SGN-1158 Introduction to Signal Processing

Test. Solutions

1. Convolve the function $\Pi(t-\frac{1}{2})$ with itself and show that the Fourier transform of the result is the square of the Fourier transform of $\Pi(t-\frac{1}{2})$. (Hints: sketch the rectangular function, inspect various scenarios for different t: (1. t<0; 2. 0<t<1; 3. 1<t<2; 4. t>2), for each of the case perform convolution operation with properly defined limits of integration:

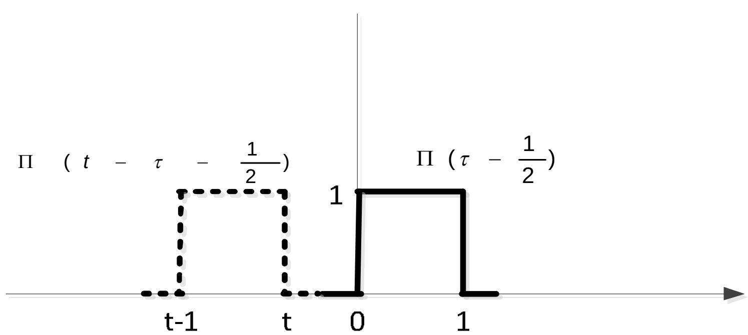
$$z(t) = \Pi\left(t-\frac{1}{2}\right) * \Pi\left(t-\frac{1}{2}\right) = \int_{-\infty}^{\infty} \Pi\left(\tau-\frac{1}{2}\right) * \Pi\left(t-\tau-\frac{1}{2}\right) d\tau,$$

Sketch the output function, find the Fourier transform using the Fourier transform pairs (Hint: $\Lambda(t)$), then check what is the Fourier transform of the rectangular function. Are they the same?)

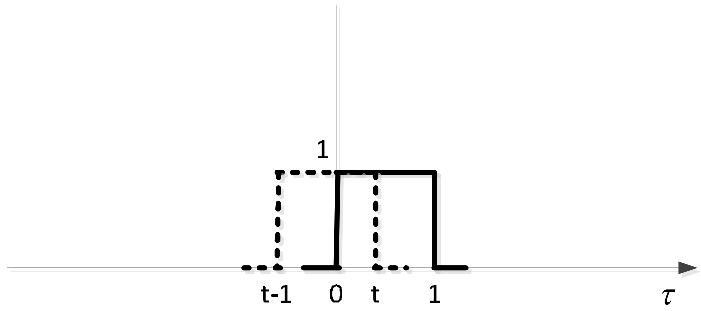
Solution:

The rectangular function can be defined as follows $\mathrm{rect}(t) = \Pi(\mathrm{t}) = \begin{cases} 0, \text{ if } |t| > \frac{1}{2} \\ \frac{1}{2}, \text{ if } |t| = \frac{1}{2} \\ 1, \text{ if } |t| < \frac{1}{2} \end{cases}$, therefore in order to obtain convolution (or to properly define the integral limits) we need to inspect various possible intersections of the shifted rectangular function $\Pi(t-\frac{1}{2})$ and $\Pi\left(t-\tau-\frac{1}{2}\right)$ for different t.

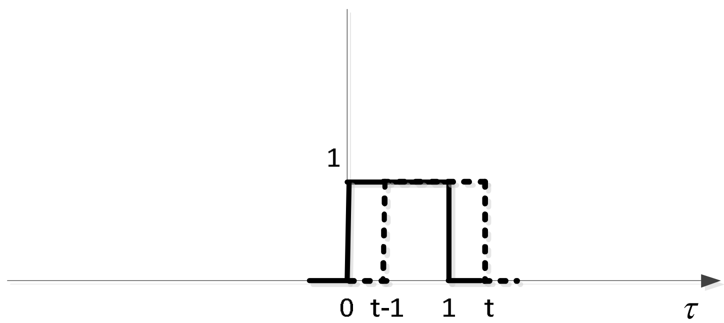
1. t<0, z(t)=0 (no intersection, see figure below)



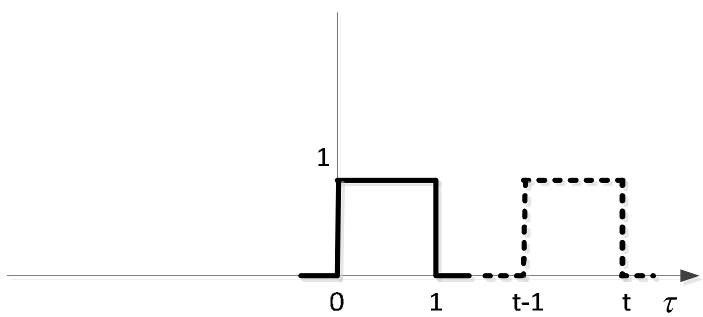
2. 0<t<1, $z(t) = \int_0^t (1\text{x}1) d\tau = [\tau]_0^t = t$.



