TLT-5806
Receiver Architectures and Signal Processing

Review of Multirate Digital Signal Processing

Goals

• **Understand the possibilities** of basic multirate signal processing techniques (sampling rate conversion by an integer factor) in the context of advanced receiver (and transmitter) architectures.

• **Understand main principles for achieving computationally efficient solutions for sampling rate conversion by integer factor.**
  - Efficient multirate implementation structure for basic FIR filters
  - Efficiency of multistage designs.
  - Polyphase structure, FIR and IIR Nth-band filters, halfband filters as an important special case.
  - Nyquist pulse shaping filtering (= separable Nth-band designs) as an application of Nth-band filters
  - CIC filter as an efficient multiplier-free structure for the first stages of the decimation chain.

• **Evaluating how the filter complexity depends on the receiver (or transmitter) system design.**

• Detailed studies on filter optimization techniques, alternative filter structures, and extensive comparisons of different solutions can be studied in the course SGN-2106 Multirate Signal Processing lectured by prof. Tapio Saramäki. Lecture notes can be found at [http://www.cs.tut.fi/~ts/](http://www.cs.tut.fi/~ts/).

• A highly (non-uniformly) decimated version of the lecture notes of that course are used in this lecture. (*Thanks, Tap!*
A practical guide for reading the multirate filter section

Main points:

- General idea of decimation and interpolation: pp. 5-43
- Efficient basic FIR structures: pp. 50-57
- “Noble identity”: pp. 61-62
- Polyphase structures: pp. 63-71 or Invocom Course 4, Lect. 2
- Multistage designs: pp. 73-74, FIR decimator examples
- Idea of Nth-band FIR filter: pp. 102-105
- Multistage Nth-band FIR filter, connection to Nyquist pulse shaping principle: p. 113
- Half-band FIR filter: pp. 118-121
- CIC filter: pp. 157-167

Also quite useful:

- Idea of Nth-band IIR filter: pp. 130-137

What else:

- Numerous examples in Tapio’s slides give insight to the characteristics of different design alternatives.
- Literacy of the strict formal representation style for design specifications, etc., is required for deeper studies in multirate signal processing.
Why multirate signal processing is a good idea in advanced transceiver architectures?

A general approach to increase the flexibility of receiver implementations is to use wideband sampling in the receiver and select the desired channel among the many digitized ones using digital filtering, using a filter optimized for the particular transmission system in use.

- In case of wideband sampling, the initial sampling rate is much higher than symbol/chip rate used in baseband processing.

The commonly used delta-sigma AD-conversion principle is also based on heavy oversampling, even in the case of narrowband sampling.

Concerning the DSP implementation complexity and power consumption, it is very crucial to use the lowest possible sampling rate at each stage of the processing chain (good examples of this will be given later during the course).

- As a rule of thumb, for given (narrowband) selectivity requirements and given input sampling rate, the computational complexity (and power consumption) is, in a well-designed multirate system, directly proportional to the output sampling rate.

- Now think about the case where you could reduce the sampling rate by a factor of 300 (could really be the case in a wideband sampling receiver).

Similar ideas can be used also in the transmitter case: synthesizing a high-rate, possibly multi-channel (also the term multicarrier is used in this context) signal using DSP would greatly improve the flexibility.
Later extensions within this course to multirate signal processing tools

- Complex (I/Q) signals
- Bandpass signals
- Sampling rate conversion by a non-integer factor
- Low-order polynomial modelling in discrete-time for modelling continuous-time signals or operators.
Part I: Basics and Motivation

- The purpose of this part is to give some motivation for multirate digital signal processing.
- First, the starting point for processing continuous-time signals with the aid of digital signal processing is considered.
- Second, the need for sampling rate alteration is discussed.
- Third, the two types of sampling rate alteration are considered:
  - Sampling rate reduction, called *decimation*.
  - Sampling rate increase, called *interpolation*.
- Fourth, some applications are considered.
**Sampling theorem:** A continuous-time signal can be reconstructed from its **sample values** if the **sampling frequency** $f_s = 1/T$ ($T$ is the sampling period) is at least two times the highest frequency component of the signal, that is, $X(j2\pi f) = 0, \ f > f_s/2$. See the figures shown below.

![Diagram of sampling theorem](image-url)
Processing of a Continuous-Time Signal with the Aid of a Digital Filter

- Based on the sampling theorem, the processing of a continuous-time signal can be performed with the aid of the discrete-time system as shown on the next page.
- In order to satisfy the conditions of the sampling theorem, the continuous-time signal has to band-limited to the frequency range $-f_s/2 \leq f \leq f_s/2$ using an anti-aliasing filter.
- Ideally, the output signal can be generated from the output samples $y(nT) \equiv Y(n)$ with the aid of the following sinc-interpolation:

$$z(t) = \sum_{k=-\infty}^{\infty} y(nT) \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}.$$  (1)

- In practice, an analog reconstruction filter is used for approximating this interpolation.
- In the frequency domain, this means that from the periodic response of the discrete-time output signal only the baseband frequencies $-f_s/2 \leq f \leq f_s/2$ are preserved.
Processing of a Continuous-Time Signal with the Aid of a Digital Filter
Needs for Altering the Sampling Rate $f_s$

- There exist several situations where the signal of interest is staying in a frequency range that is very small compared to half the sampling rate $f_s/2$ as shown in the following figure.

- In the case of this figure, the sampling rate $\hat{f}_s = f_s/3$ could be used.

- There are several advantages of using $\hat{f}_s$ as a sampling rate, instead of $f_s$, and to make it as small as possible.

Advantage 1 of Using a Lower Sampling Rate: The number of samples is reduced by a factor of $f_s/\hat{f}_s$ so that the processing workload is significantly reduced.

- For instance, if in the case of the figures on page 2, $X(j2\pi f) = 0$, $f > (f_s/2)/3$, then sampling rate $\hat{f}_s = f_s/3$ can be used.
This means that the corresponding sampling period 
\( \hat{T} = 1/\hat{f}_s = 3T \). In this case, only every third sam-
ple of the first figure are needed for carrying the 
information of the continuous-time signal!

**Advantage 2 of Using a Lower Sampling Rate:**
The discrete-time systems become easier to implement.

- As shown on the next page, the order of an FIR
  filter reduces approximately by 10 if the sampling
  rate is reduced by this factor.
- Taking into account the fact that also the number
  of data samples is only one tenth, the overall sav-
ing is approximately 10\cdot10=100!
- Pages 7 and 8 illustrate what happens in the case
  of elliptic IIR filter.
- For these filters, the order either remains the same
  or decreases by one when using a lower sampling
  rate.
- The main advantage lies in the the fact that the
  poles of the filter implemented using a lower sam-
  pling rate are further away from the unit circle.
- This means that the finite wordlength effects are
  significantly milder: significantly fewer bits are re-
  quired for both the data and coefficient respresen-
tations.
Linear-Phase FIR Filter with Passband and Stopband Edges at 250 Hz and 500 Hz and Passband and Stopband Ripples of 0.01 and 0.001 for the Amplitude Response

FIR Filter of order 216 for $f_s = 20$ kHz

FIR Filter of order 21 for $f_s = 2$ kHz
Elliptic Filter with Passband and Stopband Edges at 250 Hz and 500 Hz and Passband and Stopband Ripples of 0.2 dB and 60 dB
Elliptic Filter with Passband and Stopband Edges at 250 Hz and 500 Hz and Passband and Stopband Ripples of 0.2 dB and 60 dB

- Since the filter poles for $f_s = 2$ kHz are not so close to the unit circle, the coefficient sensitivity as well as the output noise are much lower than for $f_s = 20$ kHz.
Needs for Altering the Sampling Rate $f_s$

- In addition to the above example, there are numerous other applications where it is advantageous or even necessary to change (reduce or increase) the sampling rate, as will be seen later on in this course.
- In our example case, the sampling rate of the signal of the system of Page 4 could be reduced according to the characteristics of our input continuous-time signal.
- However, in most applications, there exist signals having different bandwidths.
- Therefore, it is preferred to study how to change the sampling rate directly in the digital domain.
The Two Basic Types for Sampling Rate Alteration

- There exist two types of sampling rate alteration, namely decimation and interpolation.
- As shown below, in the case of decimation, the number of samples is reduced.
- This means that the sampling period is increased and the sampling rate is decreased.
- In the case of interpolation, the number of samples is increased.
- This means that the sampling period is reduced and the sampling rate is increased.
Decimation by an Integer Factor $M$

- When reducing the sampling rate by an integer factor of $M$, the overall system is constructed as shown on the next page.
- The first step is to filter the input signal $x(n)$ with a transfer function $H(z)$ so that the $z$- and Fourier transforms of the filtered signal $w(n)$ are given by

$$W(z) = H(z)X(z) \quad (2a)$$

and

$$W(e^{j2\pi f/f_s}) = H(e^{j2\pi f/f_s})X(e^{j2\pi f/f_s}), \quad (2b)$$

where $f_s$ is the input sampling rate.
- The second step is to pick up every $M$th sample of $w(n)$ to form the output signal $y(m)$ (the arrow downwards followed by $M$ means this operation).
- $y(m)$ is thus related to $w(n)$ via

$$y(m) = w(mM) \quad (3)$$

so that $y(0) = w(0)$, $y(1) = w(M)$, $y(2) = w(2M)$ and so on.
- The sampling rate of $y(m)$ is thus $\hat{f}_s = f_s/M$ and the new baseband is $[0, \hat{f}_s/2] = [0, (f_s/M)/2]$. 


Block Diagram for Decimation by an Integer Factor $M$

$$x(n) \xrightarrow{f_s} H(z) \xrightarrow{f_s} w(n) \xrightarrow{M} y(m)$$

$$f_s = f_s / M$$

$|X(e^{i2\pi f/f_s})|$ 

$|H(e^{i2\pi f/f_s})|$ 

$|W(e^{i2\pi f/f_s})|$ 

$|Y(e^{i2\pi f/f_s})|$
• It can be shown that the \( z \)- and Fourier transforms of the decimated signal \( y(m) \) are given by
\[
Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} W(z^{1/M} e^{j2\pi k/M})
\]
and \( z = e^{j2\pi f/(f_s/M)} \)
\[
Y(e^{j2\pi f/(f_s/M)}) = \frac{1}{M} \sum_{k=0}^{M-1} W(e^{j(2\pi f_s)(f+kf_s/M)}).
\]

• In the above equation, the term for \( k = 0 \), that is, \( W(e^{j2\pi f/f_s}) \) is the frequency component staying in the new baseband region \( |f| \leq \hat{f}_s/2 = (f_s/M)/2 \) before the sampling rate reduction.

• The other terms \( W(e^{j(2\pi f_s)(f+kf_s/M)}) \) for \( k = 1, 2, \ldots, M - 1 \) are the undesired components aliasing from \((k - 1/2)f_s/M \leq f \leq (k + 1/2)f_s/M\) into the new baseband \( |f| \leq \hat{f}_s/2 = (f_s/M)/2 \).

• The aliasing or overlapping of several terms can be avoided by requiring that before decimation \( W(e^{j2\pi f/f_s}) \) is practically nonzero only in the new baseband \( |f| \leq \hat{f}_s/2 = (f_s/M)/2 \). In this band, it is then true that \( Y(e^{j2\pi f/(f_s/M)}) \approx W(e^{j2\pi f/f_s}) \).

• In the ideal case, it is desired that \( Y(e^{j2\pi f/(f_s/M)}) \approx e^{-j2\pi \alpha/f_s} X(e^{j2\pi f/f_s}) \), that is, \( y(m) \) is a delayed and decimated version of \( x(n) \) in the frequency band \( |f| \leq \hat{f}_s/2 = (f_s/M)/2 \). (Note that some delay is always needed when filtering a signal.)
• This is achieved by designing $H(z)$ to satisfy

$$H(e^{j2\pi f/fs}) \approx \begin{cases} e^{-j2\pi \alpha f/fs} & \text{for } f \leq (fs/2)/M \\ 0 & \text{for } (fs/2)/M \leq f \leq (fs/2). \end{cases}$$

(5)

• If the phase characteristics is not of importance, then it is desired that

$$|Y(e^{j2\pi f/(fs/M)})| \approx |X(e^{j2\pi f/fs})|$$

for $|f| \leq \hat{f}/2 = (fs/M)/2$.

• In this case, it is required that

$$|H(e^{j2\pi f/fs})| \approx \begin{cases} 1 & \text{for } f \leq (fs/2)/M \\ 0 & \text{for } (fs/2)/M \leq f \leq (fs/2). \end{cases}$$

(6)

Comment 1: In Eqs. (4) the multiplier $1/M$ comes just from the use of the mathematics. The signal levels in the time domain remain the same.

Comment 2: In Eq. (4a), instead of $e^{j2\pi k/M}$, $e^{-j2\pi k/M}$ is usually used. In this case, $(f + kf_s/M)$ becomes $(f - kf_s/M)$ in Eq. (4b). The basic reason for our selection is the fact that the explanation of aliasing becomes more straightforward.

