DESIGN OF IIR FILTERS AS A TAPPED CASCADED INTERCONNECTION OF IDENTICAL ALLPASS SUBFILTERS

- This material is based on the article T. Saramäki and M. Renfors, "A novel approach for the design of IIR filters as a tapped cascaded interconnection of identical allpass subfilters," in *Proc. IEEE International Symposium on Circuits and Systems* (Philadelphia, PA), pp. 629–632, May 1987.
- This article is very theoretical and not very easy to go through. However, this article is included in the end of this pile of lecture notes.
- The purpose of these notes is to make it easier for the reader to grasp the main idea.
- When reading this material, it is worth picking up the part of lecture notes on Digital Filtering II which is entitled Design of Recursive Filters Using Allpass Filters as Building Blocks.

BACKGROUND

- As mentioned in the above-mentioned lecture notes, it is worth implementing recursive digital filters as a parallel connection of two allpass filters.
- These filters are also called wave lattice digital filters when the allpass filters are impelemented in a certain way.
- Some of the benefits of these filters are the following:
  - The filter of order N can be implemented by using only N multipliers and N delay elements.
  - When both allpass sections are implemented as a cascade of first- and second-order sections, the overall implemention becomes very modular. This is very beneficial for VLSI and signal processor implementations.
  - This means that similar building blocks with just different coefficient values can be used. The same block is easy to copy and then just

to change the coefficient values.

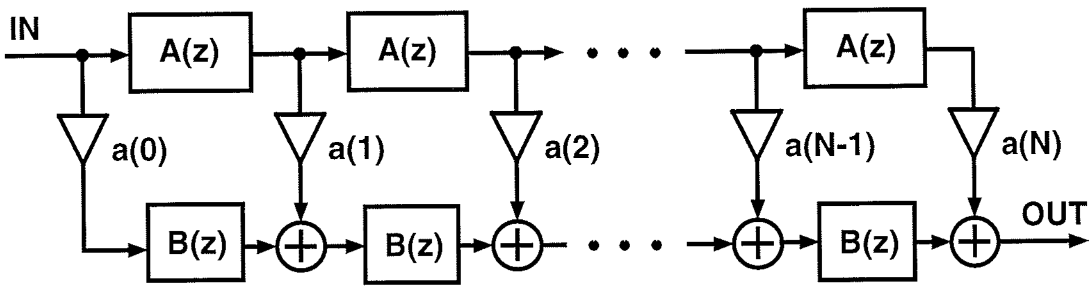
- The finite wordlength properties (output noise, limit cycles, coefficient sensitivity) are good.

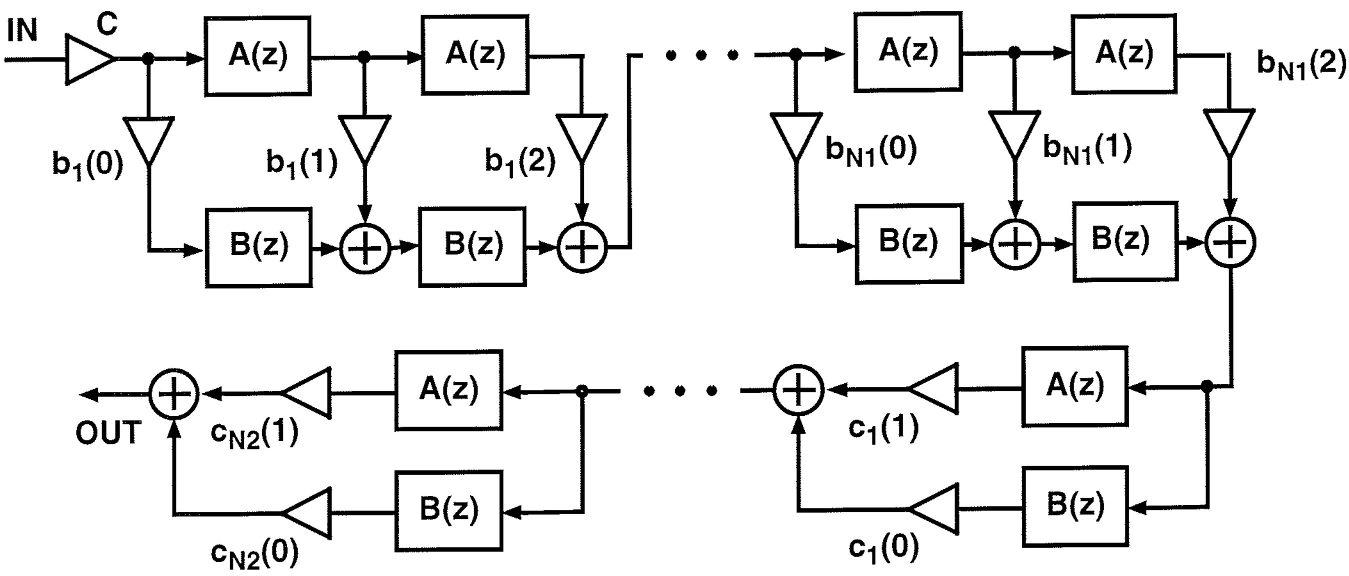
Why Then to Develop New Structures?

- When the stopband attenuation of our filter is relatively high (80–120 dB), the number of bit required for the coefficient values to keep the overall response within the given limits may exceed the number of bits available in signal processors.
- To avoid the costly multiplier elements in VLSI implementations, it is desired that all the coefficient values are representable as few powers of two; the multiplications can be performed as shifts and additions.
- These are the basic reasons for introducing the new structures to be considered in the next transparency.

Proposed Filter Structures

- The building blocks $A(z)$ and $B(z)$ in the structures shown below are **stable** allpass filters. In the following, it is assumed that the order of $B(z)$ is lower.
- The first structure is used for the design purposes and the second one for the implementation purposes.





How to Derive and Analyse the Proposed Structure?

- Consider a nonlinear-phase FIR filter

$$G(w) = \sum_{n=0}^{N} a[n]w^{-n}, \tag{1}$$

  where the $a(n)$'s are the tap coefficients in the first strucure of the previous page.

- The next step is to use the substitution

$$w^{-1} = A(z)/B(z), \tag{2}$$

  where $A(z)$ and $B(z)$ are our allpass filters in the structures of the previous page.

- This gives

$$\widehat{H}(z) = \sum_{n=0}^{N} a[n][A(z)/B(z)]^n. \tag{3}$$

- By multiplying the above transfer function by $[B(z)]^N$ yields

$$H(z) = [B(z)]^N \widehat{H}(z) = \sum_{n=0}^{N} a[n][A(z)]^n[B(z)]^{N-n}. \tag{4}$$

- **What happened? Answer:** This is the transfer function of the first structure of the previous page.

How to interpret the above result?

- Let us denote the frequency responses of our all-pass filters $A(z)$ and $B(z)$ by

$$A(e^{j\omega}) = e^{j\phi_A(\omega)}, \quad B(e^{j\omega}) = e^{j\phi_B(\omega)}. \tag{5}$$

- Based on Eqs. (3) and (4), $H(e^{j\omega})$ is expressible as

$$H(e^{j\omega}) = [e^{jN\phi_B(\omega)}][\sum_{n=0}^{N} a[n]e^{-jn[\phi_B(\omega)-\phi_A(\omega)]}]. \tag{6}$$

- The corresponding amplitude response $|H(e^{j\omega})|$ can be written as

$$|H(e^{j\omega})| = |\sum_{n=0}^{N} a[n]e^{-jn[\phi_B(\omega)-\phi_A(\omega)]}|. \tag{7}$$

- On the other hand, the amplitude response of our nonlinear-phase FIR filter with transfer function given by Eq. (1) is expressible as $(w = e^{j\Omega})$

$$|G(e^{j\Omega})| = |\sum_{n=0}^{N} a[n]e^{-jn\Omega}|. \tag{8}$$

- By comparing the above two equations, it is observed that $|H(e^{j\omega})|$ can be obtained from $|G(e^{j\Omega})|$

using the substitution

$$\Omega = \phi_B(\omega) - \phi_A(\omega). \tag{9}$$

- That is,

$$|H(e^{j\omega})| = |G(e^{j[\phi_B(\omega) - \phi_A(\omega)]})|. \tag{10}$$

Introductory Examples

- Before stating the simultaneos conditions for the allpass filters and the additional tap coefficients $a(n)$ we consider introductory examples.
- Our nonlinear-phase FIR filter is of order $N = 4$ and its impulse response coefficients are $a(0) = 0.20316651$, $a(1) = 0.52407075$, $a(2) = 0.37100043$, $a(3) = -0.02787074$, and $a(4) = -0.07796693$.
- The amplitude response $|G(e^{j\Omega})|$ of this filter oscillates within $1 \pm \delta_p$ in the region $[0,\ \Omega_p]$ and within δ_s and zero in the region $[\Omega_s,\ \pi]$ with

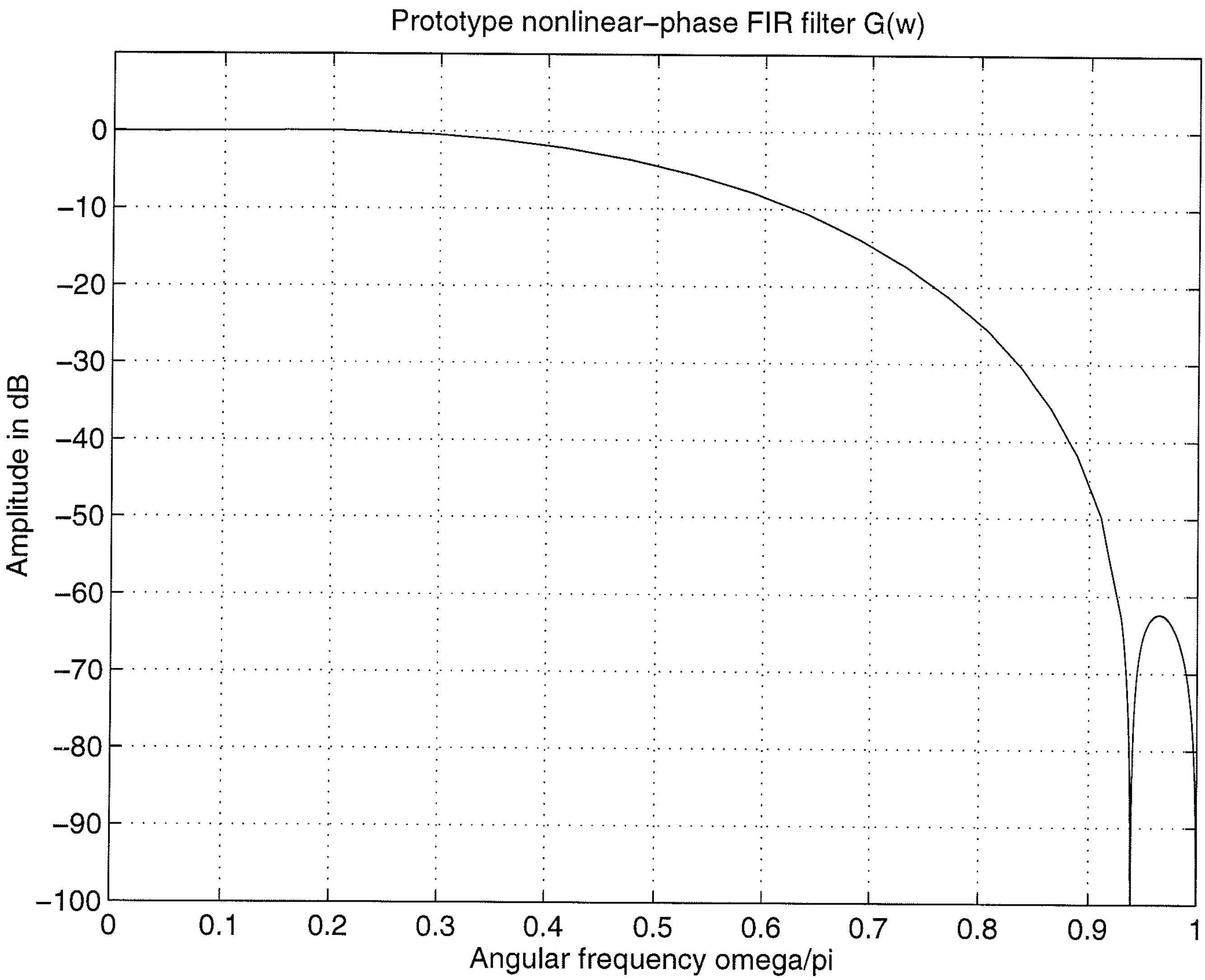
$$\delta_p = 0.0076, \quad \delta_s = 0.00076 \qquad (11a)$$

and

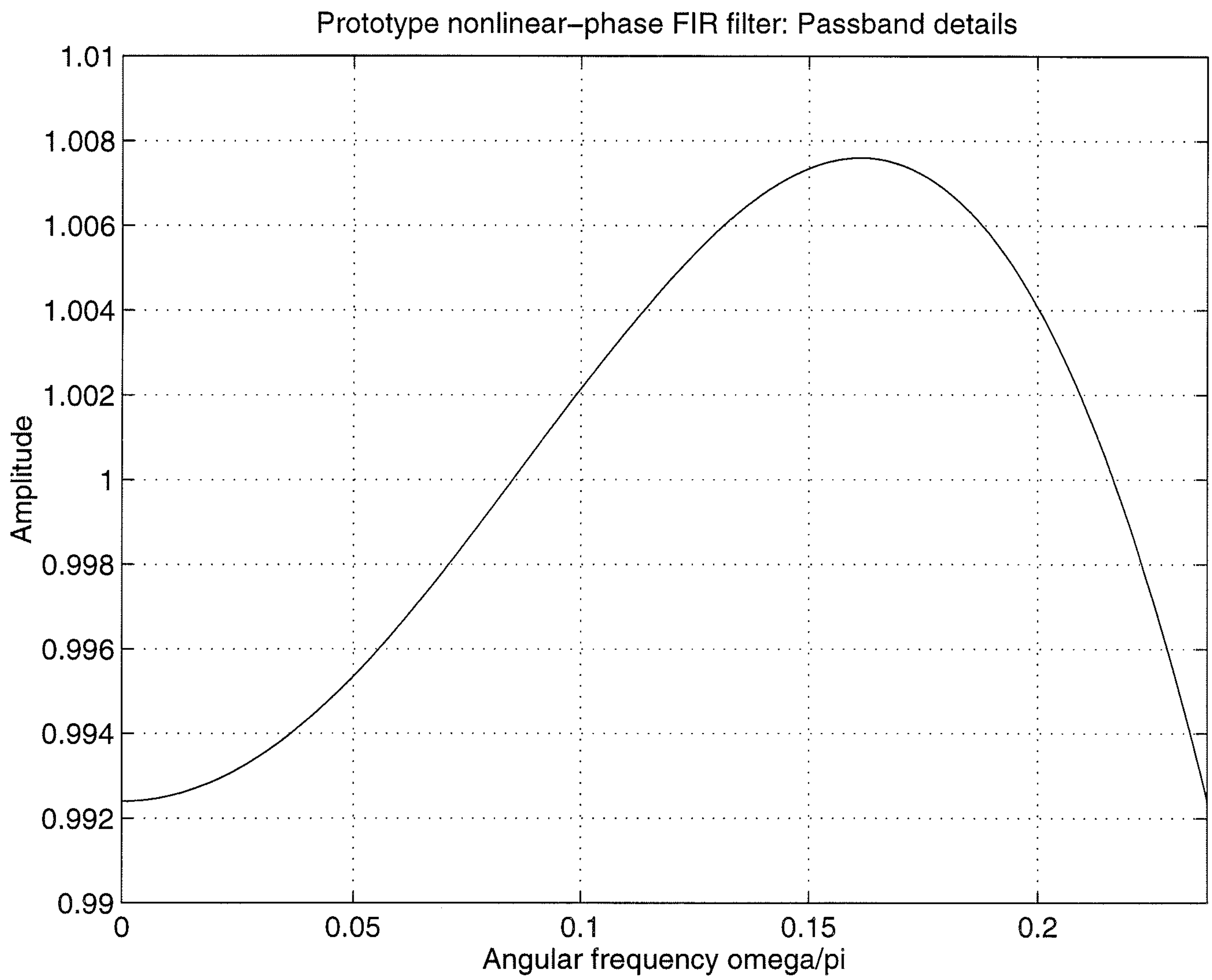
$$\Omega_p = 0.23680867\pi, \quad \Omega_s = 0.929394050\pi. \qquad (11b)$$

- The amplitude response for the above filter are depicted in the following two transparencies.
- This is a special nonlinear-phase FIR filter. Later in these lecture notes we explain how to design this filter.

Amplitude response for the nonlinear-phase FIR filter $G(w)$



Nonlinear-phase FIR filter $G(w)$: Passband details



Our examples

- Based on the above nonlinear-phase filter, it is desired to design an overall filter meeting

$$1 - \delta_p \leq |H(e^{j\omega})| \leq 1 + \delta_p \quad \text{for} \quad \omega \in X_p \qquad (12a)$$

$$|H(e^{j\omega})| \leq \delta_s \quad \text{for} \quad \omega \in X_s. \qquad (12a)$$

- We consider the following four cases:
  - Lowpass: $X_p = [0,\ 0.3\pi]$ and $X_s = [0.301\pi,\ \pi]$.
  - Highpass: $X_p = [0.7\pi,\ \pi]$ and $X_s = [0,\ 0.699\pi]$.
  - Bandstop: $X_p = [0,\ 0.15\pi] \cup [0.85\pi,\ \pi]$ and $X_s = [0.1505\pi,\ 0.845\pi]$.
  - Bandpass: $X_p = [0.35\pi,\ 0.65\pi]$ and $X_s = [0,\ 0.345\pi] \cup [0.655\pi,\ \pi]$.

Lowpass design

- In this case, the desired performance for the overall filter is achieved by properly designing $A(z)$ and $B(z)$, that is, the overall amplitude response oscillates within $1 \pm \delta_p$ with $\delta_p = 0.0076$ on $X_p = [0,\ 0.3\pi]$ and within $\delta_s = 0.00076$ and zero on $X_s = [0.301\pi,\ \pi]$.
- They are determined in such a way that their phase responses satisfy

$$-\Omega_p \leq \phi_B(\omega) - \phi_A(\omega) \leq \Omega_p \quad \text{for} \quad \omega \in X_p \qquad (13a)$$

and

$$2\pi - \Omega_s \leq \phi_B(\omega) - \phi_A(\omega) \leq \Omega_s \quad \text{for} \quad \omega \in X_s, \qquad (13b)$$

where $\Omega_p = 0.23680867\pi$ and $\Omega_s = 0.929394050\pi$.

- Because of the periodicity of $|G(e^{j\Omega})|$, it stays within $1 \pm \delta_p$ for $\Omega \in [-\Omega_p,\ \Omega_p]$ and within δ_s and zero for $\Omega \in [\Omega_s,\ 2\pi - \Omega_s]$. See transparency 15.
- Therefore, the mapping

$$\Omega = \phi_B(\omega) - \phi_A(\omega) \qquad (14)$$

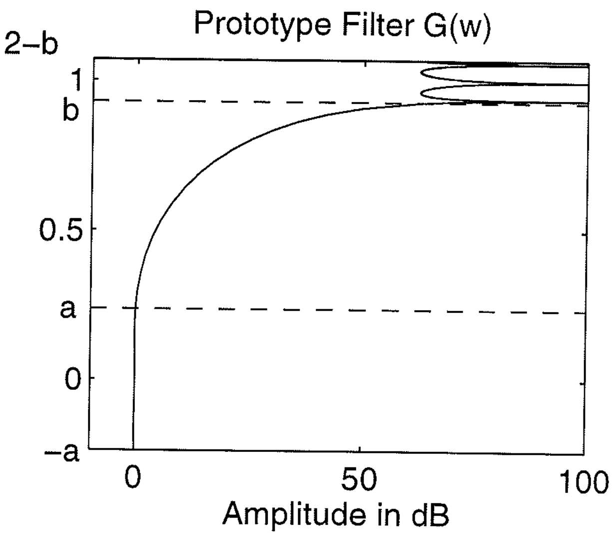
satisfying Eq. (13a) converts the passband performance of $|G(e^{j\Omega})|$ into that of the amplitude response

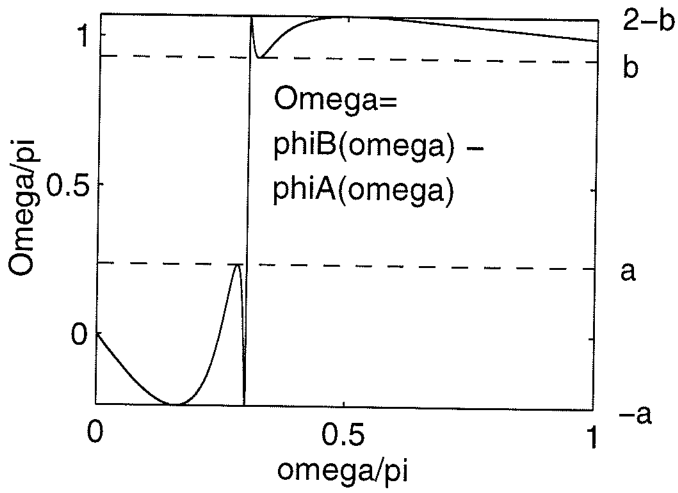
$$|H(e^{j\omega})| = G(e^{j[\phi_B(\omega)-\phi_A(\omega)]})| \qquad (15)$$

of the overall filter (see the following transparency).

- Similarly, the desired stopband performance of $G(w)$ is converted to that of $H(z)$.
- Page 16 depicts in a larger scale the mapping $\Omega = \phi_B(\omega) - \phi_A(\omega)$, whereas pages 17 and 18 show the overall response in more details.
- Note that the overall filter is implementable using the first structure of page 5. There are four identical copies of $A(z)$ and $B(z)$ with the five coefficients $a(n)$ for $n = 0, 1, \cdots, 4$ given on page 9.

Mapping a nonlinear-phase FIR filter to a lowpass filter being implementable as a tapped cascaded interconnection of two identical allpass filters.



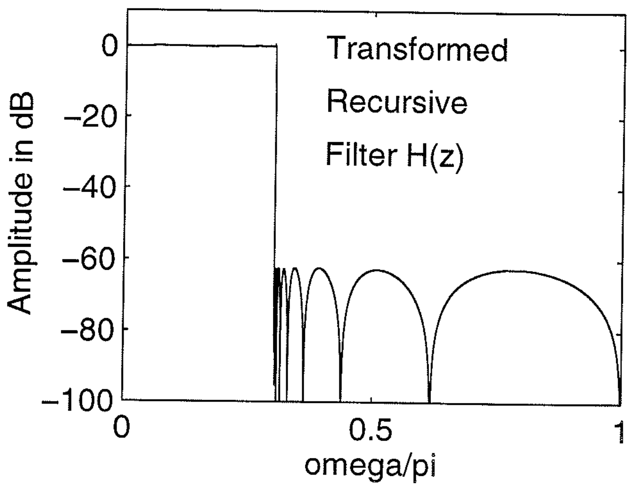


Passband edge = 0.300*pi

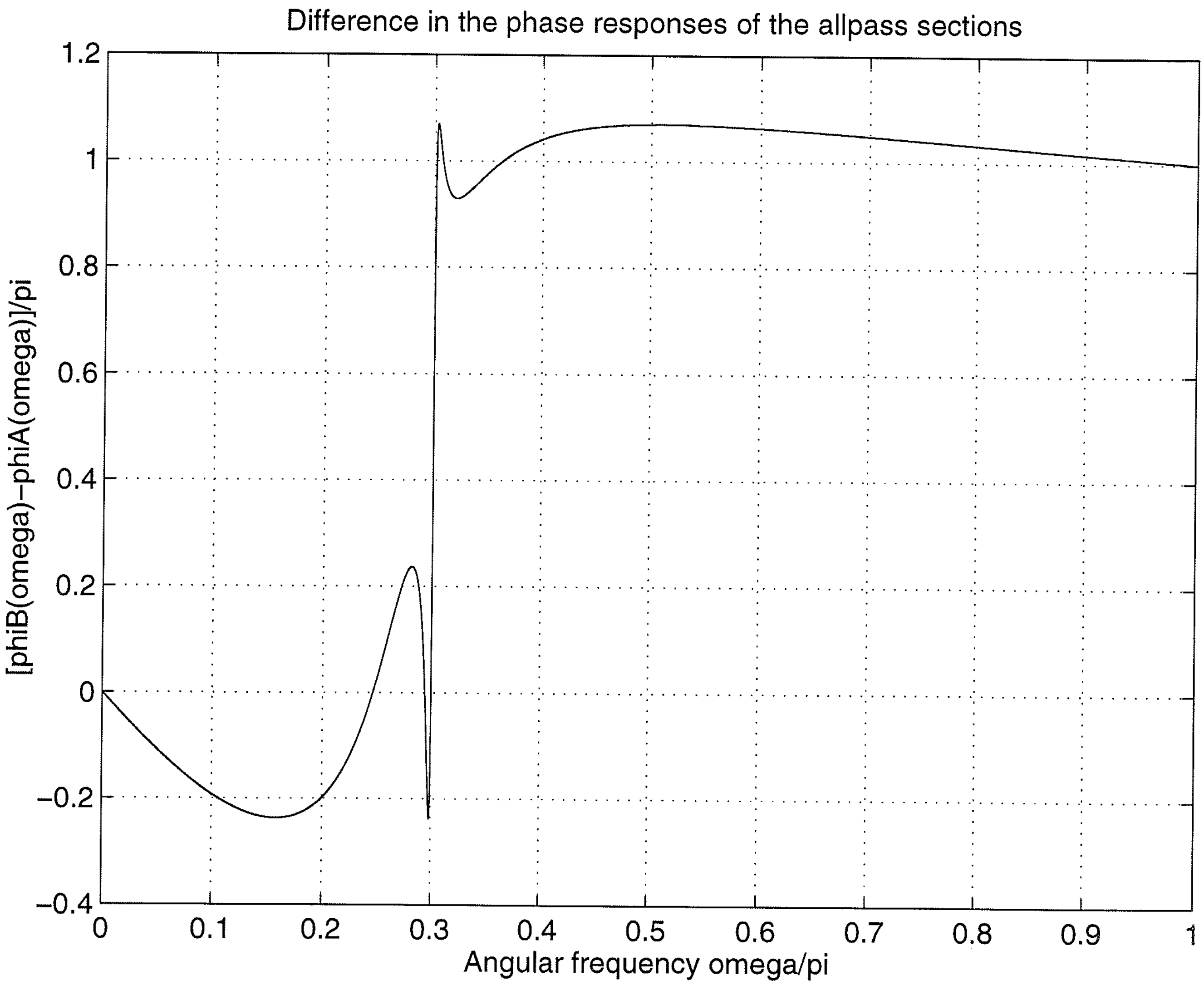
Stopband edge = 0.301*pi

a = 0.236809

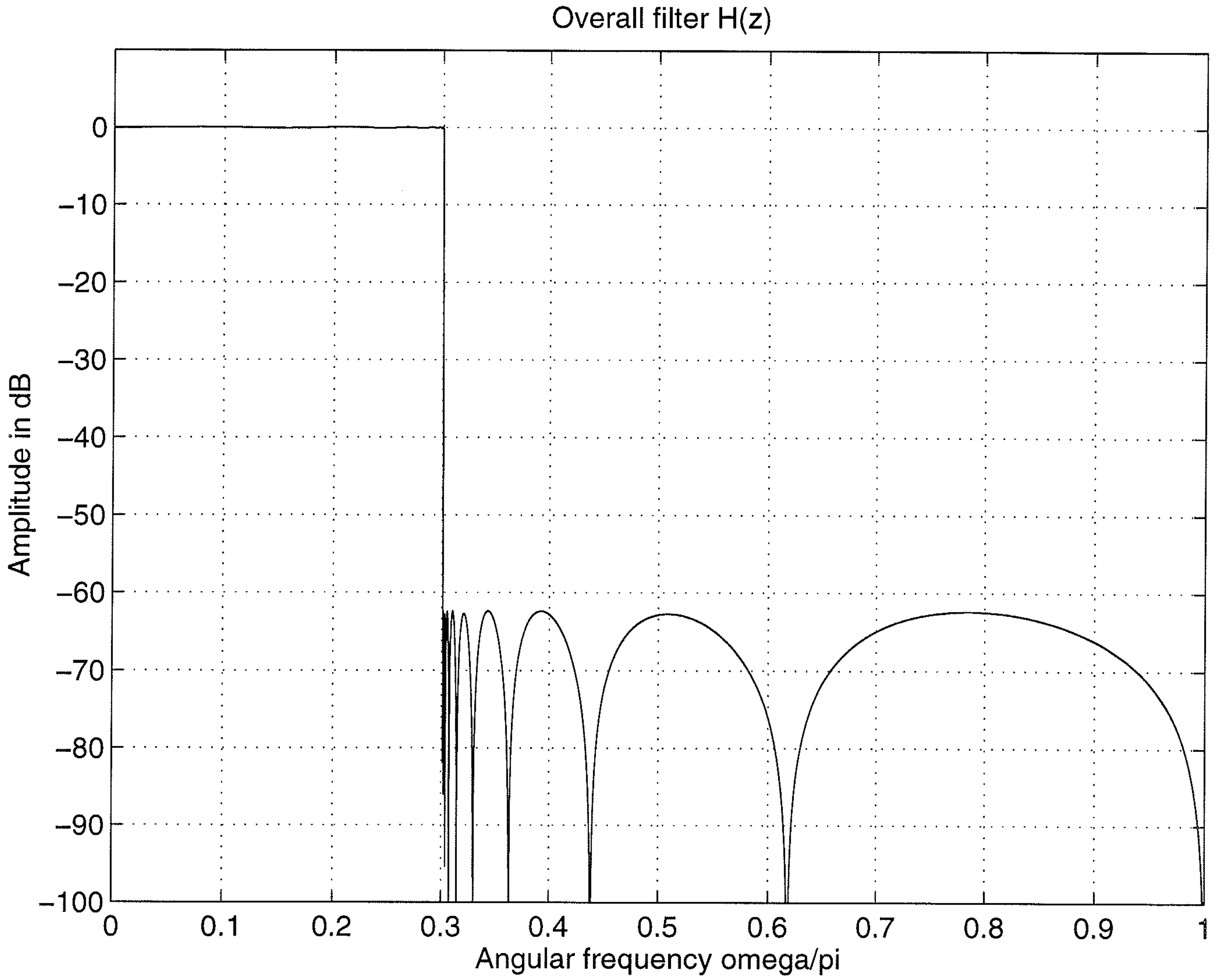
b = 0.929394



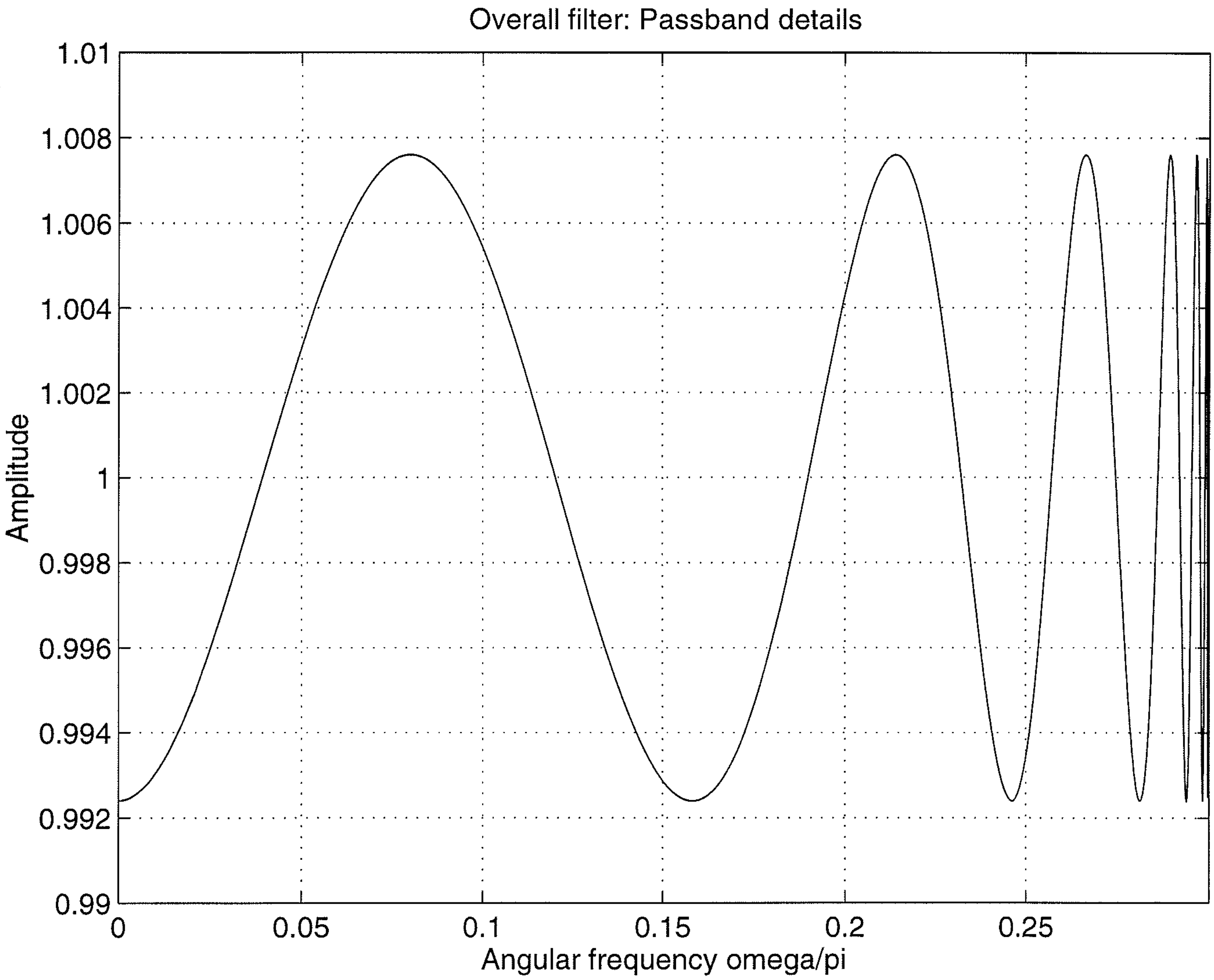
Phase difference $\phi_B(\omega) - \phi_A(\omega)$ between the all-pass filters $B(z)$ and $A(z)$



Amplitude response for the overall filter $H(z)$



Overall filter $H(z)$: Passband details



Another interpretation of the above mapping

- The allpass filters providing the desired mapping are given by

$$A(z) = \prod_{k=1}^{2} \frac{r_k^2 - 2r_k \cos\theta_k z^{-1} + z^{-2}}{1 - 2r_k \cos\theta_k z^{-1} + r_k^2 z^{-2}} \qquad (16a)$$

and

$$B(z) = \frac{-r_3 + z^{-1}}{1 - r_3^2 z^{-1}} \frac{r_4^2 - 2r_4 \cos\theta_4 z^{-1} + z^{-2}}{1 - 2r_4 \cos\theta_4 z^{-1} + r_4^2 z^{-2}}, \qquad (16b)$$

where $r_1 = 0.99810563$, $r_2 = 0.85341217$, $r_3 = 0.49990410$, $r_4 = 0.98138246$, $\theta_1 = 0.30016665\pi$, $\theta_2 = 0.27455889\pi$, and $\theta_4 = 0.29781677\pi$.

