+1)]z^{-1} + [(k-1)/(k+1)]}{[(k-1)/(k+1)]z^{-2} - [2\alpha k/(k+1)]z^{-1} + 1} \quad (42a)$$

where

$$\alpha = \frac{\cos[(\omega_{p2} + \omega_{p1})/2]}{\cos[(\omega_{p2} - \omega_{p1})/2]} \quad (42b)$$

and

$$k = \cot[(\omega_{p2} - \omega_{p1})/2] \tan(\theta_p/2) \quad (42c)$$

converts a prototype filter with passband edge at $\theta_p$ to a bandpass filter with passband edges at $\omega_{p1}$ and $\omega_{p2}$.

- The transformation

$$Z^{-1} = \frac{z^{-2} - [2\alpha/(k+1)]z^{-1} + [(1-k)/(k+1)]}{[(1-k)/(k+1)]z^{-2} - [2\alpha k/(k+1)]z^{-1} + 1} \quad (43a)$$

where

$$\alpha = \frac{\cos[(\omega_{p2} + \omega_{p1})/2]}{\cos[(\omega_{p2} - \omega_{p1})/2]} \quad (43b)$$

and

$$k = \tan[(\omega_{p2} - \omega_{p1})/2] \tan(\theta_p/2) \quad (43c)$$

converts a prototype filter with passband edge at $\theta_p$ to a bandstop filter with passband edges at $\omega_{p1}$ and $\omega_{p2}$.

**Example : Convert the Chebyshev design of pages 45–47 ($\theta_p = 0.4\pi$) to a bandpass filter with $\omega_{p1} = 0.3\pi$ and $\omega_{p2} = 0.5\pi$**

- In this case, $\alpha = 0.32491970$ and $k = 2.236067977$.
- We end up with sixteenth-order filter the following pole locations

$$0.98569053\, \exp(\pm j0.29835922\pi)$$

$$0.98227710\, \exp(\pm j0.50200212\pi)$$

$$0.95797336\, \exp(\pm j0.31130499\pi)$$

$$0.94960368\, \exp(\pm j0.48594844\pi)$$

$$0.93315440\, \exp(\pm j0.33691275\pi)$$

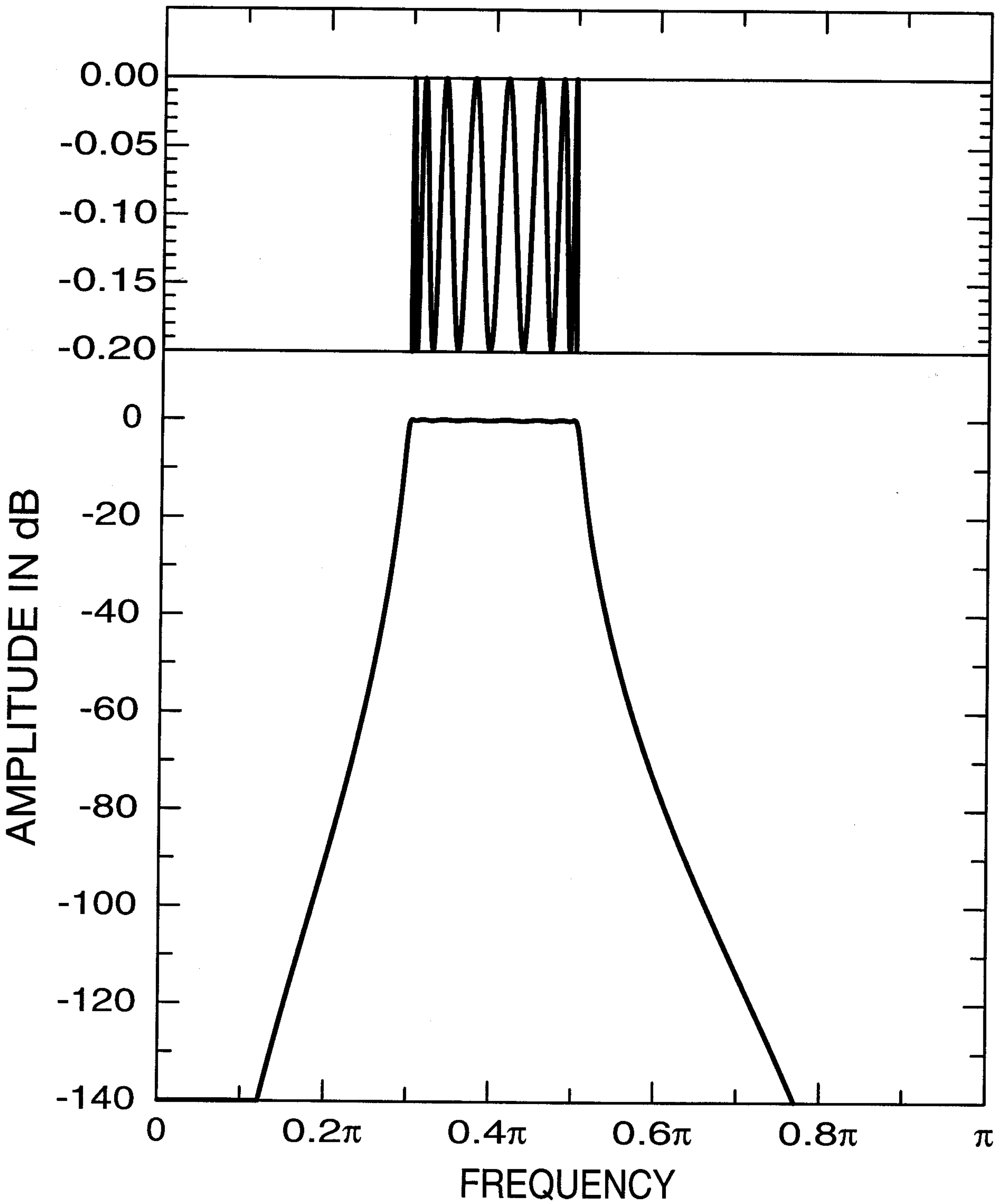
$$0.92442018\, \exp(\pm j0.45561647\pi)$$