• The role of the filter with transfer function $H(z)$ is thus similar to the analog anti-aliasing filter: the signal components of the input sequence $x(n)$ outside the range $-\hat{f}/2 \leq f \leq \hat{f}/2$ should be attenuated in order to avoid aliasing.
A Simple Way of Checking How the Frequency Components Are Aliasing

The following figure illustrates in the $M = 5$ case how the components from the range $(f_s/2)/M < f \leq f_s/2$ are aliasing into the new baseband $0 \leq f \leq (f_s/2)/M$. 
Requirements for $H(z)$ in the Decimation Case

- There exist three ways of stating the criteria for $H(z)$.
- In order to make the consideration of interpolators the same, $N \equiv M$ and the following structure are used:

$$
\begin{align*}
x(n) & \rightarrow H(z) \rightarrow w(n) \rightarrow \downarrow N \rightarrow y(m) \\
f_s & \rightarrow \hat{f}_s = \frac{f_s}{N}
\end{align*}
$$

- In terms of the angular frequency $\omega = 2\pi f / f_s$, they can be stated as

$$
1 - \delta_p \leq |H(e^{j\omega})| \leq 1 + \delta_p \quad \text{for} \quad \omega \in [0, \alpha \pi / N], \quad (7a)
$$

where $\alpha < 1$, and

$$
|H(e^{j\omega})| \leq \delta_s \quad \text{for} \quad \omega \in \Omega_s, \quad (7b)
$$

where

$$
\Omega_s = \left\{ \begin{array}{ll}
[\pi / N, \pi] & \text{for Case A} \\
\bigcup_{k=1}^{[N/2]} \left[ \frac{(2k - \alpha)\pi}{N}, \min\left(\frac{(2k + \alpha)\pi}{N}, \pi\right) \right] & \text{for Case B} \\
(2 - \alpha)\pi / N, \pi & \text{for Case C.} \\
\end{array} \right. \quad (7c)
$$

- Similarly, in terms of the 'real' frequency, these crite-
ria can be stated as

\[ 1 - \delta_p \leq |H(e^{j2\pi f/f_s})| \leq 1 + \delta_p \quad \text{for} \quad \omega \in [0, \alpha(f_s/2)/N], \]

(8a)

where \( \alpha < 1 \), and

\[ |H(e^{j2\pi f/f_s})| \leq \delta_s \quad \text{for} \quad \omega \in X_s, \]

(8b)

where

\[ X_s = \begin{cases} 
(\lfloor f_s/2 \rfloor N, f_s/2] & \text{for Case A} \\
\bigcup_{k=1}^{[N/2]} \left( \frac{(2k - \alpha)f_s/2}{N}, \min\left(\frac{(2k + \alpha)f_s/2}{N}, f_s/2\right) \right] & \text{for Case B} \\
(2 - \alpha)(f_s/2)/N, f_s/2] & \text{for Case C}. 
\end{cases} \]

(8c)

- In all the cases, the signal is preserved in the passband region \([0, \alpha(f_s/2)/N]\) with \( \alpha < 1 \).
- In Case A, all the components aliasing into the new baseband \([0, (f_s/2)/N]\) are attenuated.
- In Case B, all the components aliasing into the passband \([0, \alpha(f_s/2)/N]\) are attenuated, but aliasing is allowed into the transition band \([\alpha(f_s/2)/N, (f_s/2)/N]\).
- In Case C, aliasing is allowed into the transition band \([\alpha(f_s/2)/N, (f_s/2)/N]\) only from the band \([(f_s/2)/N, (2 - \alpha)(f_s/2)/N]\).
- Case B and C specifications can be used, for instance, in audio applications when \(f_s/N=44.1\) kHz and \(\alpha = 0.907\). In this case the transition band is between 20
kHz and 22.05 kHz, that is, outside the frequency range a human ear is able to hear.

- Page 20 shows the responses for $N = 10$, $\alpha = 0.5$, and $\delta_p = \delta_s = 0.08$ in these three cases. The passband region is thus in terms of the ‘real’ frequency $[0, 0.05(f_s/2)]$ and in terms of the angular frequencies $[0, 0.05\pi]$.

- For Case A, the stopband region is in terms of the angular frequencies $[\pi/10, \pi]$ and in terms of the ‘real’ frequency $[(f_s/2)/10, f_s/2]$.

- For Case B, the stopband region is in terms of the angular frequency the union of the bands $[1.5\pi/10, 2.5\pi]$, $[3.5\pi/10, 4.5\pi]$, $[5.5\pi/10, 6.5\pi]$, $[7.5\pi/10, 8.5\pi]$, and $[9.5\pi/10, \pi]$.

- In terms of the ‘real’ frequency, the stopband region is the union of the bands $[1.5\pi/10, 2.5\pi]$, $[3.5(f_s/2)/10, 4.5(f_s/2)]$, $[5.5(f_s/2)/10, 6.5(f_s/2)]$, $[7.5(f_s/2)/10, 8.5(f_s/2)]$, and $[9.5(f_s/2)/10, f_s/2]$.

- For Case C, the stopband region is in terms of the angular frequency $[1.5\pi/10, \pi]$ and in terms of the real frequencies $[1.5(f_s/2)/10, f_s/2]$. 
Example Case A, Case B, and Case C Specifications: $N = 10$, $\alpha = 0.5$, and $\delta_p = \delta_s = 0.08$
Interpolation by an Integer Factor $L$

- When increasing the sampling rate by an integer factor $L$, the overall system is constructed as shown on the next page.
- In the first step, $L-1$ zero-valued samples are inserted between the existing input samples (the arrow upwards followed by $L$ means this operation).
- Hence,
  \[
  w(m) = \begin{cases} 
  x(m/L), & m = 0, \pm L, \pm 2L, \ldots \\
  0, & \text{otherwise}.
  \end{cases}
  \]  
  \hspace{1cm} (9)
- The sampling rate of $w(m)$ is thus $\hat{f}_s = Lf_s$ and the new baseband is $|f| \leq \hat{f}_s/2 = Lf_s/2$, that is, $L$ times that of the input signal.
- The $z$- and Fourier transforms of the interpolated signal $w(m)$ are given by
  \[
  W(z) = X(z^L)
  \]  
  \hspace{1cm} (10a)
  and
  \[
  W(e^{j2\pi f/\hat{f}_s}) = X(e^{j2\pi f/(\hat{f}_s/L)}) = X(e^{j2\pi f/f_s}).
  \]  
  \hspace{1cm} (10b)
- As illustrated on the next page, the Fourier transform of $w(m)$ contains in the increased baseband not only the baseband frequencies of $x(n)$ (i.e., $|f| \leq f_s$) but also images of the old baseband centered at $\pm f_s = \pm \hat{f}_s/L, \pm 2f_s = \pm 2\hat{f}_s/L, \ldots$. 

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Block Diagram for Interpolation by an Integer Factor $L$

\[ x(n) \xrightarrow{f_s} L \xrightarrow{f_s = Lf_s} H(z) \xrightarrow{f_s = Lf_s} y(m) \]

\[ |X(e^{j2\pi f/f_s})| \]

\[ |W(e^{j2\pi f/f_s})| \]

\[ |H(e^{j2\pi f/f_s})| \xrightarrow{\text{Gain} = L} \]

\[ |Y(e^{j2\pi f/f_s})| \]
- The role of the filter with transfer function $H(z)$ is thus to preserve the original baseband $|f| \leq f_s/2 = (\hat{f}_s/2)/L$ and to attenuate the unwanted higher frequency components (images) in the new baseband $|f| \leq \hat{f}_s/2 = Lf_s/2$, that is, the frequencies $(\hat{f}_s/2)/L \leq |f| \leq \hat{f}_s/2$.

- In the ideal case, it is desired that $Y(e^{j2\pi f/(f_s/M)})$ approximates closely $e^{-j2\pi \alpha/f_s}LX(e^{j2\pi f/f_s})$ for $|f| \leq (\hat{f}_s/2)/L$ and zero elsewhere in the new increased baseband.

- In this case, $y(m)$ is a delayed and interpolated version of $x(n)$ in the frequency band $|f| \leq f_s/2 = (f_s/M)/2$.

- Note that the multiplier $L$ is required to preserve the existing input samples in the time domain at the same level as well as raising the inserted zero-valued samples as 'interpolated' samples.

- This is achieved by designing $H(z)$ to satisfy

$$H(e^{j2\pi f/\hat{f}_s}) \approx \begin{cases} 
Le^{-j2\pi \alpha/f_s} & \text{for } f \leq (\hat{f}_s/2)/L \\
0 & \text{for } (\hat{f}_s/2)/L \leq f \leq \hat{f}_s/2.
\end{cases}$$  \hspace{1cm} (11)

- If the phase characteristics is not of importance, then it is desired that

$$|H(e^{j2\pi f/\hat{f}_s})| \approx \begin{cases} 
L & \text{for } |f| \leq (\hat{f}_s/2)/L \\
0 & \text{for } (\hat{f}_s/2)/L \leq f \leq \hat{f}_s/2.
\end{cases}$$  \hspace{1cm} (12)
Requirements for $H(z)$ in the Interpolation Case

- In order to state the criteria for $H(z)$ in a manner similar to the case of decimation we use the following structure:

![Diagram of interpolation process]

- The main difference is that now the input and output sampling rates are $f_s/N$ and $f_s$, instead of $f_s$ and $Lf_s$. Furthermore, the interpolation ratio is $N$, instead of $L$.
- These modifications emphasize the duality between the decimator and interpolator designs.
- After designing the decimator for the integer decimation factor $N$, the corresponding interpolator can be generated as follows:
  (a) Replace the decimation block after filtering by the transfer function $H(z)$ by the corresponding interpolation block before filtering by the transfer function $NH(z)$.
  (b) Replace the input and output sampling rates $f_s$ and $f_s/N$ by $f_s/N$ and $f_s$, respectively.
• Note that if the input signal satisfies for the input sampling rate of $f_s/N \ |X(e^{j2\pi f/(f_s/N)})| \approx 0$ for $\alpha(f_s/2)/N \leq |f| \leq (f_s/2)/N$, then the Case B or Case specifications, stated earlier in the decimation case, can be directly used for attenuating the extra images.
Sampling Rate Conversion by $L/M$ with $L$ and $M$ Being Integers

- Sampling rate conversion by $L/M$ with both $L$ and $M$ being integers can be performed with the aid of a single filter.
- This is exemplified on Pages 28 and 29 in the $L = 3$ and $M = 2$ case and in the $L = 2$ and $M = 3$ case, respectively.
- Note that in the first case the sampling rate is increased by $3/2$ and in the second case decreased by $3/2$.
- In both cases, the sampling rate is first increased by $L$ so that the resulting sampling rate is $\hat{f}_s = Lf_s$.
- In the first case $L = 3 > M = 2$. Therefore, $H(z)$ acts like a filter for interpolation and satisfies

\[
|H(e^{j2\pi f/\hat{f}_s})| \approx \begin{cases} 
L & \text{for } |f| \leq (\hat{f}_s/2)/L \\
0 & \text{for } (\hat{f}_s/2)/L \leq f \leq \hat{f}_s/2.
\end{cases} \tag{13}
\]

- What is left is to reduce the sampling rate by $M = 2$, that is, to attenuate the signal components in the range $[(\hat{f}_s/2)/2, \hat{f}_s/2]$ in order to give the output sampling rate $\tilde{f}_s = \hat{f}_s/2 = (3/2)f_s$.
- As seen from Page 28, the filter with transfer function $H(z)$ has already attenuated these components.
- Therefore, every second sample at the output of this filter can be directly picked up without causing significant aliasing.
• In the $L = 2$ and $M = 3$ case, the role of $H(z)$ is different due to the fact that $M > L$.
• In this case, $H(z)$ has to attenuate the signal components in the range $[(\hat{f}_s/2)/3, \hat{f}_s/2]$ in order to give the output sampling rate $\tilde{f}_s = \hat{f}_s = (2/3)f_s$.
• Simultaneously, the images caused by interpolation are attenuated as well as a part of the original baseband, as illustrated on Page 29.
• Therefore, $H(z)$ acts like a filter for decimation with the exception that in the passband the desired value is $L$ and satisfies

$$\left| H(e^{j2\pi f/\hat{f}_s}) \right| \approx \begin{cases} L & \text{for } |f| \leq (\hat{f}_s/2)/M \\ 0 & \text{for } (\hat{f}_s/2)/M \leq f \leq \hat{f}_s/2. \end{cases}$$

(14)
Sampling Rate Conversion by $L/M$, $L = 3 > M = 2$

\[ \frac{x(n)}{f_s} \xrightarrow{L} \frac{s(k)}{f_s} = Lf_s \xrightarrow{H(z)} \frac{t(l)}{f_s} \xrightarrow{M} \frac{y(m)}{f_s} = \frac{f_s}{M} \]

\[ |X(e^{j2\pi f/f_s})| \]

\[ |S(e^{j2\pi f/f_s})| \]

\[ |H(e^{j2\pi f/f_s})| \]

\[ |T(e^{j2\pi f/f_s})| \]

\[ |Y(e^{j2\pi f/f_s})| \]

Gain = $L$

$L = 3$

$M = 2$
Sampling Rate Conversion by $L/M$, $L = 2 < M = 3$

\[ x(n) \xrightarrow{L} \hat{s}(k) \xrightarrow{H(z)} \hat{t}(l) \xrightarrow{M} y(m) \]

\[ f_s = Lf_s \]

\[ \hat{f}_s = \frac{\hat{f}_s}{M} \]

\[ |X(e^{j2\pi f/f_s})| \]

\[ |S(e^{j2\pi f/\hat{f}_s})| \]

\[ L = 2 \]

\[ |H(e^{j2\pi f/\hat{f}_s})| \]

\[ \text{Gain} = L \]

\[ |T(e^{j2\pi f/\hat{f}_s})| \]

\[ M = 3 \]

\[ |Y(e^{j2\pi f/\hat{f}_s})| \]
General Criteria for Sampling Rate Conversion by $L/M$ with $L$ and $M$ being integers

- In the most general case, the sampling rate increase by a factor of $L$ must be performed first resulting in the sampling rate $\hat{f}_s = Lf_s$.
- The sampling rate $\tilde{f}_s = \hat{f}_s/M = (L/M)f_s$ is achieved by designing $H(z)$ to satisfy

$$|H(e^{j2\pi f/\tilde{f}_s})| \approx \begin{cases} L & \text{for } |f| \leq (\hat{f}_s/2)/D \\ 0 & \text{for } (\hat{f}_s/2)/D \leq f \leq \tilde{f}_s/2, \end{cases} \quad (16a)$$

where

$$D = \max(L, M). \quad (16b)$$

- If $L$ and $M$ are very large integers or the sampling rate conversion factor is arbitrary, it is more beneficial to use the technique to be described in Part III of this course.
Partly Digital Anti-Aliasing Filter

- The requirements for the analog anti-imaging filter can be made significantly milder and a very linear phase performance can be achieved in the passband by using the following structure:

\[ \tilde{f}_s = rf_s \]

- In this case, the output sampling rate of the analog filter with transfer function \( H_c(s) \) is \( \tilde{f}_s = rf_s \), where \( f_s \) is final sampling rate.
- This filter is followed by a decimator with transfer function \( H_d(z) \) decimating by a factor of \( r \) to generate the desired output sampling rate.
- For the overall system, the frequency response is expressible as \( H_{ove}(j2\pi f) = H_c(j2\pi f)H_d(e^{j2\pi f/(rf_s)}) \).
- It is desired to design the overall system such that the maximum deviation of \( |H_{ove}(j2\pi f)| \) from unity in the passband \([0, \alpha f_s/2]\) with \( \alpha < 1 \) is less than or equal to \( \delta_p \), and the maximum deviation from zero is less than or equal to \( \delta_s \) for \( f \geq f_s/2 \).
- As shown on Page 33, the design can be accomplished in two stages:
- In the first step, \( H_c(s) \) is designed such that its am-
plitude response oscillates within $1 \pm \delta_p^c$ in the passband $[0, \alpha f_s/2]$, the maximum amplitude deviation from zero is less than or equal to $\delta_s$ for $f \geq (2r - 1)f_s/2$, and the group delay variation around the passband average is minimized.

