COMPLEX-VALUED SIGNALS AND SYSTEMS
–
BASIC PRINCIPLES AND APPLICATIONS TO RADIO COMMUNICATIONS AND RADIO SIGNAL PROCESSING

(in short)

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1. BACKGROUND AND MOTIVATION

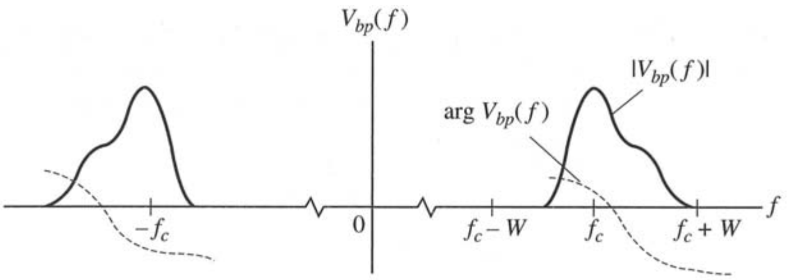
– All physical signals and waveforms are **real-valued**
  - so why bother to consider **complex-valued** signals and systems ?!?

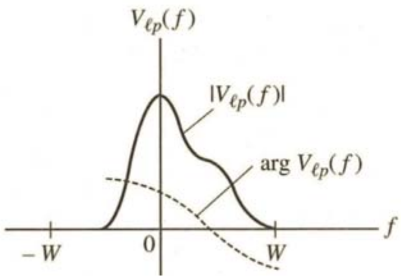
– The original complex signal concepts can be traced back to the introduction of **lowpass equivalent** notation, i.e., analysis of **bandpass signals and systems** using their lowpass/baseband equivalents
  - in general, a real-valued bandpass signal/system has a complex-valued lowpass equivalent
  - in terms of formulas

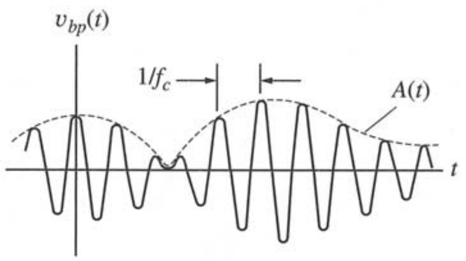
$$v_{BP}(t) = A(t)\cos(2\pi f_C t + \phi(t)) = v_I(t)\cos(2\pi f_C t) - v_Q(t)\sin(2\pi f_C t)$$
$$= \mathrm{Re}[v_{LP}(t)\exp(j2\pi f_C t)] = (v_{LP}(t)\exp(j2\pi f_C t) + v_{LP}{}^*(t)\exp(-j2\pi f_C t))/2$$

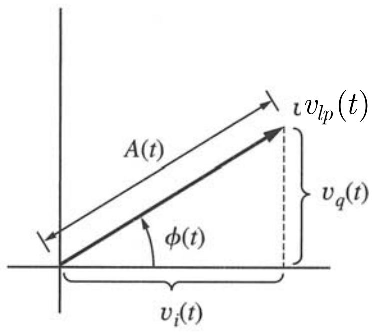
where $v_{LP}(t) = v_I(t) + jv_Q(t) = A(t)\exp(j\phi(t))$ is the corresponding lowpass or baseband equivalent signal

– spectral example + waveform + formulas:









$$v_{bp}(t) = A(t)\cos(\omega_c t + \phi(t)) = \ldots$$
$$= v_i(t)\cos(\omega_c t) - v_q(t)\sin(\omega_c t)$$

$$v_{lp}(t) = A(t)e^{j\phi(t)}$$
$$= v_i(t) + jv_q(t)$$

– Altogether complex signal notions have two important viewpoints or implications: **Communication theoretic view** and **radio implementation view**.

– Communication theoretic aspects

- for example most spectrally efficient I/Q modulation techniques (complex modulation, radio waveforms) are based on these ideas
- also modeling of the radio channel, and thereon receiver signal processing for equalization and detection, is another good example

– Radio implementation aspects

- all advanced frequency translation techniques and thus the related receiver architectures (low-IF, direct-conversion, etc.) utilize complex signals
- also sampling and efficient multirate processing (filtering) of bandpass signals form other good examples

– We try to grasp the basics here, with perhaps more emphasis here on the latter theme ! Waveforms will follow then on lectures 2 and 3.

– Some **basic notations** used in the following:

- continuous-time signals / waveforms / systems: $x(t)$, $h(t)$, etc.
- discrete-time signals / sequences / systems: $x(n)$, $h(n)$, etc.
- angular frequency with continuous-time signals (f is frequency in Hz)

$$\omega = 2\pi f$$

- normalized angular frequency with discrete-time signals (sample rate $f_S = 1/T_S$)

$$\omega = 2\pi f T_S = \frac{2\pi f}{f_S}$$

$\Rightarrow$ thus π corresponds to half the sampling frequency ($f_S/2$) in this notation

– **Notice:** Interactive demonstrations and additional (supporting) material available at

`http://bruce.cs.tut.fi/invocom/index.htm`

- short courses 1 & 2 most relevant from this material point of view

2. BASIC CONCEPTS AND DEFINITIONS

– By definition, the **time domain** waveform or sequence $x(t)$ of a **complex signal** is complex-valued, i.e.

$$x(t) = x_I(t) + jx_Q(t) = \mathrm{Re}[x(t)] + j\mathrm{Im}[x(t)]$$

– **In practice**, this is nothing more than a pair of two real-valued signals $x_I(t)$ and $x_Q(t)$ carrying the real and imaginary parts.

– Similarly, a **complex system** is defined as a system with complex-valued impulse response

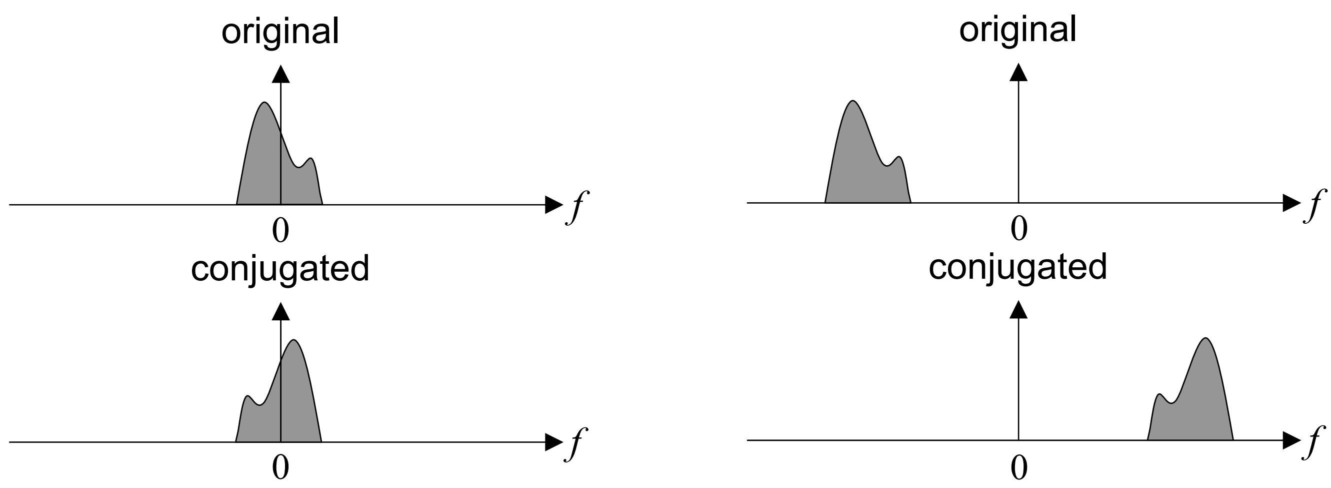
$$h(t) = h_I(t) + jh_Q(t) = \mathrm{Re}[h(t)] + j\mathrm{Im}[h(t)]$$

– In the **frequency domain**, real-valued signals/systems have always even-symmetric amplitude spectrum/response and odd-symmetric phase spectrum/response with respect to the zero frequency (origin, two-sided spectra)

- complex signals don't (need to) have any symmetry properties in general
- e.g., the spectral support (region of non-zero amplitude spectrum) can basically be anything

– One basic operation related to complex quantities is **complex-conjugation**

- if the spectrum of $x(t)$ is denoted by $X(f)$, then the spectrum of $x^*(t)$ is $X^*(-f)$
- thus the **amplitude spectra** of $x(t)$ and $x^*(t)$ are **mirror images** of each other
  - ⇒ trivial example: complex exponential $\exp(j\omega_0 t)$, impulsive spectrum at ω_0
  - ⇒ conjugation yields $\exp(-j\omega_0 t)$, impulsive spectrum at $-\omega_0$
- physically, conjugation is nothing more than changing the sign of the Q branch

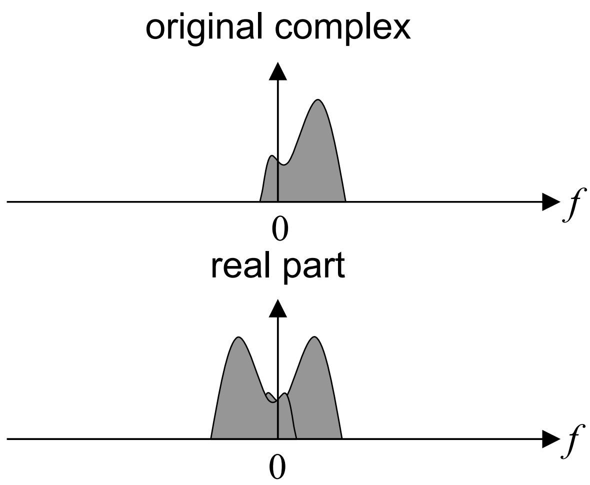


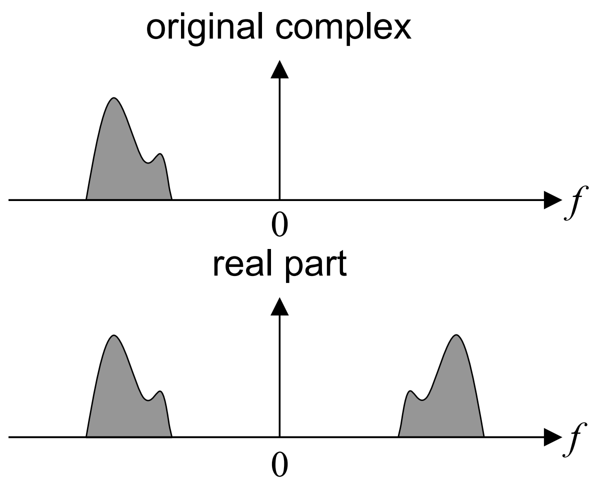
– In general, this simple-looking result related to conjugation is surprisingly useful when interpreting some properties of complex signals in the continuation

- an immediate consequence is that if you consider the **real part** of $x(t)$, i.e., $y(t) = \mathrm{Re}[x(t)] = (x(t) + x^*(t))/2$, its spectrum is $Y(f) = (X(f) + X^*(-f))/2$

  $\Rightarrow$ if $X(f)$ and $X^*(-f)$ are not overlapping, $y(t) = \mathrm{Re}[x(t)]$ contains all the information about $x(t)$

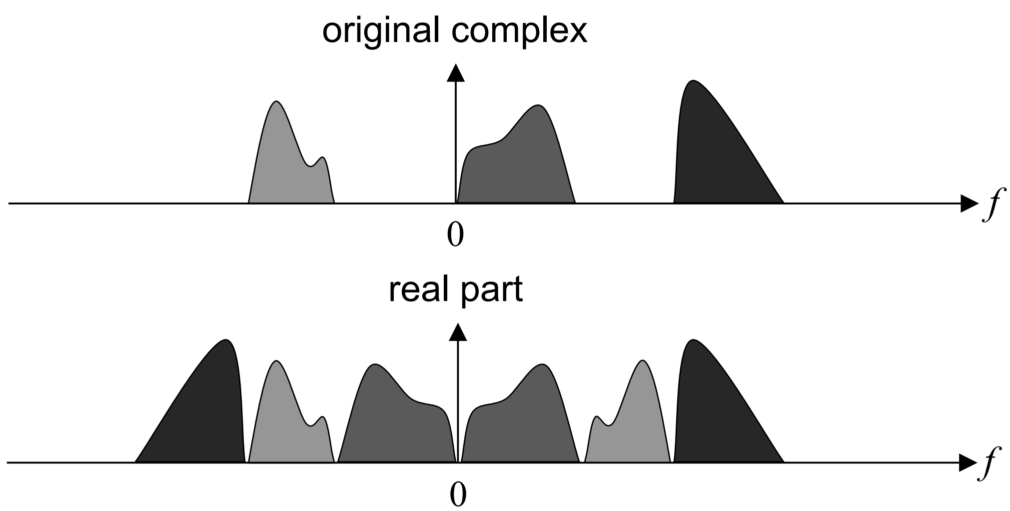
  $\Rightarrow$ this result will find good use, e.g., in understanding frequency translations





– Based on the above, it directly follows that for any complex signal $x(t)$ such that $X(f)$ and $X^*(-f)$ are not overlapping, $y(t) = \mathrm{Re}[x(t)]$ contains all the information about $x(t)$

- a general illustration given below
- …fascinating :o)



– Other two basic operations related to processing of complex signals are (*i*) **complex multiplication** and (*ii*) **complex convolution (filtering)**.