$$0.91587579\, \exp(\pm j0.37335423\pi)$$

$$0.91212767\, \exp(\pm j0.41556937\pi).$$

- Eight zeros are located at $z = 1$ and eight at $z = -1$.
- $k_0 = 3.2995443 \cdot 10^{-4}$.

# RESPONSE


0.00
-0.05
-0.10
-0.15
-0.20
0
-20
-40
-60
-80
-100
-120
-140
AMPLITUDE IN dB
0
0.2π
0.4π
0.6π
0.8π
π
FREQUENCY

**Example : Convert the Chebyshev design of pages 45–47 ($\theta_p = 0.4\pi$) to a bandstop filter with $\omega_{p1} = 0.3\pi$ and $\omega_{p2} = 0.5\pi$**

- In this case, $\alpha = 0.32491970$ and $k = 0.236067977$.
- We end up with sixteenth-order filter the following pole locations:

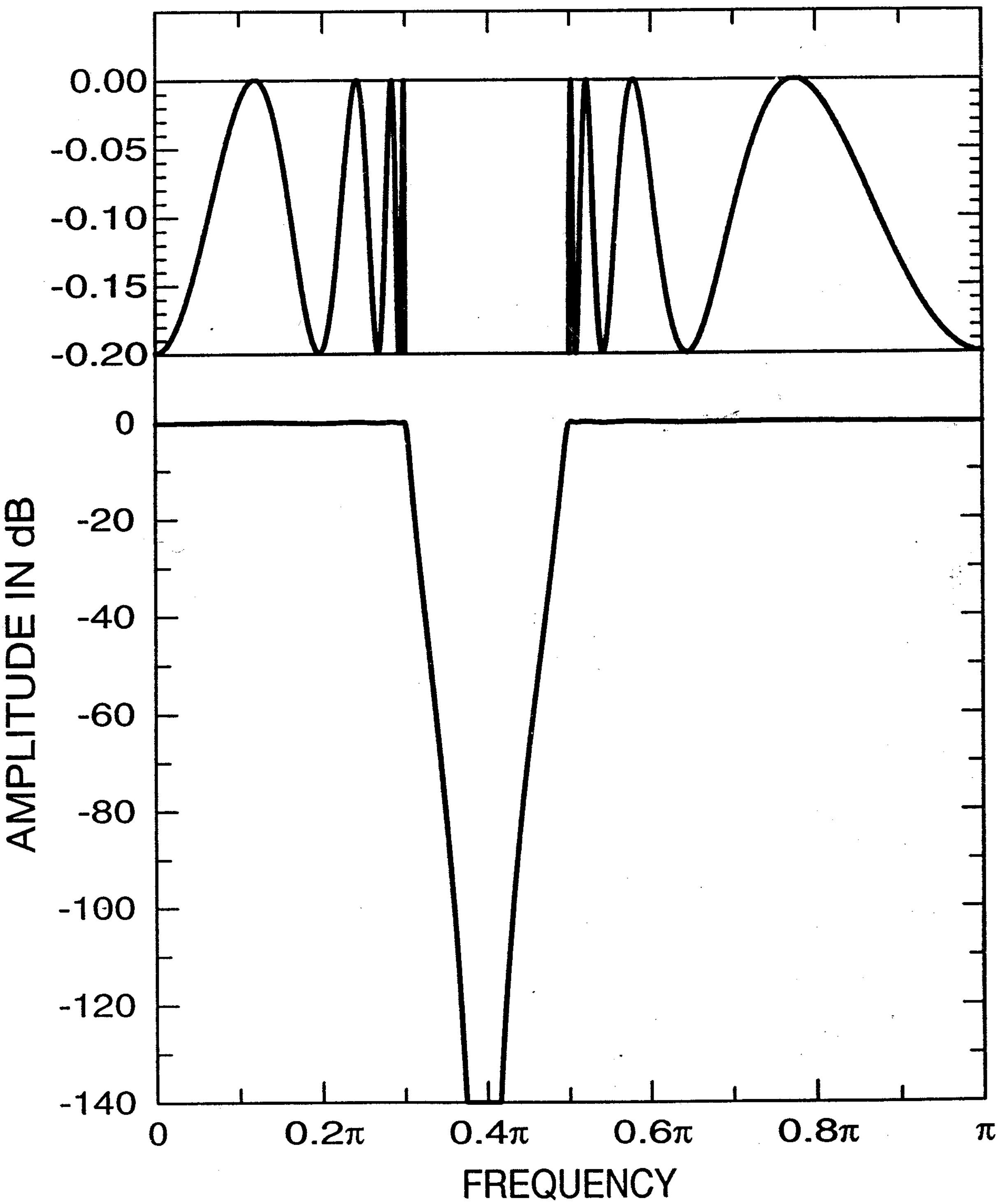
$$0.98609453 \exp(\pm j0.30178800\pi)$$
$$0.98290980 \exp(\pm j0.49776646\pi)$$
$$0.94970250 \exp(\pm j0.28958923\pi)$$
$$0.93677849 \exp(\pm j0.51263291\pi)$$
$$0.87333345 \exp(\pm j0.25771382\pi)$$
$$0.83185977 \exp(\pm j0.55201560\pi)$$
$$0.65411717 \exp(\pm j0.17585583\pi)$$
$$0.50113082 \exp(\pm j0.64602864\pi).$$