- The frequency response of the filter

$$F(z) = [A(z) + B(z)]/2 \qquad (17)$$

is given by

$$F(e^{j\omega}) = \frac{1}{2}[e^{j\phi_A(\omega)} + e^{j\phi_B(\omega)}] =$$
$$e^{j(\phi_A(\omega)+\phi_B(\omega))/2}\frac{1}{2}[e^{-j(\phi_B(\omega)-\phi_A(\omega))/2} + e^{j(\phi_B(\omega)-\phi_A(\omega))/2}]. \qquad (18)$$

- Hence, the amplitude response is expressible as

$$|F(e^{j\omega})| = |\cos[(\phi_B(\omega) - \phi_A(\omega))/2]|. \qquad (19)$$

- Therefore, the conditions given by Eq. (13) are equivalent with the conditions

$$1 - \widehat{\delta}_p \le |F(e^{j\omega})| \le 1 \quad \text{for} \quad \omega \in X_p \tag{20a}$$

$$|F(e^{j\omega})| \le \widehat{\delta}_s \quad \text{for} \quad \omega \in X_s, \tag{20b}$$

where

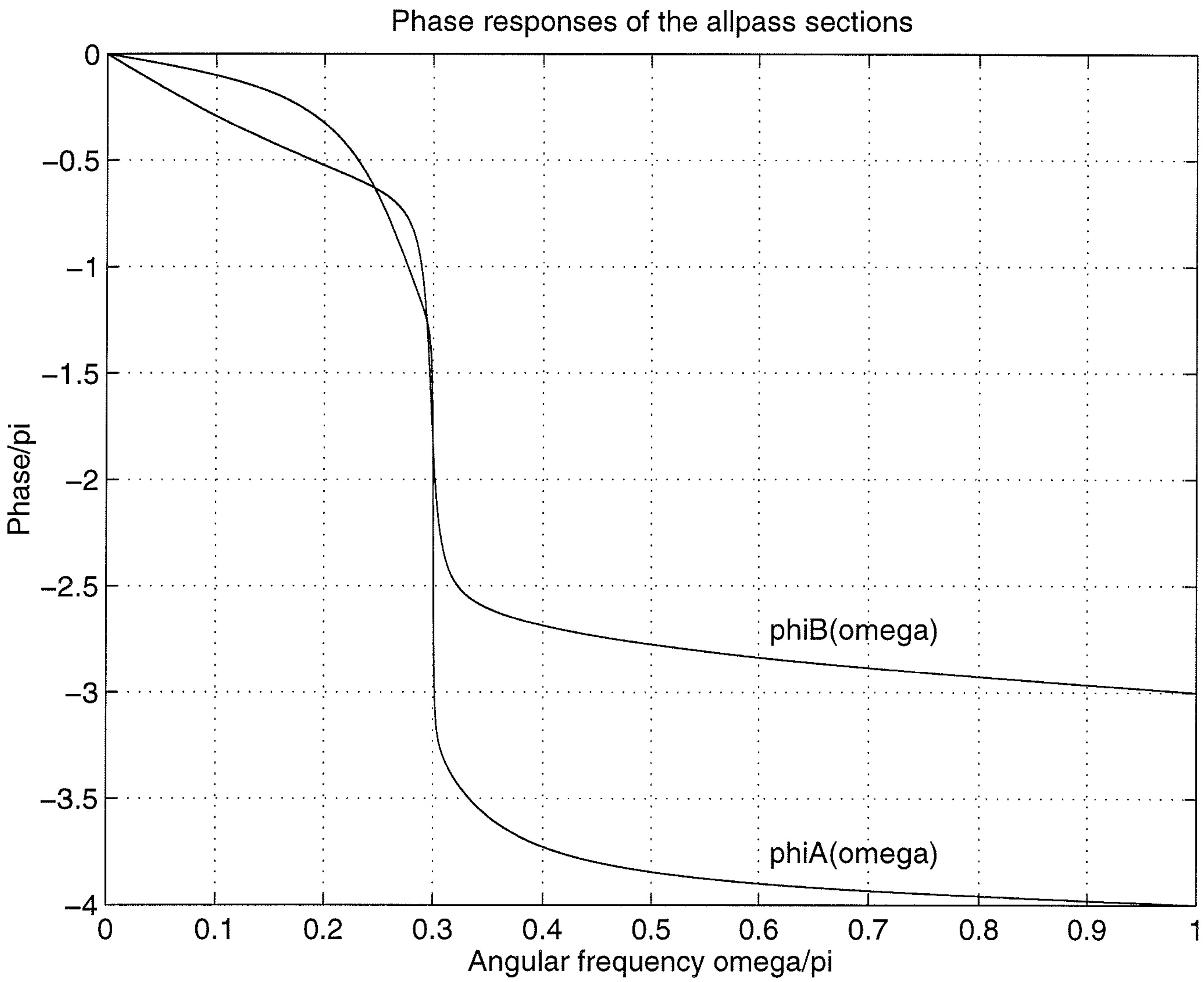
$$\widehat{\delta}_p = 1 - \cos(\Omega_p/2) = 0.06838982 \tag{21a}$$

and

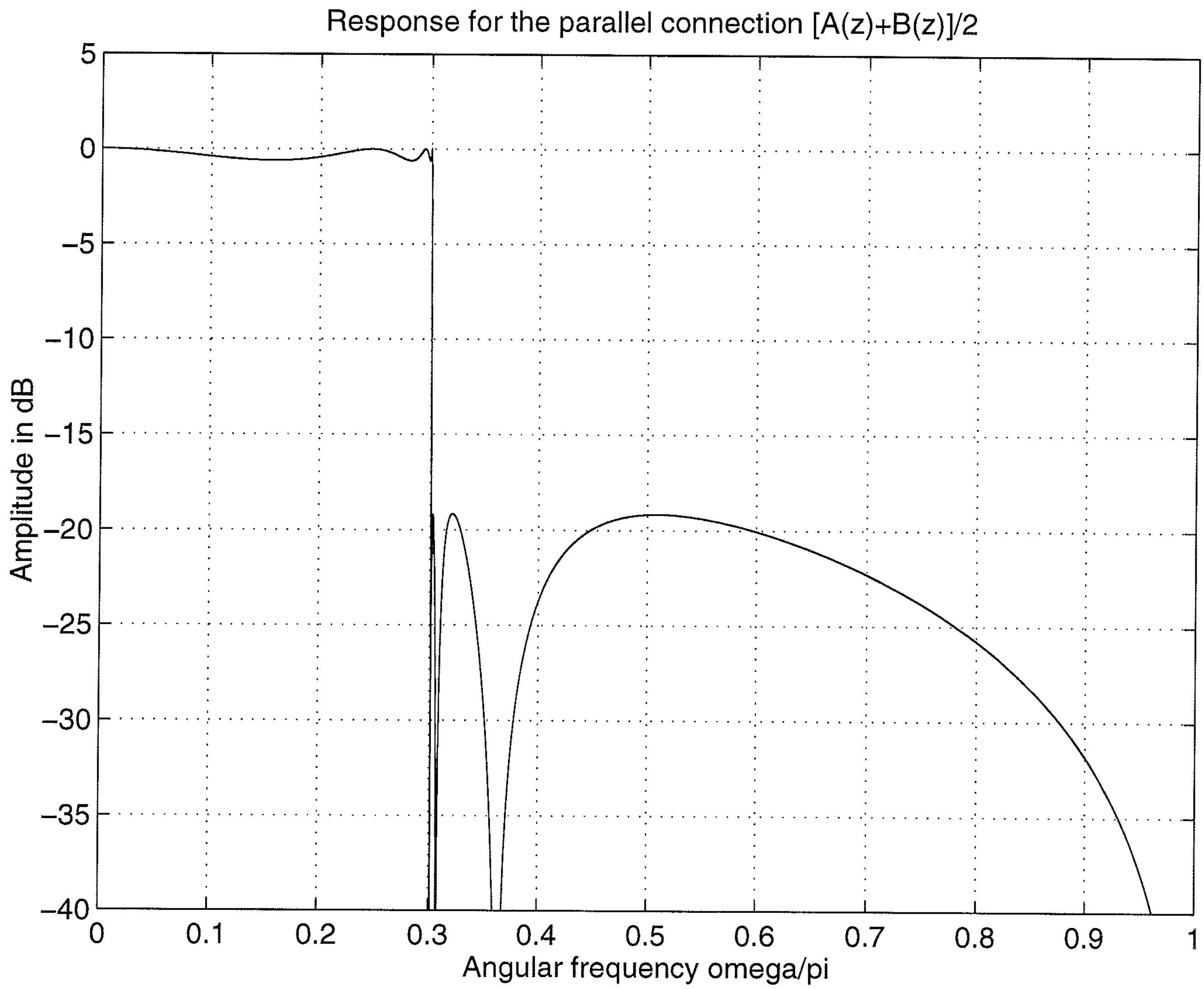
$$\widehat{\delta}_s = \cos(\Omega_s/2) = 0.11068033. \tag{21b}$$

- As a matter of fact, $F(z)$ is an elliptic lowpass filter with passband ripple of 0.61531551 dB and stopband attenuation of 19.118591 dB.
- The following three transparencies show the phase responses of $A(z)$ and $B(z)$ as well as the amplitude response of $F(z)$.
- Note that the passband and stopband regions of $F(z)$ and the overall filter $H(z)$ are identical.
- When comparing the ripple values of these two filters, it is observed that they are huge for $F(z)$.

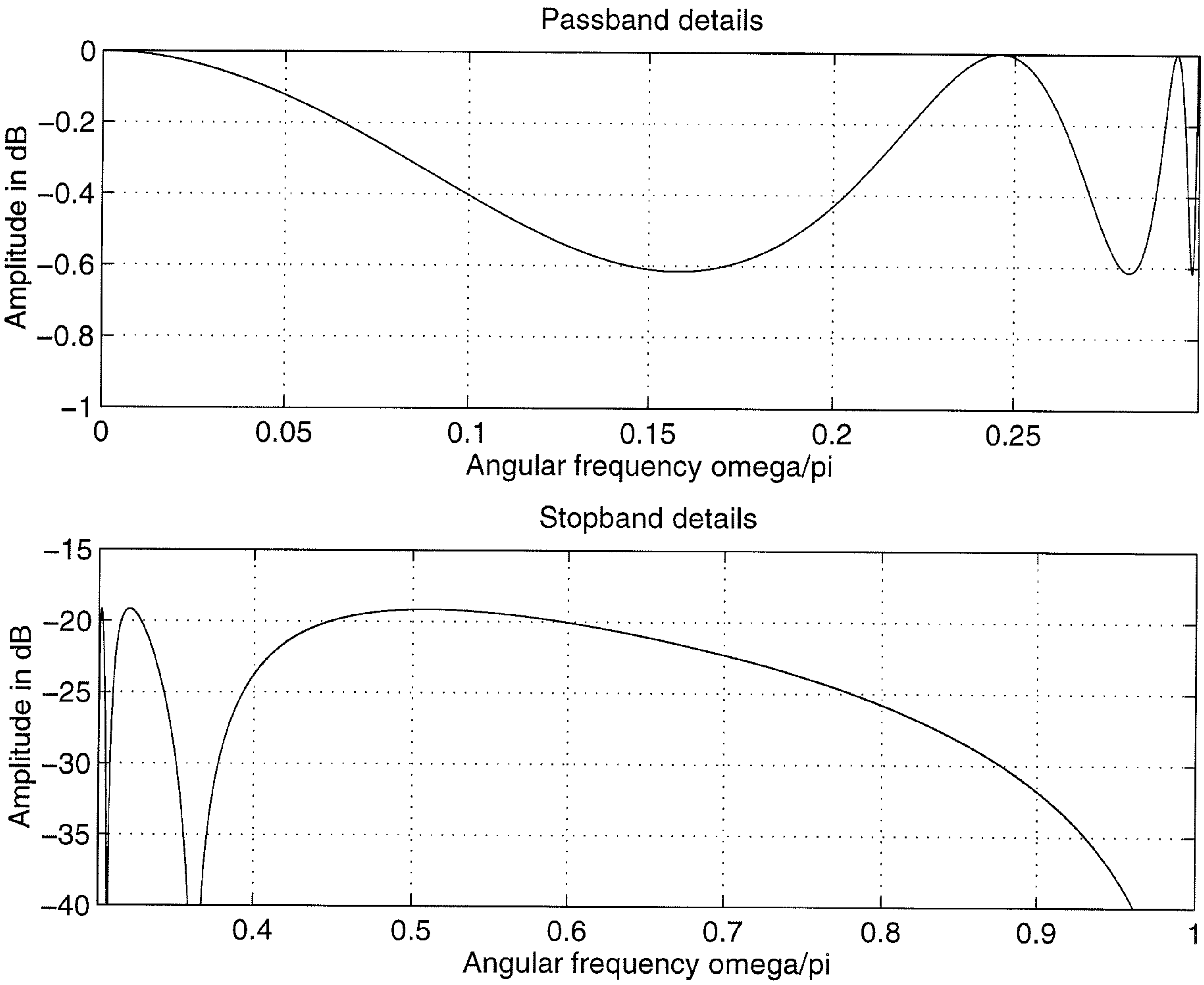
Phase responses of $A(z)$ and $B(z)$ in the low-pass case



Amplitude response for $F(z) = [A(z) + B(z)]/2$ in the lowpass case



Amplitude response for $F(z) = [A(z) + B(z)]/2$ in the lowpass case



Highpass design

- In this case, the desired performance for the overall filter is achieved by requiring that

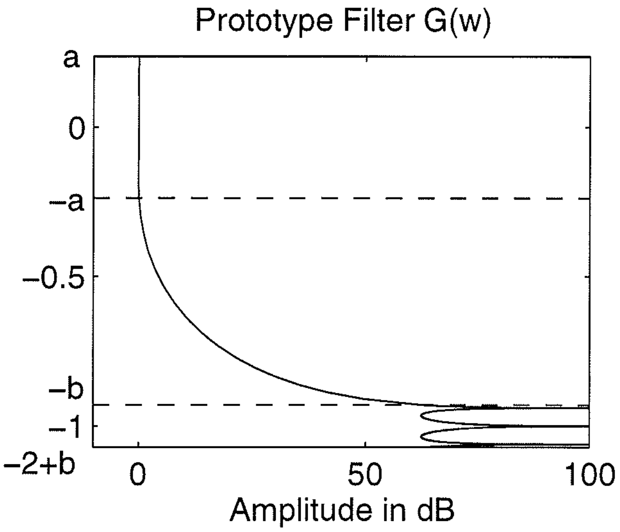
$$-\Omega_p \leq \phi_B(\omega) - \phi_A(\omega) \leq \Omega_p \quad \text{for} \quad \omega \in X_p = [0.7\pi, \ \pi] \tag{22a}$$

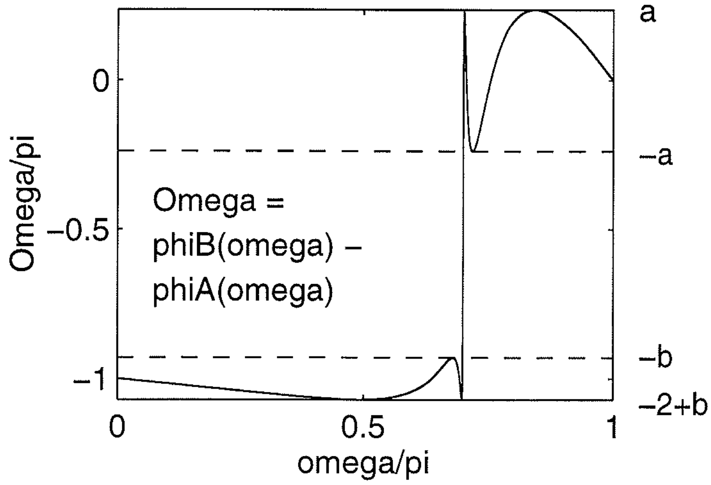
and

$$-\Omega_s \leq \phi_B(\omega) - \phi_A(\omega) \leq -2\pi + \Omega_s$$
$$\text{for} \quad \omega \in X_s = [0, \ 0.699\pi]. \tag{22b}$$

- Note that because of the periodicity, $|G(e^{j\Omega})|$ stays within $1 \pm \delta_p$ for $\Omega \in [-\Omega_p, \ \Omega_p]$ and within δ_s and zero for $\Omega \in [-\Omega_s, \ -2\pi + \Omega_s]$. See the following four transparencies.

Mapping a nonlinear-phase FIR filter to a highpass filter being implementable as a tapped cascaded interconnection of two identical allpass filters.



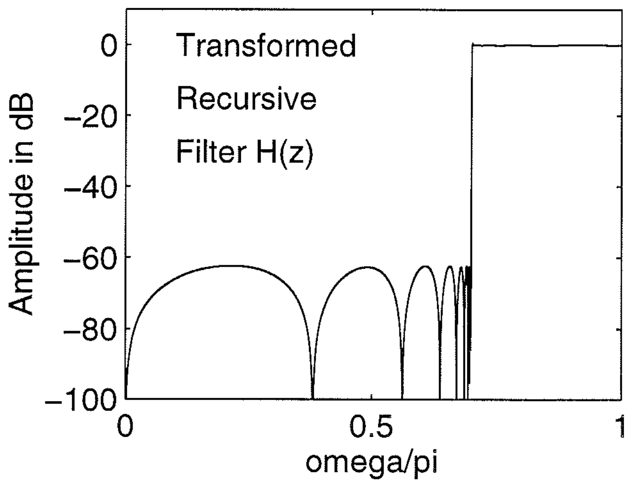


Passband edge = 0.7*pi

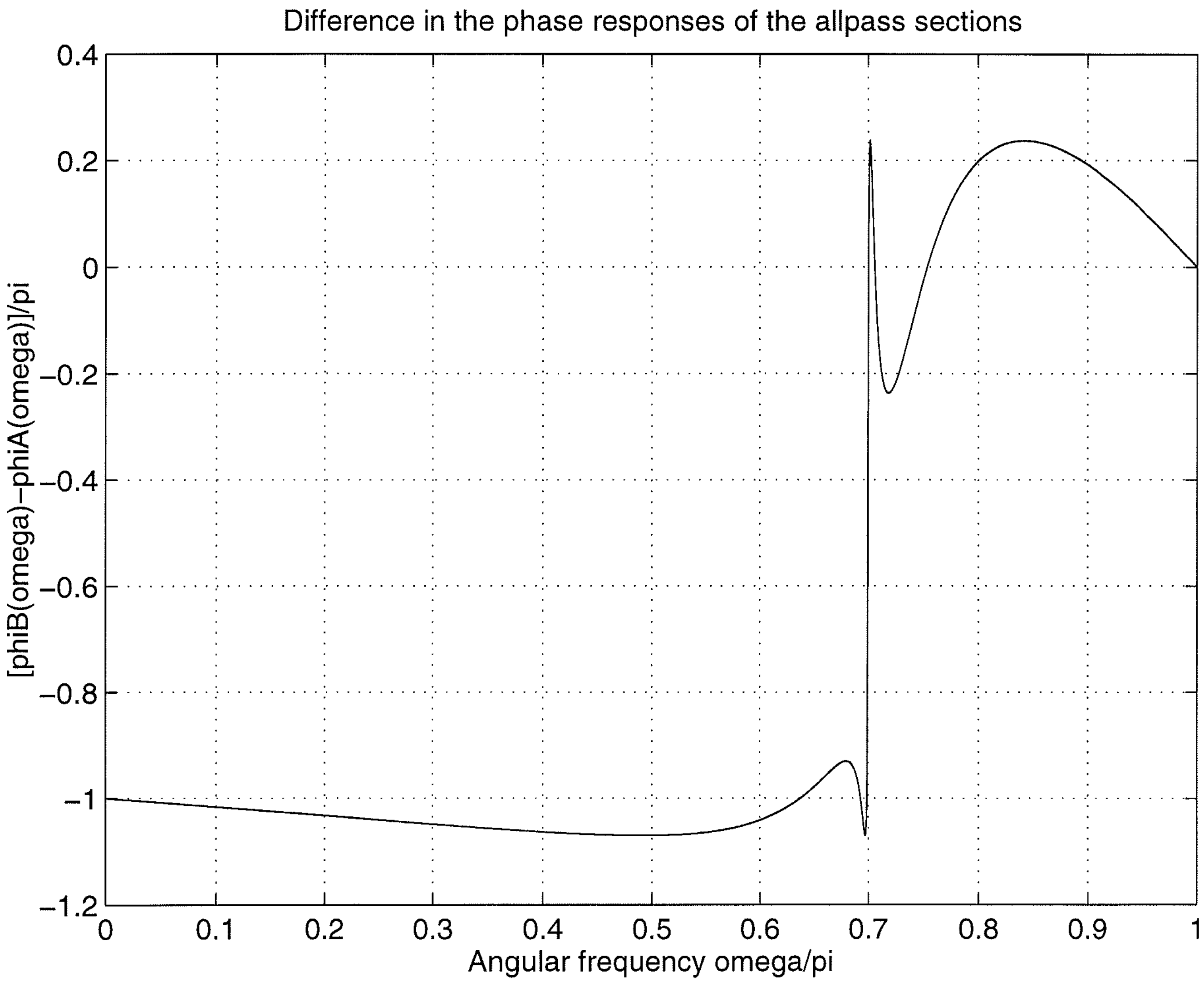
Stopband edge = 0.699*pi

a = 0.236809

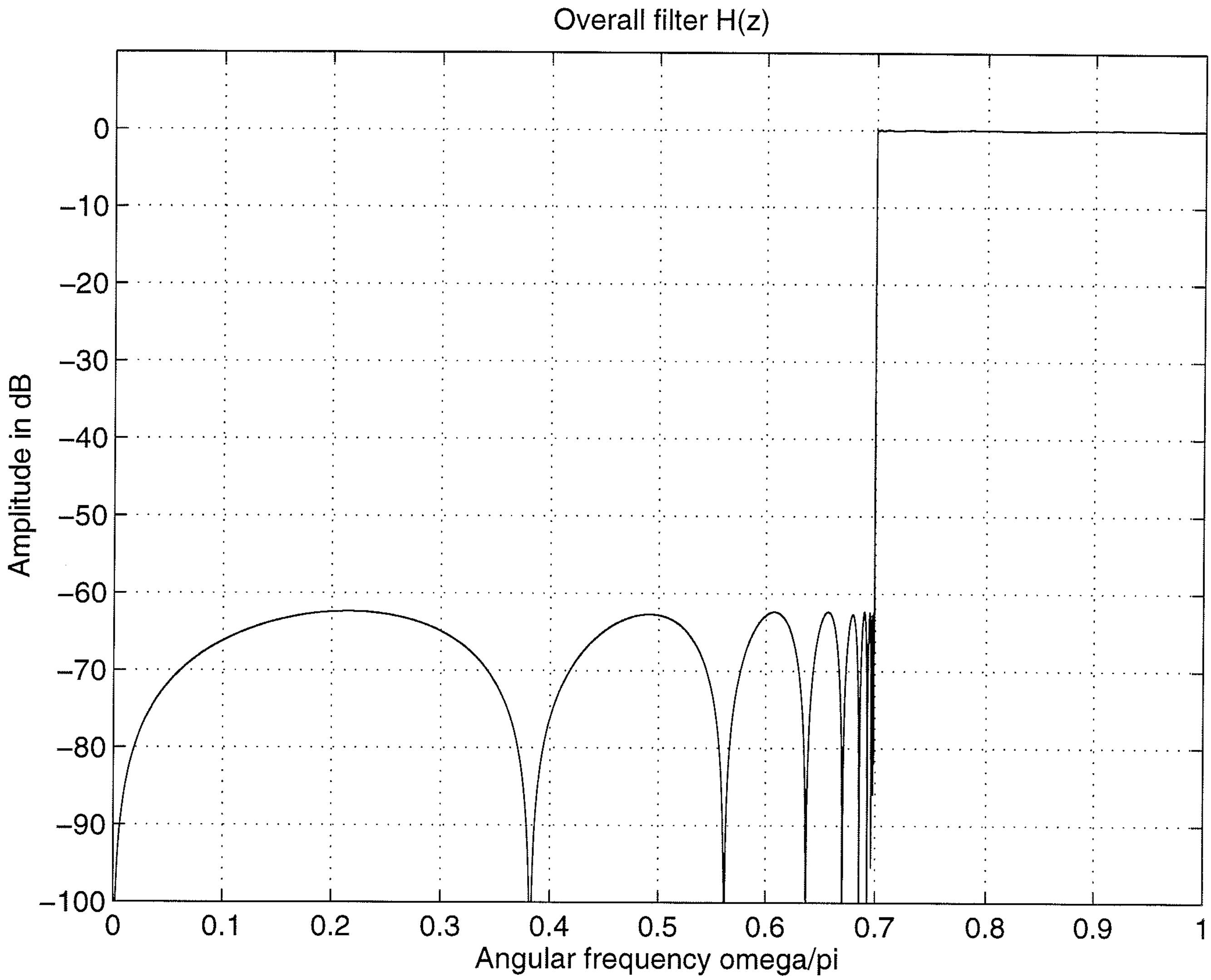
b = 0.929394



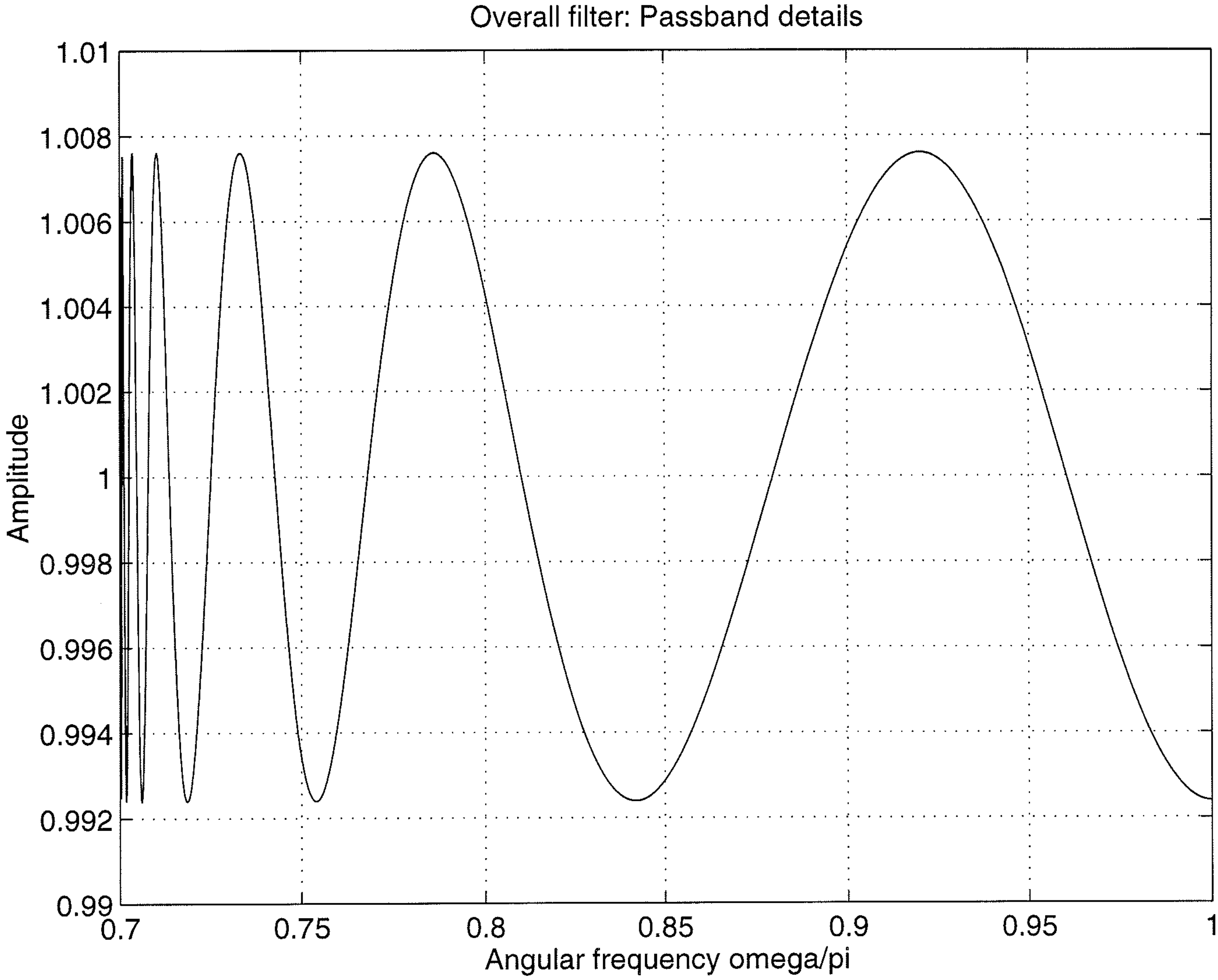
Phase difference $\phi_B(\omega) - \phi_A(\omega)$ between the allpass filters $B(z)$ and $A(z)$



Amplitude response for the overall filter $H(z)$



Overall filter $H(z)$: Passband details



Another interpretation of the above mapping

- The allpass filters providing the desired mapping are given by

$$A(z) = \prod_{k=1}^{2} \frac{r_k^2 - 2r_k \cos\theta_k z^{-1} + z^{-2}}{1 - 2r_k \cos\theta_k z^{-1} + r_k^2 z^{-2}} \qquad (23a)$$

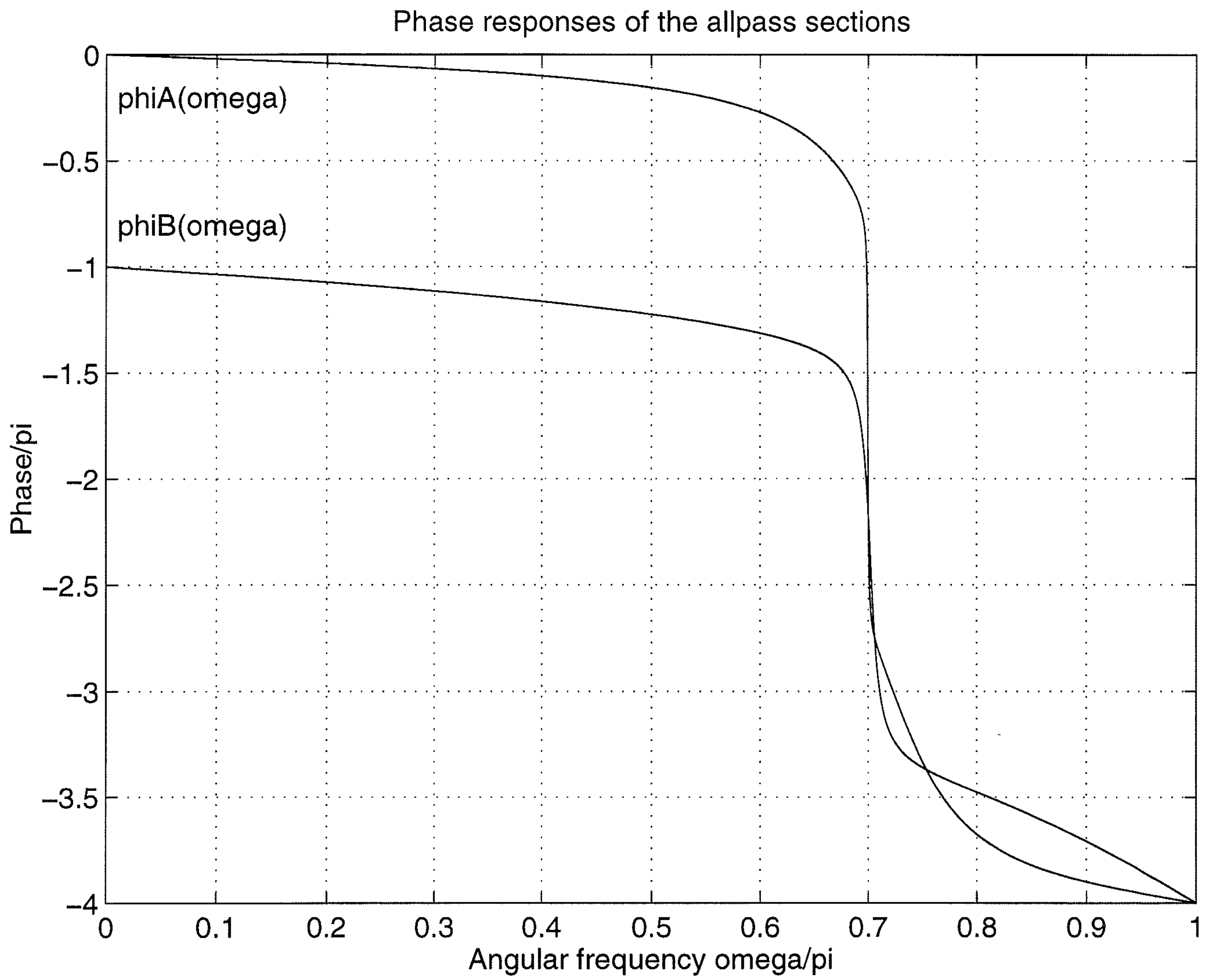
and

$$B(z) = -\frac{-r_3 + z^{-1}}{1 - r_3^2 z^{-1}} \frac{r_4^2 - 2r_4 \cos\theta_4 z^{-1} + z^{-2}}{1 - 2r_4 \cos\theta_4 z^{-1} + r_4^2 z^{-2}}, \qquad (23b)$$

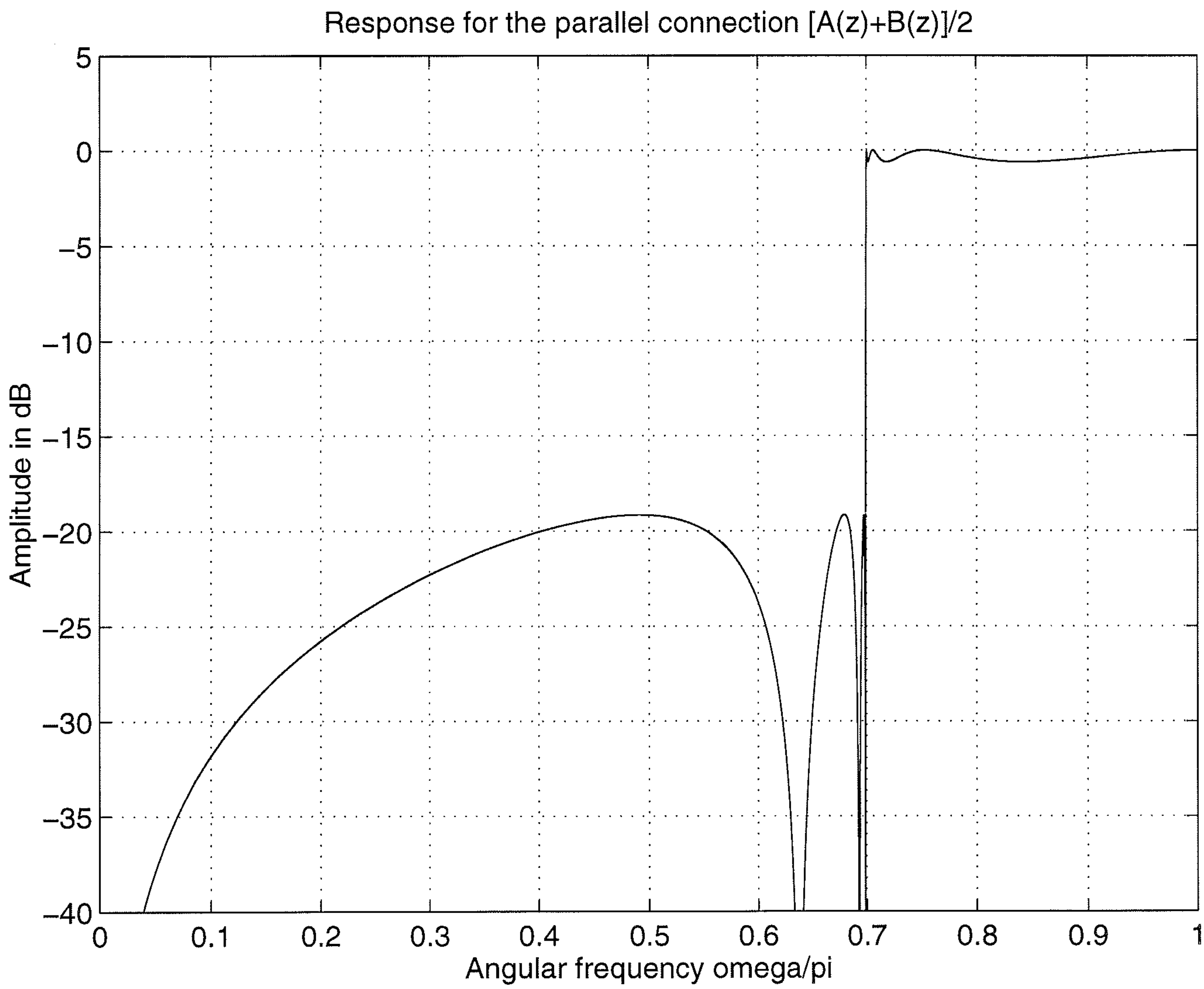
where $r_1 = 0.99810563$, $r_2 = 0.85341217$, $r_3 = -0.49990410$, $r_4 = 0.98138246$, $\theta_1 = 0.69983335\pi$, $\theta_2 = 0.72544111\pi$, and $\theta_4 = 0.7021833\pi$.