– In the general case, these can be written as (simply following complex arithmetic)

(i):
$$x(t) \times y(t) = (x_I(t) + jx_Q(t)) \times (y_I(t) + jy_Q(t))$$
$$= x_I(t) \times y_I(t) - x_Q(t) \times y_Q(t) + j(x_I(t) \times y_Q(t) + x_Q(t) \times y_I(t))$$

- thus 4 real multipliers (plus two additions) are needed in general, in the physical implementation

(ii):
$$x(t) * h(t) = (x_I(t) + jx_Q(t)) * (h_I(t) + jh_Q(t))$$
$$= x_I(t) * h_I(t) - x_Q(t) * h_Q(t) + j(x_I(t) * h_Q(t) + x_Q(t) * h_I(t))$$

- thus 4 real convolutions (plus two additions) are needed in general, in the physical implementation

– Illustrated in the following figure in terms of parallel real signals
  - here for convolution, similarly for multiplication (will be illustrated later on)

– Notice that obvious simplifications occur if either of the components is real valued
  - in these cases, only two real convolutions/multiplications needed (why?)

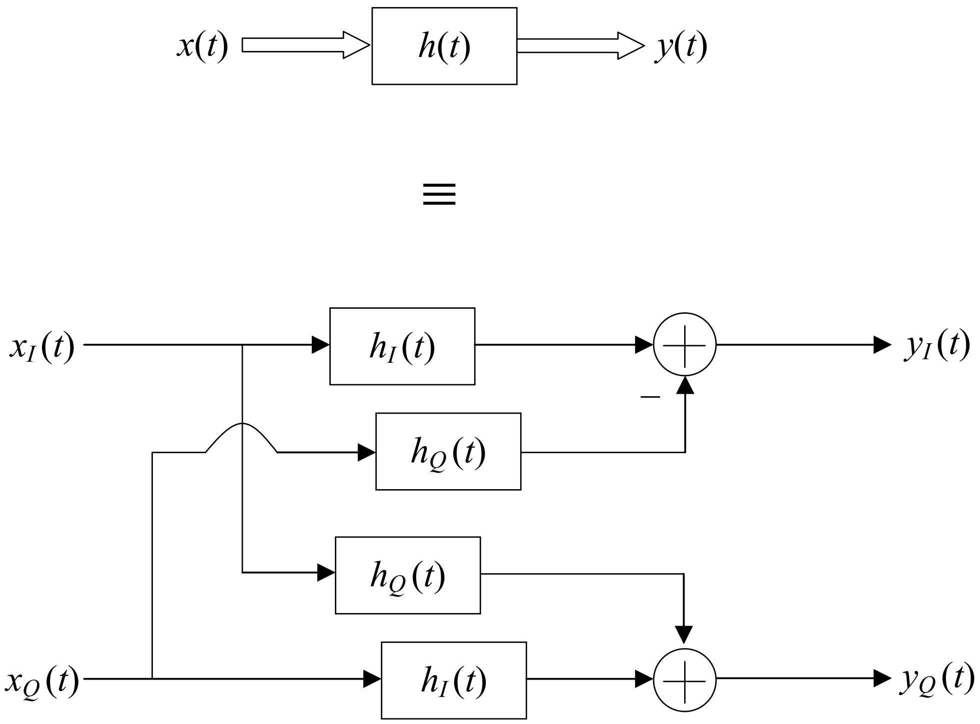


Figure: Illustration of full complex convolution $y(t) = x(t) * h(t)$ in terms of four real convolutions and two additions.

3. Analytic Signals and Hilbert Transforms

- Hilbert transformer is generally defined as an **allpass linear filter** which shifts the **phase** of its input signal by **90 degrees**.
- The (anticausal) impulse and frequency responses can be formulated as

continuous-time

$$h_{HT}(t) = \frac{1}{\pi t}$$

$$H_{HT}(f) = \begin{cases} -j, & f \geq 0 \\ +j, & f < 0 \end{cases}$$

discrete-time

$$h_{HT}(n) = \begin{cases} 0, & n \ \text{even} \\ 2/(\pi n), & n \ \text{odd} \end{cases}$$

$$H_{HT}(e^{j\omega}) = \begin{cases} -j, & 0 \leq \omega < \pi \\ +j, & -\pi \leq \omega < 0 \end{cases}$$

- In practice, this behavior can be well approximated over any finite bandwidth.
- One fascinating property related to Hilbert filters/transformers is that they can be used to construct **signals with only positive or negative frequency content**.
- This kind of signals are generally termed **analytic** and they are always **complex** (why?).

– The simplest example is to take a cosine wave $A\cos(\omega_1 t)$ whose Hilbert transform is $A\sin(\omega_1 t)$ (just a 90 degree phase shift!)
  - these together when interpreted as I and Q components of a complex signal result in $A\cos(\omega_1 t) + jA\sin(\omega_1 t) = A\exp(j\omega_1 t)$ whose spectrum has an impulse at ω_1 but nothing on the other side of the spectrum

– The "elimination" of the negative (or positive) frequencies can more generally be formulated as follows.

– Starting from an arbitrary signal $x(t)$ we form a complex signal $x(t) + jx_{HT}(t)$ where $x_{HT}(t)$ denotes the Hilbert transform of $x(t)$.

– Then the spectrum of the complex signal is $X(f) + jX_{HT}(f) = X(f)[1 + jH_{HT}(f)]$ (why?) where

$$\underline{\text{continuous-time}}$$

$$1 + jH_{HT}(f) = \begin{cases} 1 + j \times (-j), & f \geq 0 \\ 1 + j \times j, & f < 0 \end{cases} = \begin{cases} 2, & f \geq 0 \\ 0, & f < 0 \end{cases}$$

which shows the elimination of the original negative frequency content.

– Similar concepts carry on to discrete-time world and we can write

$$\underline{\text{discrete-time}}$$

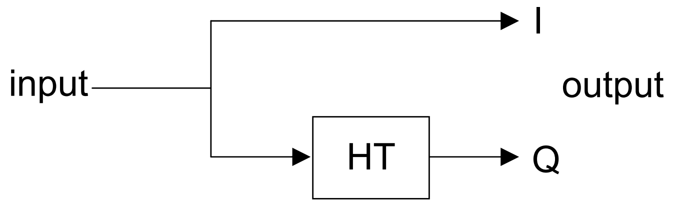
$$1 + jH_{HT}(e^{j\omega}) = \begin{cases} 1 + j \times (-j), & 0 \leq \omega < \pi \\ 1 + j \times j, & -\pi \leq \omega < 0 \end{cases} = \begin{cases} 2, & 0 \leq \omega < \pi \\ 0, & -\pi \leq \omega < 0 \end{cases}$$

– Based on this, it can easily be shown that **the I and Q** (real and imaginary parts) **of any analytic signal are always related through Hilbert transform**.

– This idea of using a Hilbert transformer to generate analytic signals is further illustrated graphically in the following figure assuming real input signal.

– In practice the Hilbert filtering causes a **delay** and a corresponding delay needs to be included also in the upper (I) branch.

– Notice also that the elimination of positive frequencies (instead of negative ones) is obtained simply by changing the sign of the imaginary part (why?)
  - i.e., $x(t) - jx_{HT}(t)$



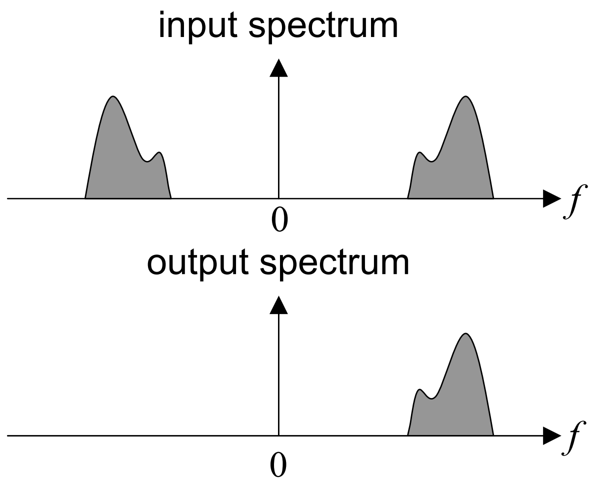
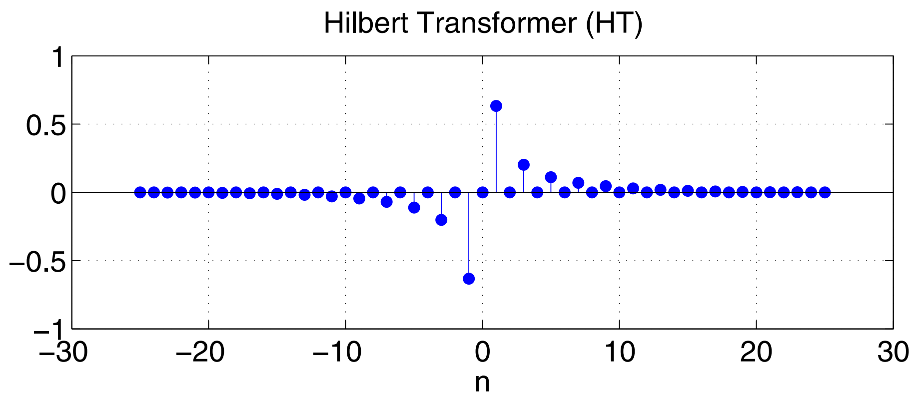
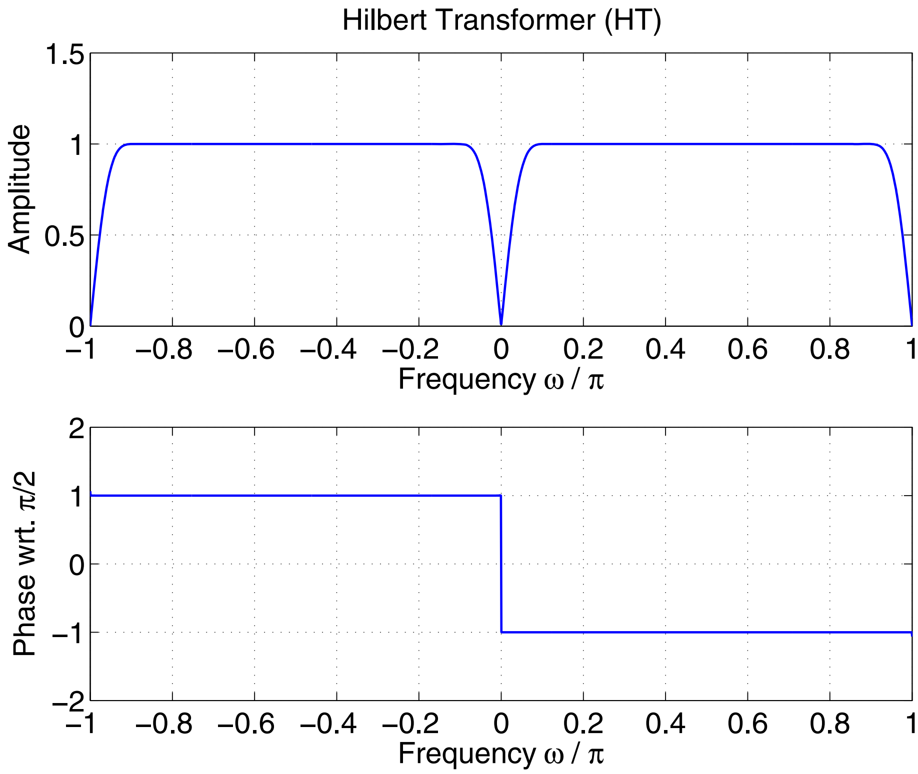