- There are eight zero pairs at

$$\exp(\pm j0.39466274\pi).$$

- $k_0 = 0.1189990$.

# RESPONSE


0.00
-0.05
-0.10
-0.15
-0.20
0
-20
-40
-60
-80
-100
-120
-140
AMPLITUDE IN dB
0
0.2π
0.4π
0.6π
0.8π
π
FREQUENCY

## Practical Design of Bandpass Filters

- For the lowpass-to-bandpass transformation the relation between $\theta$, the frequency variable of the prototype filter, and $\omega$, the frequency variable of the resulting filter, are related as

$$\theta = f(\omega) = -\pi + 2\omega + 2\text{arctan2}(y(\omega), x(\omega)), \quad (44)$$

where

$$y(\omega) = [2\alpha k/(k+1)]\sin(\omega) - [(k-1)/(k+1)]\sin(2\omega),$$

$$x(\omega) = 1 - [2\alpha k/(k+1)]\cos(\omega) + [(k-1)/(k+1)]\cos(2\omega),$$

and $\alpha$ and $k$ are given by Eqs. (42b) and (42c).

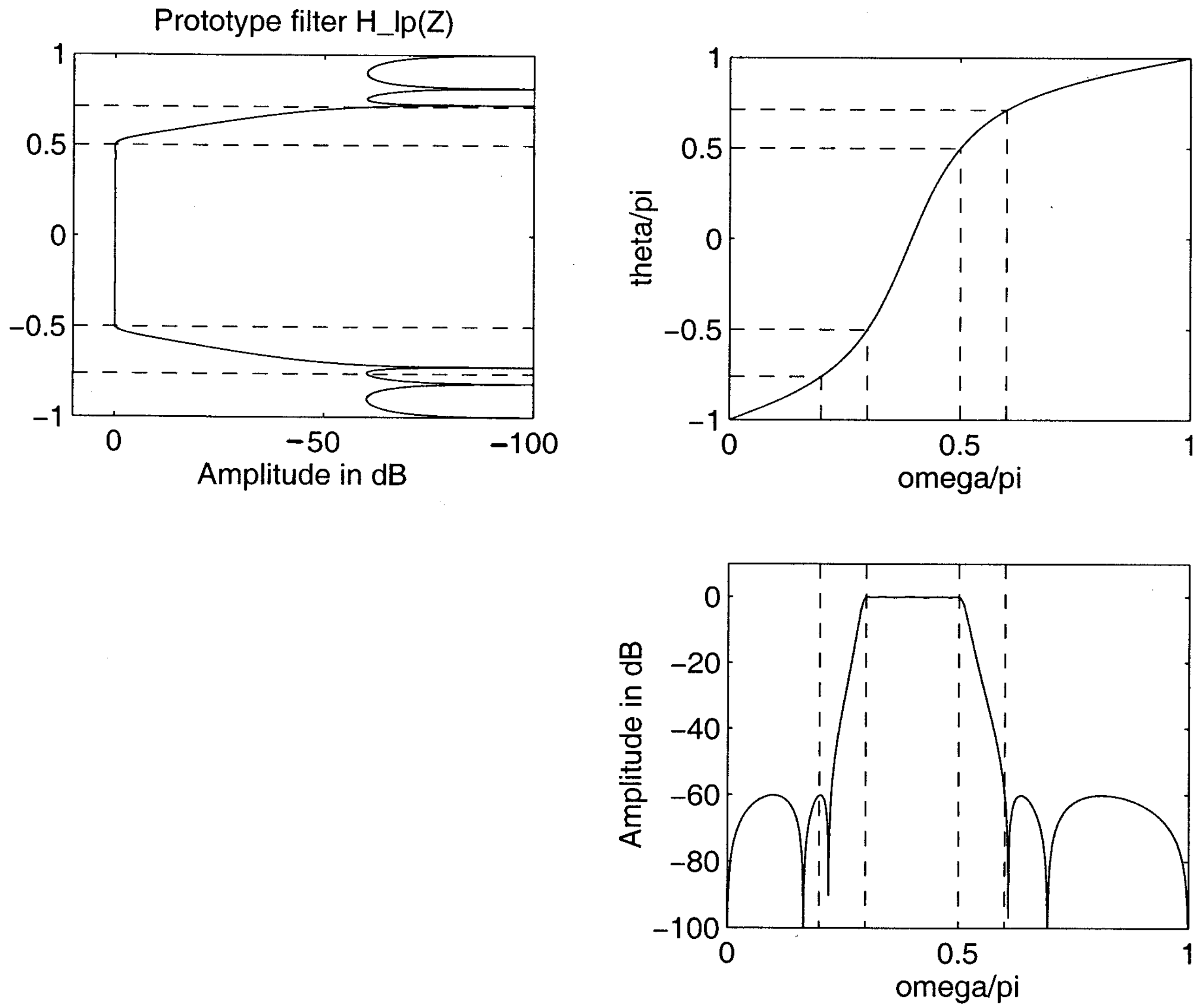
- The corresponding transformation guarantees that $\theta_p$, the passband edge of the prototype lowpass filter, are mapped to $\omega_{p1}$ and $\omega_{p2}$, the edges of the bandpass filter.

- In order to achieve the desired performance in the stopband regions $[0,\ \omega_{s1}]$ and $[\omega_{s2},\ \pi]$, the stopband edge of the prototype lowpass filter has to be

$$\theta_s = \min\{|f(\omega_{s1})|, f(\omega_{s2})\}. \quad (45)$$

**EXAMPLE:** $\omega_{s1} = 0.2\pi$, $\omega_{p1} = 0.3\pi$, $\omega_{p2} = 0.5\pi$, $\omega_{s2} = 0.6\pi$, **passband ripple = 0.1 dB, stopband ripple = 60 dB**

- Selecting $\theta_p = 0.5\pi$, $\theta_s = \min\{|-0.7608\pi|, 0.7113\pi\} = 0.7113\pi$.