- Here, $\delta_p^c$ can be significantly larger than $\delta_p$.
- When starting the stopband edge at $f = (2r - 1)f_s/2$, the frequency components aliasing to the final baseband $[0, f_s/2]$ are well attenuated, as shown on Page 33.
- There is significant aliasing into the band $[f_s/2, rf_s/2]$. These components are well attenuated by the decimator.
- In the second step, $H_d(z)$ is designed such that it provides the desired performance for the overall system for $0 \leq f \leq rf_s/2$. 
Design of a Partly Digital Anti-Aliasing Filter

(a) $|H_c(j2\pi f)|$

(b) $|H_c(j2\pi f)H_d(e^{j2\pi f/r_s})|$
Example

- Criteria: $\alpha = 0.8$, $r = 3$, $\delta_p = 0.00576$ (0.1-dB overall passband variation), $\delta_s = 0.000316$ (70-dB stopband attenuation, suitable for a 12-bit converter), the maximum allowable group delay ripple in terms of the final sampling period $T$ is $0.01T$.

- In the following, there are two pages illustrating the characteristics of the optimized overall design.

- Figures (a) and (b) show the amplitude and group delay responses for the optimized fourth-order analog filter. The desired group delay variation is achieved by selecting the passband ripple to be $\delta_p^c = 0.037$.

- Figure (c) shows the response of a linear-phase FIR decimator filter $H_d(z)$ required for the system to meet the amplitude criteria, as shown in Figure (d).

- If no decimator is used, then an elliptic filter of order nine is required to meet the same criteria. The phase response of this filter is very nonlinear in the passband. Furthermore, the tuning of the elliptic filter is significantly more difficult.
Example Partly Digital Anti-Aliasing Filter

(a) Amplitude in dB

(b) Group Delay

Log Frequency

Lin. Amp.

0.963

0.8f_s/2

1.3f_s/2

0.01f_s/2

0.1f_s/2

f_s/2

5f_s/2

10f_s/2

100f_s/2

0

-60

-80

-100

0.697T

0.717T

0.50T

0.25T

0.01f_s/2

0.1f_s/2

f_s/2

10f_s/2
Example Partly Digital Anti-Aliasing Filter

(c)

(d)
Sigma-Delta A/D Conveter

- An extreme case is an A/D converter based on the use of sigma delta modulator.
- In the case of a converter shown on the next page, \( r = 64 \) and a simple analog RC filter can be used.
A Stereo Sigma-Delta A/D-Converter with Oversampling Ratio of 64

Figure 1: AD-converter for audio use
Design of a Narrowband Linear-Phase FIR Filter

Example:  Passband edge = 0.0095 \cdot f_s/2  
Stopband edge = 0.01 \cdot f_s/2  
Passband ripple = 0.001  
Stopband ripple = 0.0001 (80 dB)

Direct-form conventional FIR filter of order 15590:  
7796 multiplications per input sample

FIR filter implemented using decimation and interpolation:

\[ \frac{y(n)}{x(n)} = H_1(z) \downarrow 15 \rightarrow H_2(z) \downarrow 6 \rightarrow H_3(z) \]

\[ \frac{y(n)}{x(n)} = 15 H_1(z) \uparrow 15 \rightarrow 6 H_2(z) \uparrow 6 \rightarrow H_3(z) \]

- \( H_1(z) \): order = 39  
- \( H_2(z) \): order = 40  
- \( H_3(z) \): order = 197

Only 4.16 multiplications per input sample
A Filter Bank for Subband Coding

The $H_k(z)$'s and $F_k(z)$'s can be designed such that $\hat{x}(n) = x(n-K)$. This filter bank is used for subband coding.

After decimation by $M$
Part II: Design and Implementation of Efficient Decimators and Interpolators

- The purpose of this part is to review a number different techniques for constructing efficient filters for decimation and interpolation purposes.
- Both efficient implementation forms as well as various methods for designing decimation and interpolation filters for these implementation forms are considered.
- This part has been divided into the following subparts:
  - II.A: One-Stage Desimation and Interpolation Structures
  - II.B: Conventional Multistage Implementations
  - II.C: Special Filter Structures
  - II.D: Nth-Band IIR Filters
  - II.E: Nth-Band FIR Filters
  - II.F: Half-Band FIR Filters
  - II.G: Half-Band IIR Filters
  - II.H: Use of Conventional and Modified Comb (Running Sum) Structures as a First Stage for Multistage Decimator Implementations
Part II.A: One-Stage Decimation and Interpolation Structures

- To emphasize the duality between the decimators and interpolators we first consider the implementation and design of decimators and interpolators using the one-stage structures as shown on the next page.
- In the case of the decimator, the input sampling rate is $f_s$ and the output sampling rate after decimation by an integer factor $N$ is $f_s/N$.
- The role of the filter to preserve the signal components for $|f| \leq (f_s/2)/N$ and attenuate the signal components aliasing from the region $(f_s/2)/N \leq |f| \leq f_s/2$ into $|f| \leq (f_s)/N$ using Case A, Case B, or Case C specifications considered in Part I.
- In the case of the interpolator, the input sampling rate is $f_s/N$ and the output sampling rate after interpolation by an integer factor $N$ is $f_s$.
- The role of the filter to preserve the original baseband components for $|f| \leq (f_s/2)/N$ and attenuate the images in the the region $(f_s/2)/N \leq |f| \leq f_s/2$ using the Case A, Case B, or Case C specifications.
- Recall that the amplitude response of $H(z)$ in the interpolation case approximates $N$ in the passband.
One-Stage (Single-Stage) Decimator and Interpolator Structures to be Considered

\[ x(n) \xrightarrow{f_s} H(z) \xrightarrow{f_s} w(n) \xrightarrow{N} y(m) \xrightarrow{f_s = f_s/N} \]

\[ \hat{x} \xrightarrow{f_s = f_s/N} N \xrightarrow{\hat{f_s} = f_s/N} w(m) \xrightarrow{NH(z)} y(m) \xrightarrow{f_s} \]
• In the sequel, it will be shown to be beneficial to carry out the sampling rate alteration in several stages.
• As will be seen later, also in these cases is it very useful to express the overall system by means of certain identities in the one-stage (single-stage) equivalent forms as shown on the previous page.
• This makes the both the synthesis and analysis of the overall system more straightforward.
• We recall that given $N$ and $\alpha$ in Cases A, B, and C the passband region is given in terms of the 'real' frequency and the angular frequency $\omega = 2\pi f/f_s$ as $[0, \alpha(f_s/2)/N]$ and $[0, \alpha\pi/N]$, respectively.
• The stopband regions are in terms of the 'real' frequency

$$X_s = \begin{cases} 
[(f_s/2)/N, f_s/2] & \text{for Case A} \\
\left\lfloor\frac{N}{2}\right\rfloor \bigcup_{k=1} \left[\frac{(2k - \alpha)f_s/2}{N}, \min\left(\frac{(2k + \alpha)f_s/2}{N}, f_s/2\right)\right] & \text{for Case B} \\
[(2 - \alpha)(f_s/2)/N, f_s/2] & \text{for Case C.}
\end{cases}$$

(1)
In terms of the angular frequency, the stopband regions are

$$\Omega_s = \begin{cases} 
[\pi/N, \pi] \\
\bigcup_{k=1}^{[N/2]} \left[ \frac{(2k - \alpha)\pi}{N}, \min\left(\frac{(2k + \alpha)\pi}{N}, \pi\right) \right] \\
[(2 - \alpha)\pi/N, \pi] 
\end{cases}$$

for Case A

for Case B

for Case C.

(2)
Direct-Form FIR Filter Realizations for Decimation

- Consider decimation by an integer factor $N$ using an FIR filter with the transfer function

$$H(z) = \sum_{n=0}^{M} h(n)z^{-n}.$$  \hspace{1cm} (3)

- The input-output relation for the filter in the time domain can be expressed as

$$w(n) = \sum_{k=0}^{M} h(k)x(n - k).$$  \hspace{1cm} (4)

- The output after decimation by a factor of $N$ is then

$$y(m) = w(Nm).$$  \hspace{1cm} (5)

- The first structure of the next page shows an implementation where all the outputs are evaluated out of which only every $N$th output sample is picked up. Therefore, for this structure, a plenty of vain computations are performed.

- The number of multiplications per input sample is in this case $(M + 1)$.

- In order to derive a more efficient implementation, Eqs. (4) and (5) are combined to give

$$y(m) = \sum_{k=0}^{M} h(k)x(Nm - k).$$  \hspace{1cm} (6)
Implementations of an FIR filter of order $M$ for Decimation by an Integer Factor $N$
• Since there are no feedback loops, only every $N$th output of the filter with transfer function $H(z)$ can be computed.
• The second structure of the previous page shows the corresponding efficient implementation where the multiplications are performed only every $N$th time instant compared to the input sampling rate.
• In this structure, the data goes through the delays like for the conventional FIR filter.
• The key idea is to pick up the data from the delays for evaluating the output only every $N$th time instant compared to the input sampling rate.
• This reduces the number of multiplications per input sample to $(M + 1)/N$, as is desired.
Efficient Implementation of a Linear-Phase FIR Filter of Order $M$ for Decimation by an Integer Factor $N$

- The multiplication rate can be further reduced using a direct-form structure exploiting the coefficient symmetry, as shown below for $M$ even and $h(M - n) = h(n)$ for $n = 0, 1, \ldots, M/2 - 1$. The number of multiplications per input sample reduces to $(M/2 + 1)/N$.
- For $M$ odd and $h(M - n) = h(n)$ for $n = 0, 1, \ldots, (M - 1)/2$, a similar structure exists, requiring $(M + 1)/(2N)$ multiplications per input sample.
Transposed Direct-Form FIR Filter Realizations for Interpolation

- Consider interpolation by an integer factor $N$ using an FIR filter with the transfer function

$$H(z) = \sum_{n=0}^{M} h(n)z^{-n}.$$  \hspace{1cm} (7)

- In order to arrive at an efficient implementation, a transposed direct-form structure is used after interpolation. This gives the first implementation of the next page.

- The input to the FIR filter is given by

$$w(m) = \begin{cases} 
  x(m/N), & m = 0, \pm N, \pm 2N, \ldots \\
  0, & \text{otherwise}.
\end{cases} \hspace{1cm} (8)$$

- Hence, only every $N$th input sample to the FIR filter is non-zero.

- Therefore, in this structure, several zero-valued input samples are multiplied by the filter coefficients without any contribution to the overall output.

- This problem can be overcome by using the second structure of the next page.

- In this structure, the data goes through the delays in the output part like for the conventional FIR filter at the higher output sampling rate.
Implementations of an FIR Filter of order $M$ for Interpolation by an Integer Factor $N$
• The key idea is to process only every $N$th time instant compared to the output sampling rate of $f_s$ the existing 'real' non-zero input data samples.
• These samples are multiplied by the coefficients and the results are fed to the adders between the delays (and to the input of the first delay).
• This corresponds to the case where the existing input samples are multiplied at the lower input sampling rate of $f_s/N$.
• This reduces the number of multiplications per output sample from $M$ to $M/N$. 
Efficient Implementation of a Linear-Phase FIR Filter of Order $M$ for Interpolation by an Integer Factor $N$

- The multiplication rate can be further decreased using a transposed direct-form structure exploiting the coefficient symmetry, as shown in the following figure for $M$ even and $h(M - n) = h(n)$ for $n = 0, 1, \cdots, M/2 - 1$. The number of multiplications per output sample is now $(M/2 + 1)/N$.
- Like in the decimation case, there exists a similar structure for $M$ odd and $h(M - n) = h(n)$ for $n = 0, 1, \cdots, (M - 1)/2$, requiring $(M + 1)/(2N)$ multiplications per output sample.
Conventional IIR Filters for Decimation and Interpolation

- Consider decimation by an integer factor $N$ using an IIR filter with the following transfer function:

$$H(z) = \frac{\sum_{n=0}^{M} a(n)z^{-n}}{1 - \sum_{n=1}^{M} b(n)z^{-n}}. \quad (9)$$

- This filter suffers from the drawback that the fact that only every $N$th output is needed cannot be exploited due to the feedback loop.