- Like in the previous lowpass case, $F(z) = [A(z) + B(z)]/2$ is an elliptic highpass filter with the same passband and stopband regions as the overall filter. The passband ripple and the stopband attenuation are 0.61531551 dB and 19.118591 dB, respectively. See the following three transparencies.

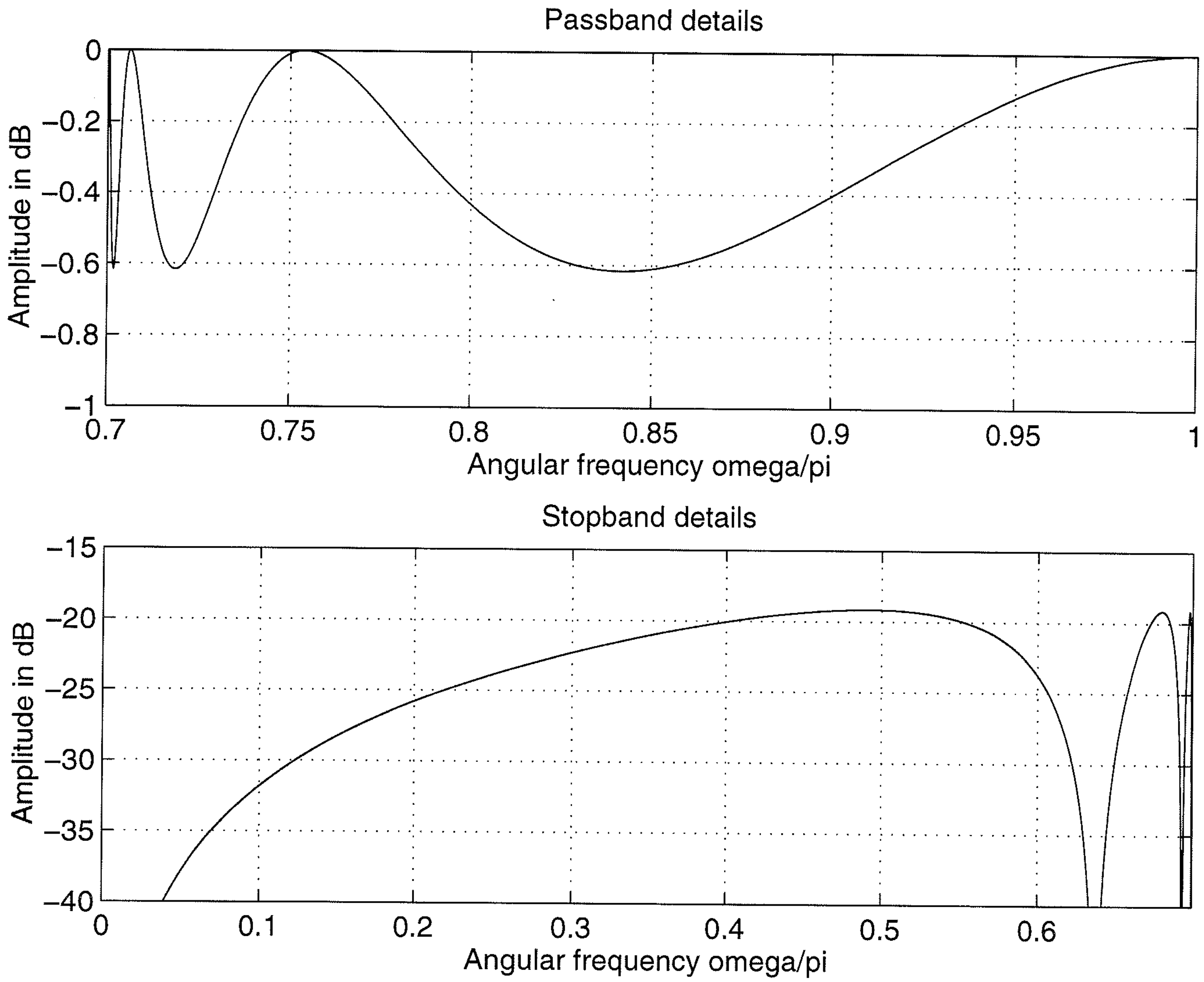
Phase responses of $A(z)$ and $B(z)$ in the high-pass case



Amplitude response for $F(z) = [A(z) + B(z)]/2$ in the highpass case



Amplitude response for $F(z) = [A(z) + B(z)]/2$ in the highpass case



Bandstop design

- In this case, the desired performance for the overall filter is achieved by requiring that

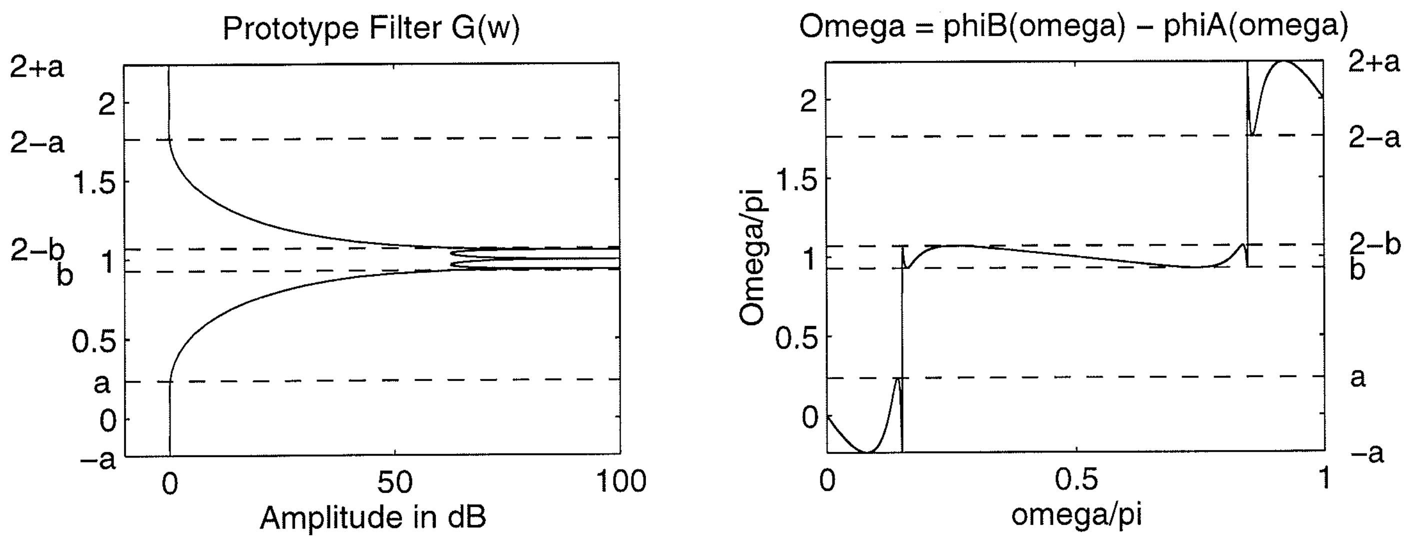
$$-\Omega_p \leq \phi_B(\omega) - \phi_A(\omega) \leq \Omega_p \quad \text{for} \quad \omega \in [0,\ 0.15\pi], \tag{24a}$$

$$\Omega_s \leq \phi_B(\omega) - \phi_A(\omega) \leq 2\pi - \Omega_s$$
$$\text{for} \quad \omega \in X_s = [0.1505\pi,\ 8445\pi], \tag{24b}$$

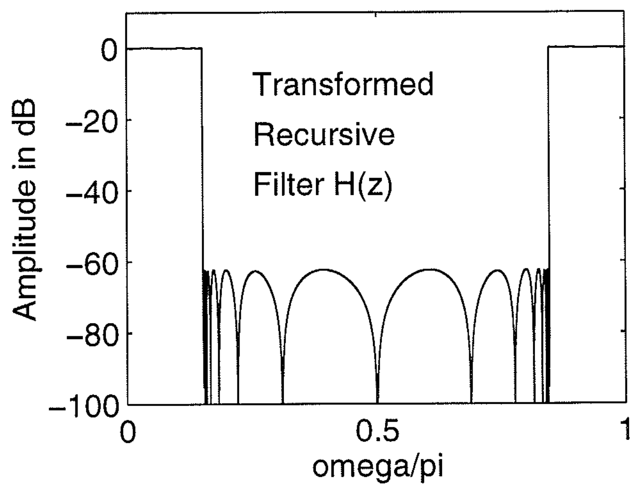
$$2\pi - \Omega_p \leq \phi_B(\omega) - \phi_A(\omega) \leq 2\pi + \Omega_p \quad \text{for} \quad \omega \in [0.85\pi,\ \pi]. \tag{24c}$$

- When Eqs. (24a) and (24c) are satisfied, the overall filter achieves the desired passband behavior. See the following four transparencies.
- When Eq. (24b) is satisfied, the overall filter achieves the desired stopband behavior.

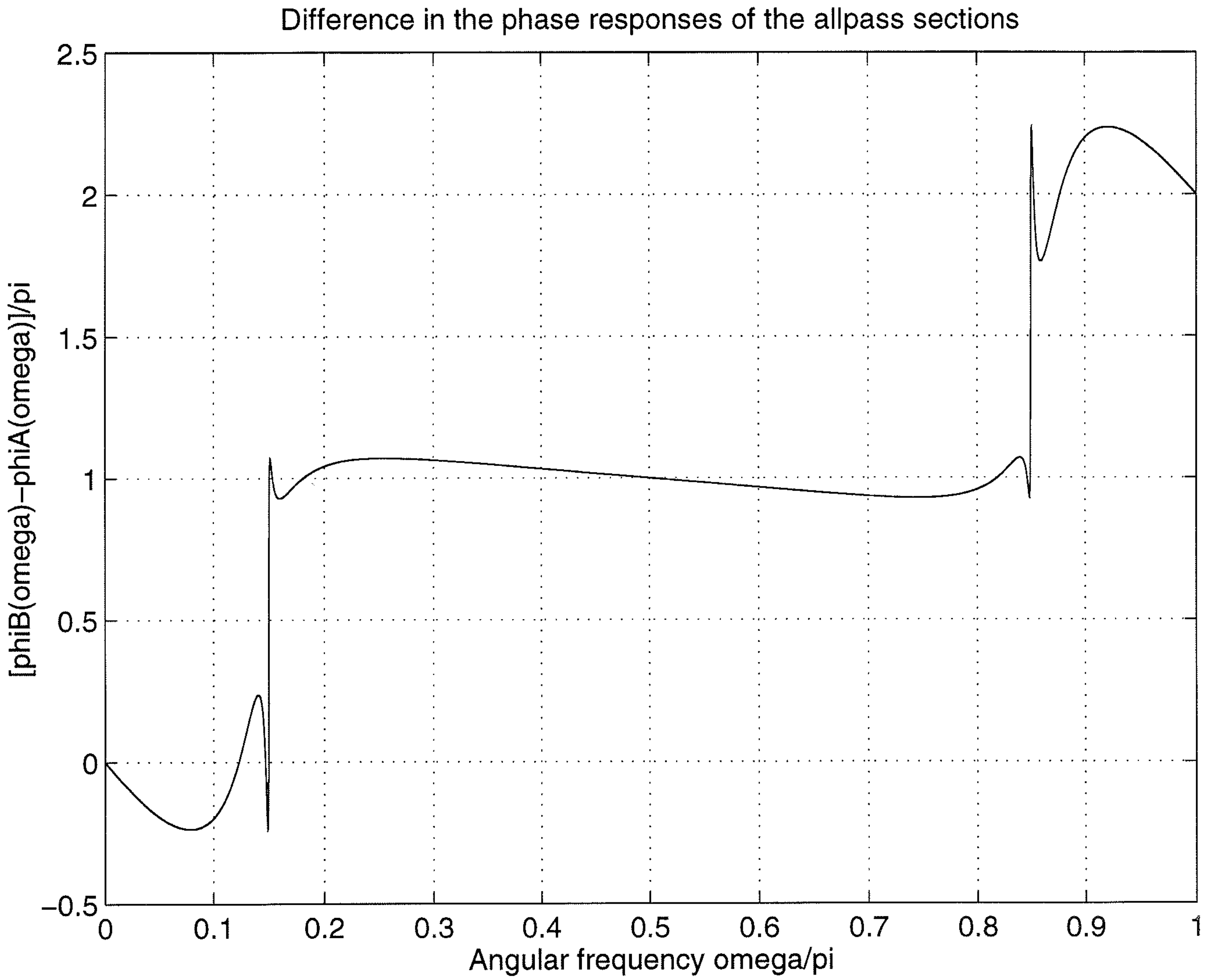
Mapping a nonlinear-phase FIR filter to a band-stop filter being implementable as a tapped cascaded interconnection of two identical all-pass filters.



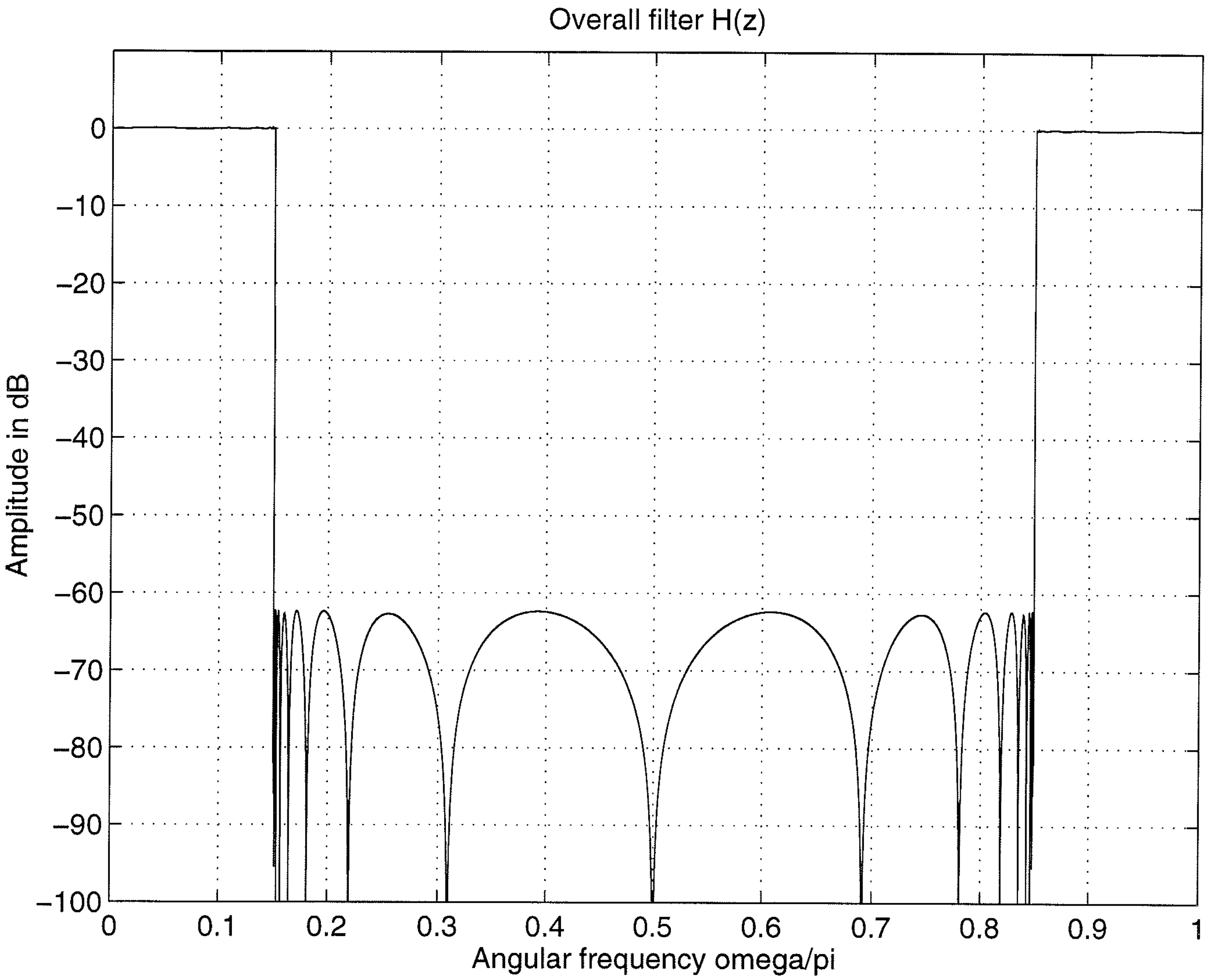
Passband edges = 0.15*pi,
0.85pi
Stopband edges = 0.1505*pi,
0.8445pi
a = 0.236809
b = 0.929394



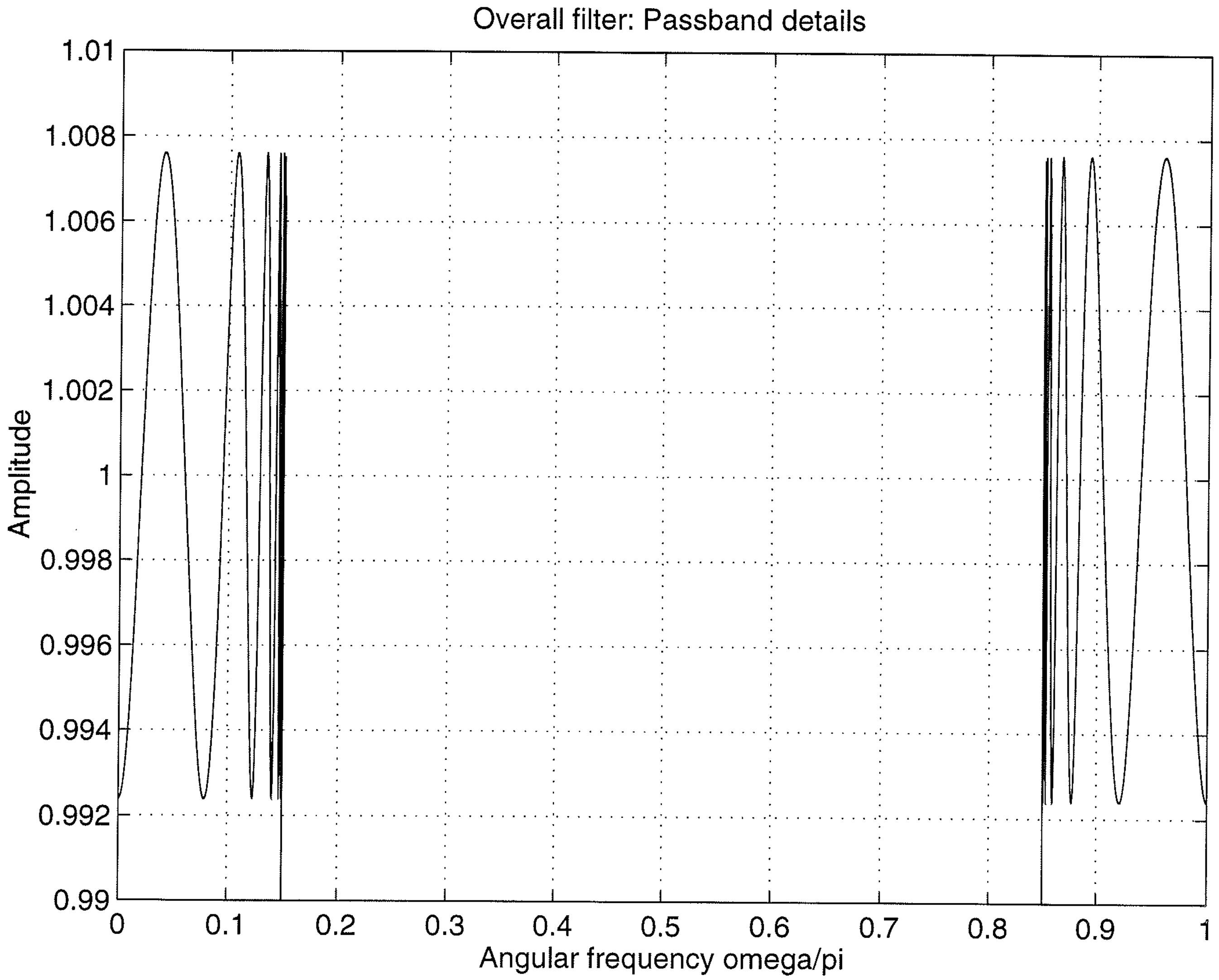
Phase difference $\phi_B(\omega) - \phi_A(\omega)$ between the all-pass filters $B(z)$ and $A(z)$



Amplitude response for the overall filter $H(z)$



Overall filter $H(z)$: Passband details



Another interpretation of the above mapping

- The allpass filters providing the desired mapping are given by

$$A(z) = \prod_{k=1}^{4} \frac{r_k^2 - 2r_k \cos\theta_k z^{-1} + z^{-2}}{1 - 2r_k \cos\theta_k z^{-1} + r_k^2 z^{-2}} \qquad (25a)$$

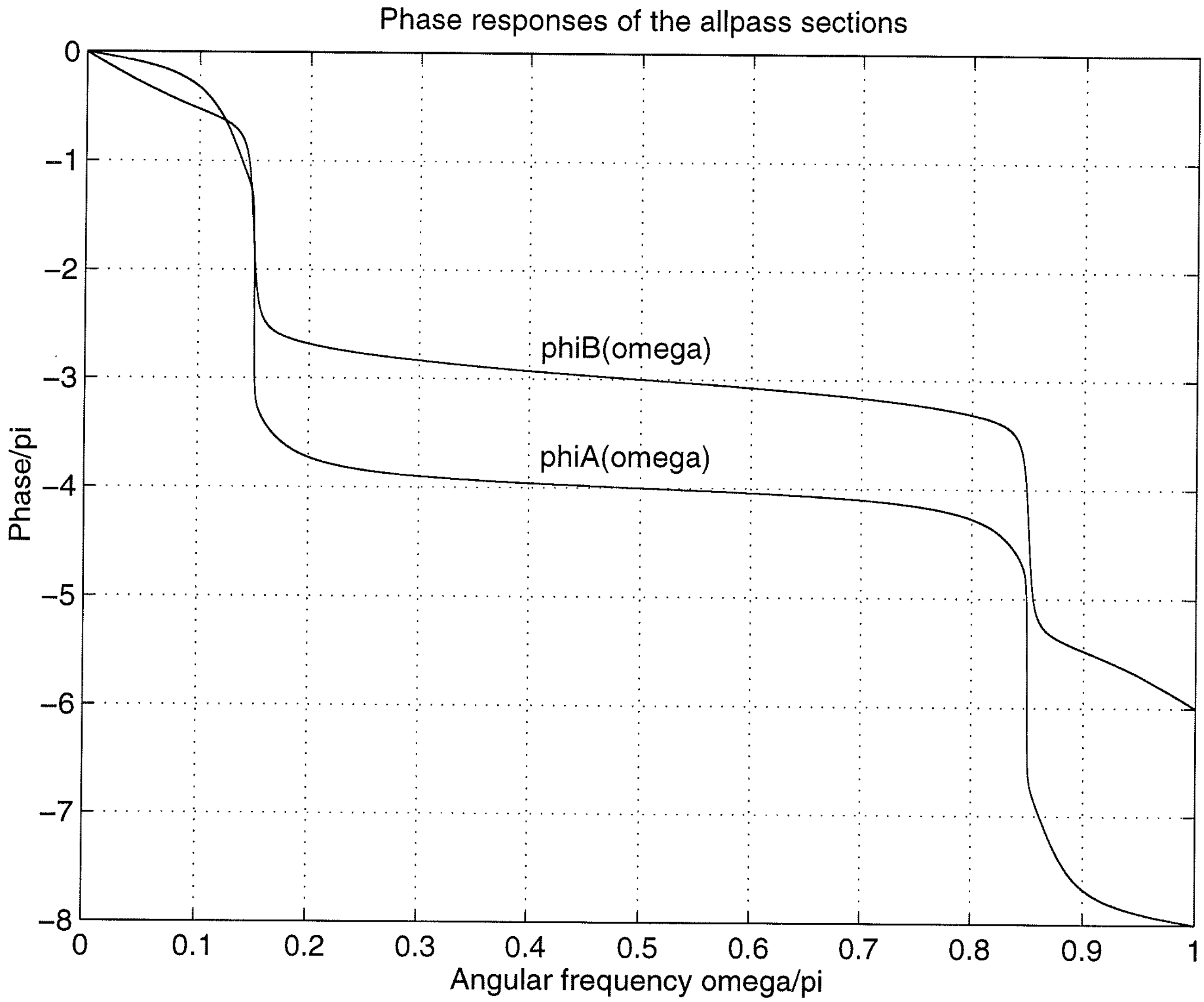
and

$$B(z) = \prod_{k=5}^{6} \frac{-r_k + z^{-1}}{1 - r_k^2 z^{-1}} \prod_{k=7}^{8} \frac{r_4^2 - 2r_4 \cos\theta_4 z^{-1} + z^{-2}}{1 - 2r_4 \cos\theta_4 z^{-1} + r_4^2 z^{-2}}, \qquad (25b)$$

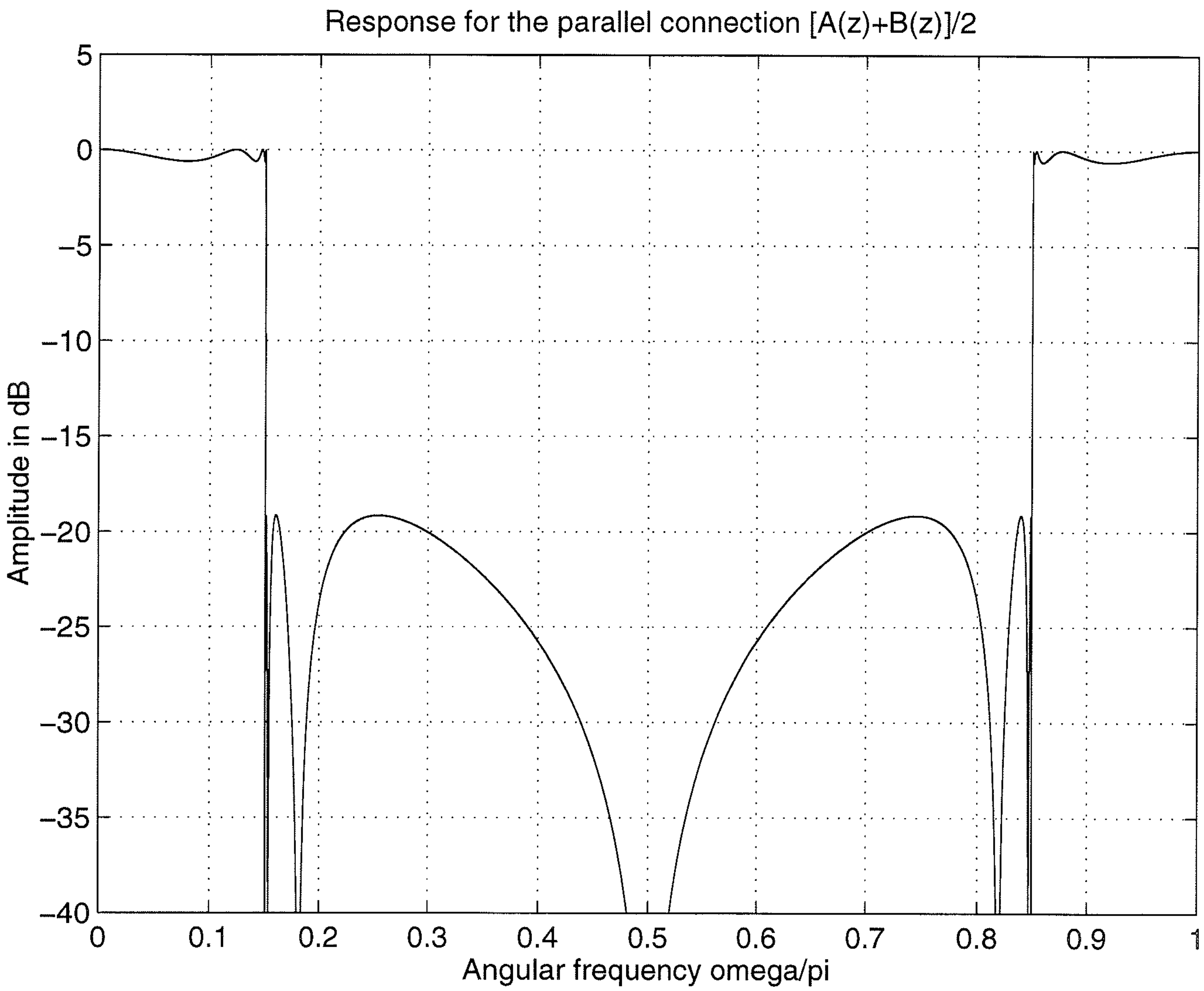
where $r_1 = r_2 = 0.99905237$, $r_3 = r_4 = 0.92380310$, $r_5 = 0.70703897$, $r_6 = -r_5$, $r_7 = r_8 = 0.99064750$, $\theta_1 = 0.15008333\pi$, $\theta_2 = \pi - \theta_1$, $\theta_3 = 0.13727944\pi$, $\theta_4 = \pi - \theta_3$, $\theta_7 = 0.14890838\pi$, and $\theta_8 = \pi - \theta_7$.

- Like in the previous cases, $F(z) = [A(z) + B(z)]/2$ is an elliptic bandstop filter with the same passband and stopband regions as the overall filter. The passband ripple and the stopband attenuation are 0.61531551 dB and 19.118591 dB, respectively. See the following three transparencies.