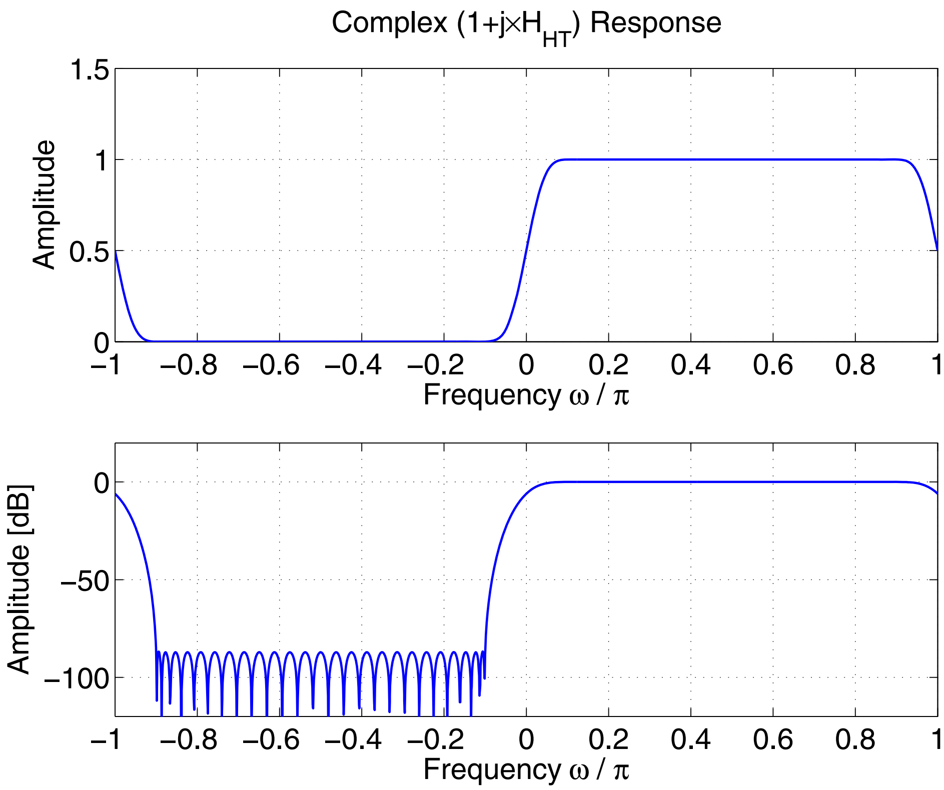
Figure: Example of eliminating negative frequencies using a Hilbert transformer with real-valued input signal. In practice the Hilbert transformer causes a delay and a similar delay element need to be included in the I branch as well.

– **Design Example:** Hilbert transformer of order 50, design bandwidth $0.1\pi \ldots 0.9\pi$ (π denotes half the sampling frequency), Remez (equiripple) design

- the selected filter order and optimization routine result in about 87 dB attenuation for the negative frequencies (wrt. corresponding positive ones)
- see `help firpm` in Matlab




Hilbert Transformer (HT)
Amplitude
Frequency ω / π
Phase wrt. π/2
Frequency ω / π


Complex (1+j×H_HT) Response
Amplitude
Frequency ω / π
Amplitude [dB]
Frequency ω / π

4. FREQUENCY TRANSLATIONS AND MIXING

4.1 Frequency Translations for Signals

– One key operation in radio signal processing is the **shifting of a signal spectrum from one center-frequency to another**

- conversions between baseband and bandpass representations and I/Q modulation and demodulation (synchronous detection) are special cases of this

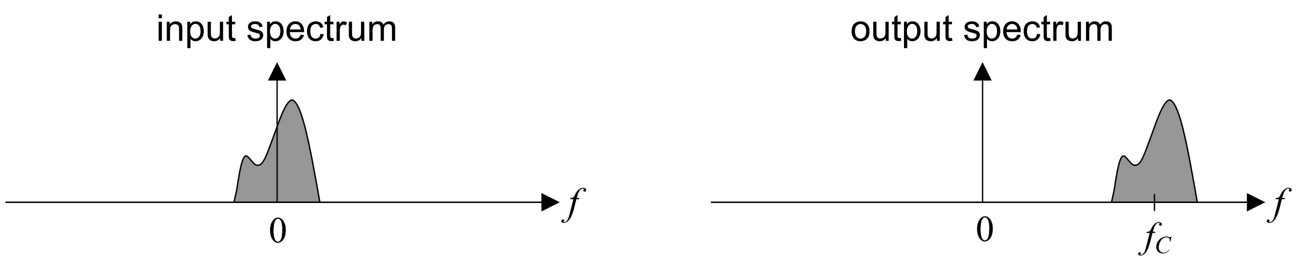
– The basis of all the frequency translations lies in multiplying a signal with a complex exponential, generally referred to as **complex or I/Q mixing**.

– This will indeed cause a pure frequency shift, i.e.,

$$y(t) = x(t)e^{j\omega_{LO}t} \quad \Leftrightarrow \quad Y(f) = X(f - f_{LO})$$

– **This forms the basis, e.g., for all the linear modulations, and more generally for all frequency translations.**

– This is illustrated in frequency domain below in the case where the input signal is at baseband.



– In general, **four real mixers** and two adders are needed to implement a full complex mixer (full complex multiplication):

$$\begin{aligned} x(t)e^{j\omega_{LO}t} &= (x_I(t) + jx_Q(t))(\cos(\omega_{LO}t) + j\sin(\omega_{LO}t)) \\ &= x_I(t)\cos(\omega_{LO}t) - x_Q(t)\sin(\omega_{LO}t) + j(x_Q(t)\cos(\omega_{LO}t) + x_I(t)\sin(\omega_{LO}t)) \end{aligned}$$

- illustrated in the following figure
- notice again that in the special case of real-valued input signal, only two mixers are needed

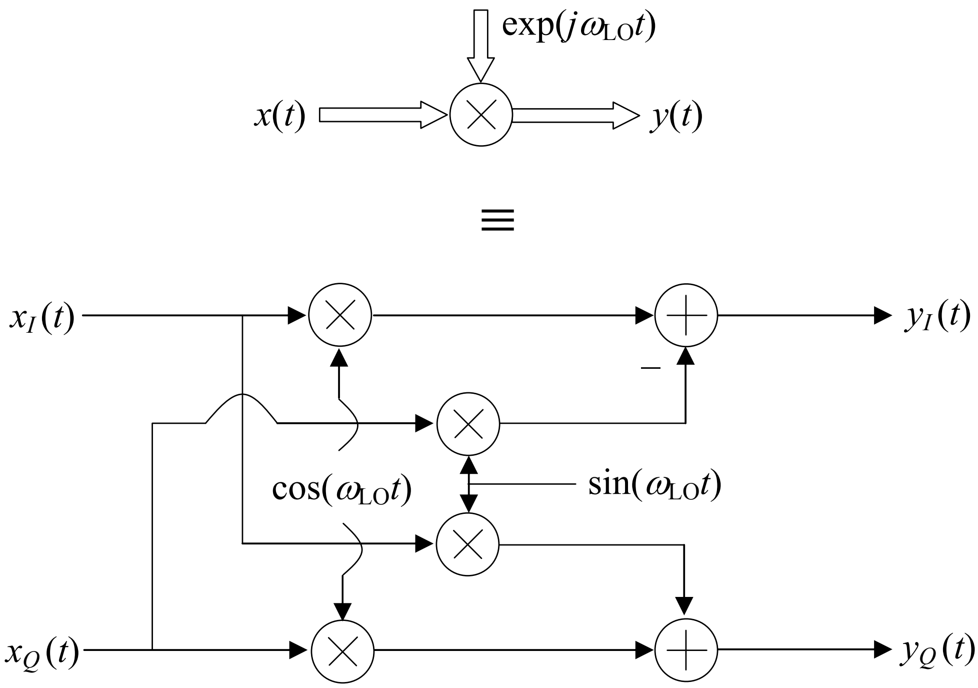


Figure: Illustration of full complex mixing (complex multiplication) $y(t) = x(t)e^{j\omega_{LO}t}$ in terms of parallel real signals (4 real mixers and 2 adders).

- **Real mixing** is obviously a special case of the previous complex one and results in two frequency translations:

$$y(t) = x(t)\cos(\omega_{LO}t)$$
$$= x(t)\frac{1}{2}(e^{j\omega_{LO}t} + e^{-j\omega_{LO}t}) \quad \Leftrightarrow \quad Y(f) = \frac{1}{2}X(f - f_{LO}) + \frac{1}{2}X(f + f_{LO})$$

- Here, the original spectrum appears twice in the mixer output, the two replicas being separated by $2f_{LO}$ in frequency.

- In receivers, this results in the so called **image signal or mirror-frequency problem** since the signals from both $f_C + f_{LO}$ and $f_C - f_{LO}$ will appear at f_C after a real mixing stage
  - if real mixing is used in the receiver, the image signal or mirror-frequency band needs to be attenuated before the actual mixer stage
  - this is the case, e.g., in the so called superheterodyne receiver (which we discussed shortly during the earlier lectures)
  - similarly also in transmitters; the other spectral replica needs to be attenuated

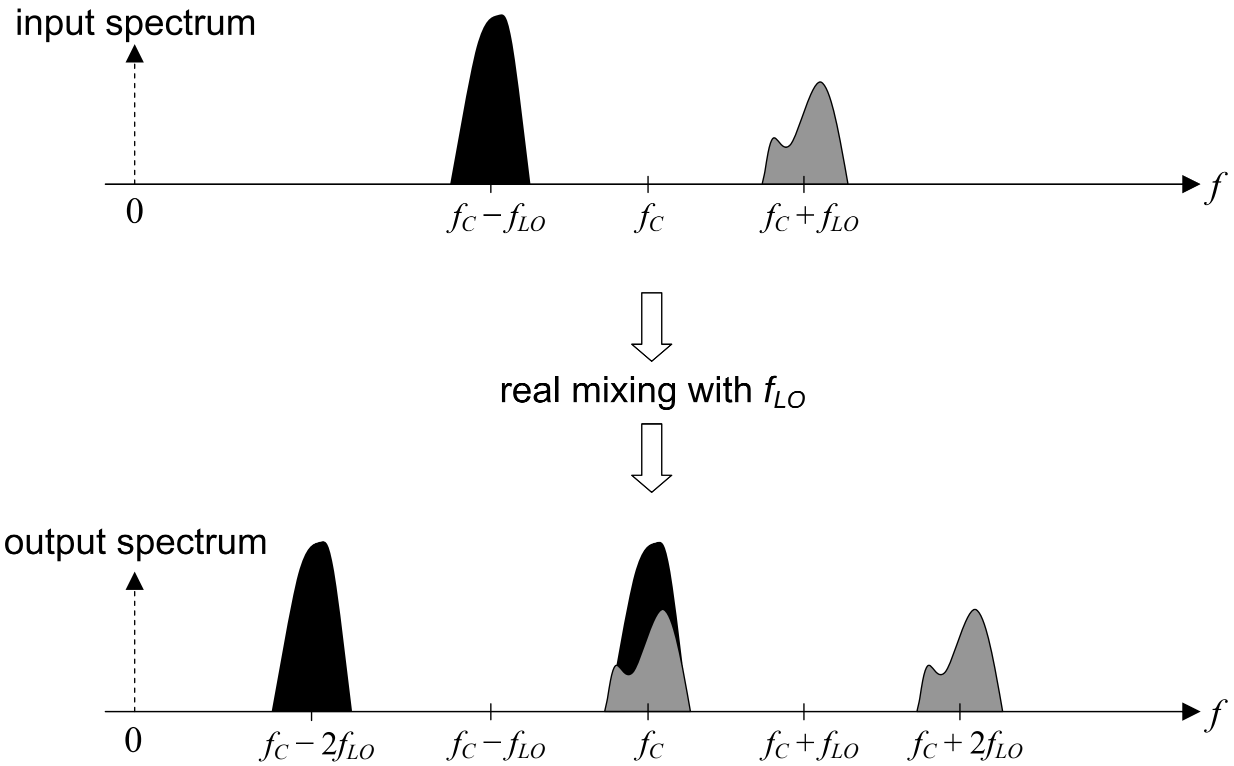


Figure: Illustration of the image signal problem in real mixing (only positive frequencies shown).