- The diagram of the next page shows a direct-form II structure, where the fact that only every $N$th output is needed has been exploited in the feedforward part.

- The time-domain equations are given by

$$w(n) = x(n) + \sum_{k=1}^{M} b(k)w(n - k) \quad (10)$$

and

$$y(m) = \sum_{k=0}^{M} a(k)w(Mm - k). \quad (11)$$

- Note that every sample value $w(n)$ is necessary needed due to the feedback loop.

- Page 16 shows the corresponding transposed direct-form II structure for the interpolator.
Implementation of a Conventional IIR Filter of order $M$ for Decimation by an Integer Factor $N$
Implementation of a Conventional IIR Filter of order $M$ for Interpolation by an Integer Factor $N$
Identities for Decimators and Interpolators

- Two important identities that are exploited in the sequel in building efficient decimator and interpolator structures as well as in analysing their performances are shown on the next page.

- If there is a transfer function $H(z)$ after decimation by $N$ [before interpolation by $N$], this transfer can be moved before the decimation block [after the interpolation block] by changing it to be $H(z^N)$ and vice versa.

- Note that $H(z^N)$ is obtained from $H(z)$ by replacing each unit delay $z^{-1}$ by $z^{-N}$, that is, a block of $N$ delays.
Identities for Decimators and Interpolators

Identity for the decimator

Identity for the interpolator
Polyphase Structures for Decimation

- Very important implementation forms for both decimators and interpolators are the so-called polyphase structures.
- When decimating by a factor of $N$, the first step is to express the overall transfer function as

$$H(z) = \sum_{k=0}^{N-1} z^{-k} G_k(z^N). \quad (12)$$

- This transfer function is a sum of $N$ branch filters with transfer functions $z^{-l} G_l(z^N)$ for $l = 0, 1, \cdots, N - 1$.
- Here, $G_l(z^N)$ is obtained from a 'conventional' transfer function $G_l(z)$ by replacing $z^{-1}$ by $z^{-N}$.
- Note that for the $l$th branch there are $l$ delay terms (for $l = 0$, there are no delays).
- The first diagram of the next page gives a very inefficient implementation form. Only the number of the additional delay terms has been minimized.
- A significant simplification can be achieved by transferring the decimation block before each $G_l(z^N)$. According to the identities considered on Page 18, the $G_l(z^N)$'s for $l = 0, 1, \cdots, N - 1$ are replaced in this case by the $G_l(z)$'s, as shown by the second diagram of the next page.
Polyphase Structures for Decimation: Intermediate Implementation Forms

Identical Structure

![Diagram of polyphase structures for decimation](image-url)
How to Perform the Decimation Effectively?

- If \( x(n) \) in the second diagram of the previous page is the input sample for \( G_0(z) \), then the input sample for the \( G_l(z) \) for \( l = 1, 2, \cdots, N - 1 \) is, due to the delays, \( x(n-l) \)'s for \( l = 1, 2, \cdots, N - 1 \).
- Therefore, if the decimation operations occur at the same time and the starting time for the first branch is \( n = 0 \), then the first samples at the inputs of the \( G_l(z) \)'s for \( l = 0, 1, \cdots, N - 1 \) are \( x(0-l) \) for \( l = 0, 1, \cdots, N - 1 \).
- When decimating for the second time, the input samples for the branches are \( x(N-l) \) for \( l = 0, 1, \cdots, N - 1 \) and for the third time \( x(2N-l) \) for \( l = 0, 1, \cdots, N - 1 \) and so on.
- Based on this fact, the overall system can be implemented using the commutative structure shown on the next page.
- This system works in such a way that the input data is first divided into the following block of \( N \) samples:
\[
\{ x(mN-(N-1)), x(mN-(N-2)), \cdots, x(mN) \}
\] for \( m = 0, 1 \cdots \).
• For the $m$th block of $N$ samples, the 'rotator' gives the first sample to $G_{N-1}(z)$, the second one to $G_{N-2}(z)$, and, finally, the last one to $G_0(z)$.
• After sharing the set of $N$ samples, each filter performs only one operation, thereby working at the lower output sampling rate of $f_s/N$.
• Finally, the outputs of the the branch filters are added to give one new output sample $y(m)$.
• Then, a new set of $N$ input samples for $m + 1$ are processed in a similar manner to generate one more output sample $y(m + 1)$. 
The corresponding commutative interpolator structure works in a similar manner as shown below.

- The main difference is that each of the branch filters gets the same input sample and perform one operation at the input sampling rate.
- The sampling rate is increased by $N$ by using the 'rotator' in such a manner that for each input sample, the first filter gives the first output sample, the second filter the second sample, and the, finally, the $N$th filter the $N$th output sample.
- This increases the sampling rate by a factor of $N$, as is desired. Note that, because of interpolation, each $G_i(z)$ is multiplied by $N$. 

\begin{align*}
\begin{tikzpicture}[node distance=1cm, auto]
  \node [block] (N0) {$NG_0(z)$};
  \node [block, below of=N0] (N1) {$NG_1(z)$};
  \node [block, below of=N1] (N2) {$NG_2(z)$};
  \node [block, below of=N2] (Nn) {$NG_{N-1}(z)$};
  \node [input, left of=N0] (in) {$x(n)$};
  \node [output, right of=Nn] (out) {$y(m)$};
  \node [scale=0.7] at (6.5, 0) {$f_s$};
  \node [scale=0.7] at (0, -1) {$f_s/N$};
  \draw [->] (in) -- (N0);
  \draw [->] (N0) -- (N1);
  \draw [->] (N1) -- (N2);
  \draw [->] (N2) -- (Nn);
  \draw [->] (Nn) -- (out);
\end{tikzpicture}
\end{align*}
Polyphase Decomposition for an FIR Filter

- The desired polyphase decomposition for FIR filters is very trivial.
- As an example, we consider the FIR transfer function of page 27:

\[ H(z) = \sum_{n=0}^{38} h(n)z^{-n}. \]  
(13)

- As shown on this page, this transfer function is expressible for \( N = 3 \) as

\[ H(z) = G_0(z^3) + z^{-1}G_1(z^3) + z^{-2}G_2(z^3), \]  
(14a)

where

\[ G_0(z^3) = \sum_{n=0}^{12} h(3n)z^{-3}, \]  
(14b)

\[ G_1(z^3) = \sum_{n=0}^{12} h(3n + 1)z^{-3}, \]  
(14c)

and

\[ G_2(z^3) = \sum_{n=0}^{12} h(3n + 2)z^{-3}. \]  
(14d)

- Note that each of these filters contain every \( N \)th (third) sample and the remaining samples are zero-valued.
- The first sample \( G_l(z^3) \) for \( l = 0, 1, 2 \) occurs at \( n = 0 \), \( n = 1 \), and \( n = 2 \), respectively.
• The filters in the commutative structures are for \( l = 0, 1, 2 \)

\[
G_l(z) = \sum_{n=0}^{12} h(3n + l)z^{-1}
\]  \hspace{1cm} (15)

• In this example, all the filters are of the same order.
• If for instance, the order of \( H(z) \) is 37, then \( G_2(z) \) is of order 11 (\( h(38) = 0 \)).
• In the general case, after knowing \( N \) and the impulse response of the FIR filter, \( G_l(z) \) simply contains the non-zero samples \( h(l + Nr) \) for \( r = 0, 1, \cdots \). Trivial!
Example Polyphase Decomposition for an FIR Filter in the $N = 3$ Case

\[
H(z) = G_0(z^3) + z^{-1}G_1(z^3) + z^{-2}G_2(z^3)
\]
Useful Polyphase Filters

• As shown earlier, FIR filters can always be expressed in the desired form.
• Nth-band FIR filters (to be considered later) are attractive since one of the branches is a pure delay.
• Half-band FIR filter extremely attractive: one of the two branches is a pure delay term and another branch is a linear-phase FIR filter so that the coefficient symmetry can be exploited.
• Nth-band IIR filters (to be considered later): the branch filters are allpass filters.
  • Best nonlinear-phase filters
  • Also approximately linear-phase filters can be designed by selecting one of the branches to be a pure delay term $z^{-K}$. 
Part II.B: Conventional Multistage Implementations

- If the overall sampling rate alteration ratio $N$ is can be factored into the product

$$N = \prod_{k=1}^{K} N_k, \quad (16)$$

where each $N_k$ is an integer, then the decimators and interpolators can be implemented using $K$ stages as shown by the first two diagrams on the next page.

- The main benefit lies in the fact that for the resulting multistage implementations the number of arithmetic operations reduce due to low-order subfilters.

- Also the number of multiplications and additions per input sample (output sample) in the decimation (interpolation) case becomes significantly lower at the expense of more control.

- For the design and analysis purposes, the first set of two diagrams can be redrawn into the equivalent single-stage forms shown by the second set of two diagrams of the next page.

- For the decimator, the transfer function of the single-stage equivalent is expressible as

$$H(z) = H_1(z)H_2(z^{N_1})H_3(z^{N_1N_2})\cdots H_K(z^{N_1N_2\cdots N_{K-1}}) \quad (17)$$
or

\[ H(z) = \prod_{k=1}^{K} H_k(z^{\tilde{N}_k}), \]  

(18a)

where

\[ \tilde{N}_1 = 1, \quad \tilde{N}_k = \prod_{l=1}^{k-1} N_l \text{ for } k = 2, 3, \ldots, K. \]  

(18b)

- For the interpolator, each \( H_k(z) \) is multiplied by \( N_k \).

---

**Single-Stage Equivalents**

\[ \begin{align*}
\frac{x(n)}{f_s} & \quad H_1(z) \quad \frac{\downarrow N_1}{N_1} \quad H_2(z) \quad \frac{\downarrow N_2}{N_2} \quad \cdots \quad H_K(z) \quad \frac{\downarrow N_K}{N_K} \\
& \quad \frac{y(m)}{f_s/N} 
\end{align*} \]

\[ \begin{align*}
\frac{x(n)}{f_{s/N}} & \quad \frac{\uparrow N_K}{N_K} \quad H_K(z) \quad \cdots \quad \frac{\uparrow N_2}{N_2} \quad H_2(z) \quad \frac{\uparrow N_2}{N_2} \quad H_1(z) \\
& \quad \frac{y(m)}{f_s} 
\end{align*} \]
Design Formulae for Cases A, B, and C

- For the single-stage equivalent, the overall frequency response is expressible as

\[ H(e^{j\omega}) = \prod_{k=1}^{K} H_k(e^{j\tilde{N}_k\omega}). \quad (19) \]

- Hence, all the filters except for the first one are periodic containing extra passbands and stopbands in the region \([0, \, f_s/2]\) in terms of 'real' frequencies or in \([0, \, \pi]\) in terms of the angular frequencies.

- Based on the periodicities, the passband and stopband regions for the \(K\) subfilters can be stated as follows in terms of angular frequencies (this fact becomes clear in connection with examples):

**Last stage (\(K\)th stage):**

1) The passband region is given in all the cases by

\[ \Omega_p^{(K)} = [0, \alpha/N_K]. \quad (20a) \]

2) The stopband region is given by

\[ \Omega_s^{(K)} = \begin{cases} 
[\pi/N_K, \, \pi] 
& \text{for Case A} \\
\left[\frac{(2l - \alpha)\pi}{N_K}, \, \min\left(\frac{(2l + \alpha)\pi}{N_K}, \, \pi\right)\right] 
& \text{for Case B} \\
[(2 - \alpha)\pi/N_K, \, \pi] 
& \text{for Case C.} \end{cases} \quad (20b) \]
**Stages for** $k = 1, 2, \cdots K - 1$:

1) The passband region is given in all the cases by

$$\Omega_p^{(k)} = [0, \alpha \hat{N}_k/N_K],$$

where

$$\hat{N}_1 = 1, \quad \hat{N}_k = \prod_{l=1}^{k-1} N_l \text{ for } k = 2, 3, \cdots, K.$$  \hspace{1cm} (21b)

2) The stopband region is given by

$$\Omega_s^{(k)} = \bigcup_{l=1}^{\lfloor N_k/2 \rfloor} \left[ \frac{(2l - \beta_k)\pi}{N_k}, \min\left(\frac{(2l + \beta_k)\pi}{N_k}, \pi\right) \right],$$

where

$$\beta_k = \begin{cases} 
\hat{N}_{k+1}/N & \text{for Case A} \\
\alpha \hat{N}_{k+1}/N & \text{for Case B} \\
(2 - \alpha) \hat{N}_{k+1}/N & \text{for Case C}. 
\end{cases}$$ \hspace{1cm} (22b)

- If the passband and stopband ripples for the overall filter are $\delta_p$ and $\delta_s$, then the corresponding ripples for the $K$ subfilters are $\delta_p/K$ and $\delta_s$.

- Note that if there are $K$ filters in cascade and all of them have the same passband with the maximum deviation from unity equal to $\delta_p/K$, then for the composite filter, $\left(1 \pm \delta_p/K\right)^K \approx 1 \pm \delta_p$.