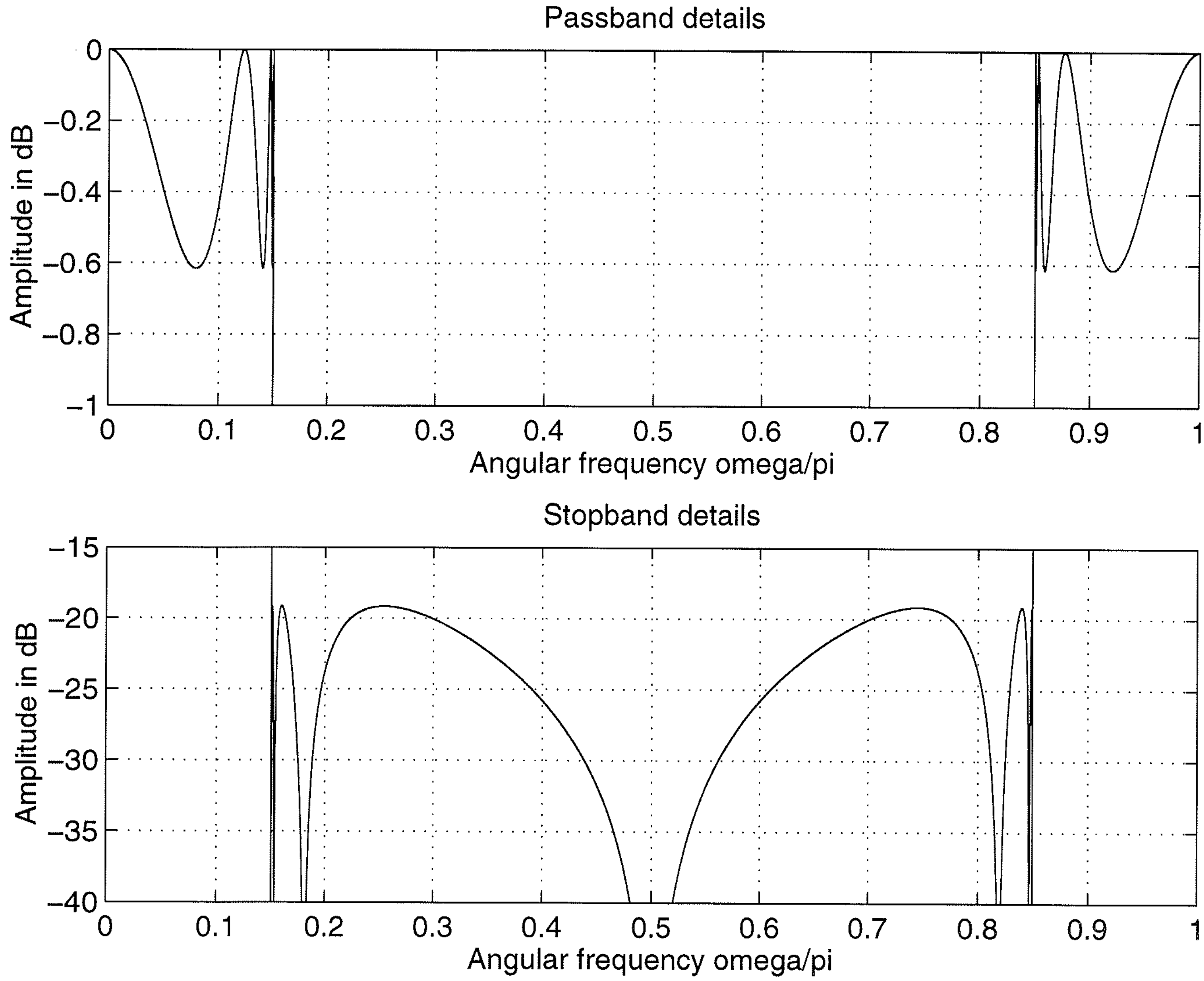
Phase responses of $A(z)$ and $B(z)$ in the band-stop case



Amplitude response for $F(z) = [A(z) + B(z)]/2$ in the bandstop case



Amplitude response for $F(z) = [A(z) + B(z)]/2$ in the bandstop case



Bandpass design

- In this case, the desired performance for the overall filter is achieved by requiring that

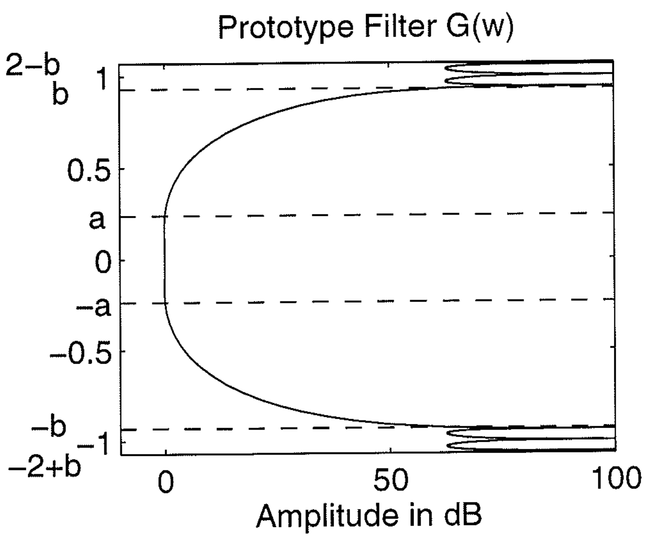
$$-2\pi+\Omega_s \le \phi_B(\omega)-\phi_A(\omega) \le -\Omega_s \quad \text{for} \quad \omega \in [0,\ 0.3495\pi], \tag{26a}$$

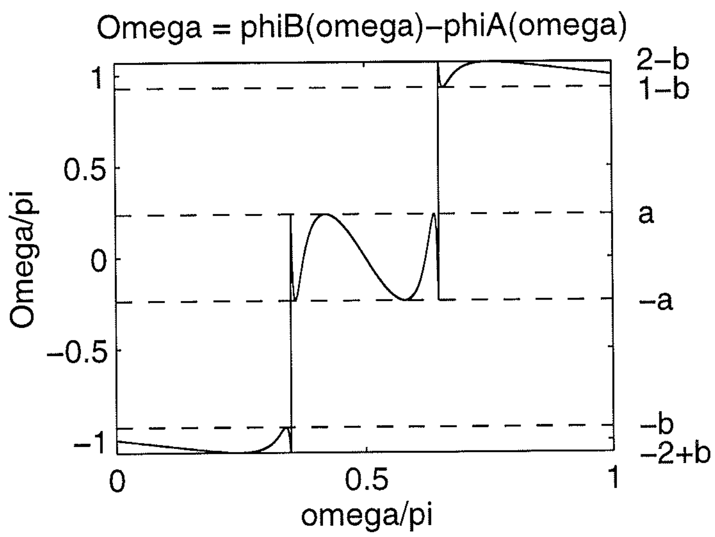
$$-\Omega_p \le \phi_B(\omega)-\phi_A(\omega) \le \Omega_p \quad \text{for} \quad \omega \in X_p = [0.35\pi,\ 0.65\pi], \tag{26b}$$

$$\Omega_s \le \phi_B(\omega)-\phi_A(\omega) \le 2\pi-\Omega_s \quad \text{for} \quad \omega \in [0.6505\pi,\ \pi]. \tag{26c}$$

- When Eqs. (26a) and (26c) are satisfied, the overall filter achieves the desired stopband behavior. See the following four transparencies.
- When Eq. (26b) is satisfied, the overall filter achieves the desired passband behavior.

Mapping a nonlinear-phase FIR filter to a bandpass filter being implementable as a tapped cascaded interconnection of two identical allpass filters.



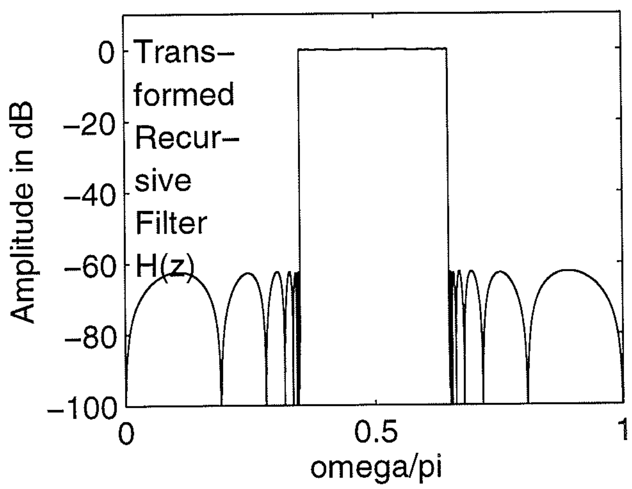


Passband edges = 0.35*pi,
0.65pi

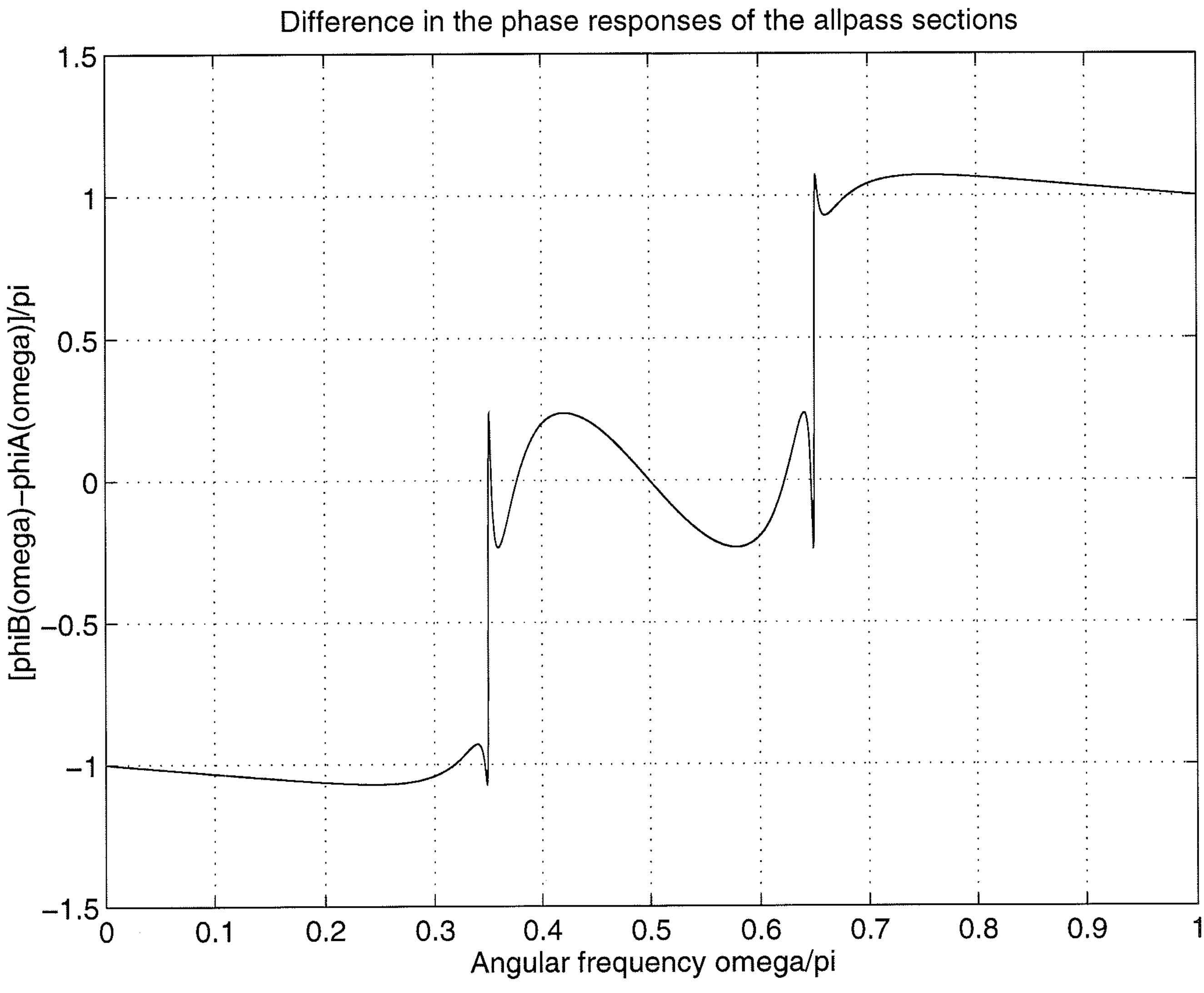
Stopband edges = 0.3495*pi,
0.6505pi

a = 0.236809

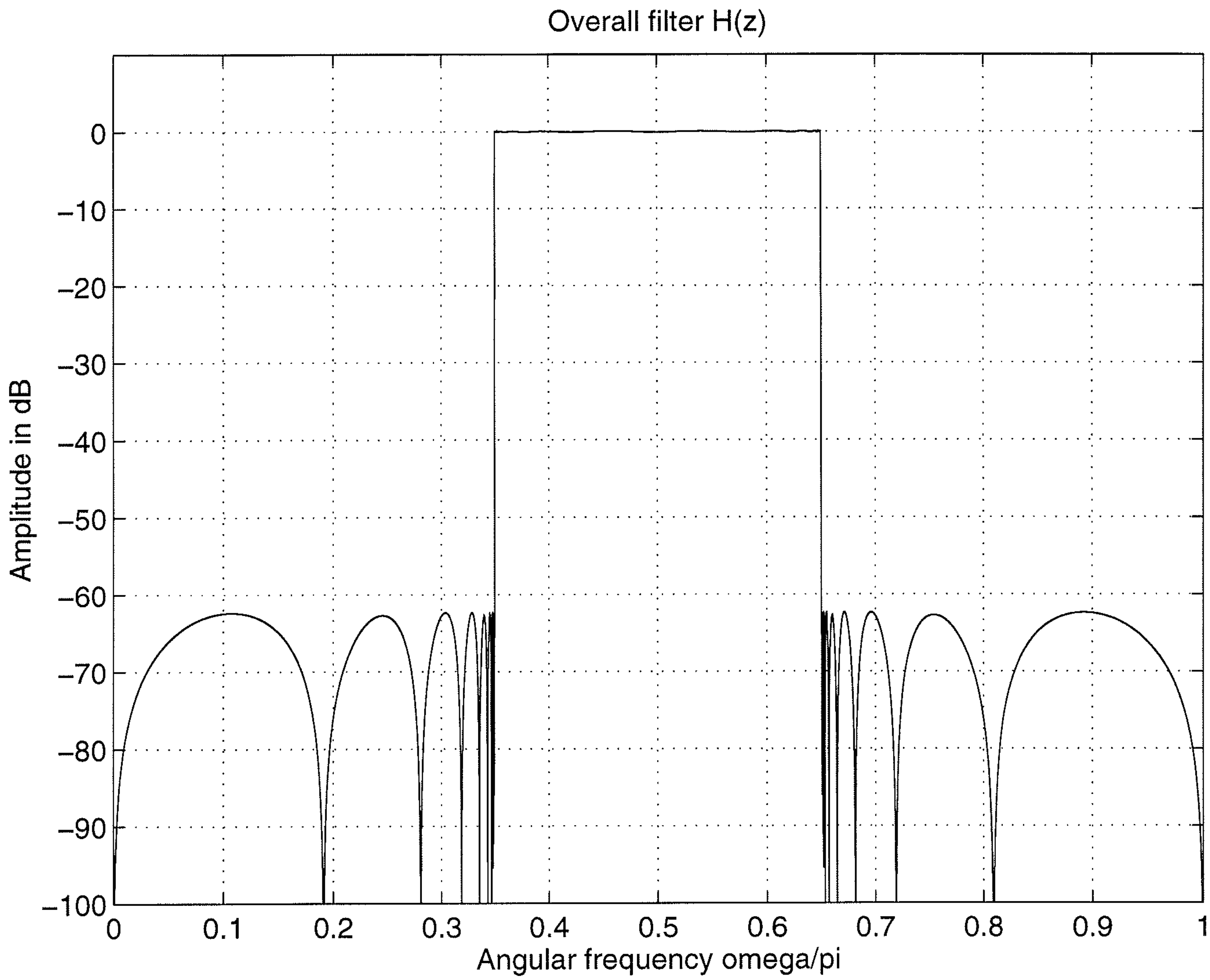
b = 0.929394



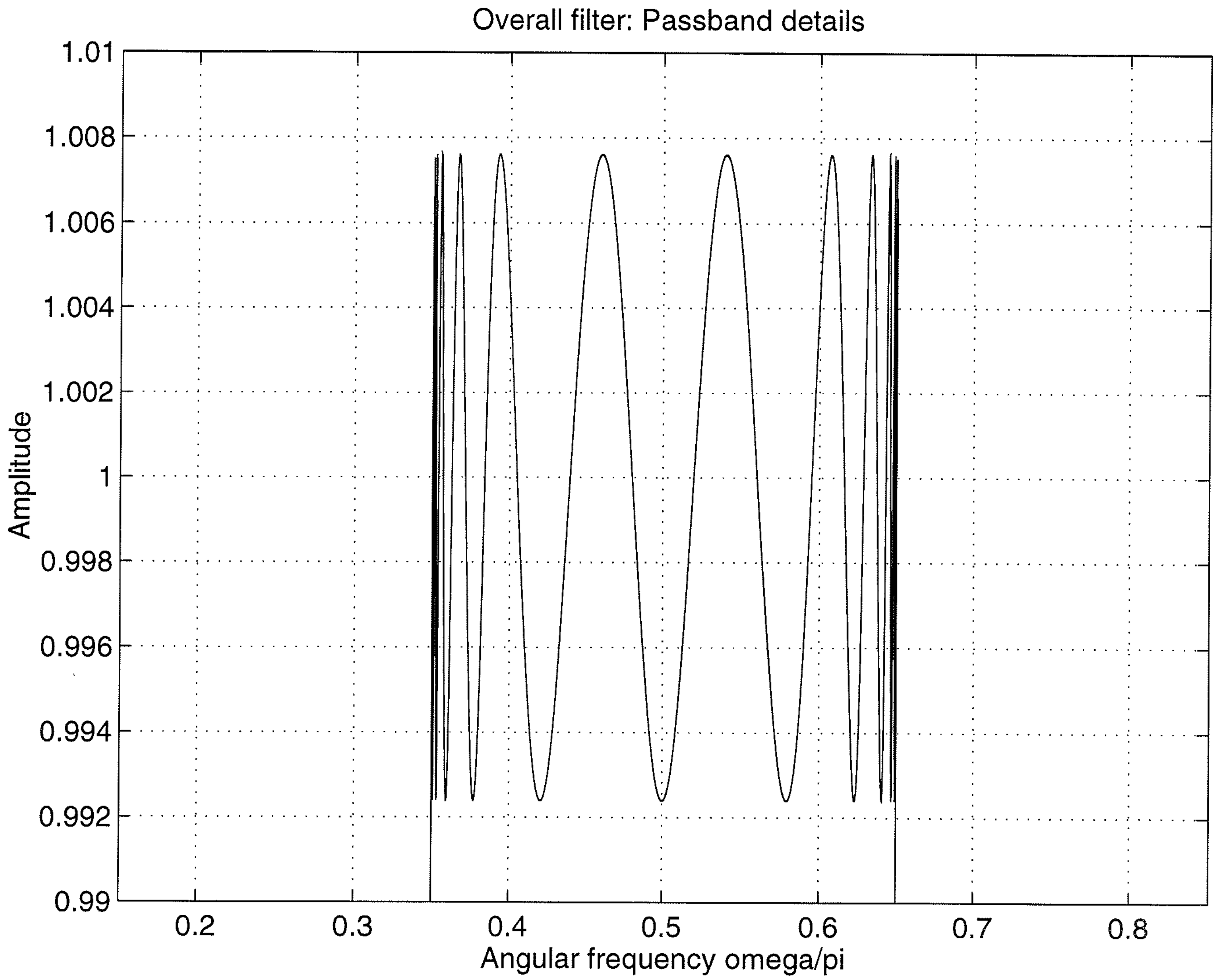
Phase difference $\phi_B(\omega) - \phi_A(\omega)$ between the all-pass filters $B(z)$ and $A(z)$



Amplitude response for the overall filter $H(z)$



Overall filter $H(z)$: Passband details



Another interpretation of the above mapping

- The allpass filters providing the desired mapping are given by

$$A(z) = \prod_{k=1}^{4} \frac{r_k^2 - 2r_k \cos\theta_k z^{-1} + z^{-2}}{1 - 2r_k \cos\theta_k z^{-1} + r_k^2 z^{-2}} \qquad (27a)$$

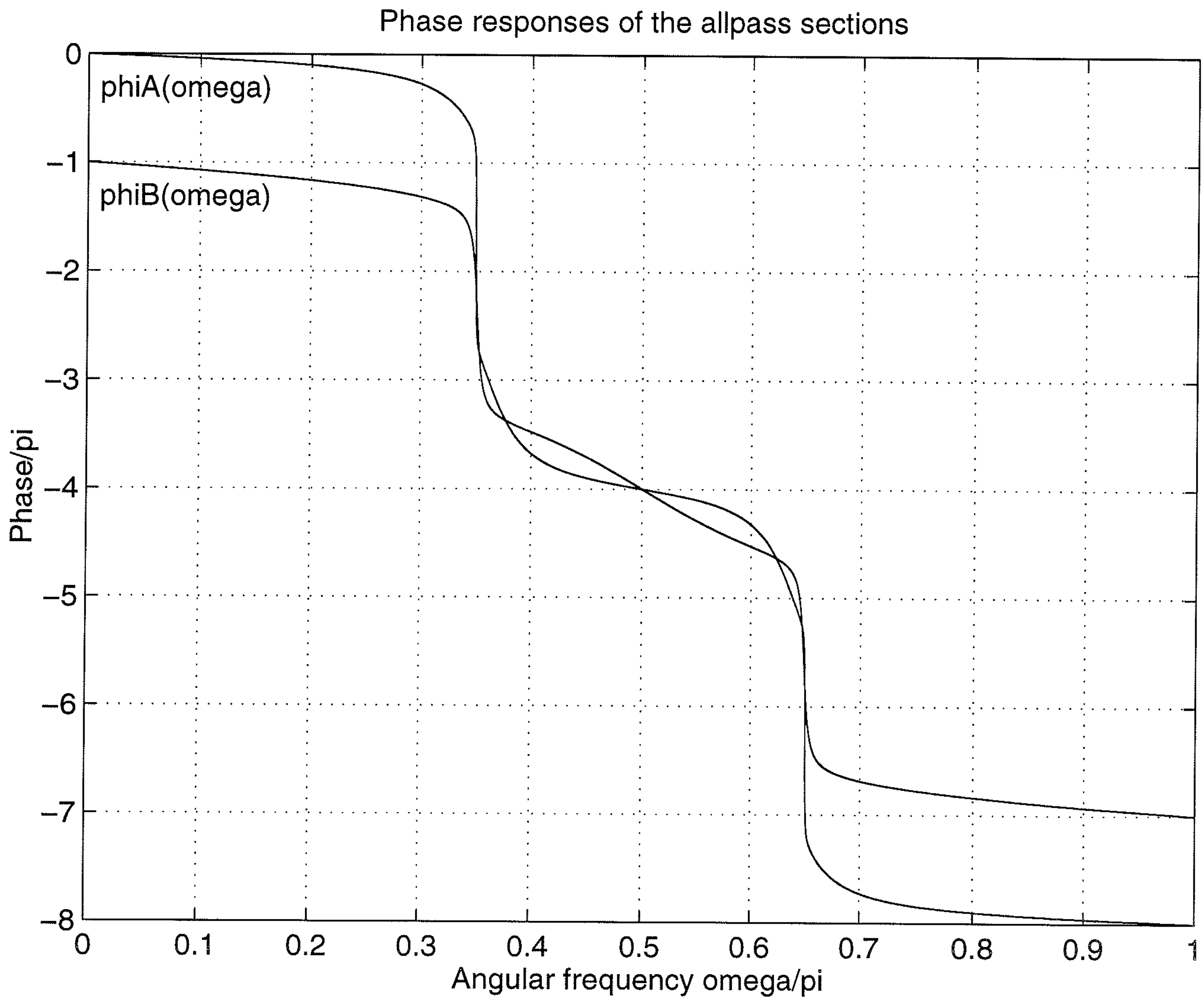
and

$$B(z) = -\prod_{k=5}^{7} \frac{r_k^2 - 2r_k \cos\theta_k z^{-1} + z^{-2}}{1 - 2r_k \cos\theta_k z^{-1} + r_k^2 z^{-2}}, \qquad (27b)$$

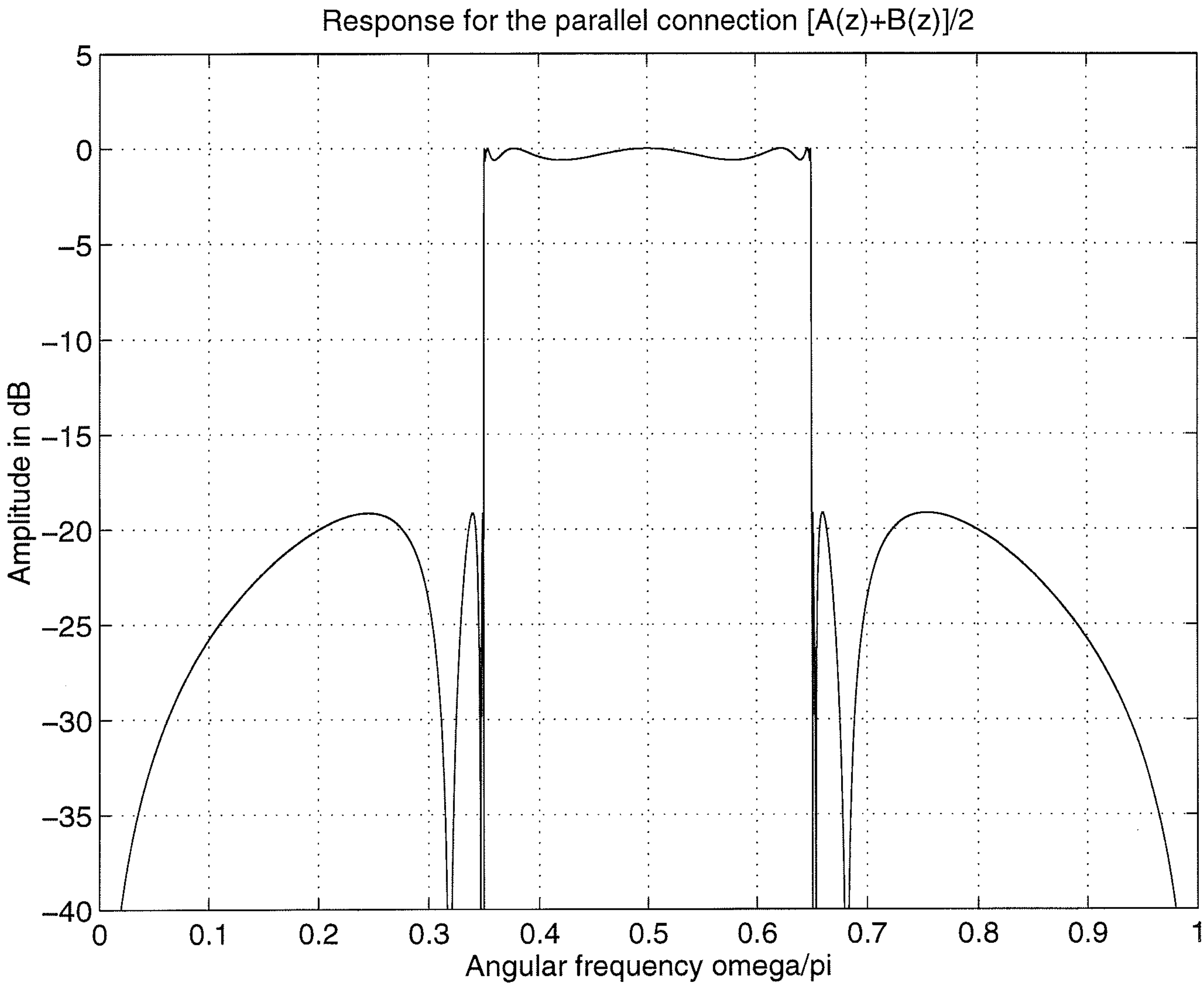
where $r_1 = r_2 = 0.99905237$, $r_3 = r_4 = 0.92380310$, $r_5 = 0.70703897$, $r_6 = r_7 = 0.99064750$, $\theta_1 = 0.34991667\pi$, $\theta_2 = \pi - \theta_1$, $\theta_3 = 0.36272056\pi$, $\theta_4 = \pi - \theta_3$,$\theta_5 = \pi/2$, $\theta_6 = 0.35109162\pi$, and $\theta_7 = \pi - \theta_6$.

- Like in the previous cases, $F(z) = [A(z) + B(z)]/2$ is an elliptic bandstop filter with the same passband and stopband regions as the overall filter. The passband ripple and the stopband attenuation are 0.61531551 dB and 19.118591 dB, respectively. See the following three transparencies.

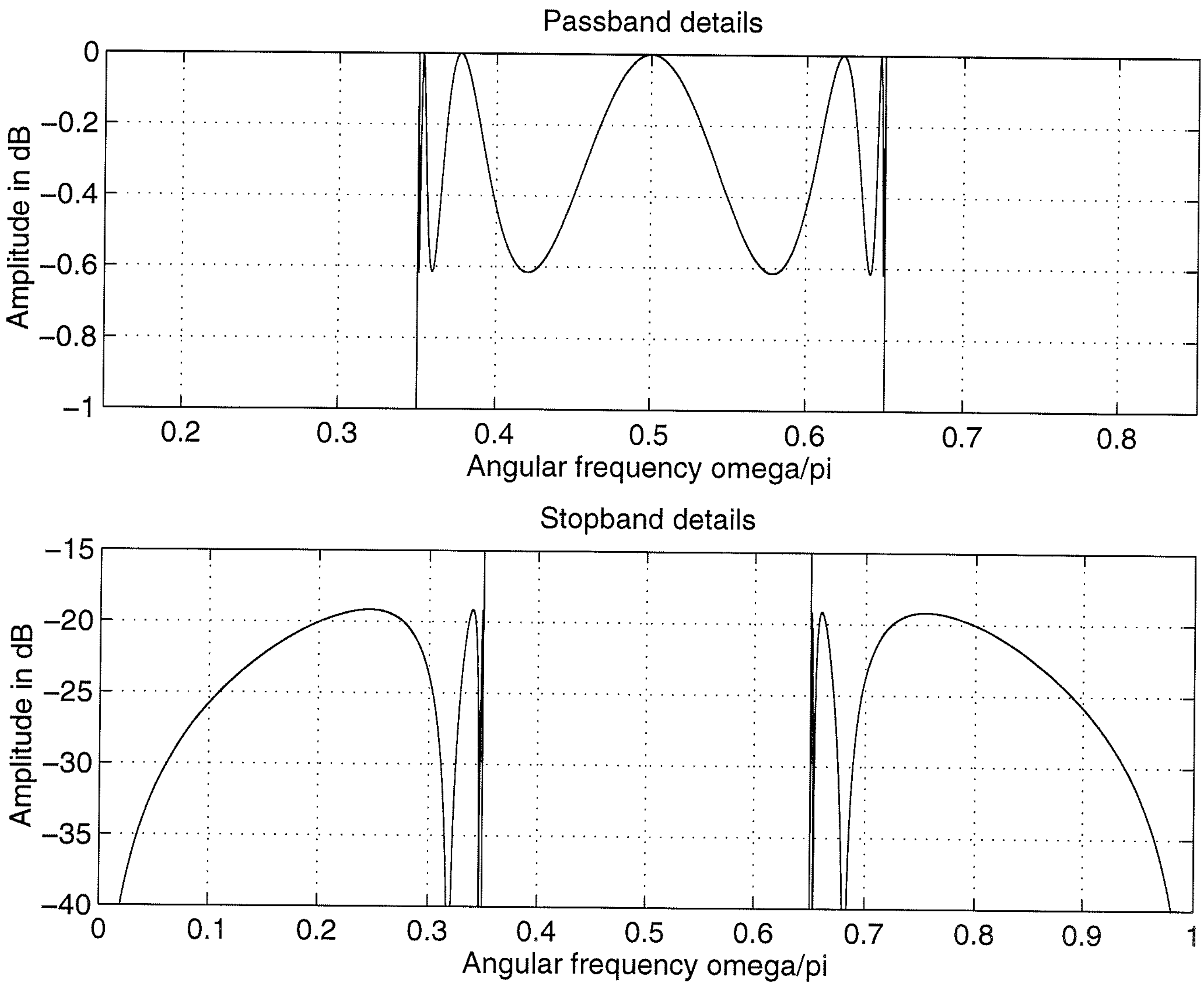
Phase responses of $A(z)$ and $B(z)$ in the band-pass case



Amplitude response for $F(z) = [A(z) + B(z)]/2$ in the bandpass case



Amplitude response for $F(z) = [A(z) + B(z)]/2$ in the bandpass case



Simultaneous Conditions for the allpass filters and the additional tap coefficients

- We recall that the transfer function of the first figure of page 5 is given by

$$H(z) = [B(z)]^N \widehat{H}(z) = \sum_{n=0}^{N} a[n][A(z)]^n [B(z)]^{N-n}. \tag{28}$$

- Let the conditions for this overall filter be given by

$$1 - \delta_p \leq |H(e^{j\omega})| \leq 1 + \delta_p \quad \text{for} \quad \omega \in X_p \quad (29a)$$

$$|H(e^{j\omega})| \leq \delta_s \quad \text{for} \quad \omega \in X_s, \quad (29b)$$

where X_p and X_s may consist of several bands.