– Linear **I/Q modulation** methods are basically just a special case of complex mixing.

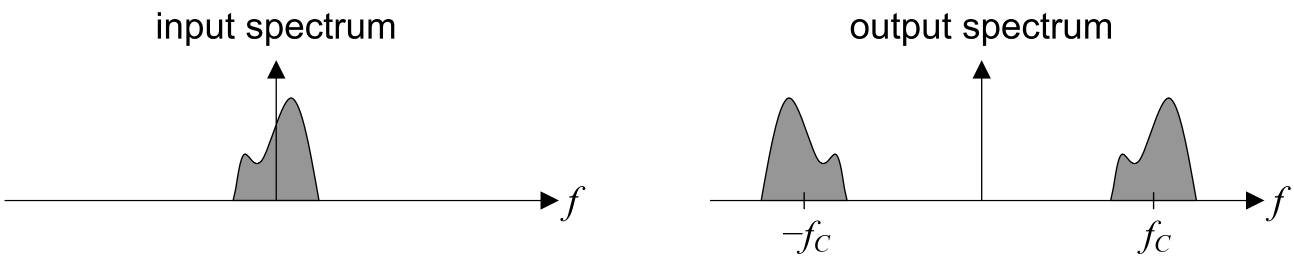
– Given a complex message signal $x(t) = x_I(t) + jx_Q(t)$, it is first modulated as $x(t)\exp(j\omega_C t)$, after which only the real part is actually transmitted (why?):

$$y(t) = \mathrm{Re}[x(t)e^{j\omega_C t}] = x_I(t)\cos(\omega_C t) - x_Q(t)\sin(\omega_C t) = \frac{1}{2}x(t)e^{j\omega_C t} + \frac{1}{2}x^*(t)e^{-j\omega_C t}$$

- interpretation #1: $x_I(t)$ and $x_Q(t)$ are modulated onto two orthogonal (cosine and sine) carriers; nice from the implementation point of view
- interpretation #2: $x(t)$ and $x^*(t)$ are modulated onto two complex exponentials $\exp(j\omega_C t)$ and $\exp(-j\omega_C t)$; key in building general spectral understanding and recovering $x(t)$ back from $y(t)$
- we will talk about this more during the waveform classes

– Notice that both terms/spectral components (at $+f_C$ and $-f_C$) contain all the original information (i.e., $x(t)$).

– This process, also termed lowpass-to-bandpass transformation, is pictured in the figure below.



– **I/Q demodulation:** In the receiver, the goal is to recover the original message $x(t)$ from the modulated signal $y(t)$.

– Based on the previous discussion, it's easy to understand that either of the signal components at $+f_C$ or $-f_C$ can be used for that purpose, while the other one should be rejected.

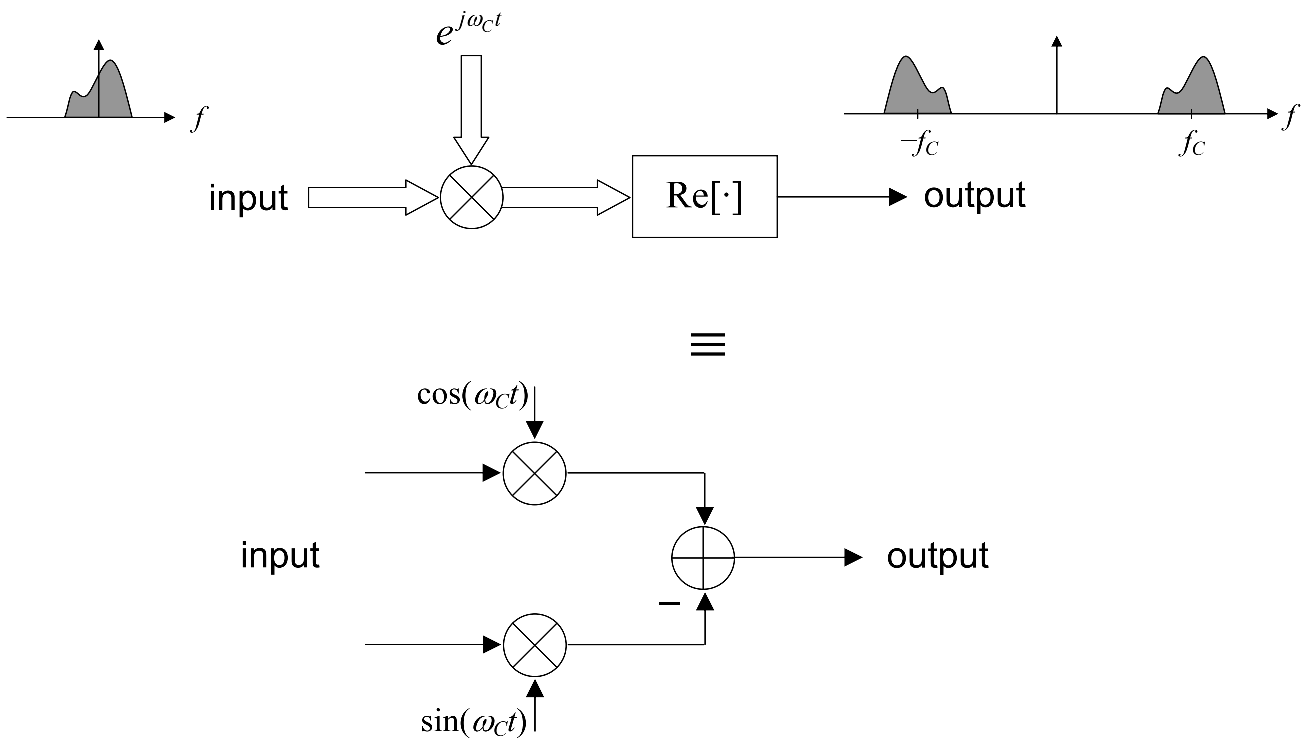
– Since

$$y(t)e^{-j\omega_C t} = (\frac{1}{2}x(t)e^{j\omega_C t} + \frac{1}{2}x^*(t)e^{-j\omega_C t})e^{-j\omega_C t} = \frac{1}{2}x(t) + \frac{1}{2}x^*(t)e^{-j2\omega_C t}$$

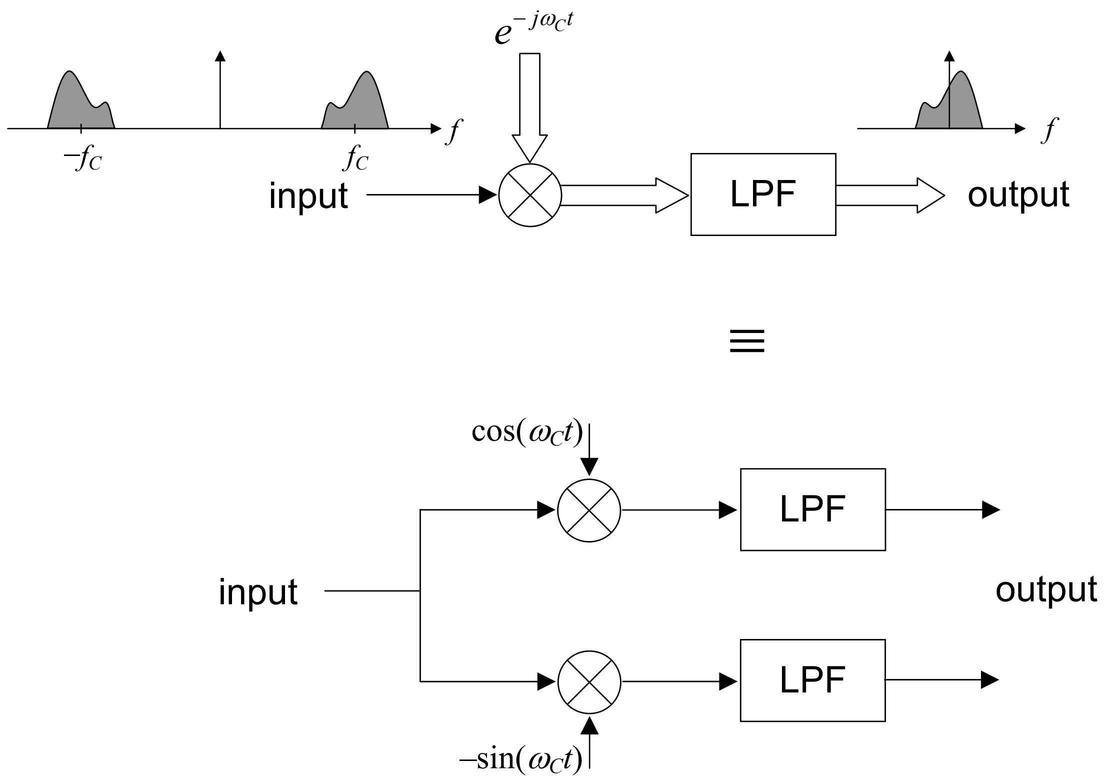
the message can be fully recovered by simply lowpass filtering the complex receiver mixer output.

– Formal block-diagrams for the modulator and demodulator in terms of complex signals as well as parallel real signals are presented below.

I/Q MODULATOR:



I/Q DEMODULATOR:



4.2 Frequency Translations for Linear Systems and Filters

– The idea of **frequency translations** can also be applied not only to signals but **linear systems or filters** as well (**why?**) !!
  - e.g., bandpass filter design through modulation of lowpass prototype
  - analytic bandpass filters – connection to Hilbert transforms

– In other words, $h(n)\exp(j\omega_0 n)$, $h(n)\cos(\omega_0 n)$, and $h(n)\sin(\omega_0 n)$ type modulated filters (modulated filter coefficients) !
  - in general these frequency translation principles apply to both **analog** and **digital filters** (focus here mostly on digital filters)

– Some interesting special cases (implementation simplicity) in case of digital filters:
  - complex modulation by $f_S/2$ => $\exp(j\pi n) = \{\ldots, +1, -1, +1, -1, +1, -1, \ldots\}$
  - complex modulation by $f_S/4$ => $\exp(j(\pi/2)n) = \{\ldots, +1, +j, -1, -j, +1, +j, -1, -j, \ldots\}$

    ⇒ more or less trivial mapping between the original and modulated filters in these cases ! (FIR filter focus here, in a sense)

    ⇒ this is one reason why $f_S/4$ is popular IF choice in receivers

– Notice also that in general, coefficient symmetry can be exploited in the implementation (assuming of course symmetric prototype)

- Why? Because of the odd/even symmetry of the "modulating" sine/cosine sequences !