- The examples to be presented illustrate the validity of the above design formulas to achieve the desired performance for the overall multistage implementation.
Example Multistage Designs

- In /home/ts/matlab/multirate, there is a program for designing multistage Case A, Case B, and Case C designs up to $K = 3$.
- As an example we consider the case with $\delta_p = 0.01, \delta_s = 0.001, N = 45$, and $\alpha = 0.5$.
- In order to emphasize the usefulness of using multistage implementations, we start with one-stage designs.
- For the one-stage designs, the filter orders are 484, 242, and 256 for Case A, Case B, and Case C, respectively.
- The six following pages illustrate the characteristics of these one-stage designs.
Characteristics of Example Case A One-Stage Decimator

One-stage Case A decimator of order 484

Amplitude in dB vs Frequency ($f_s/2$) or Angular Frequency ($\omega/\pi$)

Passband Amplitude vs Frequency ($f_s/2$) or Angular Frequency ($\omega/\pi$)
Characteristics of Example Case A One-Stage Decimator

![Impulse response graph](image1)

![Zero plot graph](image2)
Characteristics of Example Case B One-Stage Decimator

One-stage Case B decimator of order 242

Amplitude in dB vs Frequency/(f_s/2) or Angular Frequency ω/π

Passband Amplitude vs Frequency/(f_s/2) or Angular Frequency ω/π
Characteristics of Example Case B One-Stage Decimator
Characteristics of Example Case C One-Stage Decimator
Characteristics of Example Case C One-Stage Decimator

One-stage Case C decimator of order 256

Impulse response

n in samples

Zero plot for one-stage Case C decimator of order 256

Real Part

Imaginary Part
Example Case A Three-Stage Decimator

- For $K = 3$ stages, the best result is obtained by selecting $N_1 = 5$ and $N_2 = N_3 = 3$.
- In this case (see the formulae on Pages 34 and 35), $\alpha = 0.5$, $\hat{N}_1 = 1$, $\hat{N}_2 = 5$, $\hat{N}_1 = 15$.
- The passband regions for $H_1(z)$, $H_2(z)$, and $H_3(z)$ are thus in terms of the angular frequency $[0, \, 0.5\pi/45]$, $[0, \, 0.5\pi/9]$, and $[0, \, 0.5\pi/3]$.
- $\beta_1 = 5/45$ and $\beta_2 = 3/9$.
- The stopband region of $H_1(z)$ is thus a union of bands $[ (2 - 5/45)\pi/5, \, (2 + 5/45)\pi/5 ]$ and $[ (4 - 5/45)\pi/5, \, (2 + 5/45)\pi/5 ]$.
- The stopband regions for $H_2(z)$ and $H_3(z)$ are $[ (2 - 3/9)\pi/3, \, (2 + 3/9)\pi/3 ]$, and $[ \pi/3, \, \pi ]$, respectively.
- The passband and stopband ripples for the subfilters are $\delta_p/3 = 0.01/3$ and $\delta_s = 0.001$.
- The orders of $H_1(z)$, $H_2(z)$, and $H_3(z)$ to meet the criteria are 10, 10, and 37.
- When using direct-form decimator implementations, the overall number of multipliers is $6 + 6 + 19 = 31$ and the number of multiplications per input sample is $6/5 + 6/15 + 19/45 = 2.022$.
- The corresponding figures for the one-stage design of order 484 are 243 and $243/45 = 5.4$.
- The transfer function of the single-stage equivalent is
given by

\[ H(z) = H_1(z)H_2(z^5)H_3(z^{15}). \]

- The order of this equivalent is 615 compared to 484 required by a direct one-stage design.
- In the following there are seven pages illustrating the characteristics of our Case A three-stage design.
- In the first figure page it is seen that because of the periodicity of \( H_3(z^{15}) \) it takes care of the stopband shaping except for the extra unwanted passband and transition bands around the points \( 2k\pi/15 \) for \( k = 1, 2, \ldots, 7 \).
- \( H_2(z^5) \) attenuates these bands for \( k = 1, 2, 4, 5, 7 \), whereas \( H_1(z) \) takes care of the remaining bands.
- In the second figure page it is seen that \( H_3(z^{15}) \), \( H_2(z^5) \), and \( H_1(z) \) have the same passband region \([0, 0.5\pi/45]\).
- Since for all of them the passband ripple is less or equal to \( \delta_p/3 = 0.01/3 \), the passband ripple of the overall single-stage equivalent is less than or equal to \( \delta_p = 0.01 \).
Characteristics of Example Case A Three-Stage Decimator

Solid: $H_3(z^{15})$; Dashed: $H_2(z^5)$; Dot–dashed: $H_1(z)$

Single–stage equivalent $H(z) = H_1(z)H_2(z^5)H_3(z^{15})$
Characteristics of Example Case A Three-Stage Decimator

Passband: Solid: $H_3(z^{15})$; Dashed: $H_2(z^5)$; Dot-dashed: $H_1(z)$

Single-stage equivalent $H(z) = H_1(z)H_2(z^5)H_3(z^{15})$
Characteristics of Example Case A Three-Stage Decimator

$H_1(z)$ of order 10

Amplitude in dB

0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
0 20 40 60 80 100

Frequency/(f_s/2) or Angular Frequency $\omega/\pi$

Passband Amplitude

1.003 1.002 1.001 1 1.000 0.999 0.998 0.997
0 0.002 0.004 0.006 0.008 0.01
0 0.002 0.004 0.006 0.008 0.01
Characteristics of Example Case A Three-Stage Decimator

$H_2(z)$ of order 10

$H_2(z^5)$ of order 10 in $z^5$
Characteristics of Example Case A Three-Stage Decimator

\[ H_3(z) \text{ of order 37} \]

Amplitude in dB

\[ H_3(z^{15}) \text{ of order 37 in } z^{15} \]

Amplitude in dB

Passband Amplitude
Characteristics of Example Case A Three-Stage Decimator
Example Case B Three-Stage Decimator

- For $K = 3$ stages, the best result is again obtained by selecting $N_1 = 5$ and $N_2 = N_3 = 3$.
- In this case (see the formulae on Pages 34 and 35), $\alpha = 0.5$, $\hat{N}_1 = 1$, $\hat{N}_2 = 5$, and $\hat{N}_1 = 15$.
- The passband regions for $H_1(z)$, $H_2(z)$, and $H_3(z)$ are thus in terms of the angular frequencies $[0, 0.5\pi/45]$, $[0, 0.5\pi/9]$, and $[0, 0.5\pi/3]$.
- $\beta_1 = 0.5 \cdot 5/45$ and $\beta_2 = 0.5 \cdot 5/9$.
- The stopband region of $H_1(z)$ is thus a union of bands $[(2 - 2.5/45)\pi/5$, $(2 + 2.5/45)\pi/5]$ and $[(4 - 2.5/45)\pi/5$, $(2 + 2.5/45)\pi/5]$. 
- The stopband regions for $H_2(z)$ and $H_3(z)$ are $[(2 - 1.5/9)\pi/3$, $(2 + 1.5/9)\pi/3]$, and $[1.5\pi/3, 2.5\pi/3]$, respectively.
- The passband and stopband ripples for the subfilters are again $\delta_p/3 = 0.01/3$ and $\delta_s = 0.001$.
- The orders of $H_1(z)$, $H_2(z)$, and $H_3(z)$ to meet the criteria are 8, 8, and 16.
- When using direct-form decimator implementations, the overall number of multipliers is $5 + 5 + 9 = 19$ and the number of multiplications per input sample is $5/5 + 5/15 + 9/45 = 1.5333$.
- The corresponding figures for the one-stage design of order 242 are 122 and $122/45 = 2.7111$.  

• The transfer function of the single-stage equivalent is given by

\[ H(z) = H_1(z)H_2(z^5)H_3(z^{15}). \]

• The order of this equivalent is 288 compared to 242 required by a direct one-stage design.

• In the following there are seven pages illustrating the characteristics of our Case B three-stage design.

• The main difference compared to Case A is that now those frequency components aliasing in the decimation case into the region \([0, 0.5\pi/45]\) are attenuated.

• Aliasing is allowed into the region \([0.5\pi/45, \pi/45]\).
Characteristics of Example Case B Three-Stage Decimator

Solid: \( H_3(z^{15}) \); Dashed: \( H_2(z^5) \); Dot–dashed: \( H_1(z) \)

Single-stage equivalent \( H(z) = H_1(z)H_2(z^5)H_3(z^{15}) \)
Characteristics of Example Case B Three-Stage Decimator

Impulse responses

$n$ in samples

$H_1(z)$

$H_2(z)$

$H_3(z^5)$

Impulse responses

$n$ in samples

$H_1(z)$

$H_2(z^5)H_3(z^5)$
Example Case C Three-Stage Decimator

- For $K = 3$ stages, the best result is again obtained by selecting $N_1 = 5$ and $N_2 = N_3 = 3$.
- In this case (see the formulae on Pages 34 and 35), $\alpha = 0.5$, $\hat{N}_1 = 1$, $\hat{N}_2 = 5$, $\hat{N}_1 = 15$.
- The passband regions for $H_1(z)$, $H_2(z)$, and $H_3(z)$ are thus in terms of the angular frequency $[0, 0.5\pi/45]$, $[0, 0.5\pi/9]$, and $[0, 0.5\pi/3]$.
- $\beta_1 = 1.5 \cdot 5/45$ and $\beta_2 = 1.5 \cdot 5/9$.
- The stopband region of $H_1(z)$ is thus a union of bands $[(2 - 7.5/45)\pi/5, (2 + 7.5/45)\pi/5]$ and $[(4 - 7.5/45)\pi/5, (2 + 7.5/45)\pi/5]$.
- The stopband regions for $H_2(z)$ and $H_3(z)$ are $[(2 - 4.5/9)\pi/3, (2 - 4.5/9)\pi/3]$, and $[1.5\pi/3, \pi]$, respectively.
- The passband and stopband ripples for the subfilters are again $\delta_p/3 = 0.01/3$ and $\delta_s = 0.001$.
- The orders of $H_1(z)$, $H_2(z)$, and $H_3(z)$ to meet the criteria are 11, 12, and 17.
- When using direct-form decimator implementations, the overall number of multipliers is $6 + 7 + 9 = 22$ and the number of multiplications per input sample is $6/5 + 7/15 + 9/45 = 1.8667$.
- The corresponding figures for the one-stage design of order 256 are 129 and $129/45 = 2.8667$. 
• The transfer function of the single-stage equivalent is given by

\[ H(z) = H_1(z)H_2(z^5)H_3(z^{15}). \]

• The order of this equivalent is 326 compared to 256 required by a direct one-stage design.

• In the following there are seven pages illustrating the characteristics of our Case C three-stage design.

• The main difference compared to Case B is that now aliasing is allowed into the transition band \([0.5\pi/45, \pi/45]\) only from band \([\pi/45, 1.5\pi/45]\).

• Note that because the stopband edge of \(H_3(z)\) is wider than in Case A, higher filter orders for \(H_1(z)\) and \(H_2(z)\) are required to attenuate the extra unwanted passband and transition band regions of \(H_3(z^{15})\).
Characteristics of Example Case C Three-Stage Decimator

Solid: $H_3(z^{15})$; Dashed: $H_2(z^5)$; Dot-dashed: $H_1(z)$

Single-stage equivalent $H(z) = H_1(z)H_2(z^5)H_3(z^{15})$
Characteristics of Example Case C Three-Stage Decimator

Impulse responses

$H_1(z)$

$H_2(z)$

$H_3(z^5)$

$H_1(z^5)H_3(z^5)$

$n$ in samples

Impulse responses

$H_1(z)$

$H_3(z^5)$

$H_1(z^5)H_3(z^5)$

$n$ in samples
Part II.E: $N$th-Band FIR Digital Filters

- This is a pile of lecture notes on the design and properties of $N$th-band FIR digital filters.
- An interested reader should read the following articles:


Division of \textit{Nth-Band Linear-Phase FIR Filters into Subclasses}

- \textit{Nth-band linear-phase FIR filters can be divided into the following subclasses:}
  \begin{enumerate}
  \item I. Nonseparable single-stage filters
  \item II. Separable single-stage filters
  \item III. Nonseparable multistage filters
  \item IV. Separable multistage filters
  \end{enumerate}

- In the sequel, all these filter classes are considered.
- For designing these filters, there exists a MATLAB routine, \texttt{nykki.m}, in \texttt{/home/ts/matlab/multirate}.
What are Nonseparable Single-Stage Nth-Band Linear-Phase FIR Filters?

- Consider a Type I linear-phase FIR filter with transfer function \( h(2M - n) = h(n) \) for \( n = 0, 1, \ldots, M - 1 \)

\[
H(z) = \sum_{n=0}^{2M} h(n)z^{-n}
\]  

(27)

- This filter is defined to be an \( N \)-th band filter if its coefficients (see the next page for \( N = 4 \)) satisfy

\[
h(M) = \frac{1}{N} \quad (28a)
\]

\[
h(M + rN) = 0 \quad \text{for} \quad r = \pm 1, \pm 2, \ldots, \lfloor M/L \rfloor. \quad (28b)
\]

=> Reduction in implementation complexity, especially for small \( N \)!
Time-Domain and Frequency-Domain Conditions for Lowpass $N$th-Band Linear-Phase FIR Filters
Frequency-Domain Conditions for Lowpass Nth-Band Linear-Phase FIR Filters

- The time-domain conditions imply that the zero-phase frequency response as given by

\[ H(\omega) = 1/N + 2 \sum_{n=1}^{M} h(M - n) \cos(n\omega) \]  \hspace{1cm} (29)

satisfies

\[ \sum_{r=0}^{N-1} H(\omega + 2\pi r/N) = 1. \] \hspace{1cm} (30)

- This means that in the lowpass case the passband and stopband edges are related through (see the previous page for \( N = 4 \))

\[ \omega_p = (1 - \rho)\pi/N, \quad \omega_s = (1 + \rho)\pi/N, \] \hspace{1cm} (31)

where \( \rho > 0 \).

- Furthermore, \( \delta_p \leq (N - 1)\delta_s \) and a 6-dB point is approximately at \( \omega = \pi/N \).