- Based on the above introductory examples, the conditions for the allpass filters $A(z)$ and $B(z)$ as well as for the tap coefficients $a(n)$ can stated in terms of the following two transfer functions:

$$G(w) = \sum_{n=0}^{N} a[n] w^{-n} \tag{30}$$

and

$$F(z) = [A(z) + B(z)]/2. \tag{31}$$

- Here, $G(w)$ containing the additional tap coefficients $a(n)$ is a nonlinear-phase FIR filter, whereas $F(z)$ is an IIR filter being implementable as a parallel connection of our two **stable** allpass filters.
- The simultaneous conditions are given by

$$1-\delta_p \leq |G(e^{j\Omega})| \leq 1+\delta_p \quad \text{for} \quad \Omega \in [0,\ \Omega_p] \tag{32a}$$
$$|G(e^{j\Omega})| \leq \delta_s \quad \text{for} \quad \omega \in [\Omega_s,\ \pi] \tag{32b}$$

and

$$1-\widehat{\delta}_p \leq |F(e^{j\omega})| \leq 1 \quad \text{for} \quad \omega \in X_p \tag{33a}$$
$$|F(e^{j\omega})| \leq \widehat{\delta}_s \quad \text{for} \quad \omega \in X_s, \tag{33b}$$

where

$$\widehat{\delta}_p = 1-\cos(\Omega_p/2), \quad \widehat{\delta}_s = \cos(\Omega_s/2). \tag{33c}$$

- Here, the basic problem for the given value of N is to find a proper set of the values of Ω_p and Ω_s.
- The following transparencies illustrate how to determine Ω_p and Ω_s as well as $G(w)$ in the desired manner.

Design of the nonlinear-phase FIR filter $G(z)$

- We utilize the Herrmann-Schüssler technique for designing nonlinear-phase filters (see the lecture notes on Digital Filtering II). The basic difference is that the intermediate linear-phase FIR filter is an extraripple filter.

Step 1: Determine

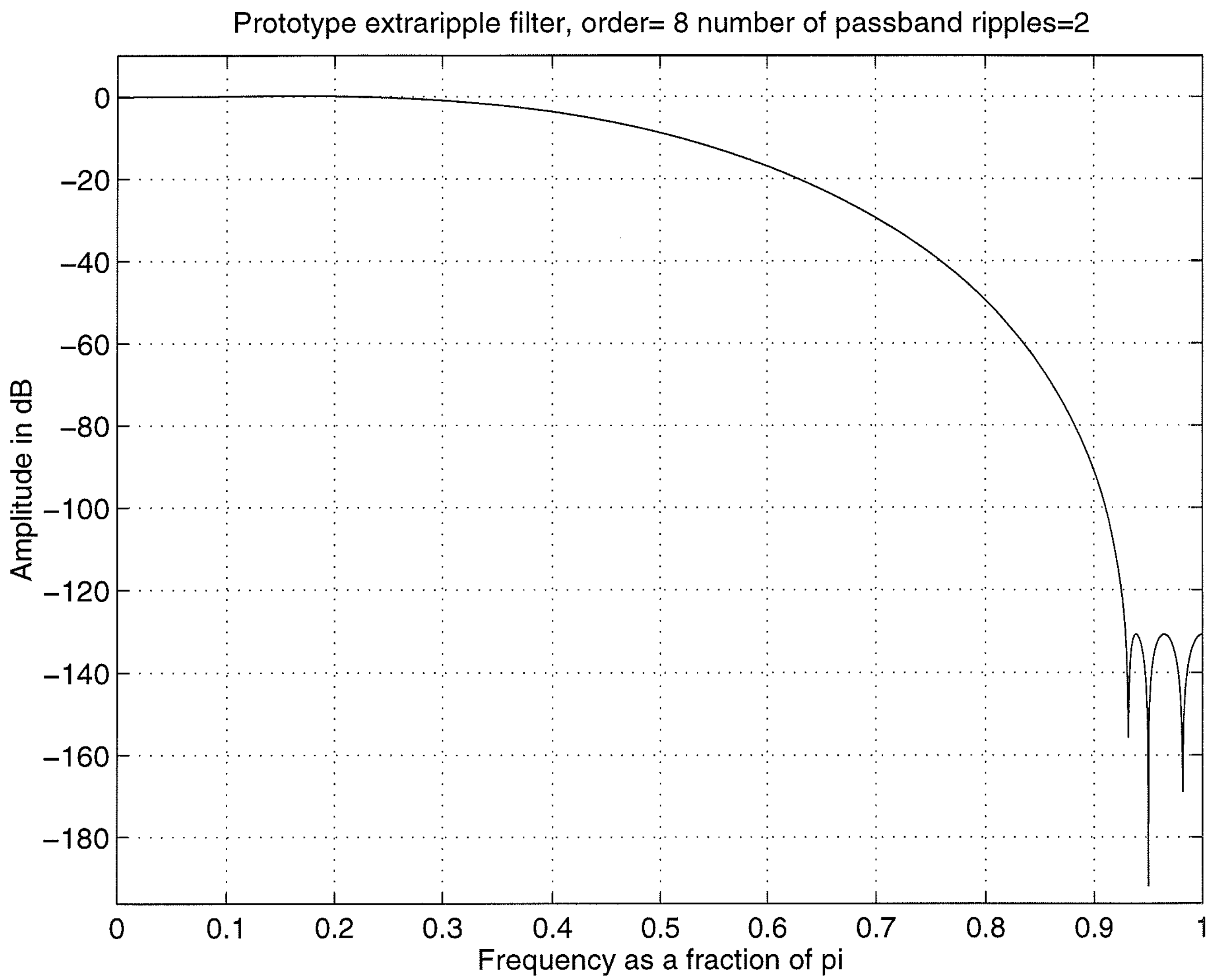
$$\widetilde{\delta}_p = \frac{2\delta_p}{1+(\delta_p)^2+(\delta_s)^2/2}, \quad \widetilde{\delta}_s = \frac{(\delta_p)^2/2}{1+(\delta_p)^2+(\delta_s)^2/2}. \tag{34}$$

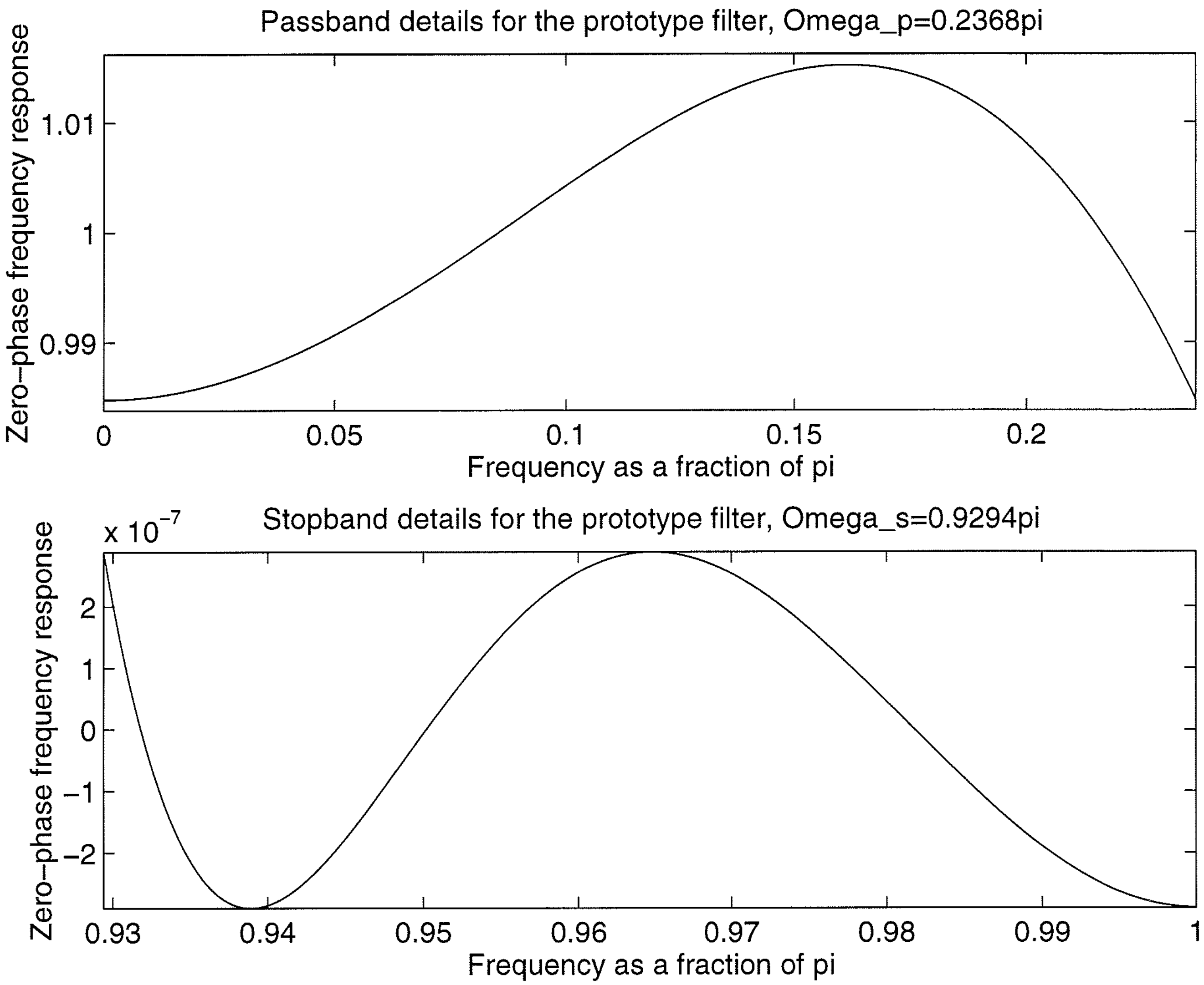
Step 2: Find the extraripple FIR filter $\widetilde{E}(w)$ of order $2N$ in such a way that in the passband its zero-phase frequency response oscillates exactly between $1 \pm \widetilde{\delta}_p$ and in the stopband exactly between $\pm\widetilde{\delta}_s$.

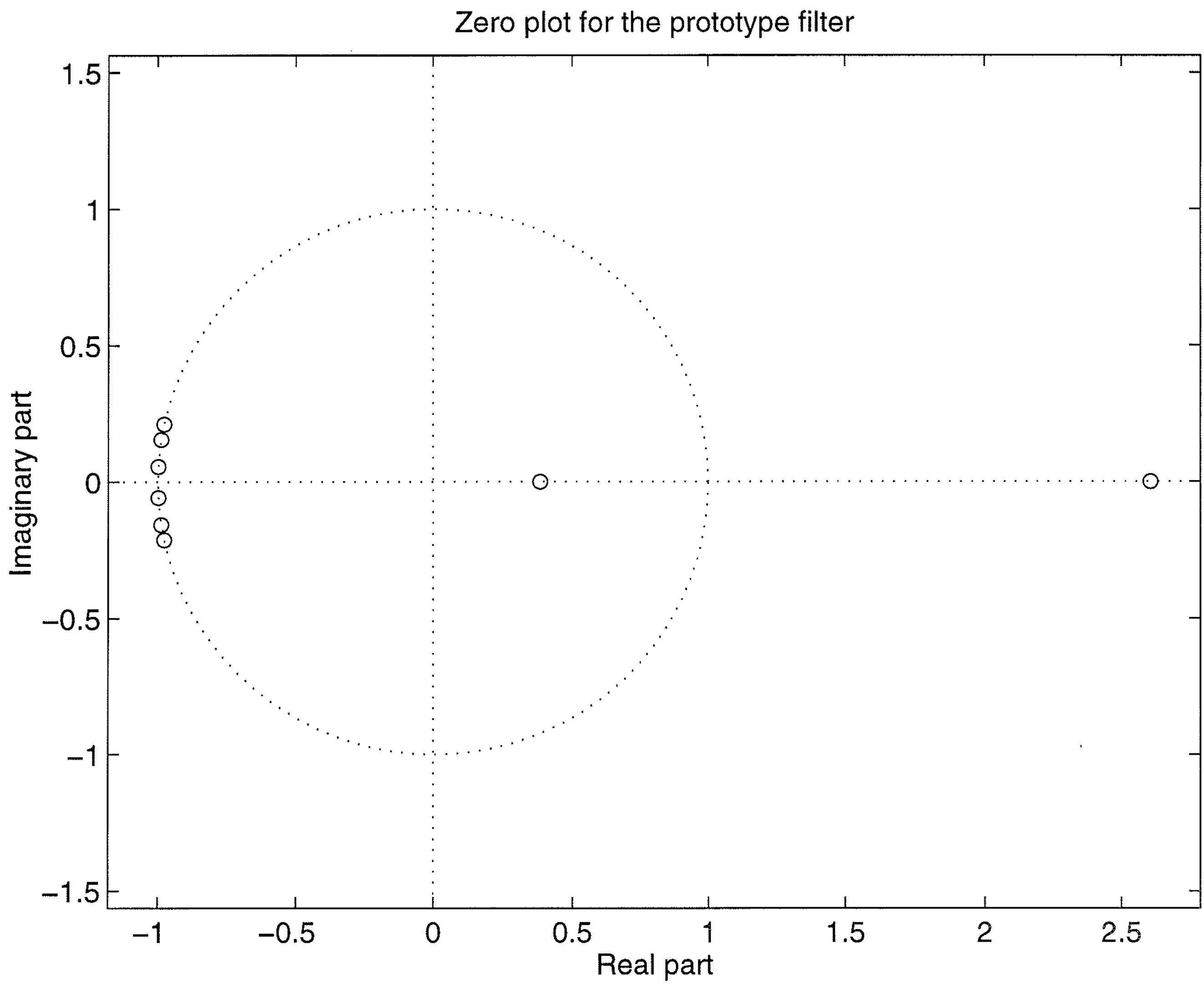
- Let the number of passband ripples be $L+1$ (the ripple at the edge is not included in this number).
- Determine Ω_p (Ω_s) to be the last (the first) extremal point where the zero-phase frequency

achieves the value $1 - \widetilde{\delta}_p$ $(\widetilde{\delta}_s)$.

- As an example, the following three transparencies show the characteristics for the extraripple design with $\delta_p = 0.0076$, $\delta_s = 0.00076$, $N = 4$, and $L + 1 = 2$.

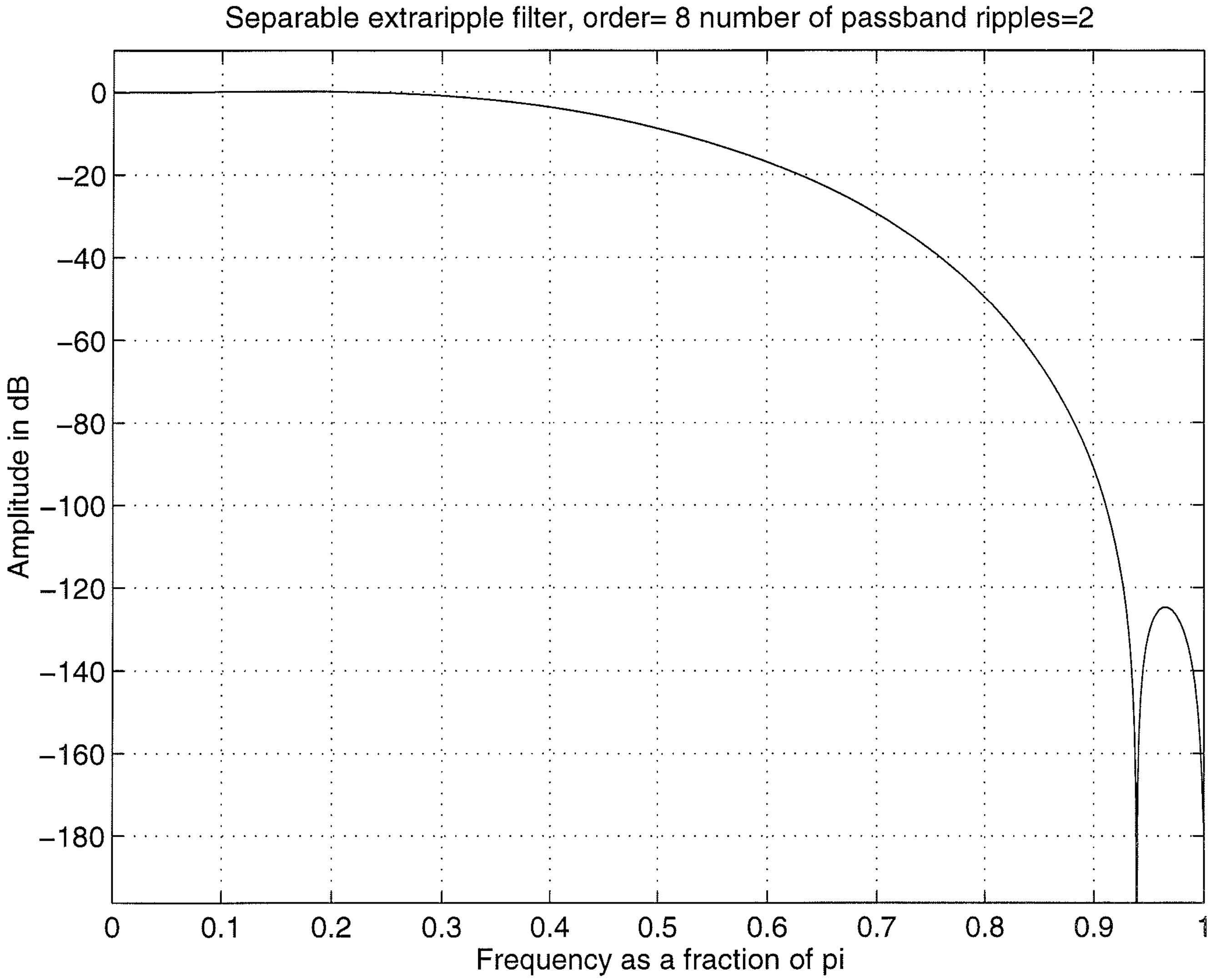

Prototype extraripple filter, order= 8 number of passband ripples=2
0
−20
−40
−60
−80
−100
−120
−140
−160
−180
Amplitude in dB
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Frequency as a fraction of pi

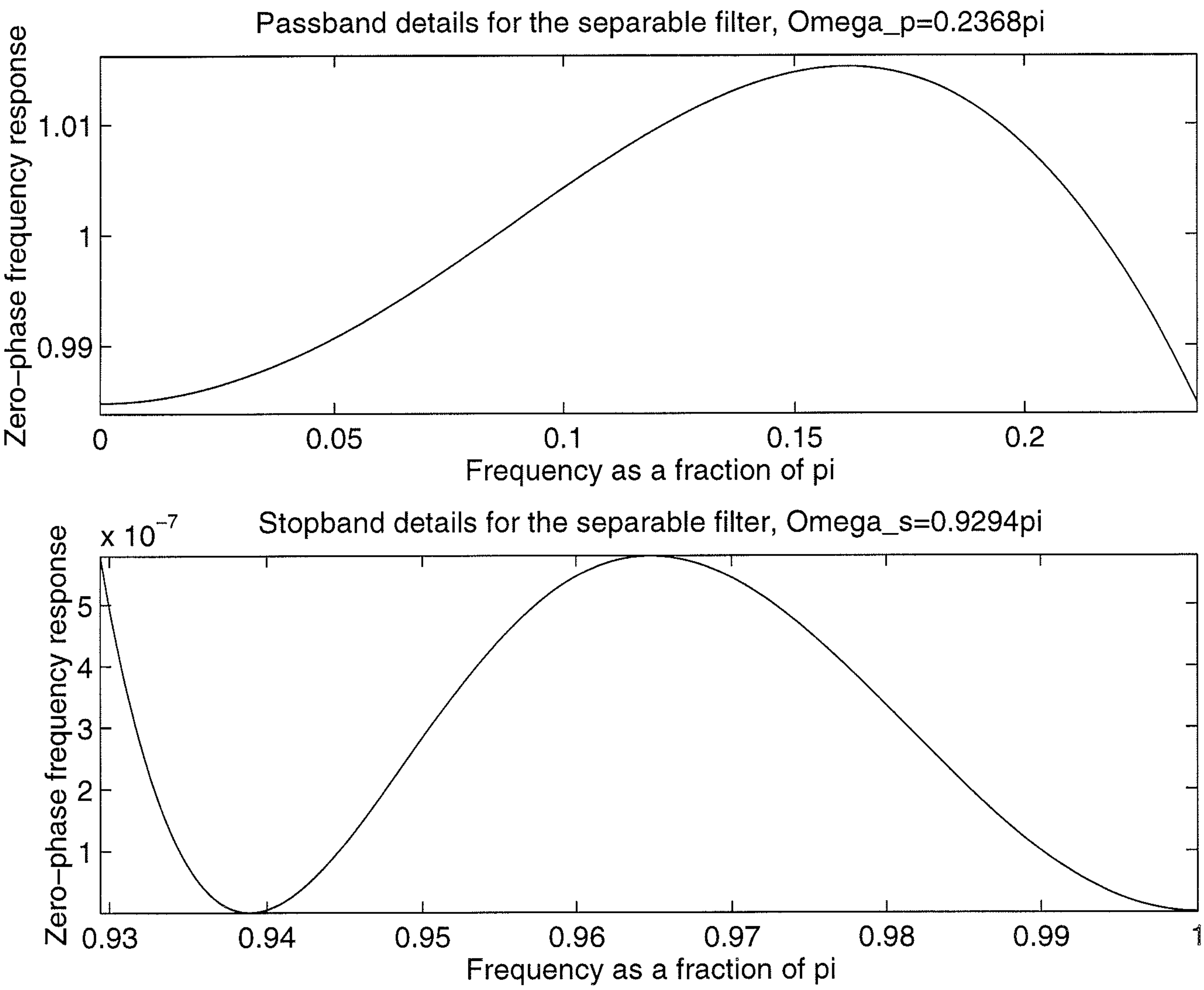

Passband details for the prototype filter, Omega_p=0.2368pi
Zero−phase frequency response
1.01
1
0.99
0
0.05
0.1
0.15
0.2
Frequency as a fraction of pi
Stopband details for the prototype filter, Omega_s=0.9294pi
x 10^-7
Zero−phase frequency response
2
1
0
−1
−2
0.93
0.94
0.95
0.96
0.97
0.98
0.99
1
Frequency as a fraction of pi

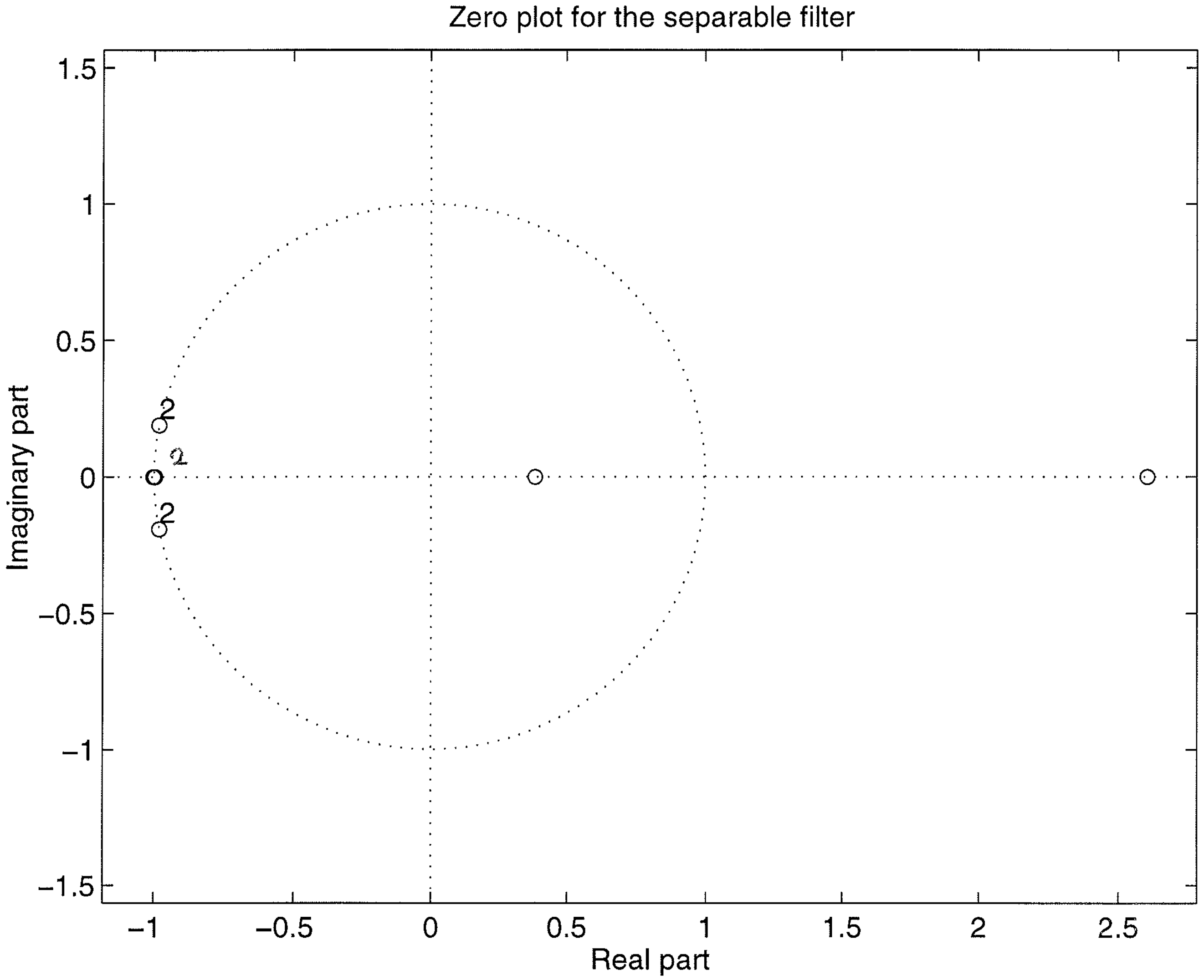

Zero plot for the prototype filter
Imaginary part
Real part
1.5
1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1
1.5
2
2.5

Step 3: Form $E(w) = \widetilde{\delta}_s w^{-N} + \widetilde{E}(w)$.

- For this filter, the zero-phase response oscillates within $1 \pm \widetilde{\delta}_p + \widetilde{\delta}_s$ in the passband $[0,\ \Omega_p]$ and within $2\widetilde{\delta}_s$ and zero in the stopband $[\Omega_s,\ \pi]$.
- Furthermore, it has double zeros on the unit circle. See the following three transparencies.


Separable extraripple filter, order= 8 number of passband ripples=2
Amplitude in dB
0
−20
−40
−60
−80
−100
−120
−140
−160
−180
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Frequency as a fraction of pi


Passband details for the separable filter, Omega_p=0.2368pi
Zero−phase frequency response
1.01
1
0.99
0
0.05
0.1
0.15
0.2
Frequency as a fraction of pi
Stopband details for the separable filter, Omega_s=0.9294pi
x 10^−7
Zero−phase frequency response
5
4
3
2
1
0.93
0.94
0.95
0.96
0.97
0.98
0.99
1
Frequency as a fraction of pi



Zero plot for the separable filter
1.5
1
0.5
0
−0.5
−1
−1.5
Imaginary part
2
2
2
−1
−0.5
0
0.5
1
1.5
2
2.5
Real part

- The resulting $E(w)$ is factorizable as

$$E(w) = e[0]E_1(w)E_2(z)[E_3(w)]^2[E_4(w)]^2, \qquad (35a)$$

where $e(0)$ is the first impulse response value and

$$\begin{aligned} E_1(w) = \prod_{k=1}^{M_1}(1 &- [2(r_k + \frac{1}{r_k})\cos\theta_k]w^{-1} \\ &+ [r_k^2 + \frac{1}{r_k^2} + 4\cos^2\theta_k]w^{-2} \\ &- [2(r_k + \frac{1}{r_k})\cos\theta_k]w^{-3} + w^{-4}), \end{aligned} \qquad (35b)$$

$$E_2(w) = \prod_{k=1}^{M_2}(1 - [\widehat{r}_k + \frac{1}{\widehat{r}_k}]w^{-1} + w^{-2}), \qquad (35c)$$

$$E_3(w) = \prod_{k=1}^{M_3}(1 - [2\cos\widehat{\theta}_k]w^{-1} + w^{-2}), \qquad (35d)$$

$$E_4(w) = [1 + w^{-1}]^{M_4}. \qquad (35e)$$

- $E_1(w)$ contains M_1 zero quadruplets at $w = r_k e^{\pm j\theta_k}$ and $w = (1/r_k)e^{\pm j\theta_k}$ for $k = 1, 2, \cdots, M_1$.
- $E_2(w)$ contains M_2 reciprocal zero pairs on the real axis at $w = \widehat{r}_k, 1/\widehat{r}_k$ for $k = 1, 2, \cdots, M_2$.
- $E_3(w)$ contains M_3 zero pairs on the unit circle at $w = e^{\pm j\widehat{\theta}_k}$ for $k = 1, 2, \cdots, M_3$.

- $E_4(w)$ contains M_4 zeros at $z = -1$.
- If there are $L + 1$ extrema in the passband (the extrema at the cutoff point $\Omega = \Omega_p$ is not included), then

$$M_1 = \lfloor L/2 \rfloor, \; M_2 = L - M_1, \tag{36a}$$

$$M_3 = \lfloor (N-L)/2 \rfloor, \; M_4 = N - L - 2M_3, \tag{36b}$$

where $\lfloor x \rfloor$ stands for integer part of x.

Step 4: Form a minimum-phase filter $G(w)$ by picking up the zeros inside the unit circle and one each of the double zeros on the unit circle.

- This gives

$$G(w) = CG_1(w)G_2(z)G_3(w)G_4(w), \qquad (37a)$$

where

$$G_1(w) = \prod_{k=1}^{M_1}(1 - (2r_k \cos\theta_k)w^{-1} + r_k^2 w^{-2}), \quad (37b)$$

$$G_2(w) = \prod_{k=1}^{M_2}(1 - \widehat{r}_k w^{-1}), \qquad (37c)$$

$$G_3(w) = \prod_{k=1}^{M_3}(1 - [2\cos\widehat{\theta}_k]w^{-1} + w^{-2}), \qquad (37d)$$

$$G_4(w) = (1 + w^{-1})^{M_4} \qquad (37e)$$

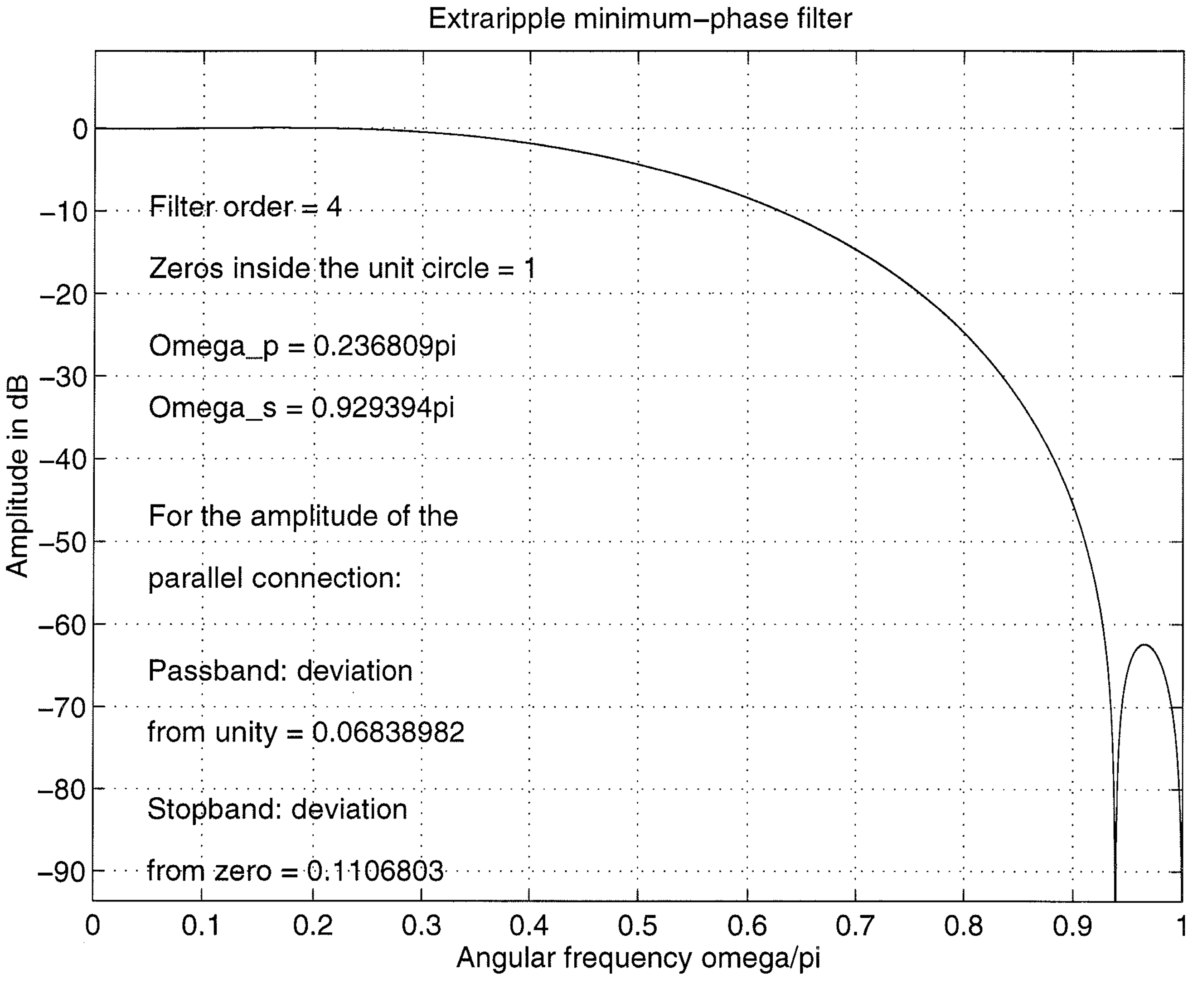
- Finally, find C in such a way that the passband average becomes equal to unity.
- The resulting $|G(e^{j\Omega})|$ oscillates in the passband $[0,\ \Omega_p]$ between the limits $1 \pm \delta_p$ and in the stopband $[\Omega_s,\ \pi]$ between the limits δ_s and zero.