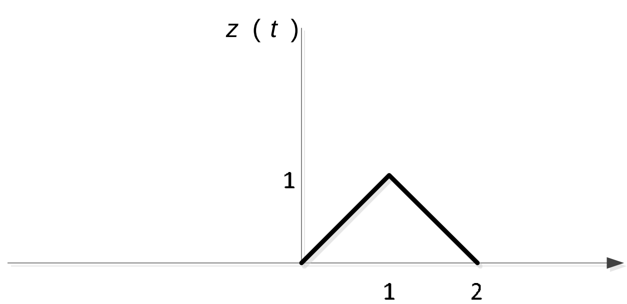
3. 1<t<2, $z(t) = \int_{t-1}^{1}(1\text{x}1)d\tau = [\tau]_{t-1}^{1} = 2 - t$



4. For t>2, *z(t)=0* (no intersection)

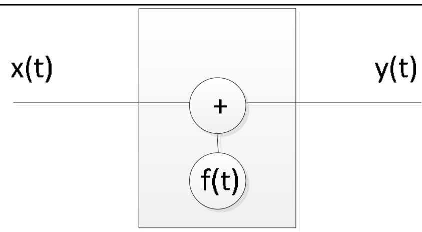


Therefore, the output of the convolution is the triangular function $\Lambda(t-1)$, which is shown in Figure below. (Note that $\Lambda(t) = \begin{cases} 1-|t|, |t|<1 \\ 0, \text{otherwise} \end{cases}$)

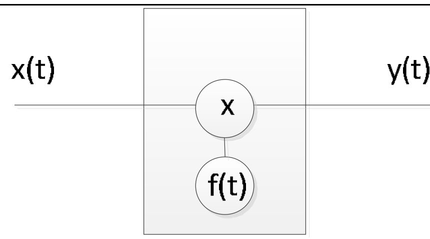


In order to obtain the Fourier transform of the found triangular function, we utilize the time delay theorem: $v(t-T) \rightarrow V(f)e^{-jwT}$ and the tables of Fourier transform pairs ($\Lambda\left(\frac{t}{\tau}\right) \rightarrow \tau sinc^2(\tau f)$) to obtain: $\Lambda(t-1) \rightarrow sinc^2(f)e^{-jw1}$. Moreover, the square of the Fourier transform of the rectangular function is $\text{FT}\{\Pi(t-\frac{1}{2})\}^2 = [sinc(f)e^{-\frac{jw}{2}}]^2 = sinc^2(f)e^{-jw}$, which is the same.

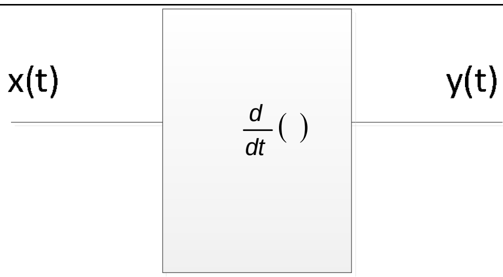
2. Demonstrate the linearity or otherwise of the systems represented by the diagrams below:



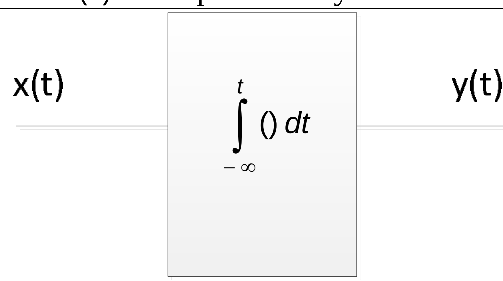
(a) Addition of a function



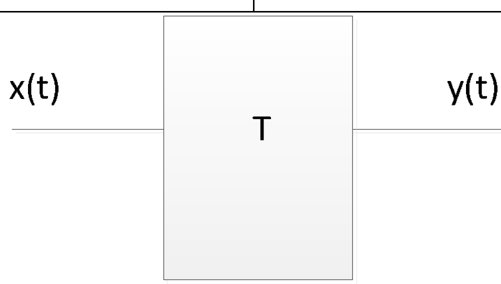
(b) Multiplication by a function



(c) Differentiation



(d) Integration



(e) Time delay (T seconds)

Solution:

(a) For the input $x_1(t)$ output is $y_1(t) = x_1(t) + f(t)$, for the input $x_2(t)$ output is $y_2(t) = x_2(t) + f(t)$, then for input $x_1(t) + x_2(t)$ output is $x_1(t) + x_2(t) + f(t) \neq y_1(t) + y_2(t)$, i. e. superposition does not hold and system (a) is not linear. (Usually addition is a linear operation, however here $f(t)$ is considered to be the part of the system, not an input).

(b) For input $x(t) = x_1(t) + x_2(t)$ the output $y(t) = f(t)[x_1(t) + x_2(t)] = f(t)x_1(t) + f(t)x_2 = y_1(t) + y_2(t)$, therefore the system is linear. (Usually multiplication is a nonlinear operation, however here $f(t)$ is considered to be the part of the system, not an input, therefore superposition holds).