– One additional key property is obtained from the **transfer function interpretation** of modulated complex filters:

- $H(z) = \sum_{n=0}^{N} h(n) z^{-n}$
- $\sum_{n=0}^{N} (h(n)\exp(j\omega_0 n)) z^{-n} = \sum_{n=0}^{N} h(n)(z^{-1}\exp(j\omega_0))^{n} = H(z)|_{z^{-1} \leftarrow z^{-1}\exp(j\omega_0)}$

– This means that the modulated filter can also be implemented by simply replacing the unit delays (z^{-1} elements) of the original filter with "generalized" elements $z^{-1}\exp(j\omega_0)$

- straight-forward also for IIR type filters
- some examples available e.g. at the demonstration web-site

– **Design Example:** Analytic FIR bandpass filter design using complex modulation

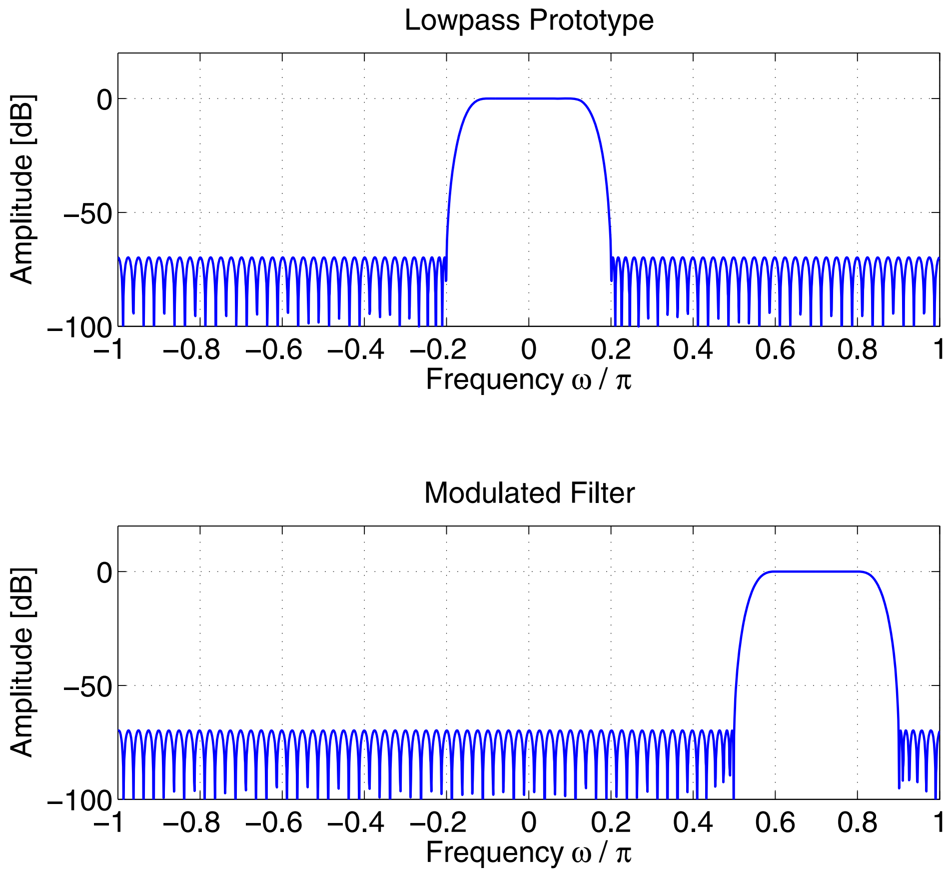
- Target: passband at $0.6\pi \ldots 0.8\pi$, order 50, remez (equiripple) design

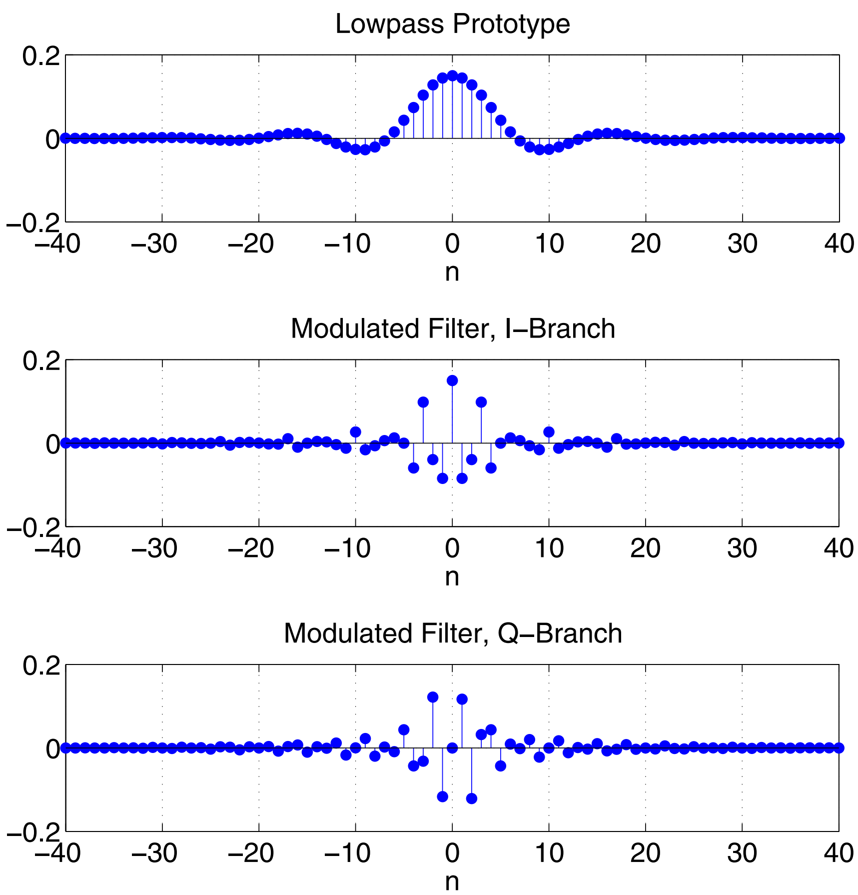
  ⇒ lowpass prototype with passband $-0.1\pi \ldots 0.1\pi$, complex modulation with $\exp(j0.7\pi n)$

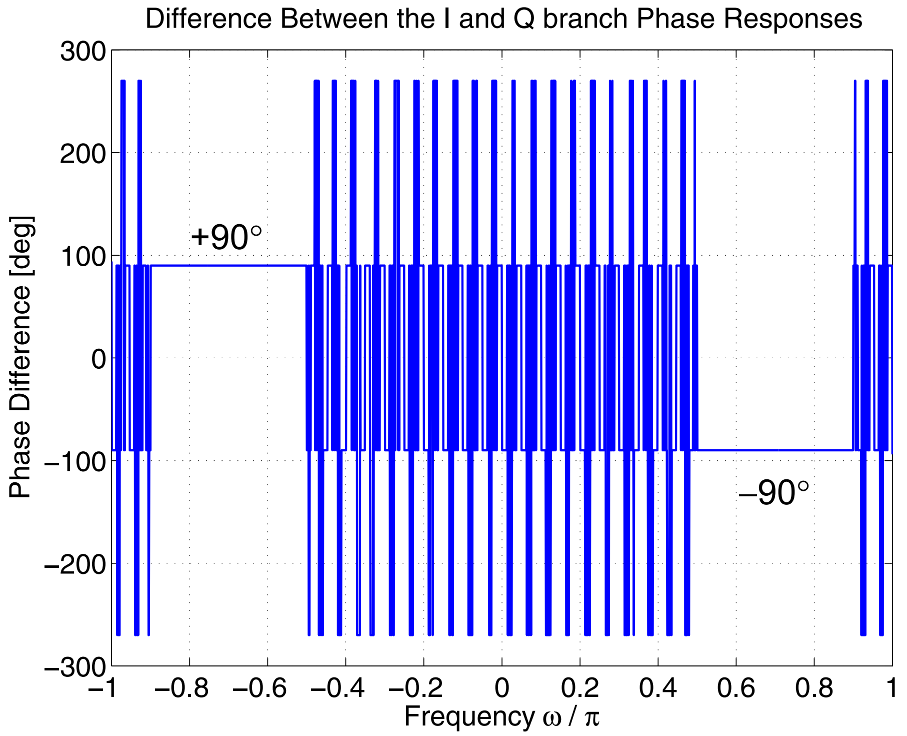
- Results illustrated in the following figures, notice the phase response behavior of the modulated filter (I and Q) – connection to Hilbert transform (phase difference exactly 90 degrees).
- This actually gives also an idea for alternative implementation of the complex analytic bandpass filter:

  ⇒ a complex filter whose real and imaginary parts are related through Hilbert transform !!

  ⇒ … more details after the figures …


Lowpass Prototype
Amplitude [dB]
0
−50
−100
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
Frequency ω / π
Modulated Filter
Amplitude [dB]
0
−50
−100
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
Frequency ω / π

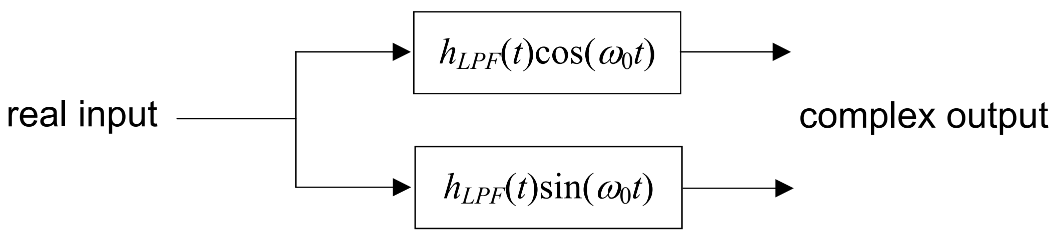

Lowpass Prototype
0.2
0
−0.2
−40
−30
−20
−10
0
10
20
30
40
n
Modulated Filter, I−Branch
0.2
0
−0.2
−40
−30
−20
−10
0
10
20
30
40
n
Modulated Filter, Q−Branch
0.2
0
−0.2
−40
−30
−20
−10
0
10
20
30
40
n

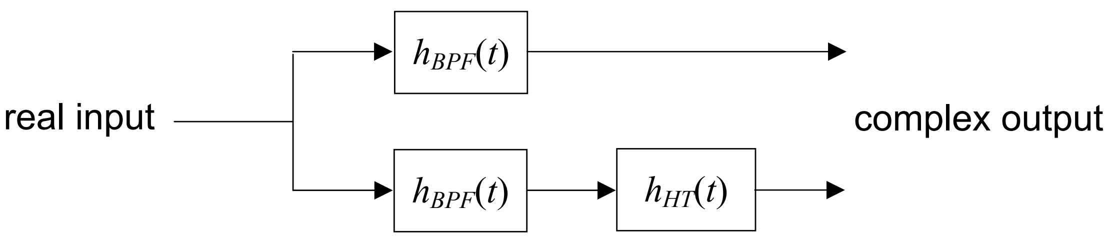


– Based on the above discussions, there are **in general two alternative implementation strategies** to implement complex (analytic) bandpass filtering:

- complex-modulated lowpass prototype
- Hilbert transformed bandpass prototype

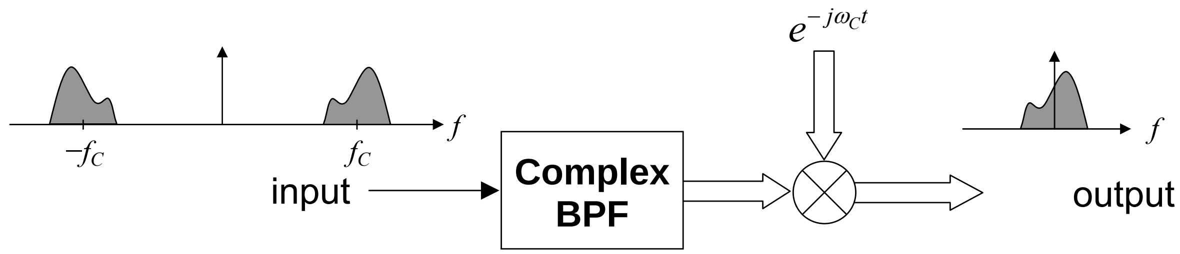
– Illustrated below for real-valued input signal and real prototype filters.