- This means that for a small stopband ripple \( \delta_s, \delta_p \), the maximum deviation of \( H(\omega) \) from unity in the passband, is guaranteed to be small.

- Therefore, in many cases, the filter optimization can concentrate on shaping the stopband response.
Approximation Criteria

- There exist the following two criteria:

**Minimax approximation:** Find the filter unknowns to minimize

$$\epsilon_\infty = \max_{\omega \in [(1+\rho)\pi/N, \pi]} |H(\omega)|. \quad (32)$$

**Least-squared approximation:** Find the filter unknowns to minimize

$$\epsilon_2 = \int_{(1+\rho)\pi/N}^{\pi} |H(\omega)|^2 d\omega. \quad (33)$$
**Example:** $\rho = 0.2$ and $N = 8$

**Minimax design:** The minimum stopband attenuation is at least 40 dB, that is, $\delta_s$ is less than or equal to 0.01.
- The minimum even order to meet the given criteria is $2M = 74$.
- By exploiting the coefficient symmetry and the fact that $h(37 \pm 8r) = 0$ for $r = 1, 2, 3$ and assuming that the implementation of $h(37) = 1/8 = 2^{-3}$ requires no multipliers, the overall number of multipliers is 32.

**Least-squared design:** It is desired to minimize the filter stopband energy for $2M = 74$.
- The following four pages show the characteristics of the optimized minimax and least-squared designs.
Example Nonseparable Minimax Eighth-Band ($N = 8$) FIR filter
Example Nonseparable Minimax Eighth-Band ($N = 8$) FIR filter
Example Nonseparable Least-Squared Eighth-Band \((N = 8)\) FIR filter
Example Nonseparable Least-Squared Eighth-Band \((N = 8)\) FIR filter
What are Multistage $N$th-Band FIR Filters?

- Let $N$ be factorizable as

$$N = N_1 \cdot N_2 \cdots N_K,$$  \hspace{1cm} (39)

where the $N_k$'s are integers and the $H_k(z)$'s for $k = 1, 2, \cdots, K$ be linear-phase $N_k$th-band filters, that is,

$$H_k(z) = \sum_{n=0}^{2M_k} h_k(n)z^{-n},$$ \hspace{1cm} (40a)

where

$$h_k(M_k) = 1/N_k$$ \hspace{1cm} (40b)

and

$$h_k(M_k \pm rN_k) = 0 \quad \text{for} \quad r = 1, 2, \cdots, \lfloor M_k/N_k \rfloor.$$ \hspace{1cm} (40c)

- Then

$$H(z) = \prod_{k=1}^{K} H_k(z^{\hat{N}_k}),$$ \hspace{1cm} (41a)

where

$$\hat{N}_1 = 1, \quad \hat{N}_k = \prod_{l=1}^{k-1} N_k, \quad k = 2, 3, \cdots, K$$ \hspace{1cm} (41b)

is an $N$th-band linear-phase FIR filter.

• The overall order of this filter is $2M = 2(\hat{N}_1M_1 + \hat{N}_2M_2 + \cdots + \hat{N}_KM_K)$

• The main advantage of the above decomposition is that the number of multipliers is significantly reduced when compared with the direct-form implementation.
Furthermore, if the overall filter is used for decimation or interpolation by a factor of $N$, then it can be implemented as shown below.

Note that in these implementations unit delays are used.
About Multi-Stage Nth-band Filters  (added by MR)

Consider a multistage FIR decimator (or interpolator), where the overall sampling rate conversion factor is partitioned as

\[ N = N_1 \cdot N_2 \cdot \ldots \cdot N_K \]

and each of the stages is an \( N_k \)-th-band filter. **It can be proven that the cascade is an \( N \)-th-band filter!**

This makes it possible to combine the benefits of multistage decimators/interpolators and \( N \)-th-band FIR filters. One example of a two-stage design is shown on the following pages.

A separable \( N \)-th-band filter is a cascade of two filters which have the same amplitude responses and complementary phase responses, such that the overall system has \( N \)-th-band characteristics and a linear phase response.

\( N \)-th-band FIR filters and Nyquist pulse shaping filters are closely related: \( N \)-th-band filter has zero intersymbol interference, i.e., it satisfies the Nyquist pulse shaping criteria, both in time and frequency domains.

- Raised-cosine filter is an \( N \)-th-band filter.
- Cascade of two root-raised-cosine filters is a special case of separable \( N \)-th-band filters, where both stages have linear phase.
- \( N \)-th-band filter design methods can be used for optimizing pulse shaping filters.
- Later, an example of a two-stage separable \( N \)-th-band filter is shown. In this design, the stages have non-symmetric impulse responses, but the cascade impulse response is symmetric.
  - This is a matched filter pair and could, in principle, be used as RX and TX filters in a communication system.
  - Such a design may result on lower filter orders for given frequency domain specifications.
  - However, non-symmetric transmit filter impulse response usually increases considerably the peak-to-average power ratio of the transmitted signal, and is thus not useful in practice.
**Example:** $\rho = 0.2$ and $N = 8$

**Minimax design with $K = 2$:** The given criteria are met by $N_1 = 4$, $N_2 = 2$, $2M_1 = 14$, and $2M_2 = 18$. The minimum stopband attenuation is at least 40 dB, that is, $\delta_s$ is less than or equal to 0.01.

- By exploiting the coefficient symmetries and the facts that $h_1(3) = h_1(11) = 0$ and $h_2(9 \pm 2r) = 0$ for $r = 1, 2, 4$ and assuming that the implementations of $h_1(7) = 1/4 = 2^{-2}$ and $h_1(9) = 1/2 = 2^{-1}$ require no multipliers, the overall design has $6 + 5 = 11$ multipliers.

- The number of delays is $2(M_1 + N_1M_2) = 86$.

- In the following, there are 5 pages illustrating the characteristics of this design.

- The corresponding one-stage design requires $2M = 74$, having 32 multipliers and 74 delays.
Nonseparable Two-Stage Minimax Eighth-Band $(N = 8)$ FIR filter
Nonseparable Two-Stage Minimax Eighth-Band $(N = 8)$ FIR filter
Separable Two-Stage Minimax Eighth-Band ($N = 8$) FIR filter

Impulse responses for single-stage equivalents

$H(z) = T(z)R(z)$

$n$ in samples
II.F: Half-Band FIR Filters

- For a half-band linear-phase FIR filter, the transfer function is of the form

\[ H(z) = \sum_{n=0}^{2M} h[n]z^{-n}, \quad h[2M - n] = h[n], \quad (49) \]

where \( M \) is odd.

- For these filters,

\[ h[M] = 1/2 \quad (50a) \]

\[ h[M + 2r] = 0 \quad \text{for} \quad r = \pm 1, \pm 2, \ldots, \pm (M - 1)/2. \quad (50b) \]
Efficient Implementation of a Half-Band Filter

- An implementation for the half-band filter as a parallel connection of $G(z^2)$ and $(1/2)z^{-M}$ is shown below.
- This implementation is very attractive as in this case the complementary highpass output having the zero-phase frequency response $1 - H(\omega)$ is obtained directly by subtracting $G(z^2)$ from $(1/2)z^{-M}$.
- The term $z^{-M}$ can be shared with $G(z^2)$.
- The number of non-zero coefficients in $G(z^2)$ is $M + 1$. By exploiting the symmetry in these coefficients, only $(M + 1)/2$ multipliers ($M$ is odd) are needed to implement a lowpass–highpass filter pair of order $2M$. 

![Diagram](image.png)
Responses for a Complementary Half-Band Filter Pair of order 34 for $\omega_p = 0.4\pi$.

- The implementation of this filter pair requires only nine multipliers.
Efficient Implementations for Half-Band FIR Decimators and Interpolators

- The transfer function of the half-band filter is expressible as the following polyphase form:

\[
H(z) = G_0(z^2) + z^{-1}G_1(z^2), 
\]

where

\[
G_0(z) = G(z) \quad \text{and} \quad G_1(z) = (1/2)z^{-(M-1)/2}. 
\]

- Based on these relations, the decimation and interpolation filters for sampling rate alteration by a factor of two can be effectively implemented using the structures shown below.

\[(M+1)/4\] multiplications per input sample needed for implementing a filter of length \(M+1\).

=> A cascade of half-band filters is often a very efficient choice for multirate signal processing!
Design of a Classical Lowpass and/or Highpass Filter as a Parallel Connection of Two Allpass Filters

- Any odd-order classical lowpass or highpass IIR filter (Butterworth, Chebyshev, inverse Chebyshev, elliptic) can be implemented as a parallel connection of two allpass filters. Also bandpass and bandstop filters can be designed, with certain other restrictions on the filter order.

- This was first discovered in the case of wave digital lattice filters, in which case the allpass filters are implemented with special structures.

Design procedure using any filter package for classical lowpass/highpass IIR filters:

1. Design a classical odd-order IIR filter of order \( 2N + 1 \).
2. Include in \( A_1(z) \) the outermost pole pair, the third outermost pole pair, the fifth outermost pole pair, and so on. The other pole pairs belong to \( A_2(z) \). In the last step a single pole, instead of a pole pair, is included in one of the allpass filters.
3. Select the structures of the allpass filters and compute the allpass filter coefficients. First- and second-order allpass filters are commonly used as building blocks:

\[
A^{(1)}(z) = \frac{a + z^{-1}}{1 + az^{-1}}, \quad A^{(2)}(z) = \frac{a_2 + a_1z^{-1} + z^{-2}}{1 + a_1z^{-1} + a_2z^{-2}}
\]

Alternating distribution of poles to the allpass subfilters:
ALLPASS FILTER

- Transfer function of order $N$

$$A_N(z) = \frac{z^{-N}D_N(z^{-1})}{D_N(z)} = \frac{a_N + a_{N-1}z^{-1} \cdots a_1z^{-(N-1)} + z^{-N}}{1 + a_1z^{-1} + \cdots + a_{N-1}z^{-(N-1)} + a_Nz^{-N}}$$

- For the stable filter, the poles [roots of $D_N(z)$] are inside the unit circle.

- $A_N(e^{j\omega}) = e^{j\phi_N(\omega)}$, $\phi_N(\omega) = \arg A_N(e^{j\omega})$.

- As $\omega$ varies from 0 to $\pi$, $\phi_N(\omega)$ decreases monotonically from 0 to $-N\pi$; $N = 4$.
BUILDING A LOWPASS FILTER

- A parallel connection of $A_N(z)$ and $A_M(z)$ with $N = M + 1$: $H_{LP}(z) = \frac{1}{2} [A_M(z) + A_N(z)]$.

- Desired responses: $|H_{LP}(e^{j\omega})| = \frac{1}{2} |e^{j\phi_M(\omega)} + e^{j\phi_N(\omega)}|$.
RESPONSES FOR SINUSOIDAL SIGNALS

INPUT SIGNAL: \( x(n) = C \sin(n\omega) \)

\[
\text{OUTPUT OF } A_M(z)/2:\ 
y_1(n) = \frac{C}{2} \sin(n\omega + \phi_M(\omega))
\]

\[
\text{OUTPUT OF } A_N(z)/2:\ 
y_2(n) = \frac{C}{2} \sin(n\omega + \phi_N(\omega))
\]

OVERALL OUTPUT
\( y(n) = y_1(n) + y_2(n) \)

\[
\phi_M(\omega) = \phi_N(\omega) \quad \rightarrow
\]
\[
y(n) = C \sin(n\omega + \phi_M(\omega))
\]
\[
y_1(n) \text{ and } y_2(n) \text{ are in phase}
\]

\[
\phi_M(\omega) = \phi_N(\omega) + \pi \quad \rightarrow
\]
\[
y(n) = 0
\]
\[
y_1(n) \text{ and } y_2(n) \text{ are out of phase}
\]
BUILDING A HIGHPASS FILTER

- A parallel connection of $A_M(z)$ and $-A_N(z)$ with $N=M+1$: $H_{HP}(z) = [A_M(z) - A_N(z)]/2$.

- Desired responses: $|H_{HP}(e^{j\omega}) = \frac{1}{2}|e^{j\phi_M(\omega)} - e^{j\phi_N(\omega)}|$.
A COMPLEMENTARY FILTER PAIR

- Implementation:

- This pair is a power-complementary filter pair since

$$|H_{LP}(e^{j\omega})|^2 + |H_{HP}(e^{j\omega})|^2 = 1.$$  

- Responses:

A complementary filter pair can be implemented (essentially) with the complexity of a single filter!
Special Case: Halfband IIR Filters

- In this case, all the poles are on the imaginary axis, and the first-order section reduces to a pure delay. The second-order allpass sections are now of the form

\[ A^{(2)}_{HB}(z) = \frac{a_2 + z^{-2}}{1 + a_2 z^{-2}} \]

This can be implemented using a first-order allpass section where each unit delay has been replaced by two delays.

- Now only N coefficients are needed to implement a filter of order \(2N+1\). => Very efficient!

Properties of halfband IIR filters:
- Transition band is symmetric around quarter of the sampling rate.
- Lowpass and highpass filters have the same stopband attenuations and pass band ripples. The passband and stopband ripples are related through the power complementarity property.
- For any reasonable stopband attenuation is very small.