(c) For input $x(t) = x_1(t) + x_2(t)$ output is $y(t) = \frac{d}{dt}[x_1(t) + x_2(t)] = \frac{d}{dt}x_1(t) + \frac{d}{dt}x_2 = y_1(t) + y_2(t)$, i. e. the system (c) is linear.

(d) For input $x(t) = x_1(t) + x_2(t)$ output is $y(t) = \int_{-\infty}^{t}[x_1(t') + x_2(t')]dt' = \int_{-\infty}^{t} x_1(t')dt' + \int_{-\infty}^{t} x_2(t')dt' = y_1(t) + y_2(t)$, i. e. the system (d) is linear.

(e) For input $x(t) = x_1(t) + x_2(t)$ output is $y(t) = x_1(t - T) + x_2(t - T) = y_1(t) + y_2(t)$, i. e. the system (e) is linear.

Here we checked only superposition condition, which basically is enough to demonstrate the behavior of the given systems, however keep in mind that there is another condition to check (the scaling).

3. For the given LTI systems determine the impulse response h[n]:

(a) Discrete-time accumulator

(b) Linear interpolator $\boldsymbol{y[n] = x[n] + \frac{1}{2}(x[n-1] + x[n+1])}$

(c) $\boldsymbol{y[n] + 0.3y[n-1] - 0.45y[n-2] + 0.6y[n-3] = 0.7x[n] - 0.5x[n-2] + 0.1x[n-3]}$ **(Determine 6 first samples).**

Solution:

(a) The impulse response of the discrete-time accumulator is obtained by making the substitution: $x[n] = \delta[n]$, which results in $h[n] = \sum_{l=-\infty}^{n} \delta[l]$, which corresponds to the unit step sequence.

(b) On making the substitution $x[n] = \delta[n]$, we obtain $h[n] = \delta[n] + \frac{1}{2}(\delta[n-1] + \delta[n+1])$, thus , the impulse response sequence is $h[n] = \{0.5, 1, 0.5\}$, when $-1 \leq n \leq 1$.

(c) Assuming that h[k]=0, if k<0, then h[0]=0.7, h[1]=-0.3h[0]=-0.3*0.7=-0.21,

h[2]=-0.3*h[1]+0.45*0.7-0.5=-0.1220 and so on. The output is h={0.7; -0.21; -0.1220; -0.3779; 0.1845;-0.1522}. Try it in Matlab.

4. Calculate the DFT of the periodic signal

The length of the signal x[n]=[-1 2 ½ 3] is 4, therefore the DFT matrix will be of the following way (check the lecture slides (L4(61-62))):

$$D_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix},$$

Next thing is to perform multiplication $X = D_4\vec{x}$, therefore

$$X \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1/2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4.5 \\ -1.5 + i \\ -5.5 \\ -1.5 - i \end{bmatrix}$$

5. Calculate the cross-correlation (with different lags) of the one-period signal from the task 4 and y = [0, 1 ,-5, 2, 3].

[0, 1, -5, 2, 3], [-1 2 ½ 3]	Lags	
[0 0 0 0]	-4	0
[3 0 0 0]	-3	0
[1/2 3 0 0]	-2	0*1/2+1*3=3
[2 1/2 3 0]	-1	0*2+1*1/2+(-5)*3=-14.5
[-1 2 1/2 3]	0	-1*0+2*1+1/2*(-5)+3*2=5.5
[0 -1 2 ½ 3]	1	1*(-1)+2*(-5)+2*1/2+3*3=-1
[0 0 -1 2 1/2]	2	(-5)*(-1)+2*2+1/2*3=10.5
[0 0 0 -1 2]	3	(-1)*2+2*3=4
[0 0 0 0 -1]	4	(-1)*3=-3

Answer: z={-3,4,10.5,-1,5.5,-14.5,3,0,0}

6. **Write an analogue signal representation with amplitude of ½ and the frequency of 50Hz. Now define the discrete version of the signal with the sampling frequency, which enables perfect reconstruction of the signal. Sketch the obtained digital signal. Reveal the topic of aliasing in 5-7 sentences (definition, Nyquist–Shannon sampling theorem, how to avoid aliasing)**

Solution:

The general equation is $y(t) = Acos(2\pi ft + \varphi)$, then the given signal in analog representation is $y(t) = \frac{1}{2}cos(2\pi 50t)$. Now, in order to avoid aliasing we use the Nyquist-Shannon sampling theorem: The Nyquist-Shannon Sampling Theorem says that a signal can be reconstructed when the sampling rate is more than twice the maximum frequency of the signal being sampled. This rule tells us that to reproduce sounds as high as human hearing 20,000 hertz we must take at least 40,001 samples per second. Therefore to get the digital signal we substitute $t = nT, T = 1/Fs$, where *Fs* is the sampling frequency, and should be more than 100 Hz. So in case, $= \frac{1}{150}$; $y(nT) = \frac{1}{2}cos\left(2\pi 50\frac{n}{150}\right) = \frac{1}{2}cos\left(\frac{2\pi n}{3}\right)$, which is shown below.