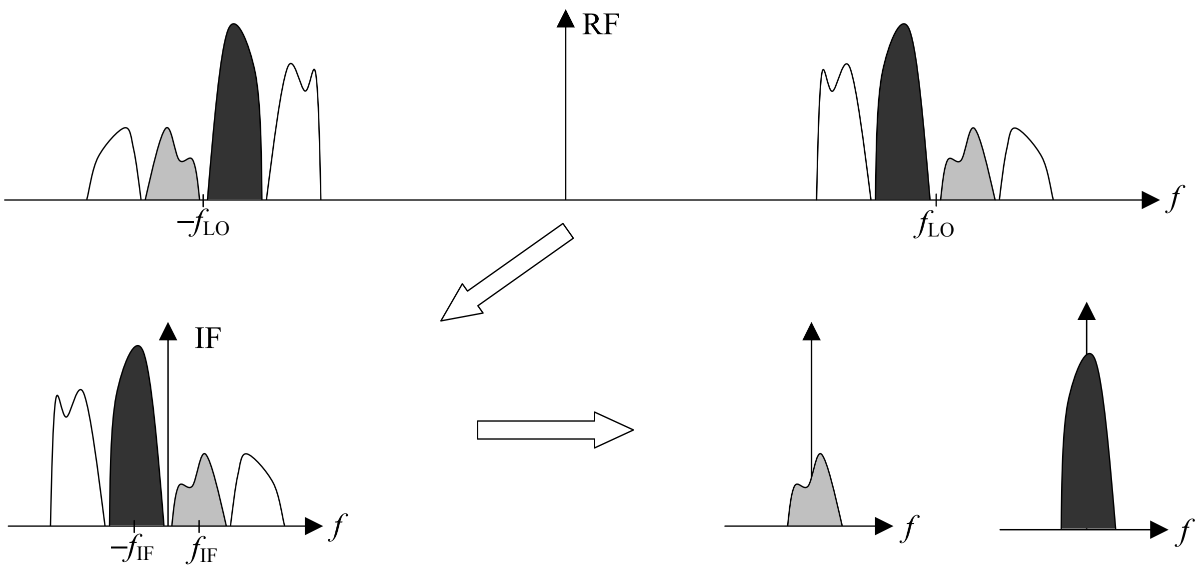


- Now after learning that we can build complex (analytic) bandpass filters, it's also easy to devise an **alternative strategy for I/Q demodulation** shown below.
- Notice that here the complex BPF creates already complex output signal and thus a true complex mixer is required (4 muls and 2 adds).
- This structure has, however, some benefits e.g. from analysis point of view, and it is also very suitable for digital I/Q demodulation combined with decimation/ down-sampling.

I/Q DEMODULATOR (v.2):

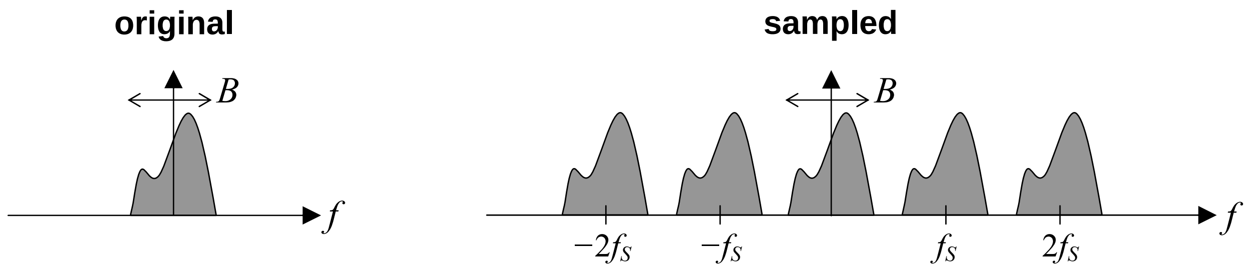


- Additional **example applications with more radio architecture focus:**
  - Digital IF-demodulation of carrier(s) in e.g. one-carrier low-IF receiver or two-carrier wideband I/Q downconversion based receiver
  - Either with complex digital BPF's + complex digital downconversions or with complex digital downconversions and real digital LPF



5. COMPLEX SIGNALS AND SAMPLING

– Basic starting point: In periodic sampling (sample rate f_S), the resulting discrete-time signal has a **periodic spectrum** where the original continuous-time spectrum is replicated around the **integer multiples of the sampling frequency**.



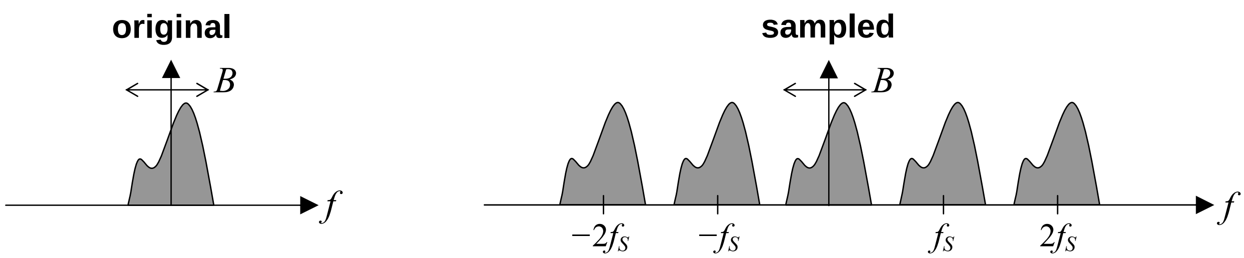
– Interestingly, any of these **spectral replicas or "images"** can be considered as the useful part and thus be used for further processing.

– Consequently, sampling (and multirate operations in general) can also be used, in addition to mixing techniques, in performing **frequency translations in radios**.

– Let B denote the **double-sided bandwidth** of a complex-valued baseband signal (i.e., the spectrum is nonzero only for $-B_{neg} \leq f \leq B_{pos}$, $B = B_{neg} + B_{pos}$)

- to avoid **harmful aliasing**, the sampling frequency f_S should simply be high enough such that the spectral images don't overlap, i.e.,

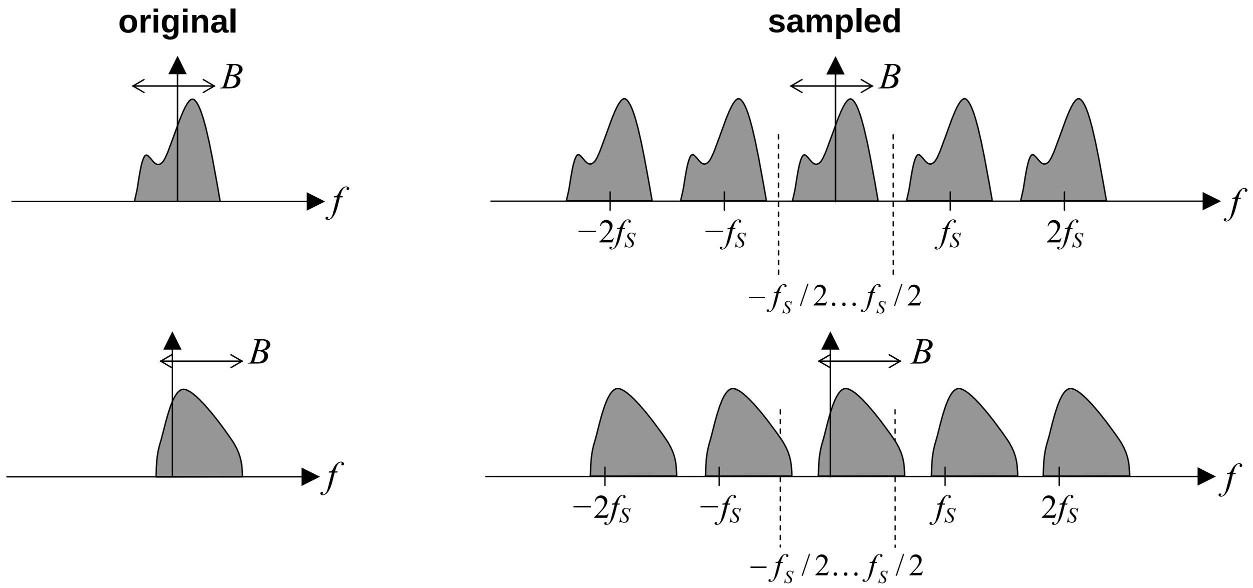
$$f_S - B_{neg} \geq B_{pos} \quad \Leftrightarrow \quad f_S \geq B_{neg} + B_{pos} \quad \Leftrightarrow \quad \underline{f_S \geq B}$$



- this is the (slightly-generalized) **traditional Nyquist sampling theorem**
- naturally, since the signal to be sampled is complex-valued, there exist two real-valued sample streams (I and Q) both at rate f_S
- if the signal to be sampled consists of multiple frequency channels, sampling rates below $f_S = B$ are possible iff only some of the channels are of interest

  $\Rightarrow$ the sampling frequency should simply be selected in such a manner that aliasing is avoided on top of those interesting frequency bands !

– Two example spectra which both have the same lower limit $f_S = B$ for the sampling frequency are depicted in the figure below.



– It should, however, be kept in mind that the "accessible" band for further discrete-time processing (with sample rate f_S) is always

$$-f_S/2 \quad \dots \quad f_S/2$$

- see the spectral contents of this band in the above examples
- in general, any of or all the spectral images can be "accessed", if so wanted, by increasing the sample rate

  ⇒ this forms the basis for multirate filtering techniques

– **Conclusion:** It really doesn't matter whether the signal is real or complex or whether it is located "symmetrically" with respect to origin

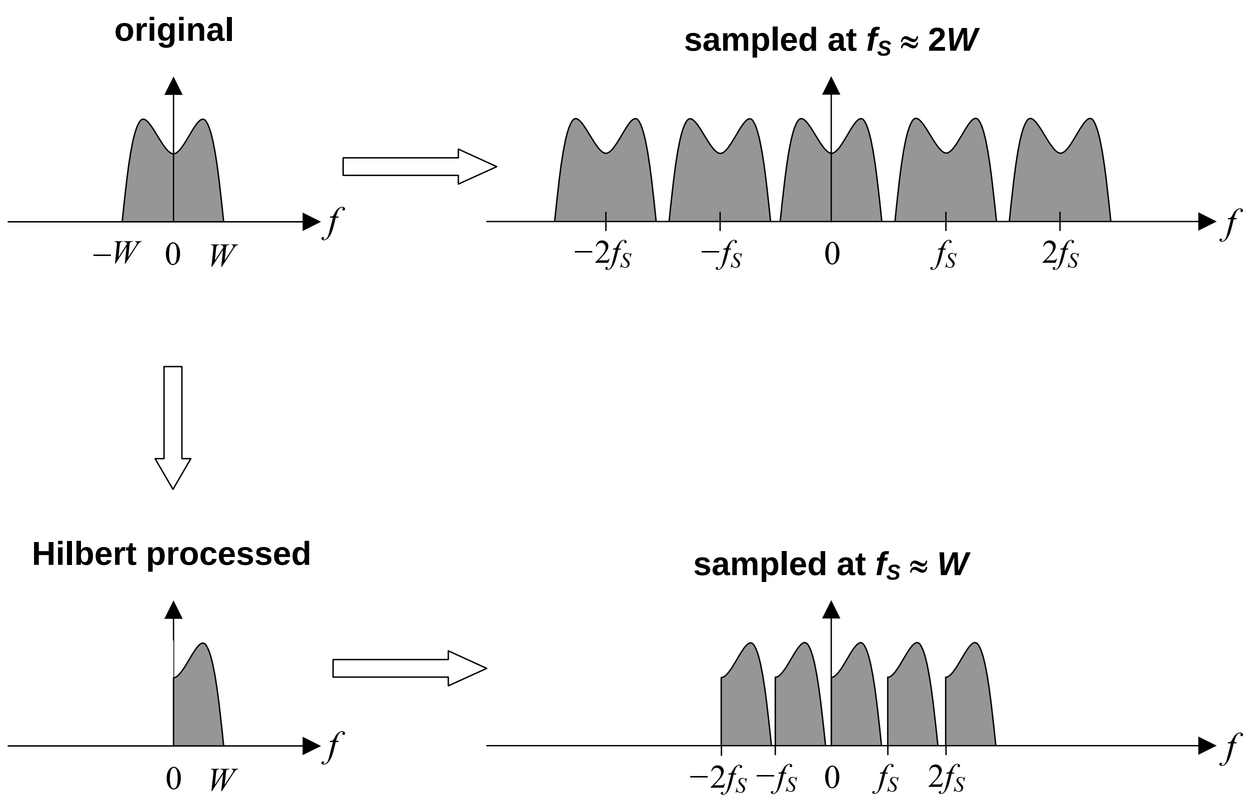
- the ultimate minimum sampling rate to avoid harmful aliasing is always

$$\underline{f_S = B}$$

– So in general the traditional statement "signal should be sampled at least at rate two times its highest frequency component" can be concluded inaccurate already now (or even misleading)