Halfband IIR filter is a special case of \(N\)-th-band IIR filters, which will be discussed next. The connection to classical filter theory is clear only in the half-band case.
Special Case 2: Hilbert Transform

- We can start with a lowpass IIR filter which can be implemented as a parallel connection of two allpass filters.
- Then we introduce a frequency translation by $\pi$ (i.e., quarter of the sampling rate):

$$ H_{HT}(z) = A_1(-z^2) + jz^{-1}A_2(-z^2) $$

- The frequency translation is achieved by replacing each delay $z^{-1}$ by $jz^{-1}$.
- In the general lowpass case, the filter becomes complex.
- Using a halfband IIR filter, the transfer function takes a special form and the implementation becomes simple (ignoring the scaling factor 0.5 for simplicity)

- The phase difference of the two allpass filters is approximately $\pi/2$ outside the transition band regions. Increasing the stopband attenuation improves the phase approximation.
- Using approximately linear phase halfband IIR filter (see later) results in a Hilbert transformer in the usual sense.
Part II.D: Nth-Band Recursive Digital Filters

- This is a very short pile of lecture notes on the design and properties of Nth-band recursive digital filters.
- An interested reader should read the articles:
- There exists an efficient FORTRAN routine for designing these filters. A MATLAB program is coming up next.
Filter Structures

- For these filters, the overall transfer function is of the following polyphase form:

\[ H(z) = \frac{1}{N} \sum_{n=0}^{N-1} z^{-n} G_n(z^N), \]  

(23a)

where

\[ G_n(z^N) = z^{-k_n N} \frac{z^{-K_n N} D_n(z^{-N})}{D_n(z^N)}. \]  

(23b)

- Here, \( G_n(z^N) \) is obtained from

\[ G_n(z) = z^{-k_n} A_n(z), \quad A_n(z) = \frac{z^{-K_n} D_n(z^{-1})}{D_n(z)} \]  

(24)

by replacing each unit delay by \( N \) delays. \( A_n(z) \) is an allpass filter of order \( K_n \).

- An implementation of the above transfer function is shown below.
Commutative Models for $N$th-Band Recursive Filters in Decimation and Interpolation Cases
Advantages of Nth-Band Recursive Filters

1. Lowest multiplication rate among known recursive decimators and interpolator.

2. Approximately linear phase designs possible by selecting one of the all-pass filters to be a pure delay.

3. Low noise and sensitivity, limit cycles can be suppressed.

4. Very modular structures, can be constructed using first- and second-order allpass filters.

5. If phase distortion is not of interest, only first-order blocks are present.
Filter Properties

- Since the $G_n(z)$'s are allpass filters, the frequency response of the overall filter is expressible as

$$H(e^{j\omega}) = \frac{1}{N} \sum_{n=0}^{N-1} e^{j\phi_n(\omega)},$$

(25a)

where

$$\phi_n(\omega) = -n\omega + \arg[G_n(e^{jN\omega})].$$

(25b)

- It can be shown that this frequency response satisfies

$$\sum_{r=0}^{N-1} |H(e^{j(\omega + 2\pi r/N)})|^2 = 1.$$  

(26)

- This states some limitations on the frequency-domain behavior of the filter as we shall see later on.

- In order to arrive at a filter with good amplitude characteristics, the orders of the branch filters have to be selected properly (see the above-mentioned articles for details)

- We start with an example and, then, generalize the results.
Example Filter: $N = 5$, $G_0(z)$, $G_1(z)$, and $G_2(z)$ are Cascades of Two First-Order Allpass Sections, whereas $G_3(z)$ and $G_4(z)$ are First-Order Allpass Sections

- $G_0(z)$: poles at $z = -0.06064045371$, $-0.7032070490$
- $G_1(z)$: poles at $z = -0.1465783514$, $-0.8191991118$
- $G_2(z)$: poles at $z = -0.2502545637$, $-0.9210697554$
- $G_3(z)$: a poles at $z = -0.3770169094$
- $G_4(z)$: a pole at $z = -0.5665661715$
- The following page shows the phase responses of the branches and the resulting amplitude response.

- In the region $[0, 0.8\pi/5]$, the phases are almost the same. $\Rightarrow |H(e^{j\omega})| = \frac{1}{5} |\sum_{n=0}^{4} e^{j\phi_n(\omega)}| \approx 1$.
- In the region $[1.2\pi/5, 2.8\pi/5]$, the differences between the consecutive phases are approximately $2\pi/5$. $\Rightarrow H(e^{j\omega}) = \frac{1}{5} \sum_{n=0}^{4} e^{j\phi_n(\omega)} \approx 0$.
- In the region $[3.2\pi/5, 4.8\pi/5]$, the differences between the consecutive phases are approximately $4\pi/5$. $\Rightarrow H(e^{j\omega}) = \frac{1}{5} \sum_{n=0}^{4} e^{j\phi_n(\omega)} \approx 0$. 

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Responses for an Example Filter

First: $z^{-3}G_1(z^2)$
Second: $z^{-2}G_3(z^5)$
Third: $G_0(z^5)$
Fourth: $z^{-1}G_2(z^5)$
Fifth: $z^{-2}G_2(z^5)$

Phase responses of the branches as fractions of $\pi$
Filter Properties in General

- The passband edge must satisfy $\omega_p < \pi/N$.
- The filter has stopbands of width $2\omega_p$ around the points $k2\pi/N$ for $k = 1, 2, \ldots$
- The filter has always peaks around the points $(2k+1)\pi/N$ for $k = 1, 2, \ldots$ and the stopband edge cannot be located at $\pi/N$.
- The filter satisfies automatically the Case B specifications considered earlier.
- The Case A specifications can be met by using in the decimation (interpolations) case at the filter output (input) an extra filter stage, as we shall see later on.
Example Nonlinear-Phase Filter: \( N = 5 \), 
\( \omega_p = 0.8\pi/N \) and at Least a 60-dB Attenuation in the Stopbands

- These criteria are met as follows:
- \( G_0(z) \) consists of three first-order allpass filters with poles at \( z = -0.03247627480, z = -0.4519480048, z = -0.9477051753 \).
- \( G_1(z) \) has two first-order allpass filters with poles at \( z = -0.08029157130 \) and \( z = -0.5548998293 \).
- \( G_2(z) \) has two first-order allpass filters with poles at \( z = -0.1417079348 \) and \( z = -0.6883346404 \).
- \( G_3(z) \) has two first-order allpass filters with poles at \( z = -0.2320513100 \) and \( z = -0.7961481351 \).
- \( G_4(z) \) has two first-order allpass filters with poles at \( z = -0.3532045984 \) and \( z = -0.8755417392 \).
- The next page shows the responses for this design.
Responses for the Nonlinear-Phase Design
Comments

- If aliasing is allowed into the transition band 
  \([0.8\pi/5, \pi/5]\), then this design can be directly used.
- The peaks in the stopband region occur in the re-
gions that alias into this band, whereas regions aliasing
into the passband \([0, 0.8\pi/5]\) are well attenuated.
- The Case B specifications are thus met.
Approximately Linear-Phase Design

- These criteria are met as follows:

- $G_0(z)$ consists of two first-order allpass filters with poles at $z = 0.3539559551$ and $z = -0.600954816$; and two second-order sections with poles at $z = 0.36045530 \exp(\pm j 0.32083089 \pi)$ and at $z = 0.39292469 \exp(\pm j 0.63853828 \pi)$.

- $G_1(z)$ consists of two first-order allpass filters with poles at $z = 0.3362484694$ and $z = -0.7296322306$; and two second-order sections with poles at $z = 0.34509052 \exp(\pm j 0.32657981 \pi)$ and at $z = 0.38698306 \exp(\pm j 0.65132553 \pi)$.

- $G_2(z)$ consists of two first-order allpass filters with poles at $z = 0.2969941345$ and $z = -0.8290479$; and two second-order sections with poles at $z = 0.30739674 \exp(\pm j 0.33448924 \pi)$ and at $z = 0.35542291 \exp(\pm j 0.67048455 \pi)$.
• $G_3(z)$ consists of two first-order allpass filters with poles at $z = 0.2422225263$ and $z = -0.9169533323$; and two second-order sections with poles at $z = 0.2528221 \exp(\pm j0.34583305\pi)$ and at $z = 0.30084497 \exp(\pm j0.6997103\pi)$.

• $G_4(z) = z^{-5}$

• The following two pages show the responses of this design.

• Note that by selecting one branch filter to be a pure delay term makes the phase response very linear in the passband at the expense of increased orders for other filter branches.
Responses for the Approximately Linear-Phase Design
Responses for the Approximately Linear-Phase Design

- Note the extremaly small phase error.
Conventional Half-Band IIR Filters

- The transfer function is expressible as

\[ H(z) = (1/2)[A_0(z^2) + z^{-1}A_1(z^2)], \quad (59a) \]

where

\[ A_0(z) = \prod_{k=1}^{K_0} \frac{a_k^{(0)} + z^{-1}}{1 + a_k^{(0)} z^{-1}} \quad (59b) \]

and

\[ A_1(z) = \prod_{k=1}^{K_1} \frac{a_k^{(1)} + z^{-1}}{1 + a_k^{(1)} z^{-1}} \quad (59c) \]

are allpass filters consisting of first-order sections.

- All the \(a_k^{(0)}\)'s and \(a_k^{(1)}\)'s are negative so that the poles of the overall filter are located at the imaginary axis. One pole is at the origin.
- The overall order of this filter is \(2(K_0 + K_1) + 1\).
- Here, \(K_0 = K_1\) or \(K_0 = K_1 + 1\) for a lowpass filter.
- For a power-complementary highpass filter (\(|H(e^{j\omega})|^2 + |G(e^{j\omega})|^2 = 1\)), the transfer function is simply given by

\[ G(z) = (1/2)[A_0(z^2) - z^{-1}A_1(z^2)]. \quad (60) \]
Example and Properties of Conventional Half-Band IIR Filters

- Consider a power-complementary filter pair with $K_0 = K_1 = 2$, $a_1^{(0)} = 0.07986644637$, $a_2^{(0)} = 0.5453236405$, $a_1^{(1)} = 0.2838293419$, and $a_2^{(1)} = 0.8344118932$.
- In the following, there are two pages illustrating the performance of both $H(z) = (1/2)[A_0(z^2) + z^{-1}A_1(z^2)]$ and $G(z) = (1/2)[A_0(z^2) - z^{-1}A_1(z^2)]$.
- $H(z)$ is a special elliptic filter of order $2(K_0 + K_1) + 1 = 9$ designed to have the stopband edge at $\omega_s = 0.6\pi$:
  1) The passband edge is located at $\omega_p = \pi - \omega_s = 0.4\pi$.
  2) $|H(e^{j\omega_0})|^2 + |H(e^{j(\pi-\omega_0)})|^2 = 1$ for any $\omega_0 \in [0, \pi]$.
  3) $|G(e^{j\omega})|^2 = 1 - |H(e^{j(\pi-\omega)})|^2$.
  4) If the maximum deviation of $|H(e^{j\omega})|^2$ from zero in the stopband is $\delta$, then the maximum deviation from unity in the passband (the maximum value is unity) is also $\delta$. Hence, the passband variation is very small.
  5) All the poles of $H(z)$ and $G(z)$ are located on the imaginary axis.
  6) Only $K_0 + K_1 = 4$ multipliers are required to implement an elliptic filter of order $2(K_0 + K_1) + 1 = 9$ when using the lattice wave digital filter structures.
- An efficient structure for simultaneously implementing $H(z)$ and $G(z)$ is shown below.
Example Conventional Half-Band IIR Filter

Solid: $A_0(z^2)$; Dashed: $z^{-1}A_1(z^2)$

Phase responses as fractions of $\pi$

Angular frequency $\omega$

Responses for the power-complementary filter pair

Amplitude in dB

Angular frequency $\omega$
Example Conventional Half-Band IIR Filter

\[ H(z) = A_0(z^2) + z^{-1}A_1(z^2) \]

\[ G(z) = A_0(z^2) - z^{-1}A_1(z^2) \]
Approximately Linear-Phase Half-Band IIR Filters

- The transfer function is expressible as
  \[ H(z) = (1/2)[A_0(z^2) + z^{-K_1}], \quad (61a) \]
  where
  \[ A_0(z) = \prod_{k=1}^{K_0^{(1)}} \frac{a_k + z^{-1}}{1 + a_k z^{-1}} \prod_{k=1}^{K_0^{(2)}} \frac{c_k + b_k z^{-1} + z^{-2}}{1 + b_k z^{-1} + c_k z^{-1}} \quad (61b) \]
  is an allpass filters consisting of first-order and second-order sections.
- This filter is a special case where the second allpass filter is a pure delay term.
- The overall order of this filter is \( K_0 + 2(K_1 + 2K_1) \).
- Here, \( K_1 = 2(K_1^{(1)} + 2K_1^{(2)}) - 1 \).
- For a power-complementary highpass filter \(|H(e^{j\omega})|^2 + |G(e^{j\omega})|^2 = 1\), the transfer function is simply given by
  \[ G(z) = (1/2)[A_0(z^2) - z^{-K_1}]. \quad (62) \]
Example and Properties of Approximately Linear-Phase Half-Band IIR Filters

- Consider a power-complementary filter pair with $K_1 = 13$, $K_0^{(1)} = 1$, $K_0^{(2)} = 3$, $a_1 = 0.7977649139$, $b_1 = -0.6270566870$, $c_1 = 0.1207727840$, $b_2 = -0.1697974960$, $c_2 = 0.1300746053$, $b_3 = 0.4897869030$, and $c_3 = 0.1663758012$.

- In the following, there are three pages illustrating the performance of both $H(z) = (1/2)[A_0(z^2) + z^{-13}]$ and $G(z) = (1/2)[A_0(z^2) - z^{-13}]$.