- This is quaranteed by selecting the ripples of the prototype linear-phase extraripple filter according to Eq. (34).
- Note that the above $G(w)$ is expressible in the form

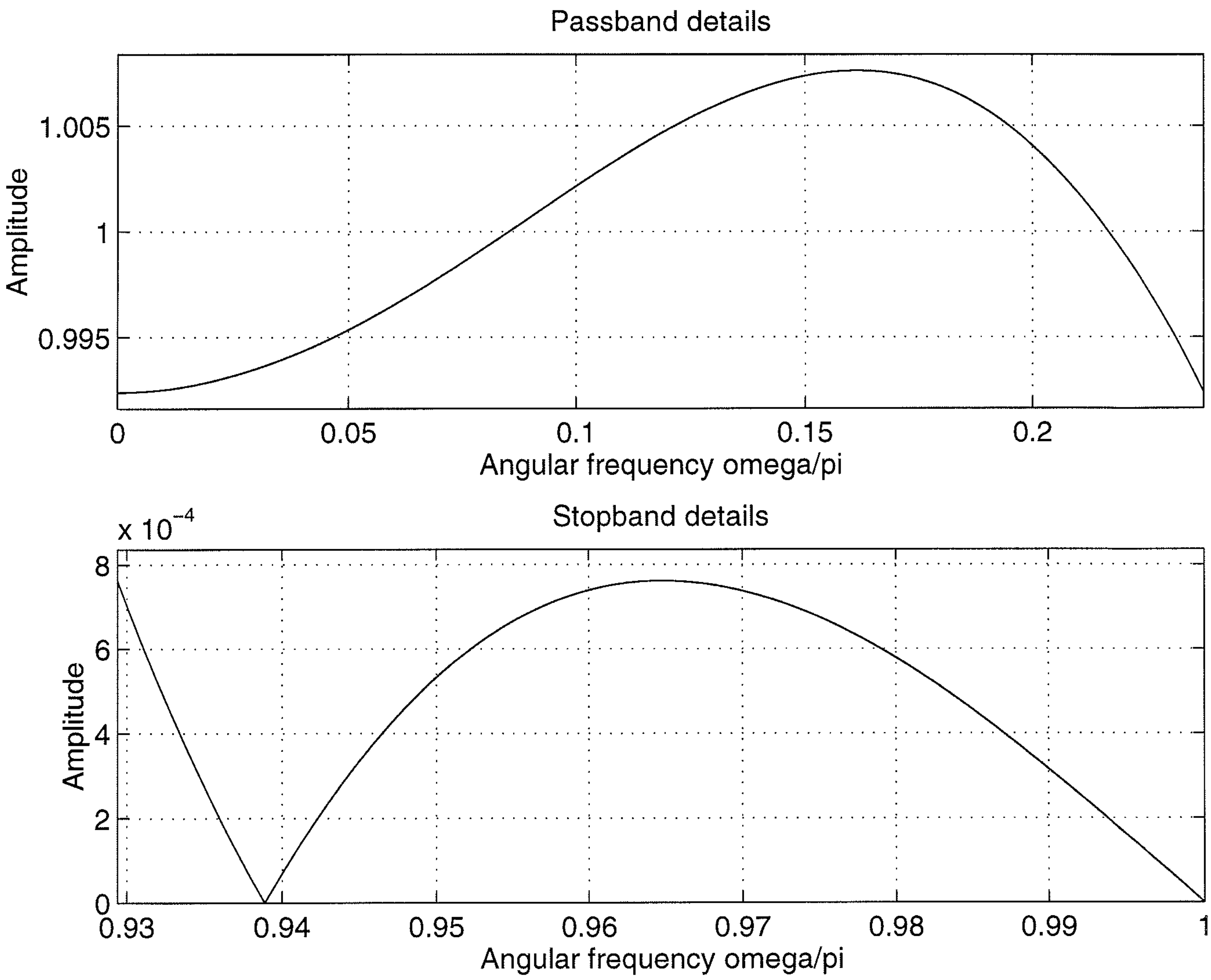
$$
\begin{aligned}
G(w) =& C \prod_{k=1}^{N_1} (b_k(0) + b_k(1)w^{-1} + b_k(2)w^{-2}) \times \\
& \prod_{k=1}^{N_2} (c_k(0) + c_k(1)w^{-1}),
\end{aligned}
\tag{38}
$$

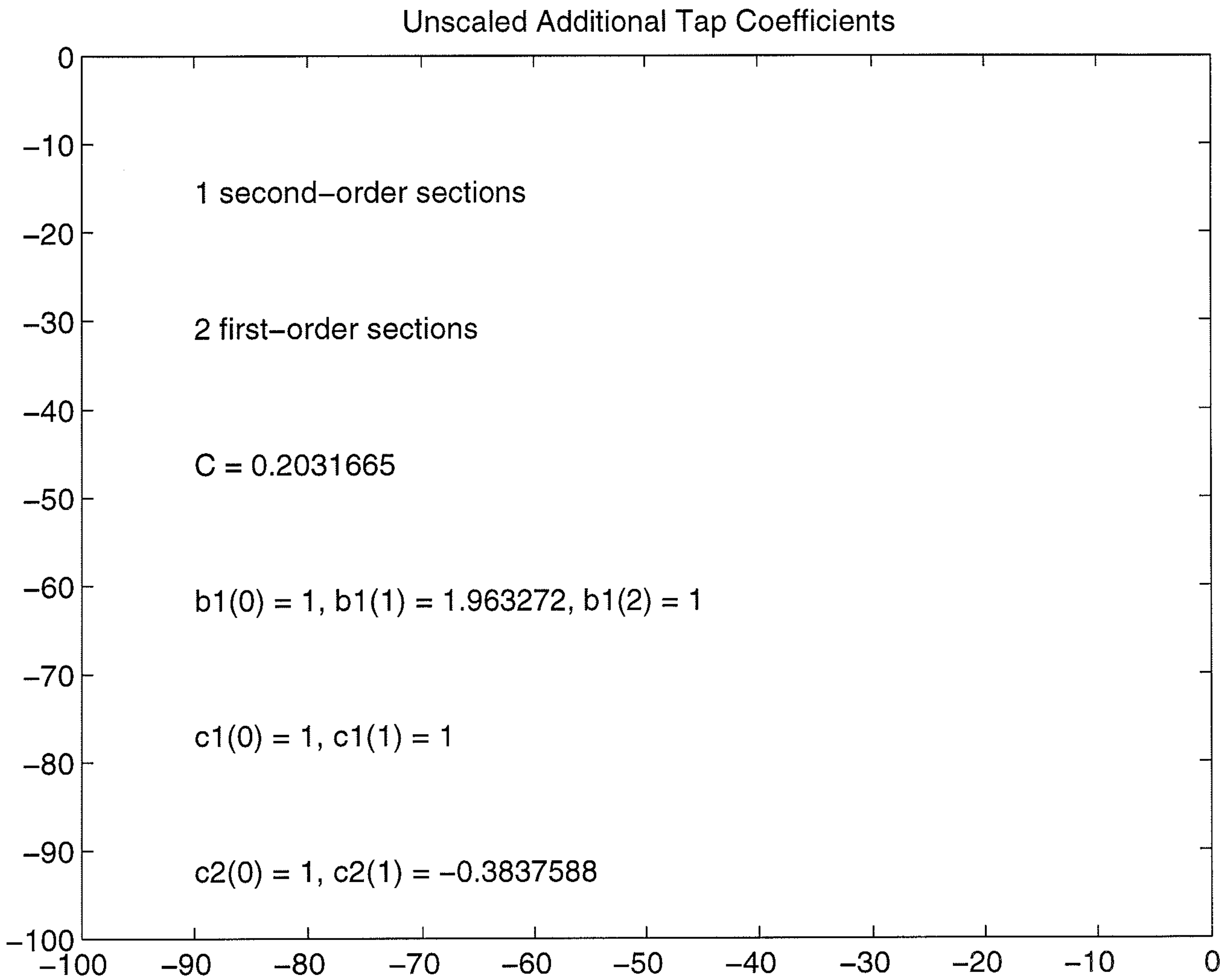
where the $b_k(0)$'s and $c_k(0)$'s are equal to unity.

- These are directly the additional tap coefficients in the second structure of page 5.
- The following transparencies show the characteristics of the resulting $G(w)$ in our example case.
- After these transparencies, a Matlab-file extramin.m is introduced for automatically performing the above procedure for us.



Extraripple minimum−phase filter
Amplitude in dB
0
−10
−20
−30
−40
−50
−60
−70
−80
−90
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Angular frequency omega/pi
Filter order = 4
Zeros inside the unit circle = 1
Omega_p = 0.236809pi
Omega_s = 0.929394pi
For the amplitude of the
parallel connection:
Passband: deviation
from unity = 0.06838982
Stopband: deviation
from zero = 0.1106803


Passband details
1.005
1
0.995
Amplitude
0
0.05
0.1
0.15
0.2
Angular frequency omega/pi
Stopband details
x 10^-4
8
6
4
2
0
Amplitude
0.93
0.94
0.95
0.96
0.97
0.98
0.99
1
Angular frequency omega/pi


Unscaled Additional Tap Coefficients
1 second−order sections
2 first−order sections
C = 0.2031665
b1(0) = 1, b1(1) = 1.963272, b1(2) = 1
c1(0) = 1, c1(1) = 1
c2(0) = 1, c2(1) = −0.3837588
0
−10
−20
−30
−40
−50
−60
−70
−80
−90
−100
−100
−90
−80
−70
−60
−50
−40
−30
−20
−10
0

Matlab-file extramin.m

- Given δ_p, δ_s, N (the number of subfilters), and $L+1$ (the number of passband extrema), this routine automatically finds out the corresponding extraripple minimum-phase solution.
- It gives $G(w)$ in the form of Eq. (38), where the $b_k(0)$'s and $c_k(0)$'s are equal to unity.
- It gives also Ω_s and Ω_p as well as the ripple values for $F(z)$, which are given by Eq. (33c).
- What is left is to determine $L+1$ in such a way that the criteria for $F(z)$ become as mild as possible so that they can be met by the minimum filter order.
- In the lowpass and highpass cases, we can use elliptic filters of odd order (the orders of the allpass sections $A(z)$ and $B(z)$ must differ by one.)
- If an elliptic filter is used in the bandpass and bandstop cases, the order must be two times an odd integer (the difference in the orders must be two).

- As we noticed in the course Digital Filtering II, there are more sophisticated design techniques for bandpass and bandstop filters and all what is needed is that the difference in the orders is two.
- For N subfilters, we can select $L+1$ between 1 and N. In most cases, $L+1=1$ or $L+1=2$ gives the mildest criteria for $F(z)$.
- extramin.m saves some data in file minfir for further use.
- You can find this routine in the following transparencies.

```
% Matlab m-file (extramin.m) for determining extraripple
% minimum-phase FIR filter solutions for the given ripple
% values for the given order N and for the given number of
% ripples in the passband. The number of zeros off the unit
% circle is the number of passband ripples - 1.
% A modified Remez-routine, called remmin.m, for determing
% the starting point linear-phase FIR filter solution is called.
% Also files, called fircasam.m and firam.m, are needed.
% Tapio Saram"aki 5.3.1996
% All the files can be found in Sun's: ~ts/matlab/sldsp
 clear all
 close all
 disp('Hi there')
 disp('I am an program for designing extraripple')
 disp('minimum-phase lowpass FIR filters')
 dp=input('Passband ripple= ');
 ds=input('Stopband ripple= ');
 IP=input('Number of passband ripples= ');
 N=input('Number of subfilters= ');
 den=1+dp^2-ds^2/2;
 Dp=2*dp/den;Ds=(ds^2/2)/den;
 NN=2*N
% Find the starting-point linear-phase FIR filter
%
% xx and yy contain the extremal points
 [xx,yy,h,cut]=remmin(NN, [Dp Ds], IP, [0 1], [1 1], 1);
 cut=2*cut;
 dbarp=max(yy(1:N+1))-1;dbars=-min(yy(1:N+1));
 A=2/(sqrt(1+dbarp+dbars)+sqrt(1-dbarp+dbars));
 for k=1:N+1
  ome(k)=acos(xx(k))/pi;
  y(k)=A*A*(yy(k)+dbars);
 end
%
% The value at DC for the minimum-phase design
%
 MINDC=sqrt(y(1));
%
% NZ, the number of zeros on the unit circle, is N-IP-1
% NP, number of zeros off the unit circle, is IP-1
%
 NP=IP-1;NZ=N-(IP-1);
```

```
%
% For NZ odd, there is a zero at z=-1 and NU=(NZ-1)/2 zero
% pairs on the unit circle at frequencies ome(N-2*k) for
% k=1,2,...,NU
% For NZ even, there is no zero at z=-1 and NU=NZ/2 zero
% pairs on the unit circle at frequencies ome(N+1-2*k) for
% k=1,2,...,NU
% NZ1 is the number of zeros at z=-1
 NZ1=1;
 if 2*floor(NZ/2)==NZ NZ1=0;end
 NZU=floor(NZ/2);
 for k=1:NZU
  OM(k)=ome(N+2-NZ1-2*k);
 end
%
% For NZ1=1, there is a filter part with impulse response
% gz1(0)=gz1(1)=1
%
 if NZ1==1 gZ1(1)=1; gZ1(2)=1;end
%
% There are also NU filter parts 1/2(1+w^(-2))-cos(pi*OM(k))w^(-1)
% for k=1,2,..., NU
%
 for k=1:NZU
  gZU(k,1)=1;gZU(k,3)=1;gZU(k,2)=-2*cos(pi*OM(k));
 end
% For NP > 0, there are zeros inside the unit circle.
% For this purpose, we use the Lagrange interpolation
% formula and the points ome(k) for k=1,...,IP for generating
% the squared-magnitude function corresponding to these zeros.
%
 NIN1=0;NIN2=0;
 if NP> 0
  for k=1:IP
    A=1;
    if NZ1==1 A=2*cos(pi*ome(k)/2);end
    for l=1:NZU
      A=A*2*(cos(pi*ome(k))-cos(pi*OM(l)));
    end
   yy(k)=y(k)/(A*A);end
% Next we find the zero-phase frequency response corresponding
% to the squared-magnitude function of the filter part having
```

```
% zeros inside the unit circle
% What is now remaining is to evaluate the impulse
% response coefficints of the resulting filter.
% To do this, we evaluate the zero-phase  frequency
% response of our filter at 2^l > 2(IP-1)+1 equally spaced
% frequencies and use the IFFT.
   l=log2(IP);l=round(l)+1;k=2^l;
   w=0:2*pi/k:2*(k-1)*pi/k;
   B=zeros(size(w));
   for k=1:IP
     C=ones(size(w));
     D=1;
     for l=1:IP
       CC=cos(w)-cos(pi*ome(l));
       if l==k CC=ones(size(w));end
       DD=cos(pi*ome(k))-cos(pi*ome(l));
       if l==k DD=1;end
       C=C.*CC;
       D=D*DD;
      end
      B=B+yy(k)*C/D;
    end
    b=ifft(B);
    b=real(b);
    b=fftshift(b);
    for k=1:2*(IP-1)+1
      hh(k)=b(2^(l-1)+1-(IP-1+1)+k);end
%
% Find the zeros inside the unit circle
%
   r=sort(roots(hh));
   rmi=[ ];
   for i=1:length(r)
     if abs(r(i))< 0.999, % roots inside the unit circle
     rmi(length(rmi)+1)=r(i);end
   end
%
% Sort the zeros according to the increasing angle
%
   [Y,I]=sort(angle(rmi));
   rmi=rmi(I);
%
```

```
%
% For NP odd, there is a real zero at z=rmi(1) and
% NIN2=(NP-1)/2 zero pairs inside the unit circle
% pairs on the unit circle at z=rmi(2*k), rmi(2*k+1)
% for k=1,2,...,NIN2
% For NZ even, there is no real zeros and NIN2=NZ/2 zero
% pairs inside the unit circle at at z=rmi(2*k-1),
% rmi(2*k) for k=1,2,...,NIN2
% NIN1 is the number of real zeros
  NIN1=1;
  if 2*floor(NP/2)==NP NIN1=0;end
  NIN2=floor(NP/2);
%
% For NIN1=1, there is a filter part 1-rmi(1)w^(-1)
%
  if NIN1==1 gIN1(1)=1;gIN1(2)=-rmi(1);end
%
% Form NIN2 second-order filter parts
%
  for k=1:NIN2
   ang=angle(rmi(2*k));
   rr=abs(rmi(2*k));
   gIN2(k,1)=1;gIN2(k,2)=-2*rr*cos(ang);
   gIN2(k,3)=rr*rr;
  end
 end
%
% Determine the scaling constant scale
%
 AA=1
 if NZ1==1 AA=AA*2;end
 if NZU > 0
  for k=1:NZU
   AA=AA*(gZU(k,1)+gZU(k,3)+gZU(k,2));
  end
 end
 if NIN1==1 AA=AA*(gIN1(1)+gIN1(2));end
 if NIN2 > 0
  for k=1:NIN2
   AA=AA*(gIN2(k,1)+gIN2(k,2)+gIN2(k,3))
  end
 end
```

```
 scale=MINDC/AA;
%
% Form the minimum-phase FIR filter
%
 hmin(1)=1;
 if NZ1==1 hmin=gZ1;end
 if NZU > 0
  for k=1:NZU
   hu=gZU(k,1:3);hmin=conv(hmin,hu);
  end
 end
 if NIN1==1 hmin=conv(hmin,gIN1);end
 if NIN2 > 0
  for k=1:NIN2
   hu=gIN2(k,1:3);hmin=conv(hmin,hu);
  end
 end
 hmin=scale*hmin;
%
% Find the passband edge, that is, the frequency point
% where the value 1-dp is achieved
%
 xc=(cut(2)+cut(1))/2;
 xd=cut(2)-xc;
 amdes=1-dp;
%
% the edge is dsired to be determined with accuracy
% less than or equal to 10^(-12)*pi
%
  kk=ceil(log10((10^12/(cut(2)-cut(1))/1000))/2);
  for k=1:kk
   [AA,ww]=fircasam(scale,NZ1,NZU,NIN1,NIN2,gZ1,gZU,...
        gIN1,gIN2,xc-xd,xc+xd,1000);
   [Y,I]=sort(abs(AA-amdes));
   ww=ww(I);xc=ww(1)/pi;xd=xd/1000;
  end
  cut1=xc;
%
% Find the passband edge, that is, the frequency point
% where the value 1-dp is achieved
%
 xc=(cut(2)+cut(1))/2;
```

```
xd=cut(2)-xc;
amdes=ds;
 for k=1:kk
  [AA,ww]=fircasam(scale,NZ1,NZU,NIN1,NIN2,gZ1,gZU,...
       gIN1,gIN2,xc-xd,xc+xd,1000);
  [Y,I]=sort(abs(AA-amdes));
  ww=ww(I);xc=ww(1)/pi;xd=xd/1000;
 end
 cut2=xc;
figure(1)
[AA,ww]=fircasam(scale,NZ1,NZU,NIN1,NIN2,gZ1,gZU,...
            gIN1,gIN2,0,1,10000);
As=30*log10(ds);
plot(ww/pi,20*log10(AA));grid;axis([0 1 As -.1*As]);
ylabel('Amplitude in dB');
xlabel('Angular frequency omega/pi');
title('Extraripple minimum-phase filter')
text(0.05, .1*As, ['Filter order = ', num2str(N)]);
text(0.05, .18*As, ...
['Zeros inside the unit circle = ', num2str(IP-1)]);
text(0.05, .28*As, ['Omega_p = ', num2str(cut1,6),'pi']);
text(0.05, .36*As, ['Omega_s = ', num2str(cut2,6),'pi']);
text(.05,.5*As,'For the amplitude of the');
text(.05,.58*As,'parallel connection:');
hde1=1-cos(pi*cut1/2);hde2=cos(pi*cut2/2);
text(0.05, .7*As,'Passband: deviation');
text(0.05, .78*As,['from unity = ', num2str(hde1,7)]);
text(0.05, .88*As,'Stopband: deviation');
text(0.05, .96*As,['from zero = ', num2str(hde2,7)]);
figure(2)
subplot(211)
[AA,ww]=fircasam(scale,NZ1,NZU,NIN1,NIN2,gZ1,gZU,...
        gIN1,gIN2,0,cut1,10000);
plot(ww/pi,abs(AA));grid;axis([0 cut1 1-1.1*dp 1+1.1*dp]);
ylabel('Amplitude');xlabel('Angular frequency omega/pi');
title('Passband details')
subplot(212)
[AA,ww]=fircasam(scale,NZ1,NZU,NIN1,NIN2,gZ1,gZU,...
        gIN1,gIN2,cut2,1,10000);
plot(ww/pi,abs(AA));grid;axis([cut2 1 0 1.1*ds]);
ylabel('Amplitude');xlabel('Angular frequency omega/pi');
title('Stopband details')
```

```
figure(3)
z1(1)=0;z2(1)=0;
plot(z1,z2);axis([-100 0 -100 0]);
M1=NZ1+NIN1;M2=NZU+NIN2;step=-100*2/(2*(M1+M2+3)+1);
title('Unscaled Additional Tap Coefficients');
text(-90, step, [num2str(M2), ' second-order sections']);
text(-90, 2*step, [num2str(M1), ' first-order sections']);
text(-90, 3*step, ['C = ', num2str(scale,7)]);
if NZU > 0
 for k=1:NZU
   b1=gZU(k,1);
   b2=gZU(k,2);
   b3=gZU(k,3);
   l=3+k
text(-90,l*step,...
['b',num2str(k),'(',num2str(0),') = ',num2str(b1,7),...
', b',num2str(k),'(',num2str(1),') = ',num2str(b2,7),...
', b',num2str(k),'(',num2str(2),') = ',num2str(b3,7),...
]);
 end
end
if NIN2 > 0
 for k=1:NIN2
   b1=gIN2(k,1);
   b2=gIN2(k,2);
   b3=gIN2(k,3);
   l=3+k+NZU;m=k+NZU;
text(-90,l*step,...
['b',num2str(m),'(',num2str(0),') = ',num2str(b1,7),...
', b',num2str(m),'(',num2str(1),') = ',num2str(b2,7),...
', b',num2str(m),'(',num2str(2),') = ',num2str(b3,7),...
]);
  end
end
if NZ1==1;
 b1=gZ1(1);
 b2=gZ1(2);
 l=3+NIN2+NZU+1;m=1;
text(-90,l*step,...
['c',num2str(m),'(',num2str(0),') = ',num2str(b1,7),...
', c',num2str(m),'(',num2str(1),') = ',num2str(b2,7),...
]);
```

```
end
if NIN1==1
  b1=gIN1(1);
  b2=gIN1(2);
  l=3+NIN2+NZU+NZ1+1;m=1+NZ1;
text(-90,l*step,...
['c',num2str(m),'(',num2str(0),') = ',num2str(b1,7),...
', c',num2str(m),'(',num2str(1),') = ',num2str(b2,7),...
]);
end
bb(1)=scale;bb(2)=NZ1;bb(3)=NZU;bb(4)=NIN1;bb(5)=NIN2;
bb(6)=cut1;bb(7)=cut2;
if NZ1==1;
  bb(length(bb)+1)=gZ1(1);
  bb(length(bb)+1)=gZ1(2);
end
if NZU > 0
  for k=1:NZU
    bb(length(bb)+1)=gZU(k,1);
    bb(length(bb)+1)=gZU(k,2);
    bb(length(bb)+1)=gZU(k,3);
  end
end
if NIN1==1
  bb(length(bb)+1)=gIN1(1);
  bb(length(bb)+1)=gIN1(2);
end
if NIN2 > 0
  for k=1:NIN2
    bb(length(bb)+1)=gIN2(k,1);
    bb(length(bb)+1)=gIN2(k,2);
    bb(length(bb)+1)=gIN2(k,3);
  end
 end
 hui=rot90(rot90(rot90(bb)));
 save minfir hui -ascii -double
```

Example 1: Lowpass filter: $\omega_p = 0.3\pi$, $\omega_s = 0.301$, $\delta_p \leq 0.01$, $\delta_s \leq 0.001$

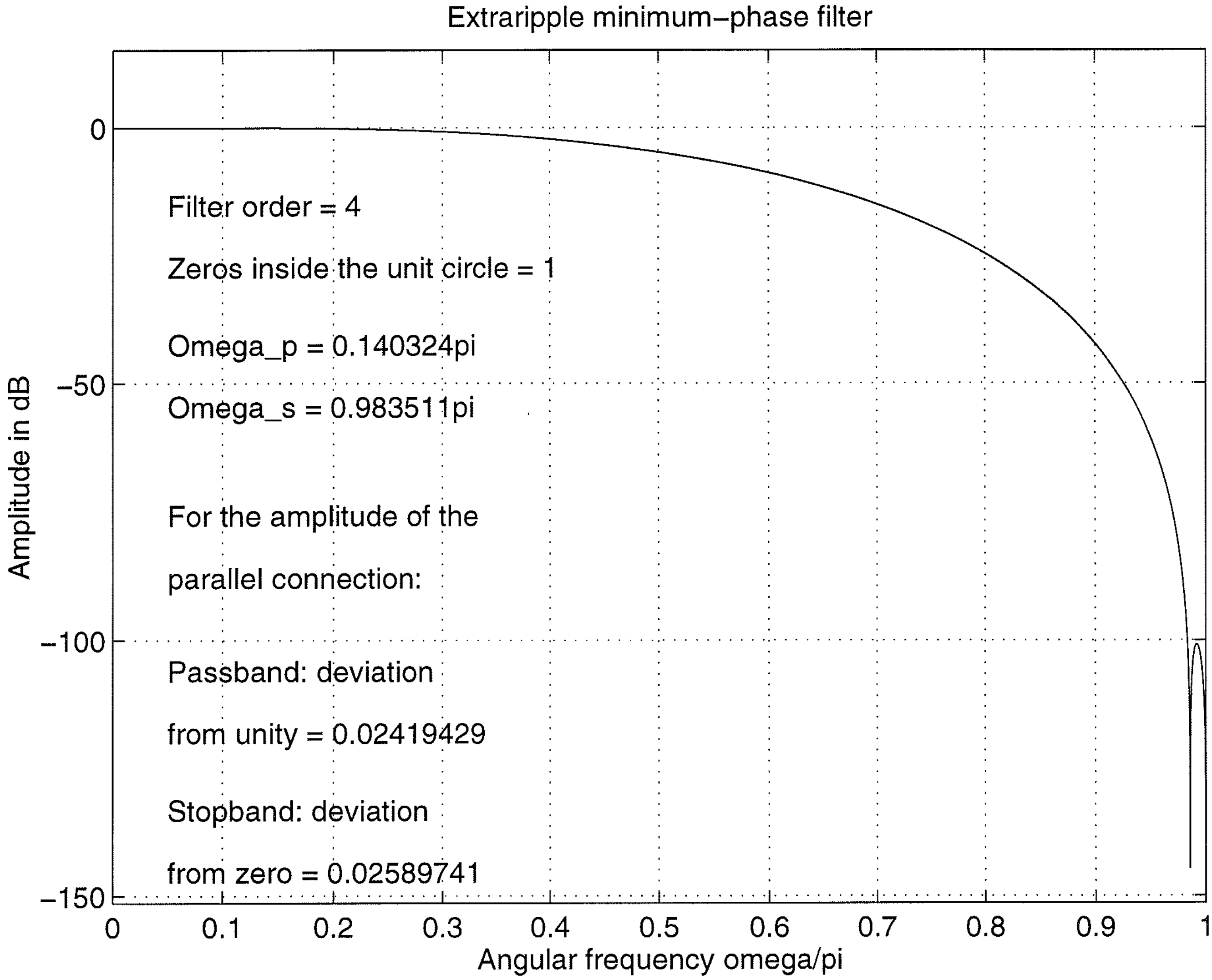
- It is desired to use $N = 4$ subfilters and meet the given criteria as well as possible such that $\delta_p/\delta_s = 10$.
- For $\delta_p = 0.01$ and $\delta_s = 0.001$, in the $L + 1 = 2$ case $F(z)$ meets the given criteria with the minimum odd order equal to 7 (the orders of $A(z)$ and $B(z)$ are 4 and 3, respectively).
- $F(z)$ meets just the given criteria for $\delta_p = 0.0076$ and $\delta_s = 0.00076$.
- The introductory lowpass example corresponds to this case.
- Furthermore, the example illustrating the design of $G(w)$ corresponds to this case.

Example 2: Lowpass filter: $\omega_p = 0.4\pi$, $\omega_s = 0.42$, $\delta_p \leq 0.001$, $\delta_s \leq 0.00001$

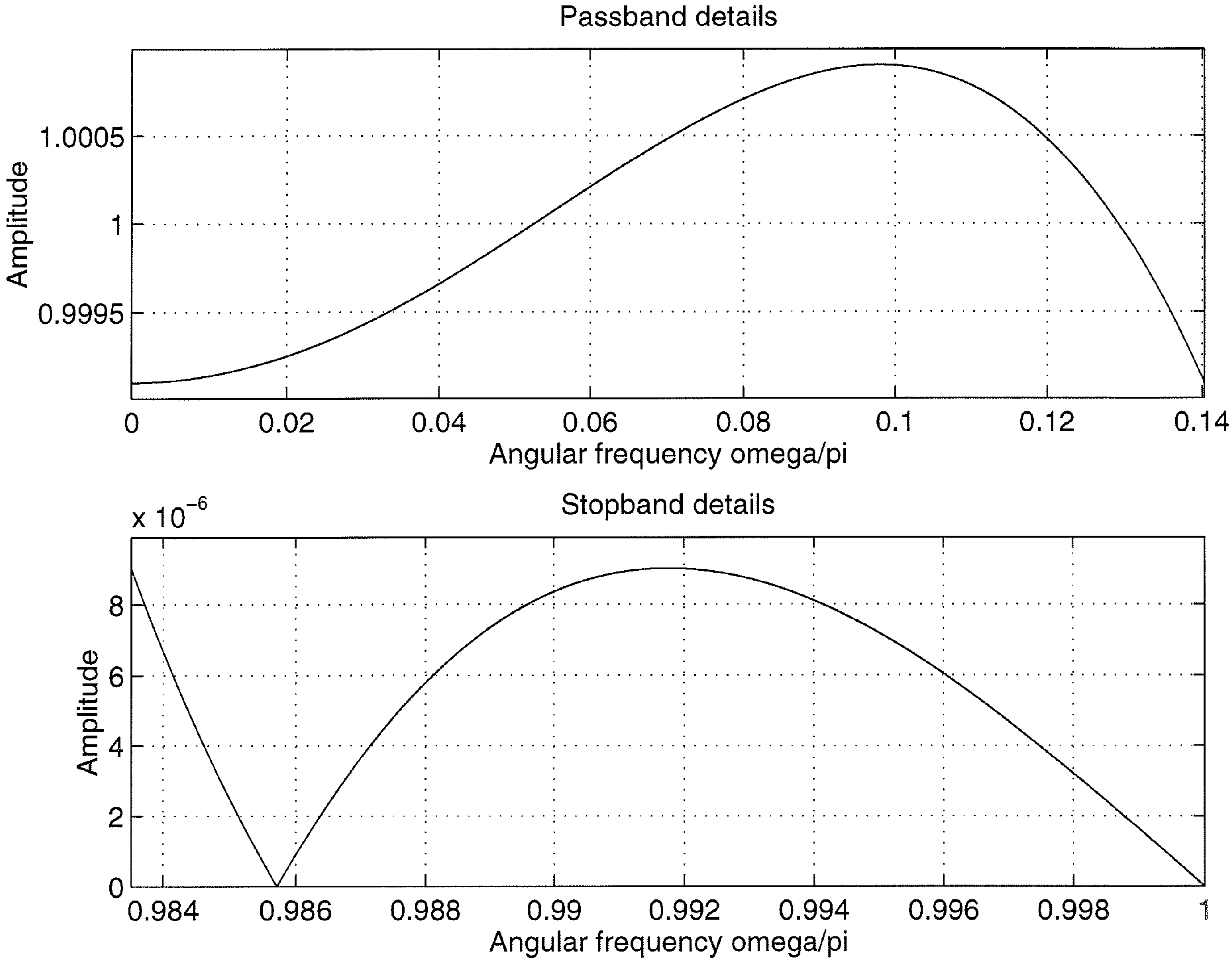
- It is desired to design an overall filter meeting these criteria without general multipliers.
- In order to achieve easily implementable tap coefficients for the second structure of page 5, we use first $\delta_p = 0.0009$ and $\delta_s = 0.000009$.
- In this case, a good selection is $N = 4$ and $L+1 = 2$ in the file extramin.m.
- The following three transparencies give the results.
- In this case, $G(w)$ is expressible as

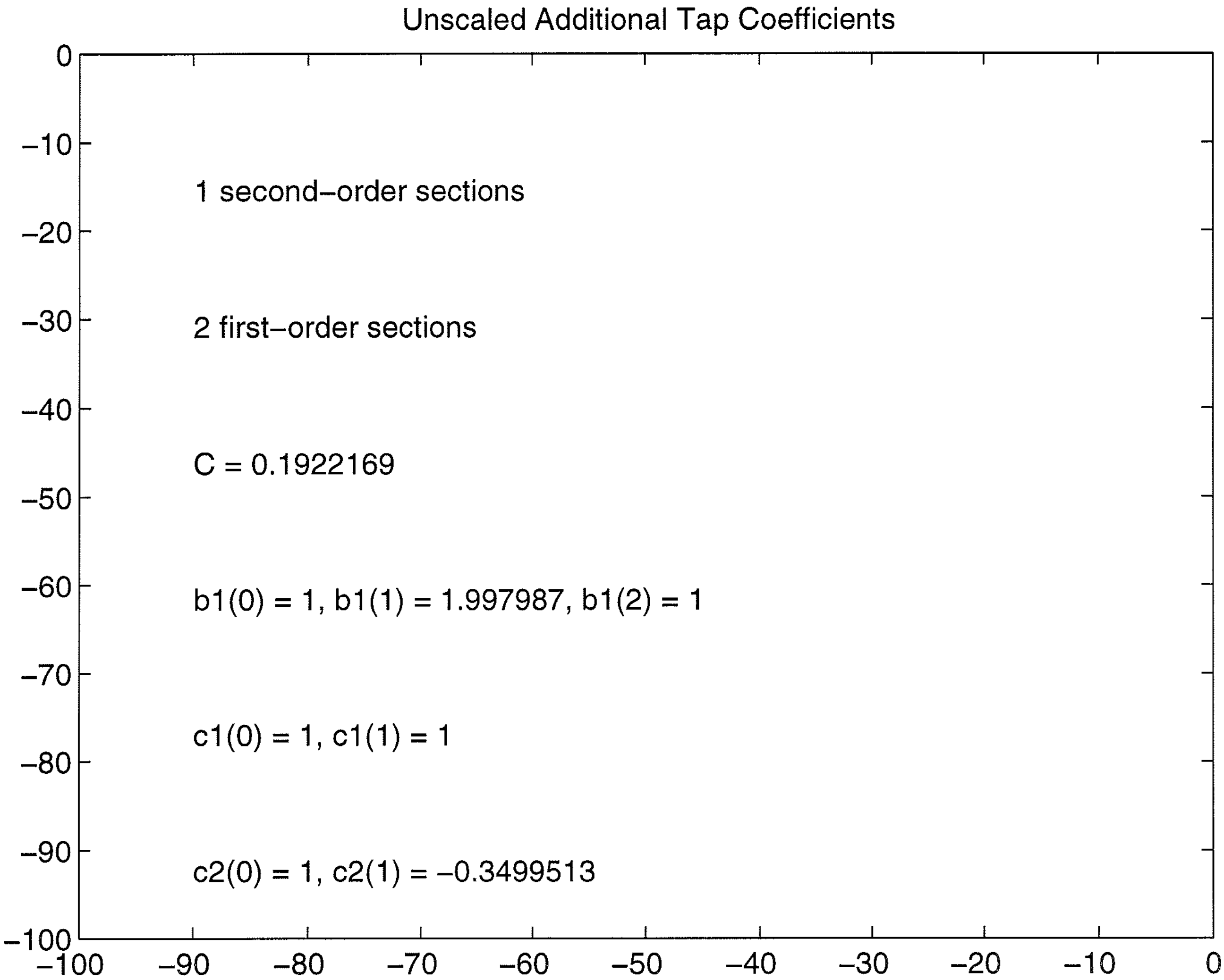
$$G(w) = [2^{-2}(1 + w^{-2}) + d_0][2^{-1}(1 + w^{-1})][d_1 + d_2 w^{-1}], \tag{39}$$

  where $d_0 = 0.49949677$, $d_1 = 1.5377355$, and $d_2 = -0.53813250$.
- Here, the blocks of the second structure of page 5 are scaled properly and the scaling constant C is included in the last term ($d_0 = b_1(1)/4$, $d_1 = 8C$, $d_2 = 8C * c_2(1)$).