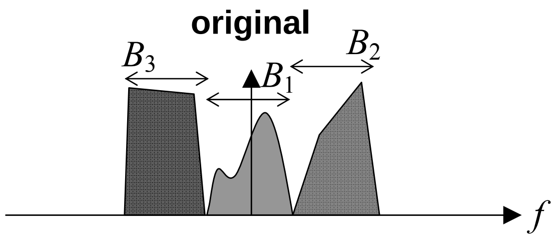
- what really matters is the **double-sided bandwidth**
- see the lower subfigure on the previous page, as an example
- even more dramatic examples soon when sampling bandpass signals

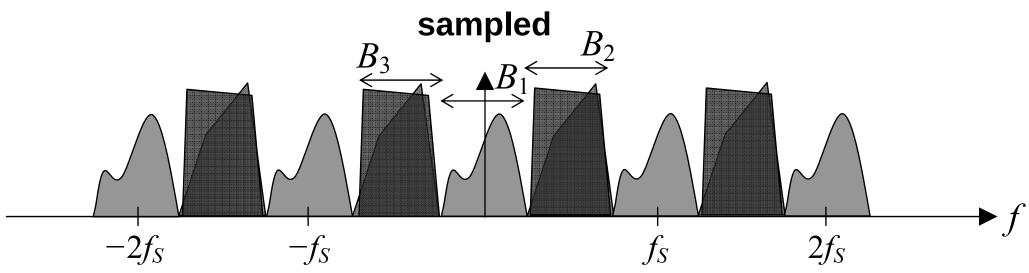
– Another good example is the sampling of a real-valued lowpass signal, say $x(t)$, with spectral support $-W \dots W$
  - when sampled directly, the minimum sampling rate is $f_S = 2W$
  - as an alternative, you can form an analytic signal $x(t) + jx_{\mathrm{HT}}(t)$, where $x_{\mathrm{HT}}(t)$ denotes the Hilbert transform of $x(t)$, for which the minimum sampling rate is only $f_S = W$ (even though the highest frequency component present in both signals is W)
  - illustrated in the following figure (next page)



– Below is an additional illustration of a case where only a part of the overall spectrum is of interest (sub-band B_1), so aliasing can be allowed on top of the other (non-interesting) parts of the spectrum

- sampling rate $f_S < B_1 + B_2 + B_3$ (but of course $f_S > B_1$)





Note on aliasing #1 – "harmful" vs. "harmless" aliasing

- (commonly misunderstood concepts, even in text books)
- fact: given a signal with maximum frequency component f_{MAX}, there will always be aliasing if sampling at any rate below $2f_{MAX}$, in the sense that the original frequency components present in the signal will appear at other (lower) frequencies

  $\Rightarrow$ see the previous examples

- but: as long as the induced spectral images DO NOT overlap (at least the interesting part of the spectrum), this aliasing is harmless, in the sense that all the information about the original (interesting) signal is still present in the samples

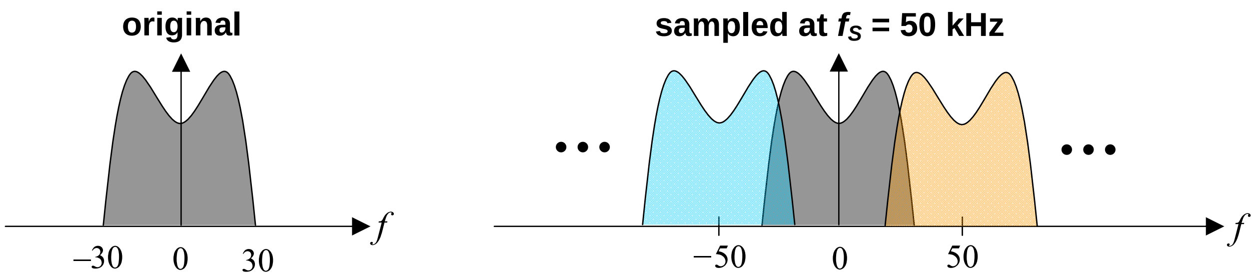
  $\Rightarrow$ see again the previous examples

- see the figures on previous pages

– **Note on aliasing #2 – exact aliasing frequencies**

- another common misunderstanding is related to how the frequencies actually alias in sampling
- typical (and strictly-speaking incorrect) interpretation: any frequency above $f_S/2$ "folds back" symmetrically with respect to $f_S/2$
- this is not exactly the case, or the above holds only for real-valued signals but not exactly for complex signals
- to see exactly how frequencies alias, all you need to remember is the periodic nature of the spectral images
- let's take a couple of simple examples

- **example 1:** real baseband signal with bandwidth 30 kHz, sampling at 50 kHz

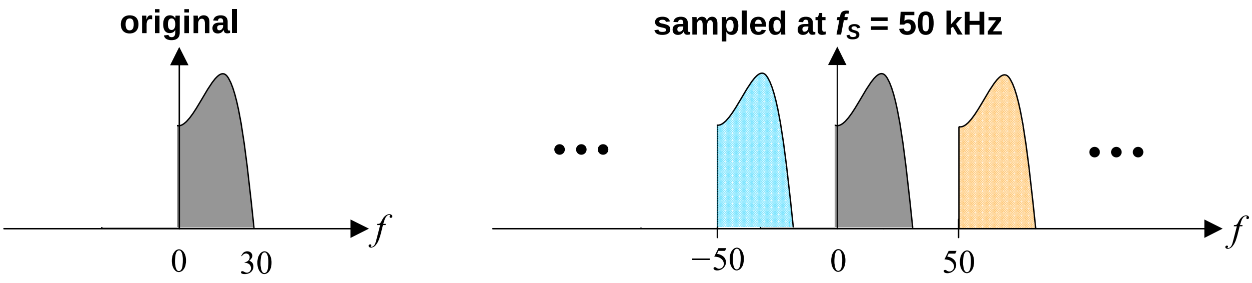


⇒ there clearly is harmful aliasing taking place

⇒ also considering the aliasing frequencies, e.g., a frequency component of +27 kHz appears at –23 kHz (not at +23 kHz)

- but since the signal is real-valued, the role of frequencies –23 kHz and +23 kHz (or any other symmetric positive/negative frequency pair) is identical, and this is usually ignored
- this is, however, not the case with complex signals (see the next example)

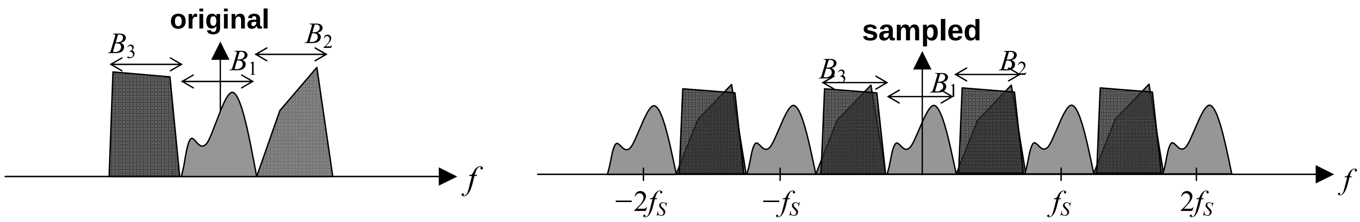
- **example 2:** analytic signal with bandwidth 30 kHz, sampling at 50 kHz



⇒ now obviously there's no harmful aliasing

⇒ also, e.g., the frequency component of +27 kHz really explicitly appears at –23 kHz (not at +23 kHz !)

⇒ important to understand when, e.g., sampling a wideband I/Q mixer output



6. SAMPLING OF BANDPASS SIGNALS

– Starting point is the traditional Nyquist sampling theorem: Any signal occupying the band $-B_{neg} \ldots B_{pos}$ [Hz] is completely characterized by its discrete-time samples given that the sampling rate is at least $B_{neg} + B_{pos}$ (two-sided bandwidth).

– People commonly interpret this that if the highest frequency component in a signal is f_{MAX}, you need to take at least $2f_{\mathrm{MAX}}$ samples per second

- strictly speaking, this is inaccurate (like we just concluded before)
- i.e., sampling at or above rate $2f_{\mathrm{MAX}}$ is clearly always sufficient but e.g. in case of **bandpass signals** we can also use (usually much) lower sample rate
- more specifically, sampling at rate below $2f_{\mathrm{MAX}}$ will indeed result in aliasing but as long as all the information about the original signal is present in the samples, we are doing good (only harmless aliasing)

  ⇒ keep in mind also that the Nyquist ("accessible") band for any sample rate f_S is $-f_S/2 \ldots f_S/2$, so with below $2f_{\mathrm{MAX}}$ sampling rates it is really one of the images that appear on this band !!!

- these kind of techniques are generally referred to as **subsampling**

– The one and only principle to remember in sampling is that the resulting signal has a periodic spectrum and any part of that spectrum can be selected/used for further processing.

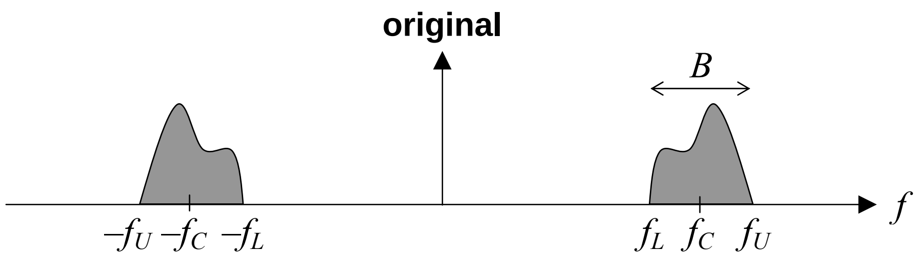
– More specifically, in communications receivers, **aliasing due to sub-sampling can be taken advantage of to bring the signal closer to baseband.**

– We consider two cases; starting from a real-valued bandpass signal, the resulting sample stream is either

- 1) real-valued or
- 2) complex-valued

6.1 Real Subsampling

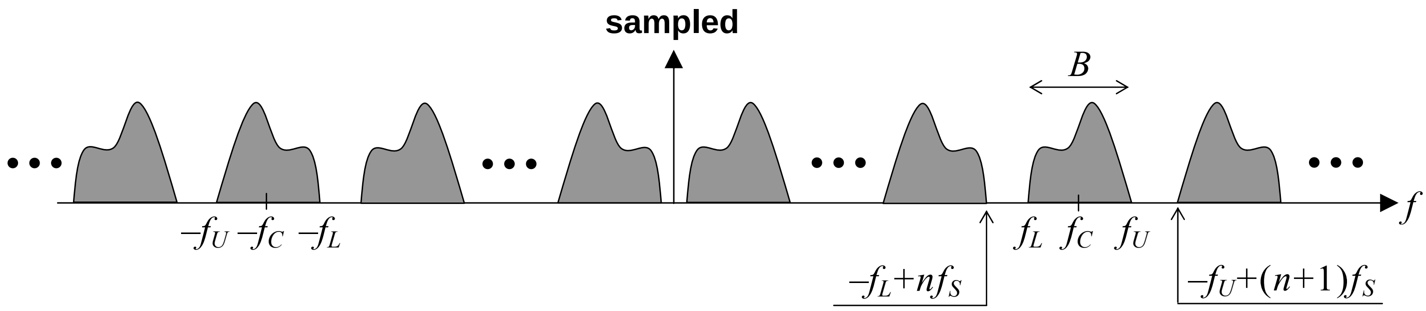
– Basic setup: real-valued bandpass signal, bandwidth B, center-frequency f_C, upper band-edge $f_U = f_C + B/2$ and lower band-edge $f_L = f_C - B/2$.