- The following figure exemplifies the relations between the prototype filter and the resulting bandpass filter.



## Practical Design of Bandstop Filters

- For the lowpass-to-bandstop transformation the relation between $\theta$, the frequency variable of the prototype filter, and $\omega$, the frequency variable of the resulting filter, are related as

$$\theta = f(\omega) = 2\omega + 2\text{arctan2}(y(\omega), x(\omega)), \qquad (46)$$

where

$$y(\omega) = [2\alpha/(k+1)]\sin(\omega) - [(1-k)/(k+1)]\sin(2\omega),$$

$$x(\omega) = 1 - [2\alpha/(k+1)]\cos(\omega) + [(1-k)/(k+1)]\cos(2\omega),$$

and $\alpha$ and $k$ are given by Eqs. (43b) and (43c).

- The corresponding transformation guarantees that $\theta_p$, the pasband edge of the prototype lowpass filter, are mapped to $\omega_{p1}$ and $\omega_{p2}$, the edges of the bandstop filter.

- In order to achieve the desired performance in the stopband region $[\omega_{s1},\ \omega_{s2}]$, the stopband edge of the prototype lowpass filter has to be

$$\theta_s = \min\{f(\omega_{s1}), 2\pi - f(\omega_{s2})\}. \qquad (47)$$

**EXAMPLE:** $\omega_{p1} = 0.2\pi$, $\omega_{s1} = 0.3\pi$, $\omega_{s2} = 0.5\pi$, $\omega_{p2} = 0.6\pi$, **passband ripple = 0.1 dB, stopband ripple = 60 dB**

- Selecting $\theta_p = 0.5\pi$, $\theta_s = \min\{0.7856\pi, 2\pi - 1.3081\pi\} = 0.6919\pi$.

- The following figure exemplifies the relations between the prototype filter and the resulting bandpass filter.