- $H(z)$ has the stopband edge at $\omega_s = 0.6\pi$ and the required stopband attenuation is 60 dB:
  1) The passband edge is located at $\omega_p = \pi - \omega_s = 0.4\pi$.
  2) $|H(e^{j\omega_0})|^2 + |H(e^{j(\pi-\omega_0)})|^2 = 1$ for any $\omega_0 \in [0, \pi]$.
  3) $|G(e^{j\omega})|^2 = 1 - |H(e^{j(\pi-\omega)})|^2$.
  4) If the maximum deviation of $|H(e^{j\omega})|^2$ from zero in the stopband is $\delta$, then the maximum deviation from unity in the passband (the maximum value is unity) is also $\delta$. Hence, the passband variation is very small.
  5) The poles of $H(z)$ and $G(z)$ are located either on the imaginary axis are they are located symmetrically with respect to the imaginary axis. $K_1 = 13$ poles are at the origin.
  6) Only $K_0^{(1)} + 2K_0^{(2)} = 7$ multipliers are required to implement this filter of order $K_1 + 2(K_0^{(1)} + 2K_0^{(2)}) = 23$.
when using the lattice wave digital filter structures.

7) The passband phase deviations of both $H(z)$ and $G(z)$ from $-K_1 \omega = -13 \omega$ are very small.

8) The impulse responses of both $H(z)$ and $G(z)$ achieve the value of half at $n = K_1 = 13$. At other odd values of $n$, the impulse-response values are zero.

9) At even values of $n$, the impulse-response values of $H(z)$ and $G(z)$ have opposite signs.

- An efficient structure for simultaneously implementing $H(z)$ and $G(z)$ is shown below.
Example Approximately Linear-Phase Half-Band IIR Filter

Solid: $A_0(z^2)$; Dashed: $z^{-13}$

Phase responses as fractions of $\pi$

Angular frequency $\omega$

Responses for the power–complementary filter pair

Amplitude in dB

Angular frequency $\omega$
Example Approximately Linear-Phase Half-Band IIR Filter

Solid: $H(z) = A_0(z^2) + z^{-13}$; Dashed: $G(z) = A_0(z^2) - z^{-13}$

Angular frequency $\omega$

Phase deviations from $-120^\circ$ in degrees

Impulse responses

$H(z)$

$n$ in samples

$G(z)$

$n$ in samples
Decimator and Interpolator Implementations of Half-Band IIR Filters

- The figure below shows the decimator and interpolator implementations for $H(z) = (1/2)[A_0(z^2) + z^{-1}A_1(z^2)]$ and $H(z) = (1/2)[A_0(z^2) + z^{-K_1}]$. 

![Diagram of decimator and interpolator implementations for half-band IIR filters](image-url)
Part II.H: Use of Conventional and Modified Comb (Running Sum) Structures as a First Stage for Multistage Decimator Implementations

- This part shows how to use a cascade of comb filters (or running sum or sinc filters) as a first decimation or last interpolation stage in multistage implementations.
- Also a modified structure introduced by Saramäki and Ritonäemi is considered.
- Moreover, practical examples are included.
What is a Comb Filter (a Running Sum or a Sinc Filter)?

• The transfer function of a comb or running sum filter of order $K - 1$ (length $K$) is given by

$$E(z) = 2^{-P} G(z),$$  \hspace{1cm} (63a)

where

$$G(z) = \sum_{n=0}^{K-1} z^{-n} = \frac{1 - z^{-K}}{1 - z^{-1}}$$  \hspace{1cm} (63b)

and $P$ is integer satisfying

$$2^{-P} \leq 1/K.$$  \hspace{1cm} (63c)

• Page 270 shows efficient implementations for the above transfer function.

• Both implementations require no multipliers and only two adders. The first structure has the following attractive property:

• If modulo arithmetic (e.g., 1’s or 2’s complement arithmetic) and the worst-case scaling (corresponds to peak scaling in this case) are used ($2^{-P} \leq 1/K$), the output of $E(z)$ is correct even though internal overflows may occur.

• This implementation is very attractive as, in this case, the system does not need initial resetting and the effect of temporary miscalculations vanishes automatically from the output in a finite time.
• For the second structure, resetting is needed and temporary miscalculations are not allowed.
Efficient Structures for Implementing a Comb Filter
Cascaded Comb Filter for Decimation or Interpolation by an Integer Factor $K$

- By cascading $M$ comb filters, we end up with the following transfer function:
  \[ E(z) = 2^{-P}[G(z)]^M, \]  
  \[ (64a) \]
  where
  \[ G(z) = \frac{1 - z^{-K}}{1 - z^{-1}} \]  
  \[ (64b) \]
  and $P$ is integer satisfying
  \[ 2^{-P} \leq (1/K)^M. \]  
  \[ (64c) \]

- Pages 273 and 274 show efficient decimator and interpolator implementations for this transfer function.
- The second implementation form for the decimator has the attractive property that only $2M$ delay elements and adders are required regardless of the value of $K$.
- In the third implementation form for the interpolator the digital zero-order hold increases the sampling rate by $K$ in such a way that it repeats the input sample $K$ times, thus increasing the sampling rate by $K$.
- In this case, the number of feedforward and feedback loops is decreased by one. Therefore, only $2(M - 1)$ delay elements and adders are required regardless of the value of $K$.
- For the interpolator, it is required that \[ 2^{-P} \leq (1/K)^{M-1} \] since the transfer function must be
multiplied by $K$ in order to keep the signal energy the same.

- For the interpolator, resetting is necessary and no miscalculations are allowed. Therefore, after a thunderstorm, resetting is needed. Otherwise, the output of the overall system may be meaningless!

- If it is desired that the output noise due to the multiplication roundoff errors correspond to rounding at the output of the overall filter, $P$ extra bits are required for internal calculations for both the decimator and interpolator.
Structures for Cascaded Comb Filters for Decimation by an Integer Factor $K$
Structures for Cascaded Comb Filters for Interpolation by an Integer Factor $K$
When to Use a Cascaded Comb Filter for Decimation or Interpolation Purposes?

- $E(z)$ can be used as a first stage (as a last stage) when the overall decimation (interpolation) ratio $N$ is factorizable as

$$N = K \cdot L. \quad (65)$$

- The overall implementations as well as the corresponding single-stage equivalents are shown below.
If the transfer function for the second decimation stage (the first interpolation stage) is denoted by $T(z)$, then the transfer functions for the single-stage equivalents in the decimation and interpolation cases are given by

$$H(z) = T(z^k)E(z)$$  \hspace{1cm} (66a)

and

$$H(z) = NT(z^k)E(z),$$  \hspace{1cm} (66b)

respectively.
How to Design the Overall Filter?

- The zero-phase frequency response of $E(z)$ is given by
  \[ E(\omega) = 2^{-P} \left[ \frac{\sin(N\omega/2)}{\sin(\omega/2)} \right]^M \]  
  (67)
  and provides $M$ zero pairs at $\omega = 2k\pi/K$ for $k = 1, 2, \cdots, \lfloor K/2 \rfloor$.
- To illustrate the design of the overall decimator, we consider the following criteria: $N = 10$, $\omega_p = 0.05\pi$, $\omega_s = 0.1\pi$, $\delta_p = 0.01$, $\delta_s = 0.001$.
- As shown on the next page, the given critria are met by $K = 5$, $M = 5$, $P = 12$, $L = 2$ and $T(z)$ of order 21.
- In this case, $T(z^5)$ takes care of the overall frequency response in the range $0 \leq \omega \leq \pi/5$, whereas $E(z)$ attenuates the unwanted extra transition bands and stopbands of $T(z^5)$ around $2\pi/5$ and $4\pi/5$ by providing 5 zero pairs at these frequencies.
- The only adjustable parameter for $E(z)$ is $M$ and $M = 5$ is the minimum value of $M$ required to attenuate the extra transition bands and stopbands to the desired level of 60 dB.
Responses for an Example Two-stage Decimator with the First Stage Being a Cacade of $M = 5$ Comb Filters of Length $K = 5$
Modified Comb Filter

- The cascaded comb filter structure suffers from the drawback that the only adjustable parameter is $M$, the number of zeros at the centers of the extra unwanted passbands of $T(z^K)$.

- In order to get around this problem, Saramäki and Ritoniemi have introduced a modified comb filter structure. For this structure, the overall transfer function is given by

$$E(z) = 2^{-P} E_1(z) E_2(z), \quad (66a)$$

where

$$E_1(z) = \left[ z^{-1} \frac{1 - z^{-K}}{1 - z^{-1}} \right]^M \quad (66b)$$

and

$$E_2(z) = \left[ z^{-1} \frac{1 - z^{-K}}{1 - z^{-1}} \right]^{2N} + \sum_{r=1}^{N} a_r z^{-r(K+1)} \left[ z^{-1} \frac{1 - z^{-K}}{1 - z^{-1}} \right]^{2(N-r)} \quad (66c)$$

- The next page shows an efficient implementation for decimating by an integer factor $K$ in the case where $a_k$'s are integers.

- Also in this case, the output of this implementation is correct for 2's complement arithmetic even though internal overflows may occur, provided that $2^{-P} \leq \sqrt{[E_1(1)E_2(1)]}$.
Implementation of the Modified Comb Filter for Decimation by $K$ in the Case where the $a_k$’s are Integers
Example

- We consider the same criteria as for the cascaded comb filter structure, that is, \( N = 10, \ \omega_p = 0.05\pi, \ \omega_s = 0.1\pi, \ \delta_p = 0.01, \) and \( \delta_s = 0.001. \)
- As shown on the next page, the given criteria are met by \( K = 5, \ M = 2, \ N = 1, \ P = 10, \ a_1 = 1, \) and \( T(z) \) of order 21.
- It is interesting to observe that the modified comb filter provides a zero pair before and after \( \omega = 2\pi/5. \)
- This explains why the overall number of feedback and feedforward loops reduces from 5 to 4 compared to the cascaded comb filter structure considered earlier.
- Also the number of additional bits from internal calculations reduces from \( P = 12 \) to \( P = 10. \)
Responses for an Example Two-stage Decimator with the First Stage Being a Modified Comb Filter with $K = 5$, $M = 2$, and $N = 1$
Illustrative Example

• It is desired to design a highly selective decimator after a sigma-delta modulator working at the sampling rate being 64 times the output sampling rate.
• The final output sampling rate is $F_s = 44.1 \text{ kMz}$. 
• The passband edge is 20 kHz = 0.4535$F_s$ and it is desired that the passband ripple for the amplitude is less than or equal to 0.0001 and the components aliasing into the passband are attenuated at least 120 dB.
• Aliasing is allowed to the frequency range between 20 kHz and $F_s/2 = 22.05 \text{ kHz}$ from the range between $F_s/2 = 22.05 \text{ kHz}$ and $F_s - 20 \text{ kHz} = 24.1 \text{ kHz}$. A normal human being is not able to hear this aliasing.
• In terms of the angular frequency the criteria are thus: $N = 64$, $\omega_p = 0.907\pi/64$, $\omega_s = 1.093\pi/64$, $\delta_p = 0.0001$, and $\delta_s = 10^{-6}$. 
• The overall filter has been synthesized using the following structure:

![Diagram](image)

**Single-stage equivalent**

![Diagram](image)

• As shown above, the transfer function for the single-stage equivalent is given by

$$H(z) = E(z)T_1(z^{16})T_2(z^{16})T_3(z^{32}).$$  \(67\)

• The given criteria are met by designing \(E(z)\) to be a modified comb filter with \(K = 16\), \(M = N = 2\), \(a_1 = -2^3\), \(a_2 = 2^4\), and \(P = 24\).

• \(T_1(z)\) is a linear-phase FIR filter of order 4, whereas \(T_2(z)\) and \(T_3(z)\) are half-band filters of orders 26 and 162, respectively.

• Various responses for the overall design are depicted on Pages 287, 288, and 289.

• The roles of the subfilters are the following:

1) \(T_2(z^{16})\) and \(T_3(z^{32})\) take care of providing the desired attenuation for the overall transfer function, as
given by Eq. (67), from $1.093F_s/2$ to $2F_s$.

2) $E(z)T_1(z^{16})$ attenuates the extra passbands and transition bands of $T_2(z^{16})T_3(z^{32})$ around $4kF_s$ for $k = 1, 2, \ldots, 8$.

- Since $T_2(z)$ and $T_3(z)$ are half-band filters, their passband ripples are very small and they cannot be used for compensating the passband distortion caused by $E(z)$. 
• The roles of $E(z)$ and $T_1(z^{16})$ in generating $E(z)T_1(z^{16})$ are the following:

1) $E_1(z)$ provides concentrates on attenuating the extra passbands and transition bands of $T_2(z^{16})T_3(z^{32})$.

2) $T_1(z^{16})$ compensates the passband distortion caused by $E(z)$. Because of periodicity, it only slightly decreases the stopband attenuation provided by $G(z)$.

• For the modified comb filter, the number of feedback and feedforward loops is 6 and $P = 24$, whereas for the corresponding direct cascade, the corresponding figures are 8 and 32.

• The modified comb filter has been used as a first stage for decimating a one-bit stream from a sigma-delta modulator, and the estimated saving in the overall silicon area provided by this filter over its cascaded counterpart is 50%.
Responses for the Example Multistage Decimator
Responses for the Example Multistage Decimator

Dot-dashed: $T_2(z^{16})$, Solid: $T_3(z^{32})$

$T_2(z^{16})T_3(z^{32})$

Amplitude in dB

Freqeency as a fraction of $F_s$

Solid: $T_2(z^{16})T_3(z^{32})$, Dot-dashed: $E(z)T_1(z^{16})$

$E(z)T_1(z^{16})T_2(z^{16})T_3(z^{32})$

Amplitude in dB

Frequency as a fraction of $F_s$
Responses for the Example Multistage Decimator

Low-Frequency Details for $E(z)T_1(z^{16})T_2(z^{16})T_3(z^{32})$

Solid: $T_2(z^{16})T_3(z^{32})$, Dashed: $E(z)T_1(z^{16})$

$G(z)T_1(z^{16})T_2(z^{16})T_3(z^{32})$