– Now sampling at any rate f_S results in a signal where the previous spectrum is replicated at integer multiples of the sampling rate (the basic effect of sampling).

– With $f_S < 2f_U$, aliasing will take place but as long as the aliasing components don't fall on top of each other, everything is OK !!

– So an example spectrum of the sampled signal could look like in the figure below, when there is no harmful aliasing and yet $f_S < 2f_U$.



– Based on the above figure, it is easy to formulate the regions of **allowable sampling rates**. These are in general of the form

$$\frac{2f_C+B}{n+1} \leq f_S \leq \frac{2f_C-B}{n} \quad \text{where} \quad 0 \leq n \leq floor(\frac{2f_C-B}{2B})$$

– (See also Exercise 9, Problem 3.)

– Comments:
- as can be seen, the possible values of the sampling rate depend on both the **bandwidth** B and the **center-frequency** f_C
- for $n=0$ we get $f_S \geq 2f_C + B = 2(f_C + B/2) = 2f_U$ which is the traditional Nyquist sampling theorem (the upper limit becomes infinity)
- for $n>0$ we are really sampling at lower frequency than given by the traditional Nyquist theorem
- for $n>0$ **aliasing does occur** but with given values of f_S, not on top of the desired signal band (**no harmful aliasing**)
- the lowest possible sampling rate is in general given by

$$f_S \geq \frac{2f_C+B}{n_{\max}+1} = \frac{2f_C+B}{floor(\frac{2f_C-B}{2B})+1} = \frac{2f_C+B}{floor(\frac{2f_C-B}{2B}+1)} = \frac{2f_C+B}{floor(\frac{2f_C+B}{2B})}$$

$\Rightarrow$ the "ultimate" sampling rate $f_S=2B$ is utilizable <u>iff</u> $\frac{2f_C+B}{2B}$ is an integer (then and only then $floor(\frac{2f_C+B}{2B}) = \frac{2f_C+B}{2B}$)

– **Numerical example:** $f_C = 20$ kHz and $B = 10$ kHz, so

- $0 \le n \le floor(\frac{40-10}{20}) = floor(1.5) = 1$ and the possible values for f_S are
- $n = 0$: $50 \text{ kHz} \le f_S \le \infty$
- $n = 1$: $25 \text{ kHz} \le f_S \le 30 \text{ kHz}$

  $\Rightarrow$ try e.g. with $f_S = 27$ kHz and you see that no harmful aliasing occurs

– **Another example:** FM broadcasting band 88 MHz … 108 MHz

- total bandwidth $B = 20$ MHz, center-frequency $f_C = 98$ MHz
- then: $0 \le n \le floor(\frac{196-20}{40}) = floor(4.4) = 4$

  $\Rightarrow n = 0$: $216 \text{ MHz} \le f_S \le \infty$

  $\Rightarrow n = 1$: $108 \text{ MHz} \le f_S \le 176 \text{ MHz}$

  $\Rightarrow n = 2$: $72 \text{ MHz} \le f_S \le 88 \text{ MHz}$

  $\Rightarrow n = 3$: $54 \text{ MHz} \le f_S \le 58.6667 \text{ MHz}$

  $\Rightarrow n = 4$: $43.2 \text{ MHz} \le f_S \le 44 \text{ MHz}$

– In above, we used graphical intuition and the periodic nature of sampled signals.

– Another way is to use a bit of math, real-valued bandpass input signal at f_C:

$$r(t) = \mathrm{Re}[z(t)e^{j\omega_C t}] = z_I(t)\cos(\omega_C t) - z_Q(t)\sin(\omega_C t)$$

- Sampling at rate $f_S = 1/T_S$

$$\begin{aligned} r_n &= r(nT_S) = z_I(nT_S)\cos(\omega_C nT_S) - z_Q(nT_S)\sin(\omega_C nT_S) \\ &= z_I(nT_S)\cos(2\pi n f_C T_S) - z_Q(nT_S)\sin(2\pi n f_C T_S) \end{aligned}$$

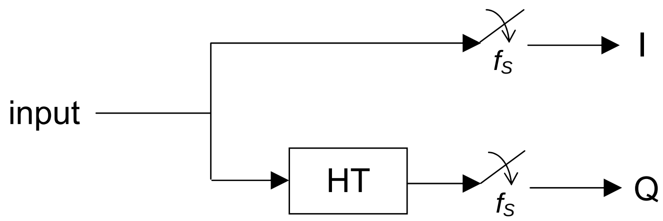
- Now in case of sub-sampling, $f_C >> f_S$ so $f_C T_S >> 1$ (take e.g. the previous FM broadcast example with f_C = 98MHz and f_S = 43.5MHz => $f_C T_S \approx 2.253$)
- Thus since cos(.) and sin(.) functions "wipe out" any integer multiple of 2π in their arguments, the sampled signal can also be written as

$$\begin{aligned} r_n &= z_I(nT_S)\cos(2\pi n f_C T_S) - z_Q(nT_S)\sin(2\pi n f_C T_S) \\ &= z_I(nT_S)\cos(2\pi n f'_C T_S) - z_Q(nT_S)\sin(2\pi n f'_C T_S) \end{aligned}$$

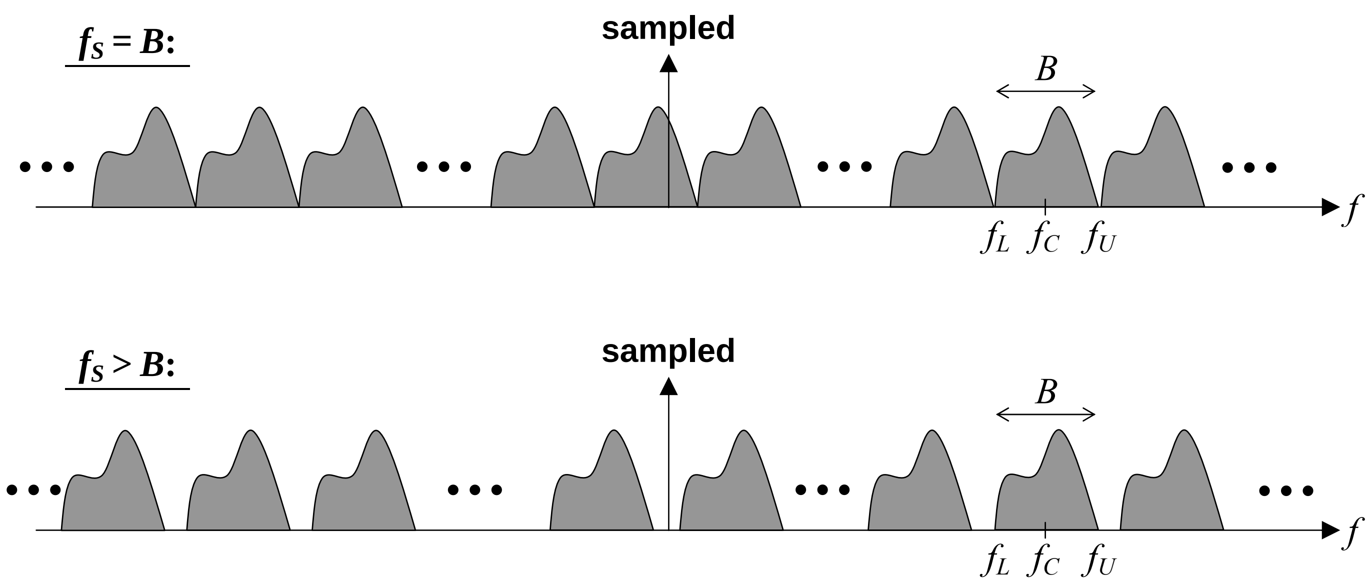
  where $f'_C T_S < 1/2$ (i.e. $f'_C < f_S/2$). But these are simply samples of a bandpass signal at (typically much) lower center-frequency f'_C !!

6.2 Complex Subsampling

- Instead of sampling directly the real-valued signal, the idea is to sample the corresponding **analytic signal** !!!
- So the sampling structure looks like (HT denotes Hilbert transformer)



- Now since the analytic signal is free from negative frequency components, sampling frequency of $f_S = B$ (or any rate above) is always (independently of the center-frequency f_C !) sufficient to avoid harmful aliasing !!!
  - No such limitations as in real sub-sampling
- Some example spectral figures below with the same input signal as in the previous subsection.



- **Notice:** If the center-frequency f_C is an integer multiple of the sample rate f_S (i.e., $f_S = f_C / k$), the center-frequency of the k-th spectral replica will coincide with zero frequency and a direct **bandpass-to-lowpass transformation** is obtained !!!

- This is easy to understand based on spectral interpretations but can also be seen using math as follows:

  ⇒ the real bandpass input, say $r(t)$, can be written in terms of its baseband equivalent $z(t)$ as

$$r(t) = \mathrm{Re}[z(t)e^{j\omega_C t}] = \frac{1}{2}z(t)e^{j\omega_C t} + \frac{1}{2}z^*(t)e^{-j\omega_C t}$$

  ⇒ then the corresponding analytic signal is of the form

$$\begin{aligned} r(t) + jr_{HT}(t) &= \frac{1}{2}z(t)e^{j\omega_C t} + \frac{1}{2}z^*(t)e^{-j\omega_C t} + j(-j\frac{1}{2}z(t)e^{j\omega_C t} + j\frac{1}{2}z^*(t)e^{-j\omega_C t}) \\ &= z(t)e^{j\omega_C t} \end{aligned}$$

  ⇒ thus sampling at $f_S = f_C / k$ (with k integer) results in

$$r(nT_S) + jr_{HT}(nT_S) = z(nT_S)e^{j\omega_C nT_S} = z(nT_S)e^{j2\pi f_C nT_S} = z(nT_S)e^{j2\pi nk} = z(nT_S)$$

  which are indeed just **samples of the baseband equivalent !**

- More generally, if there's no direct integer relation between f_S and f_C, we get

$$r(nT_S) + jr_{HT}(nT_S) = z(nT_S)e^{j\omega_C nT_S} = z(nT_S)e^{j2\pi f_C nT_S} = z(nT_S)e^{j2\pi f'_C nT_S}$$

  where again typically $f_C T_S >> 1$ while $f'_C T_S < 1/2$

  - o (Residual) frequency shift due to $e^{j2\pi f'_C nT_S}$, which can be removed using a digital complex mixer $e^{-j2\pi f'_C nT_S}$

– **So in conclusion:** complex sub-sampling and harmless aliasing can be used efficiently to do frequency translations to lower frequencies without the earlier restrictions related to real sub-sampling !

- One of the main **practical limitations** to the "heavy" use of subsampling in radio receivers, at least using today's circuits and electronics, is related to the **small random fluctuations or errors in the sampling instants, caller jitter**
  - due to, e.g., "instability" of the used sampling clock / clock generator
- In other words, typical RF carrier frequencies in wireless / radio systems are commonly in the 1-5 GHz range
  - thus even though the signal bandwidth and thus the needed sampling frequency would be rather modest (e.g., a couple of tens of MHz), the absolute frequencies in the signal to be sampled are anyway in the GHz range
  - therefore, even a really small displacement or error **(jitter)** in the actual sampling instant would result in severe error in the actual sample value !!
  - this has limited the use of subsampling techniques so far to the **IF sections of the receivers**, which are typically in the couple of tens or hundreds MHz range

    ⇒ but anyway the previous principles apply directly !

7. Some Useful Literature

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