Extraripple minimum-phase filter
Filter order = 4
Zeros inside the unit circle = 1
Omega_p = 0.140324pi
Omega_s = 0.983511pi
For the amplitude of the
parallel connection:
Passband: deviation
from unity = 0.02419429
Stopband: deviation
from zero = 0.02589741
Amplitude in dB
0
−50
−100
−150
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Angular frequency omega/pi


Passband details
1.0005
1
0.9995
Amplitude
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
Angular frequency omega/pi
Stopband details
x 10^-6
8
6
4
2
0
Amplitude
0.984
0.986
0.988
0.99
0.992
0.994
0.996
0.998
1
Angular frequency omega/pi


Unscaled Additional Tap Coefficients
1 second–order sections
2 first–order sections
C = 0.1922169
b1(0) = 1, b1(1) = 1.997987, b1(2) = 1
c1(0) = 1, c1(1) = 1
c2(0) = 1, c2(1) = –0.3499513
0
–10
–20
–30
–40
–50
–60
–70
–80
–90
–100
–100
–90
–80
–70
–60
–50
–40
–30
–20
–10
0

Example 2: Quantized tap coefficients

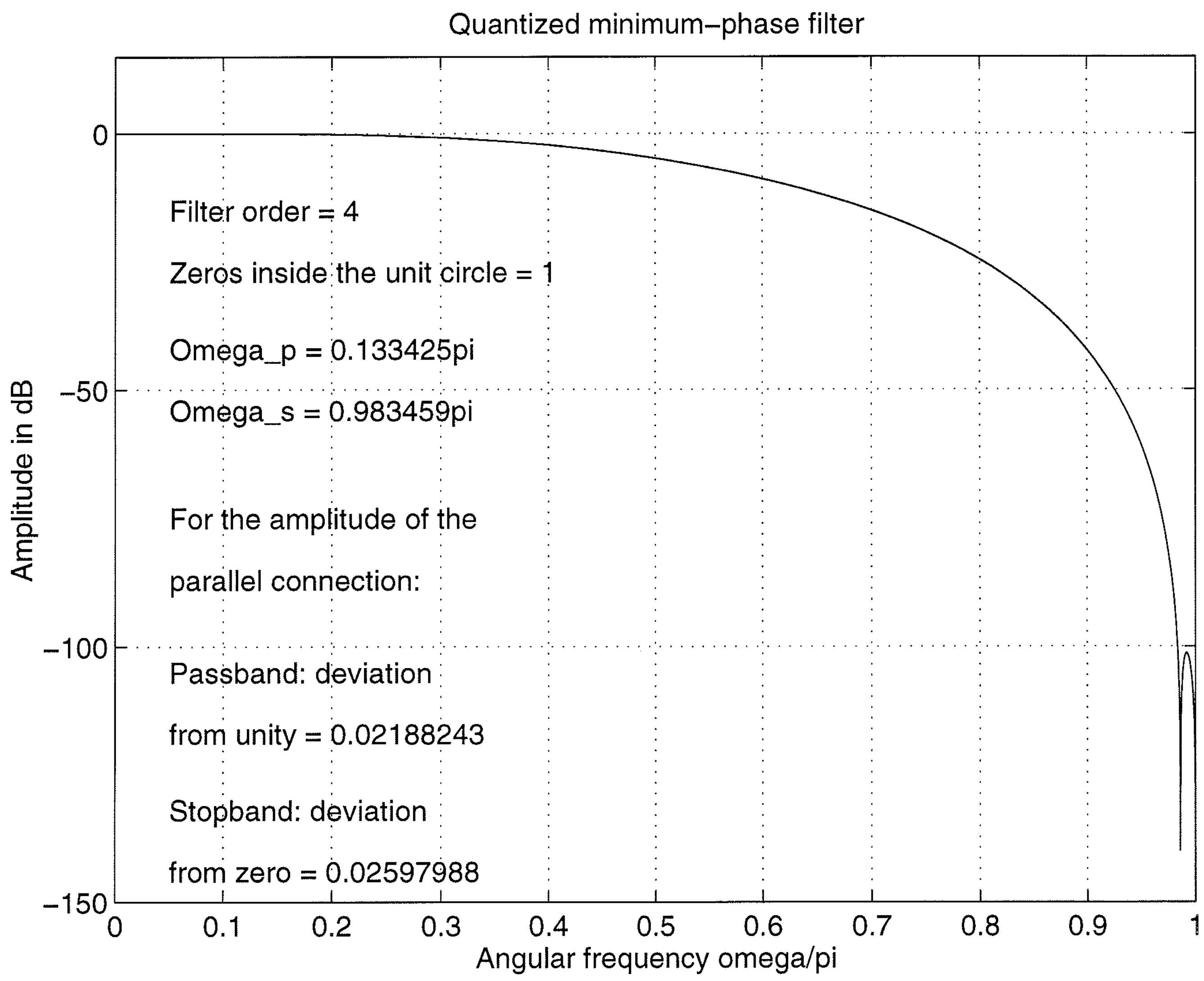
- $G(w)$ meets still the ripple requirements after quantizing d_0, d_1, and d_2 in Eq. (39) to values

$$d_0 = 2^{-1} - 2^{-11},$$

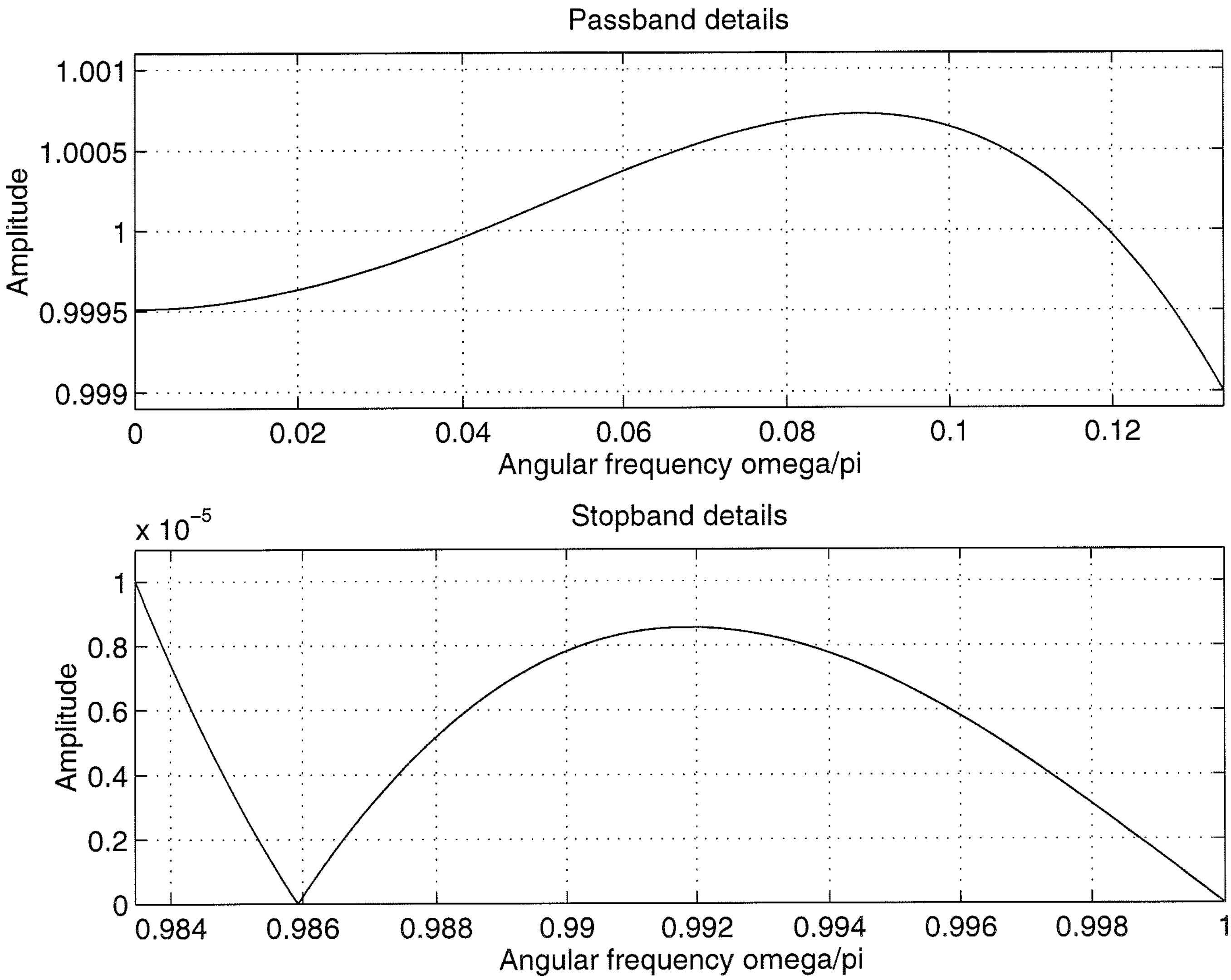
$$d_1 = 2^0 + 2^{-1} + 2^{-5},$$

$$d_2 = -2^{-1} - 2^{-5}.$$

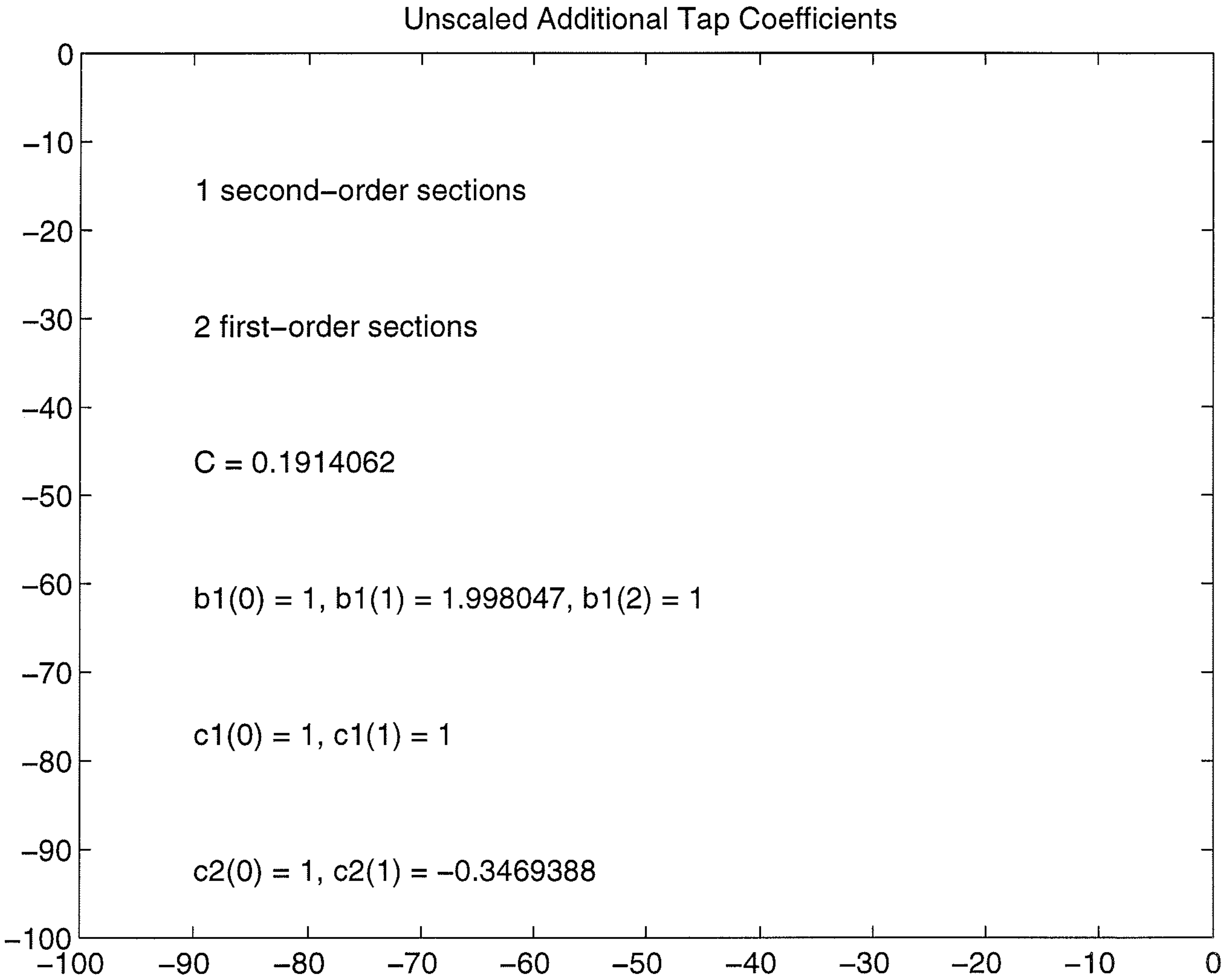
- The following three transparencies show the characteristics of this $G(w)$. These results are obtained by using extramex.m
- For $F(z) = [A(z) + B(z)]/2$, the maximum allowable deviation of the amplitude response from unity in the passband is $\widehat{\delta}_p = 0.021882428$. The maximum stopband deviation is $\widehat{\delta}_s = 0.025979878$. The passband and stopband ripples are thus 0.19217878 dB and 31.707258 dB, respectively.
- The remaining problem is to find $A(z)$ and $B(z)$ in such a way that they require no general multipliers. This will be considered next.



Quantized minimum−phase filter
Filter order = 4
Zeros inside the unit circle = 1
Omega_p = 0.133425pi
Omega_s = 0.983459pi
For the amplitude of the
parallel connection:
Passband: deviation
from unity = 0.02188243
Stopband: deviation
from zero = 0.02597988
Amplitude in dB
0
−50
−100
−150
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Angular frequency omega/pi



Passband details
Amplitude
1.001
1.0005
1
0.9995
0.999
0
0.02
0.04
0.06
0.08
0.1
0.12
Angular frequency omega/pi
Stopband details
x 10^-5
Amplitude
1
0.8
0.6
0.4
0.2
0
0.984
0.986
0.988
0.99
0.992
0.994
0.996
0.998
1
Angular frequency omega/pi



Unscaled Additional Tap Coefficients
1 second–order sections
2 first–order sections
C = 0.1914062
b1(0) = 1, b1(1) = 1.998047, b1(2) = 1
c1(0) = 1, c1(1) = 1
c2(0) = 1, c2(1) = –0.3469388
0
–10
–20
–30
–40
–50
–60
–70
–80
–90
–100
–100
–90
–80
–70
–60
–50
–40
–30
–20
–10
0

Example 2: Allpass filters with quantized coefficient values

- The criteria for $F(z)$ are met by an elliptic filter of order 7. Hence, $A(z)$ is of order 4 and $B(z)$ is of order 3.
- Using wave lattice filters and a simple quantization scheme to be described later on in these lecture notes, the given criteria are met by the following $A(z)$ and $B(z)$:
- $A(z)$ contains two second-order sections with adaptor coefficients

$$\gamma_1 = -2^{-1} + 2^{-5} + 2^{-6}, \;\; \gamma_2 = 2^{-1} - 2^{-6} - 2^{-7}$$

  and

$$\gamma_1 = -2^{0} + 2^{-5} + 2^{-7}, \;\; \gamma_2 = 2^{-2} + 2^{-5} + 2^{-6},$$

  respectively.
- $B(z)$ consists of one first-order section with the adaptor coefficient

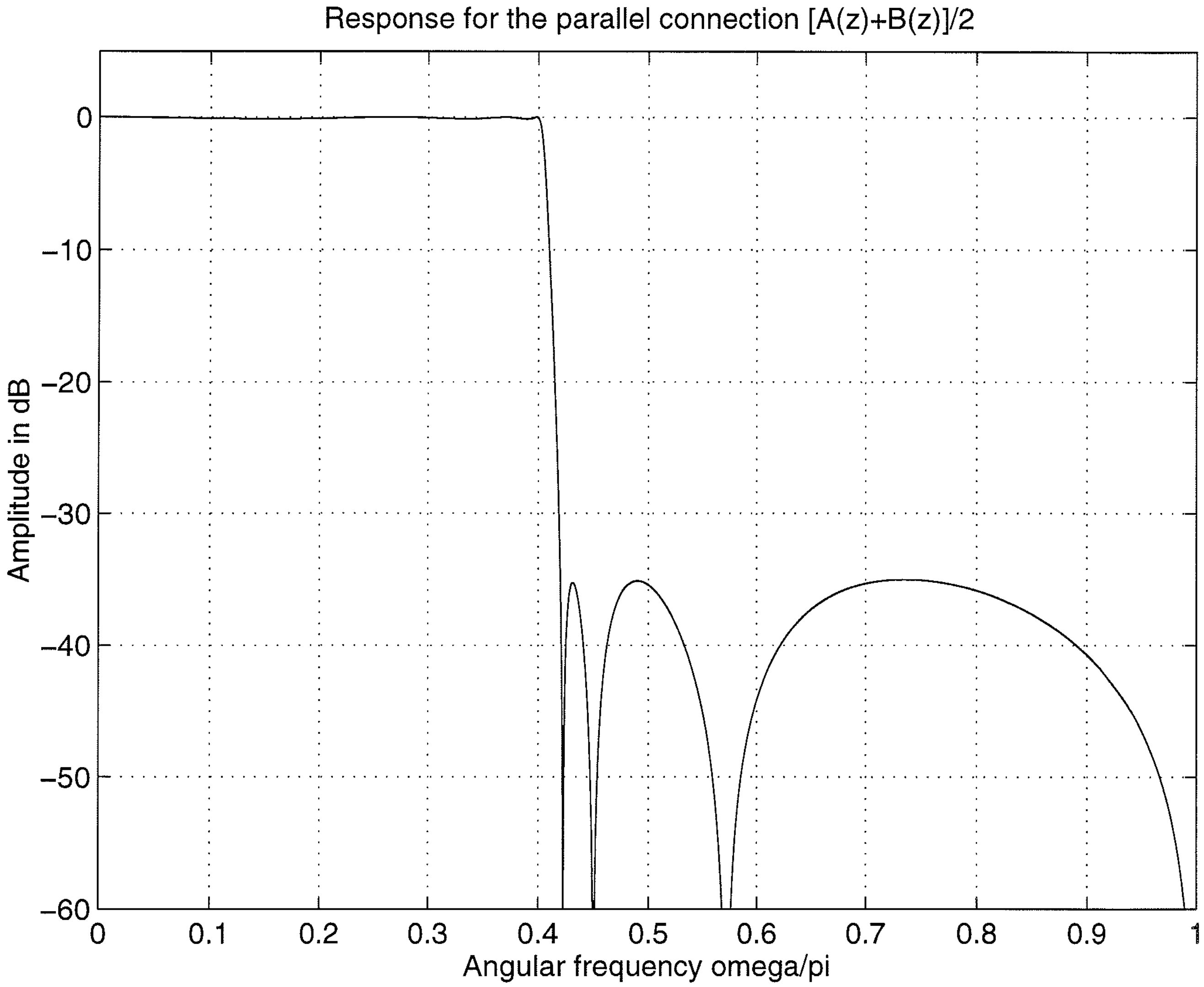
$$\gamma = 2^{-2} + 2^{-3} + 2^{-7}$$

  and one second-order section with adaptor coefficients

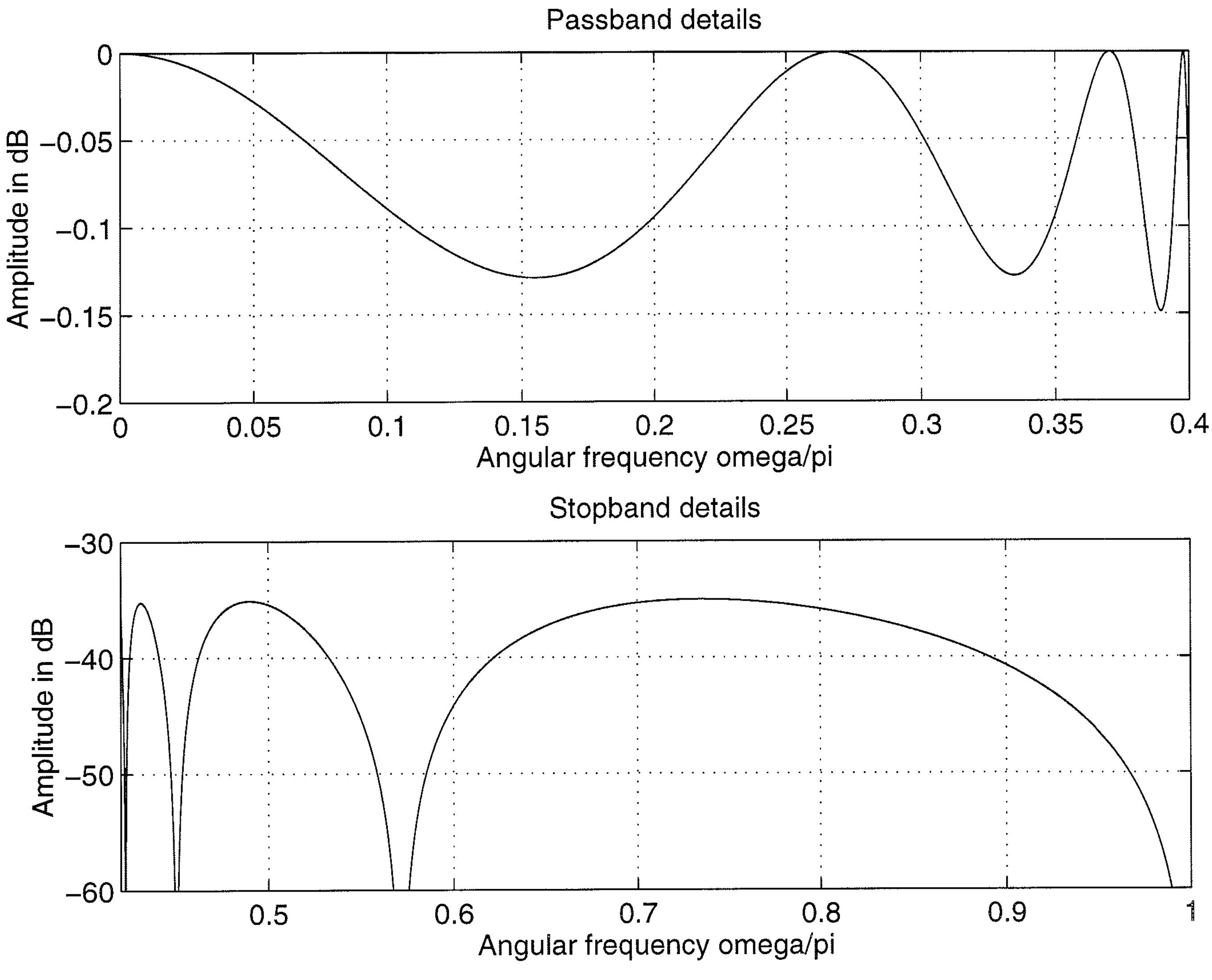
$$\gamma_1 = -2^{-1} - 2^{-2} - 2^{-4},$$

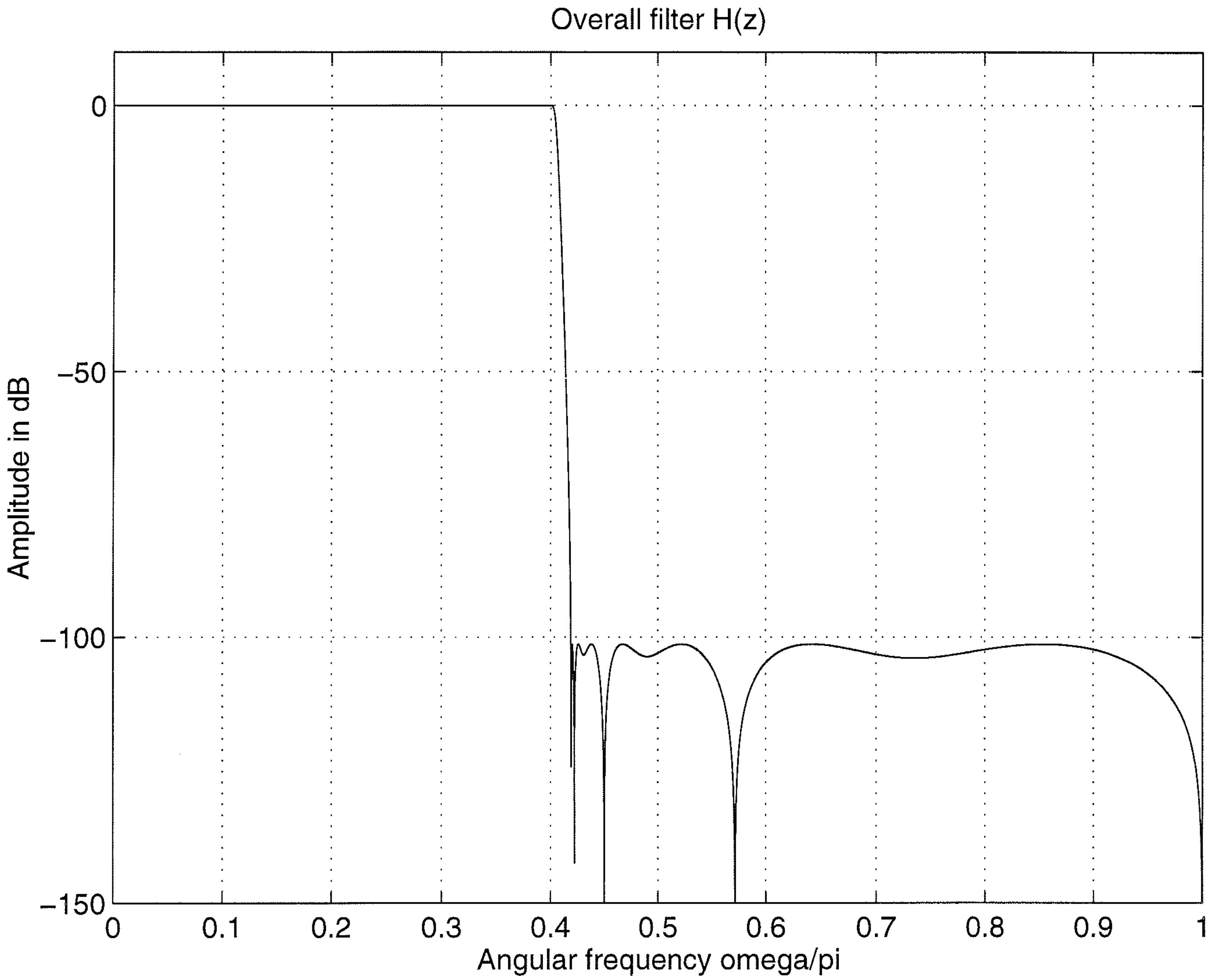
$$\gamma_2 = 2^{-2} + 2^{-4} + 2^{-6} + 2^{-7}.$$

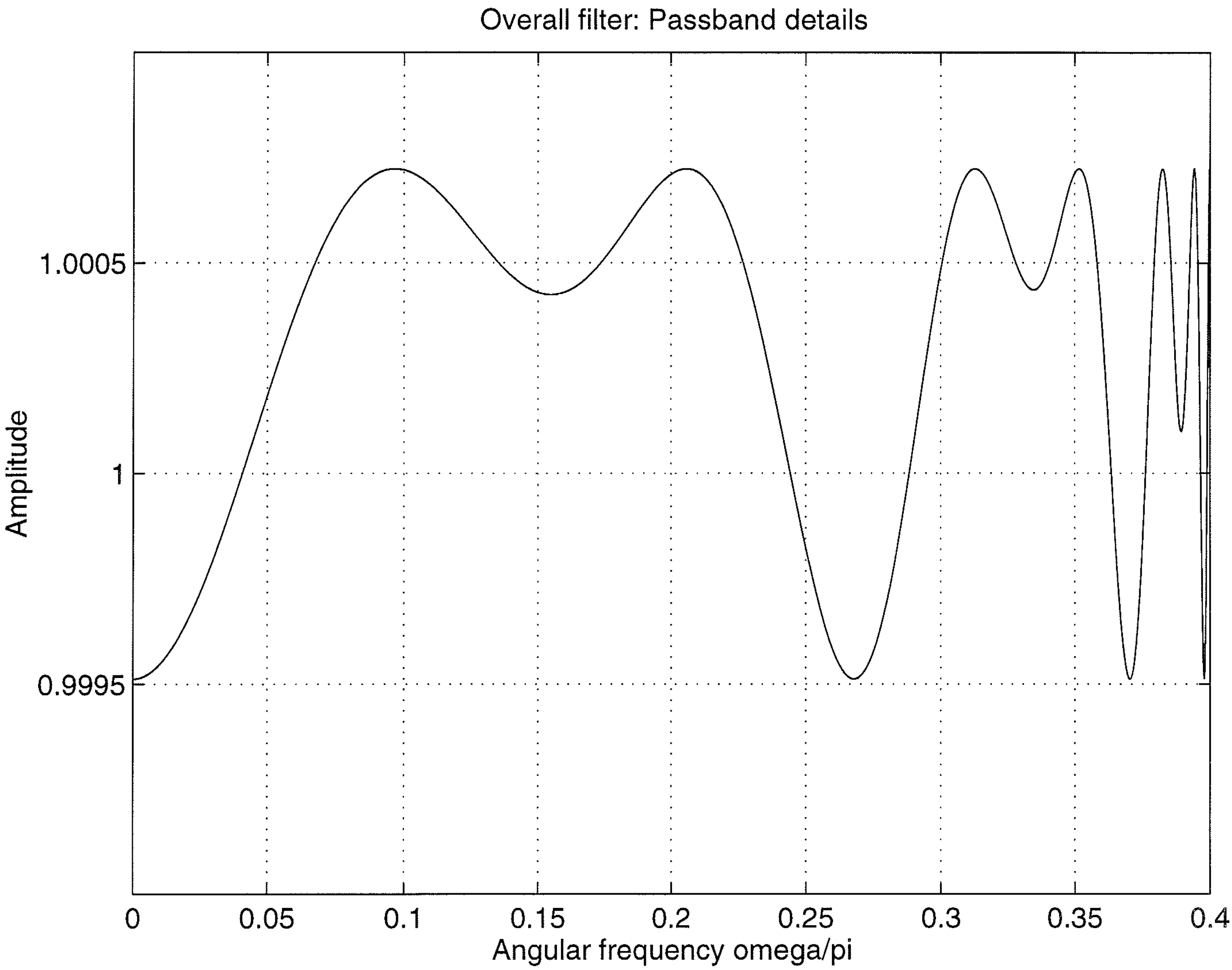
- The following four transparencies show the amplitude responses of $F(z) = [A(z) + B(z)]/2$ and the overall filter.
- These plots can be generated by first using extramex.m and then suballe.m. Please try.
- The plots for our introductory examples can be generated by first using extramin.m with $\delta_p = 0.0076$, $\delta_s = 0.00076$, number of passband extrema $= 2$, and number of subfilters $= 4$. Then use files suball1.m, suball2.m, suball3.m, and suball4.m.



Response for the parallel connection [A(z)+B(z)]/2
Amplitude in dB
0
−10
−20
−30
−40
−50
−60
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Angular frequency omega/pi


Passband details
Amplitude in dB
0
−0.05
−0.1
−0.15
−0.2
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
Angular frequency omega/pi
Stopband details
Amplitude in dB
−30
−40
−50
−60
0.5
0.6
0.7
0.8
0.9
1
Angular frequency omega/pi


Overall filter H(z)
Amplitude in dB
0
−50
−100
−150
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Angular frequency omega/pi


Overall filter: Passband details
Amplitude
1.0005
1
0.9995
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
Angular frequency omega/pi

Example 2: Comparison with the direct parallel connection of two allpass filters

- If a direct parallel connection of two allpass filters is used, then the minimum odd order of an elliptic filter to meet the overall criteria is 17.
- In this case, $A(z)$ is of order 9 and $B(z)$ is of order 8.
- The first disadvantage of this direct design compared to the proposed design is that it requires 18 bits for the coefficient representation (including the sign bit).
- The second disadvantage is that for the direct design the radius of the outermost pole is approximately 0.9948, whereas that of the subfilter is 0.9802.
- This means that the multiplication roundoff noise generated by the pure elliptic filter is significantly higher and it requires a significantly longer internal data wordlength.
- Furthermore, the proposed design requires only 7 distinct multipliers, whereas the elliptic design has 17 multipliers.

- This fact can be exploited by using multiplexing and implementing $A(z)$ and $B(z)$ only once.
- The overall order (number of delays in the overall implementation) of the proposed design is higher than that of the direct allpass connection ($4 \cdot 7 = 28$ compared to 17).
- For more examples, see the enclosed article.
- OK, it seems that you like to see one more example!

Example 3: Lowpass filter: $\omega_p = 0.1\pi$, $\omega_s = 0.2$, $\delta_p \leq 0.05$, $\delta_s \leq 0.0001$

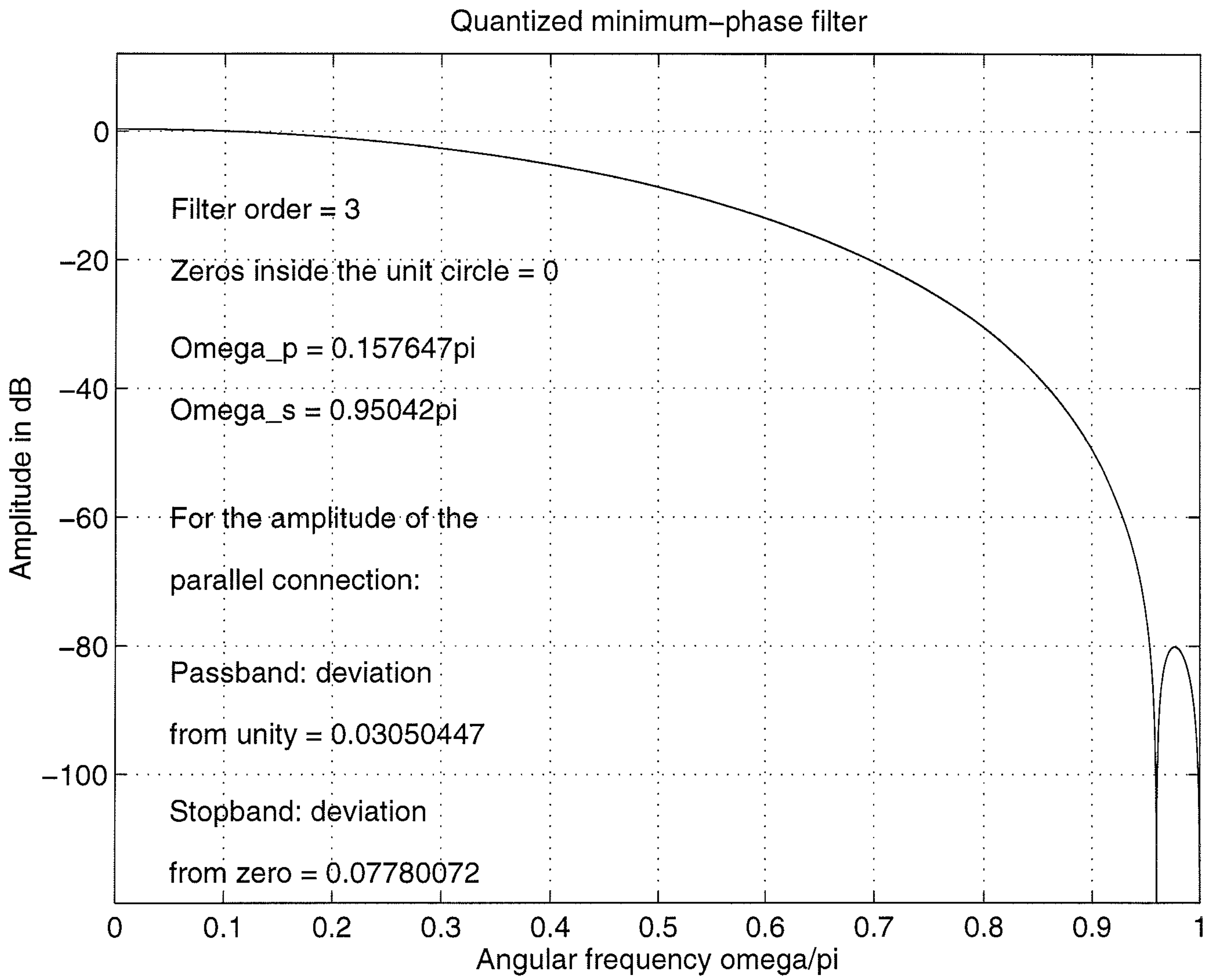
- In this case, $G(w)$ with quantized coefficients can be written in the following simple form:

  $$G(w) = C[2^{-1}(1+w^{-1})][2^{-2}(1+w^{-2})+(2^{-1}-2^{-8})w^{-1}],$$

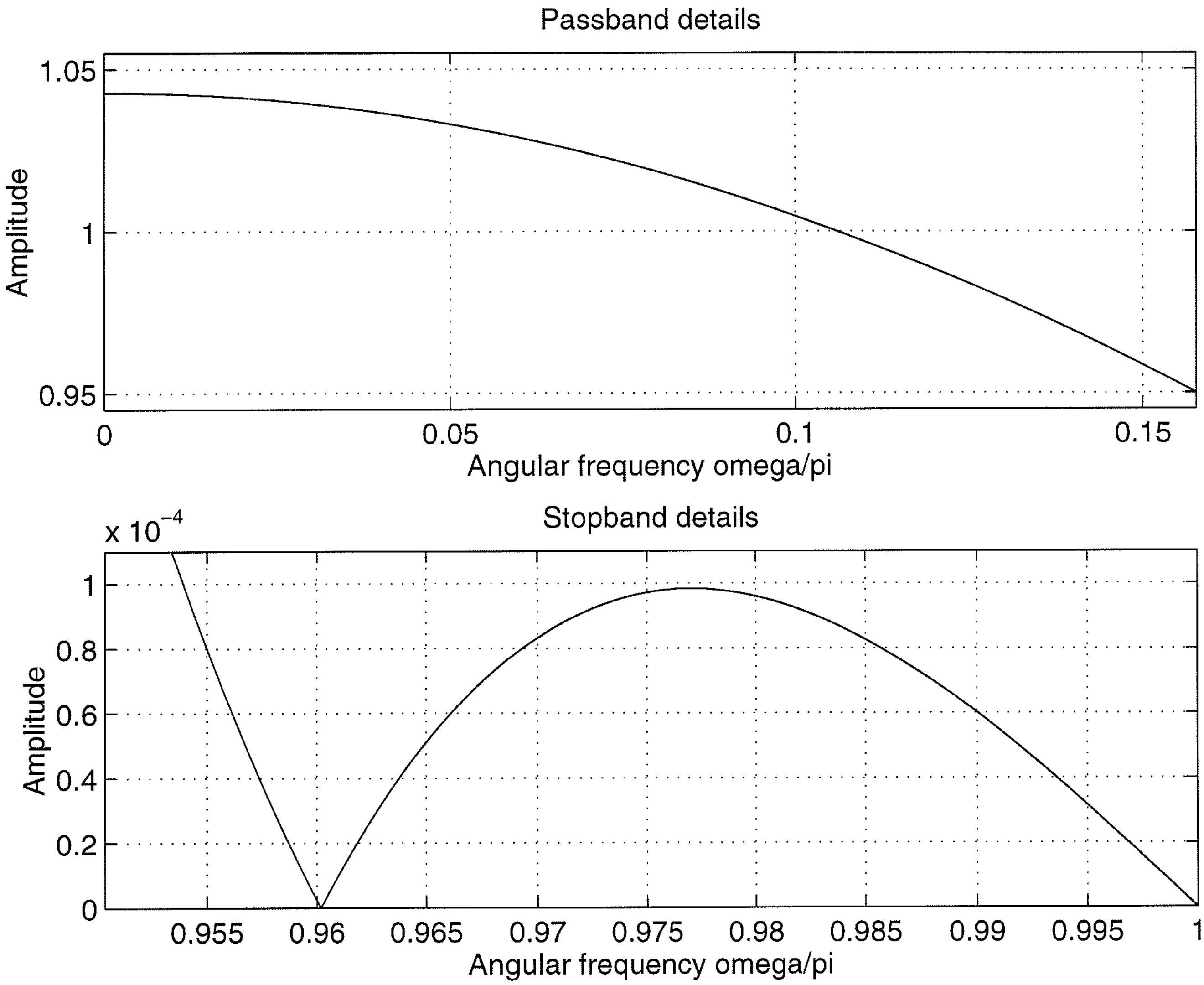
  where
  $$C = 2^0 + 2^{-4} - 2^{-6}.$$

- The following three transparencies show the characteristics of this $G(w)$. These results are obtained by using extramexx.m
- For $F(z) = [A(z) + B(z)]/2$, the maximum allowable deviation of the amplitude response from unity in the passband is $\widehat{\delta_p} = 0.030504468$. The maximum stopband deviation is $\widehat{\delta_s} = 0.077800720$. The passband and stopband ripples are thus 0.26908376 dB and 22.180327 dB, respectively.
- These criteria are met by $A(z)$ of order 2 and $B(z)$ of order 1 (the order of $F(z)$ is 3). The number of bits for the coefficient representation is 6.
- For the direct parallel connection the minimum

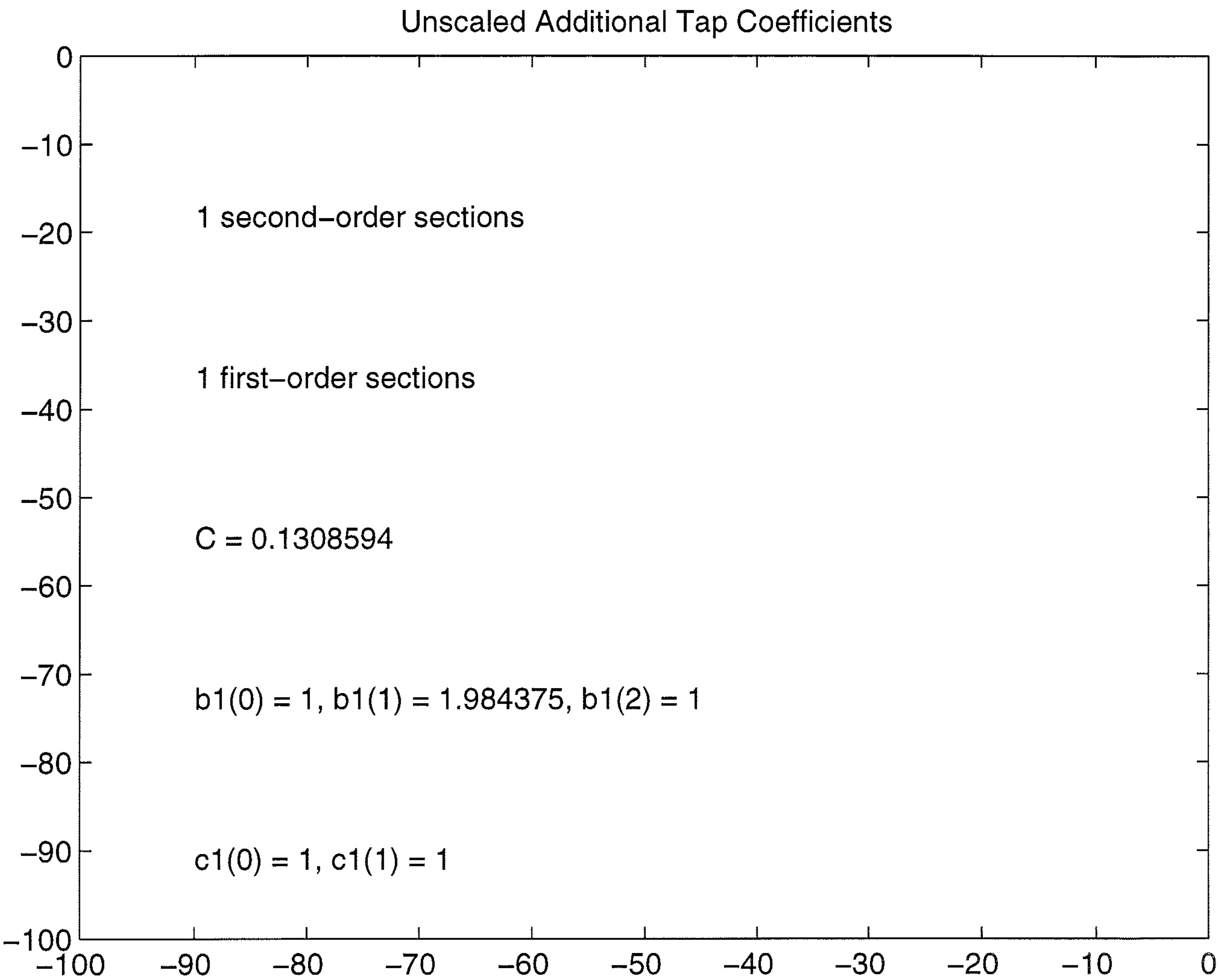
odd order is 7 and 14 bits are required for the coefficient representation.



Quantized minimum−phase filter
Filter order = 3
Zeros inside the unit circle = 0
Omega_p = 0.157647pi
Omega_s = 0.95042pi
For the amplitude of the
parallel connection:
Passband: deviation
from unity = 0.03050447
Stopband: deviation
from zero = 0.07780072
Amplitude in dB
0
−20
−40
−60
−80
−100
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Angular frequency omega/pi



Passband details
1.05
1
0.95
Amplitude
0
0.05
0.1
0.15
Angular frequency omega/pi
x 10^-4
Stopband details
1
0.8
0.6
0.4
0.2
0
Amplitude
0.955
0.96
0.965
0.97
0.975
0.98
0.985
0.99
0.995
1
Angular frequency omega/pi


Unscaled Additional Tap Coefficients
1 second–order sections
1 first–order sections
C = 0.1308594
b1(0) = 1, b1(1) = 1.984375, b1(2) = 1
c1(0) = 1, c1(1) = 1
0
–10
–20
–30
–40
–50
–60
–70
–80
–90
–100
–100
–90
–80
–70
–60
–50
–40
–30
–20
–10
0

A Novel Approach for the Design of IIR Filters as a Tapped Cascaded Interconnection of Identical Allpass Subfilters

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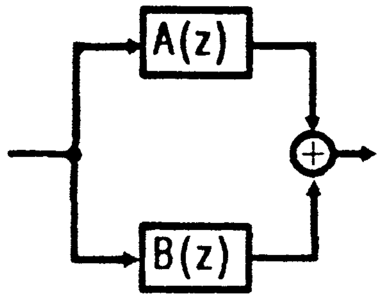
BACKGROUND

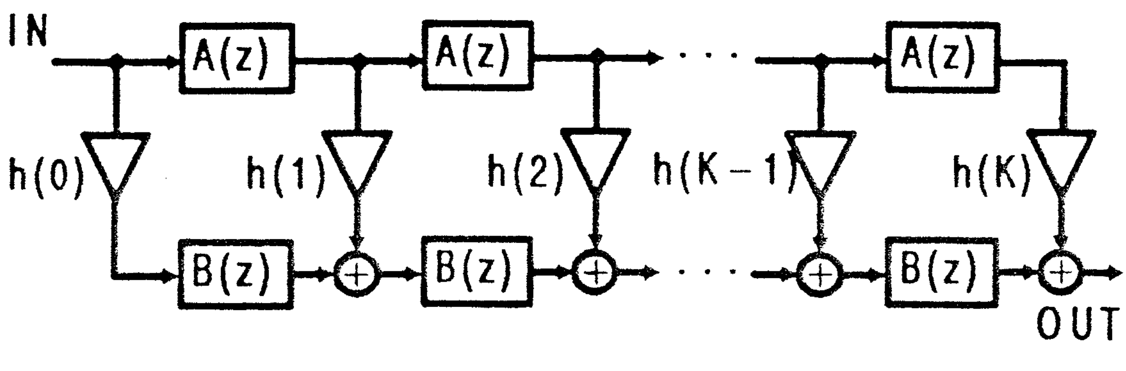
Realization of digital filters as a parallel connection of two allpass filters (Lattice WDF's)



- Good finite wordlength properties
- Stopband sensitivity may be a problem when high stopband attenuation is required.

Better stopband sensitivity can be achieved by using

- a cascade of lower order filters or
- tapped cascaded structure:



Advantages:

- Better finite wordlength performance
- Implementation using multiplexed hardware

Earlier work:

Sharpening the response of FIR filters

- Kaiser and Hamming [1977]
- Nakamura and Mitra [1982]
- Saramäki [1984]

IIR filters

- Claasen and Mecklenbräuker [1979]

DERIVING THE FILTER STRUCTURE

Prototype nonlinear phase FIR filter

$$\widetilde{H}(w) = \sum_{n=0}^{K} h(n) w^{-n}$$

Substitution

$$w^{-1} = \frac{A(z)}{B(z)}$$

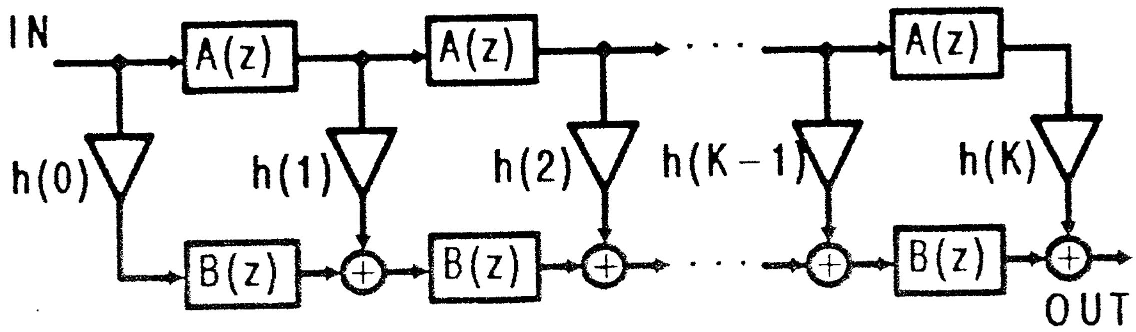
gives

$$H'(z) = \sum_{n=0}^{K} h(n) [\frac{A(z)}{B(z)}]^{n}$$

where $A(z)$ and $B(z)$ are stable allpass filters.

The proposed filter

$$H(z) = [B(z)]^{K} H'(z) = \sum_{n=0}^{K} h(n) [A(z)]^{n} [B(z)]^{K-n}$$



FREQUENCY RESPONSE OF THE PROPOSED FILTER

$$H'(z) = \sum_{n=0}^{K} h(n)[\frac{A(z)}{B(z)}]^{n}$$

$$H'(e^{j\omega}) = \sum_{n=0}^{K} h(n)e^{-jn(f_B(\omega)-f_A(\omega))}$$

where

$$f_A(\omega) = \arg[A(e^{j\omega})] \qquad f_B(\omega) = \arg[B(e^{j\omega})]$$

$|H(e^{j\omega})|$ is obtained from

$$|\widetilde{H}(e^{j\Omega})| = |\sum_{n=0}^{K} h(n)e^{-jn\Omega}|$$

through the frequency transformation

$$\Omega = f_B(\omega) - f_A(\omega)$$

SIMULTANEOUS FREQUENCY-DOMAIN CONDITIONS FOR THE ALLPASS SECTIONS AND PROTOTYPE FILTER

Overall specifications:

$$1-\delta_p \le |H(e^{j\omega})| \le 1+\delta_p \quad \text{for} \quad \omega \in I_p$$

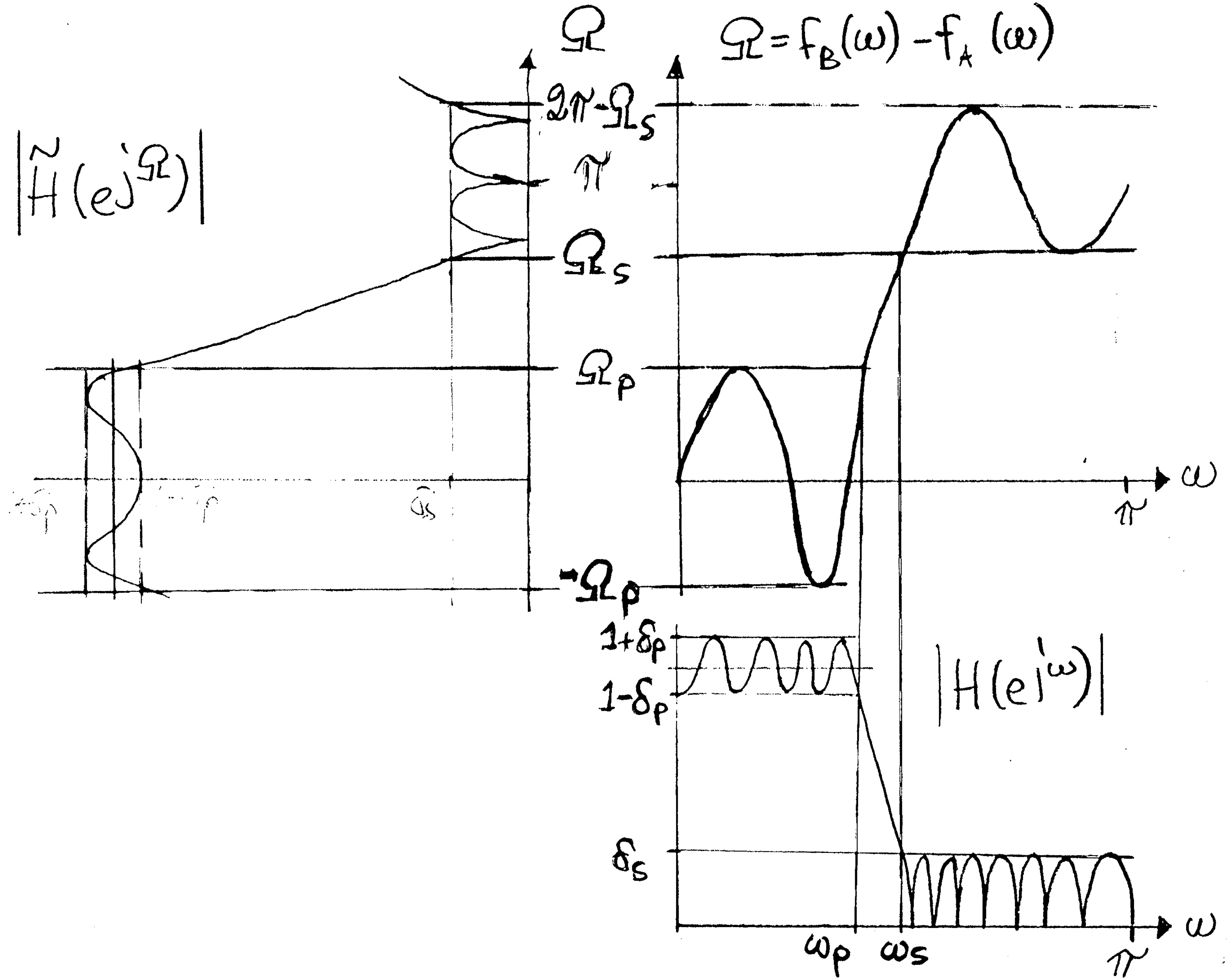
$$|H(e^{j\omega})| \le \delta_s \quad \text{for} \quad \omega \in I_s$$

Specifications for the lowpass nonlinear phase FIR filter:

$$1-\delta_p \le |\widetilde{H}(e^{j\Omega})| \le 1+\delta_p \quad \text{for} \quad 0 \le \Omega \le \Omega_p$$

$$|\widetilde{H}(e^{j\Omega})| \le \delta_s \quad \text{for} \quad \Omega_s \le \Omega \le \pi$$

The parameters Ω_p and Ω_s can be chosen freely



$|\tilde{H}(e^{j\Omega})|$
Ω
$2\pi-\Omega_s$
π
Ω_s
Ω_p
$-\Omega_p$
$\Omega = f_B(\omega) - f_A(\omega)$
$1+\delta_p$
$1-\delta_p$
$|H(e^{j\omega})|$
δ_s
ω_p
ω_s
π
ω

Specifications for the allpass filters:

$$f_B(\omega) - f_A(\omega) \in [2r\pi - \Omega_p,\ 2r\pi + \Omega_p] \quad \text{for } \omega \in I_p$$

$$f_B(\omega) - f_A(\omega) \in [2r\pi + \Omega_s\ ,\ 2r\pi + (2\pi - \Omega_s)] \quad \text{for } \omega \in I_s$$

Equivalent specifications using the amplitude response of the parallel connection of the all-pass filters:

$$\begin{aligned} |F(e^{j\omega})| &= \frac{1}{2}|[A(e^{j\omega}) + B(e^{j\omega})]| \\ &= |\cos\{[f_B(\omega) - f_A(\omega)]/2\}| \end{aligned}$$

$$1 - \widehat{\delta}_p \leq |F(e^{j\omega})| \leq 1 \quad \text{for} \quad \omega \in I_p$$

$$|F(e^{j\omega})| \leq \widehat{\delta}_s \quad \text{for} \quad \omega \in I_s$$

where

$$\widehat{\delta}_p = 1 - \cos(\Omega_p/2), \quad \widehat{\delta}_s = \cos(\Omega_s/2)$$

DESIGN OF THE ALLPASS SECTIONS

- Odd-order elliptic lowpass and highpass filters
- Bandpass and bandstop filters using frequency transformations

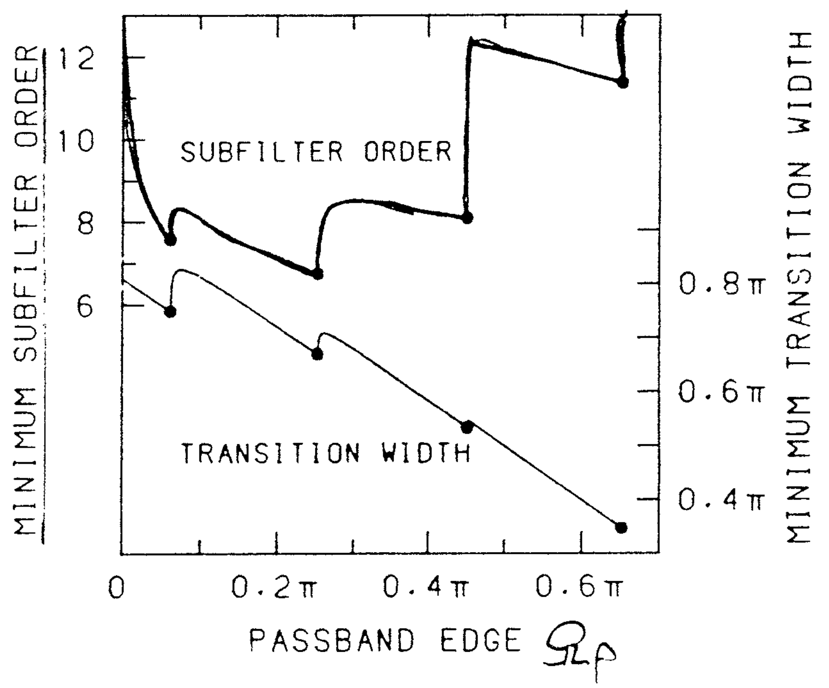
DESIGN OF THE TAP COEFFICIENTS

Design of a nonlinear phase FIR filter of order K

- Can be converted into the design of a linear phase filter of order $2K$
- Factorizing the non-negative zero-phase transfer function into minimum phase and maximum phase transfer functions

(1) Given the number of subfilters, find the minimum subfilter order such that it is implementable using allpass filters

Single-variable optimization problem where Ω_p is the primary unknown



- The local minima of the order estimate correspond to extraripple solutions for the prototype FIR filter
- These extraripple solutions can be found directly using the algorithm of Hofstetter *et al.*

(2) Finding the optimum overall solution

Using the specified values for δ_p and δ_s leaves no margin for coefficient quantization.

The optimum solution minimizes the passband ripple of the overall filter for the given ripple ratio.

It is found by reducing the passband and stopband ripples of the best extraripple prototype FIR filter until the minimum-order elliptic subfilter still meets the specifications. This is a single-variable optimization problem with respect to δ_p.

DESIGN EXAMPLE

$$\omega_p = 0.3\pi \qquad \omega_s = 0.301\pi$$

$$\delta_p = 0.01 \qquad \delta_s = 0.001 \text{ (60 dB)}$$

$$\omega_p = 0.3\pi \qquad \omega_s = 0.301\pi$$

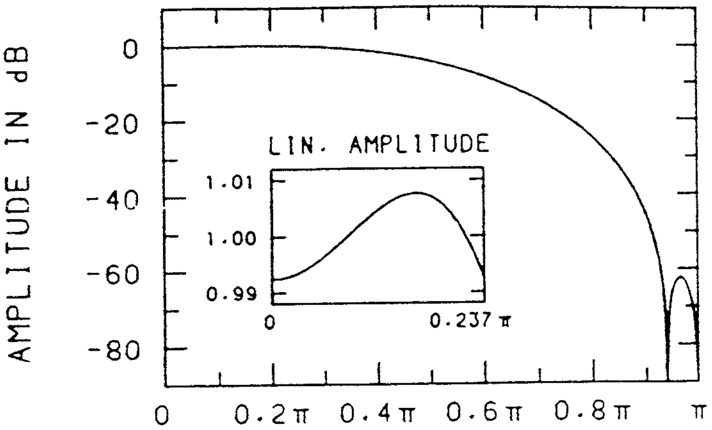
$$\delta_p = 0.01 \qquad \delta_s = 0.001$$

Number of subfilters = 4

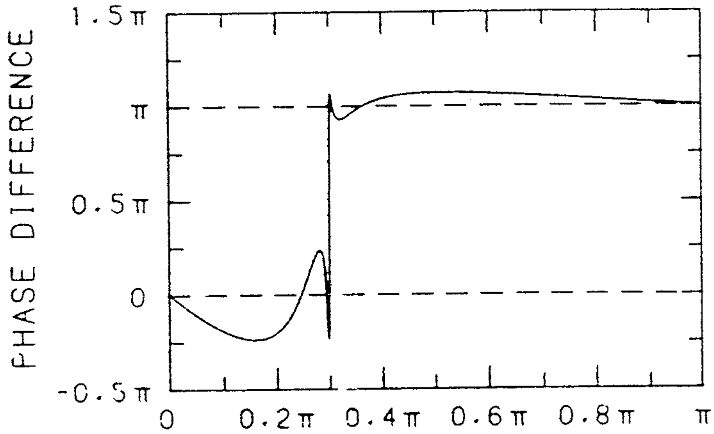
Required Subfilter crder = 7.

DESIGN EXAMPLE

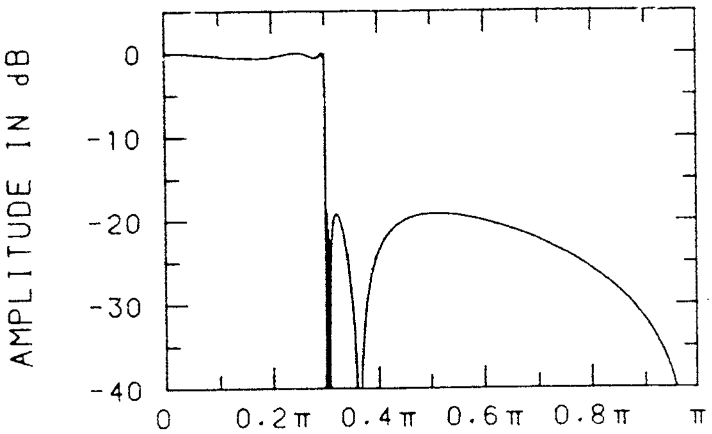
Prototype nonlinear phase FIR filter



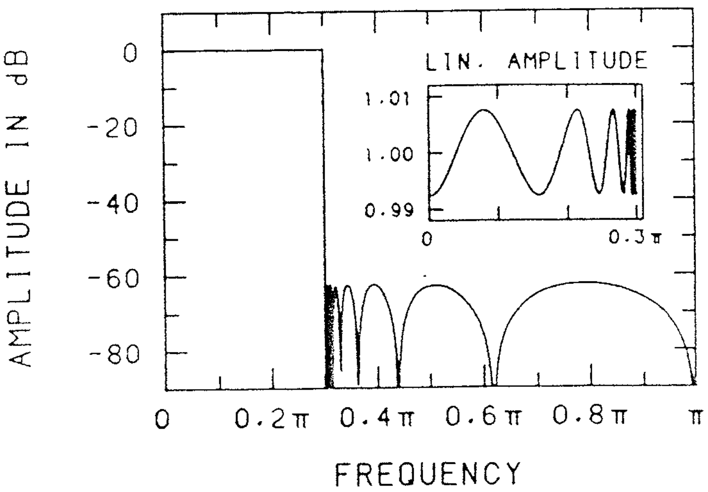
Elliptic subfilter



Elliptic subfilter



Overall filter



EXAMPLE 1

$$\omega_p = 0.4\pi \qquad \omega_s = 0.42\pi$$

$$\delta_p = 0.001 \qquad \delta_s = 0.00001 \ (100 \text{ dB})$$

Order for elliptic filter = 15.3

Subfilter order	Number of subfilters	
	Optimized	Cascade
17	1	1
9	2	3
7	4	6
5	6	Not possible

Coefficient wordlengths:

- 18 bits for direct elliptic design
- 8 bits for the 7th-order elliptic subfilter with $K = 4$

EXAMPLE 2

$$\omega_p = 0.1\pi \qquad \omega_s = 0.2\pi$$

$$\delta_p = 0.05 \qquad \delta_s = 0.0001 \ (80 \text{ dB})$$

Elliptic filter of order 7 requires 14 bit coefficient wordlength.

Design with 3 subfilters

Prototype FIR filter:

$$\widetilde{H}(w) = [2^{-1}(1 + w^{-1})] \cdot [2^{-2}(1 + w^{-2}) + (2^{-1} - 2^{-8})w^{-1}]$$

Elliptic subfilter:

- Minimum order =3
- Parallel connection of a first-order and a second-order allpass filter
- Coefficient wordlength = 6 bits