Iterative Methods for Designing Orthogonal and Biorthogonal Two-channel FIR Filter Banks with Regularities

Robert Bregović and Tapio Saramäki
Signal Processing Laboratory
Tampere University of Technology
P. O. Box 553, FIN-33101 Tampere, Finland
e-mail: bregovic@cs.tut.fi and ts@cs.tut.fi

Abstract

Efficient iterative methods are described for designing orthogonal and biorthogonal two-channel perfect-reconstruction FIR filter banks in such a way that for the analysis and the synthesis lowpass filter the number of zeros at $z = -1$ are fixed and the energies in the given filter stopband regions are minimized. The regularity of the analysis and the synthesis filter banks resulting when using only half of the tree structure (only the low-pass branch is split into two branches) are roughly proportional to the number of fixed zeros at $z = -1$ (vanishing moments) in the analysis and the synthesis filter, respectively. The frequency selectivity of these banks, in turn, is reciprocally related to the energies in the filter stopband regions. These two parameters are contradictory. By increasing the number of fixed zeros, the frequency selectivity of the overall filter bank is decreasing and vice versa. Since the selection of these two contradictory parameters depend on the application for which the filter bank is designed, it is necessary to find compromise solutions between them for every particular case. Using the proposed methods with different design requirements enables us to generate, for both orthogonal and biorthogonal filter banks, all possible combination between the maximally flat filter banks (maximum number of vanishing moments in the analysis and the synthesis filter) and standard frequency selective filter banks (no regularity requirements). Comparing the proposed method with some existing methods, for a given number of fixed zeros, filter banks with increased regularity and decreased stopband energies are obtained. The efficiency and flexibility of the proposed synthesis techniques are illustrated by means of several examples.

Keywords: FIR, Two-channel filter banks, Wavelets, Octave filter banks, Least-squared error, Perfect reconstruction, Vanishing moments, Regularity, De-noising, Orthogonal, Biorthogonal

1 Introduction

During the last two decades, two-channel perfect-reconstruction (PR) FIR filter banks have been studied intensively due to their numerous applications [1]–[3]. They are widely used as building blocks for generating filter banks of the following three basic types. First, it is straightforward to generate multi-channel filter banks by building up tree structures using such two-channel filter banks as basic blocks. For these filter banks, the number of filters in both the analysis and synthesis bank is a power of two and the filter bandwidths are equal. In this case, the equal bandwidths are achieved by using building-block two-channel filter banks of different lengths. If only half of the tree structure is used (only the low-pass branch is split into lowpass and highpass components), octave filter banks are obtained. Usually, at all stages,
the same two-channel filter bank is used. These octave filter banks are structurally identical with discrete-time wavelet banks being the third type of filter banks.

Despite of the same structure, the design of the building-block two-channel filter bank is very different for conventional octave filter banks and discrete-time wavelet banks. For octave filter banks, it is desired that the frequency selectivity of the filters in the overall bank as well as in the building-block two-channel filter banks are high. For wavelet banks, due to their different applications, instead of the frequency selectivity, such properties as the regularity and the number of vanishing moments are of great importance [4], [5]. The number of vanishing moments is directly the number of zeros being located at $z = -1$ for the lowpass analysis (synthesis) filter. For wavelet banks, the regularity is connected with the number of vanishing moments. It gives the number of continuous derivatives of the corresponding continuous-time “mother wavelet”. This function can be generated using the procedure described by Rioul in [6]. In practice, the regularity is difficult to determine exactly. A good estimate is the lower bound and the upper bound of the Hölder regularity as reported in [6].

For PR orthogonal wavelets, the regularities of the analysis and the synthesis part are equal whereas for biorthogonal wavelets, the regularity of the analysis part depends on the lowpass analysis filter and the regularity of the synthesis part depends on the lowpass synthesis filter.

Typically, the regularity of discrete-time wavelets is maximized by designing the analysis (synthesis) lowpass filter to have the maximum possible number of zeros at $z = -1$, resulting in a very poor selectivity for the filters in the bank. On the other hand, for selective octave banks the analysis (synthesis) lowpass filter is not forced to have any zeros at $z = -1$ and the regularity becomes unacceptable. The high frequency selectivity and the high regularity (large number of vanishing moments) are thus conflicting requirements. Many authors have shown that in order to achieve a good overall performance a proper compromise between these conflicting measures of ’goodness’ is needed.

Rioul has studied in [7] the effects of the regularity, the frequency selectivity, and the phase linearity of the subfilters on the performance of the overall filter bank in image compression. According to his observations, the regularity is the most important property among the above-mentioned three characteristics, that is, the greater the regularity is, the better compression is achieved. For the same application, Villasenor, Belzer, and Liao [8] have performed a survey over 4300 different biorthogonal FIR filter banks with less than 36 taps in the analysis/synthesis pair. Their results show that filter banks with the highest regularity do not achieve the best performances. Therefore, the regularity alone is not the best or at least not the sufficient measure of ‘goodness’ for designing filter banks providing good image compression properties.

Furthermore, it is known that the regularity is roughly directly connected with the number of vanishing moments, but, as has been shown by Lang and Heller in [9], a system with the maximum number of vanishing moments does not always result in the largest regularity value. To achieve the maximum possible regularity for given filter lengths, the use of optimization is necessary. Since the number of vanishing moments is easier to include in to the design procedure, Balasingham and Ramstad have investigated in [10] how the effectiveness of image compression depends on the number of vanishing moment. They have shown that the best results are again not obtained using filter banks with maximum number of vanishing moments (maximal flat filters), but with some intermediate solutions between the maximal flat and frequency-selective filter banks. Similar results can be also obtained in other filter bank applications. Generally, it is not clear what requirements must be taken into account and in what manner to optimize a filter (wavelet) bank to achieve best performances for some particular application. In most cases, it is a question of making a proper compromise between the regularity and the frequency selectivity.
The purpose of this paper is to introduce fast methods for synthesizing building-block two-channel PR orthogonal and biorthogonal FIR filter banks in such a way that the resulting octave filter banks provide tradeoffs between the regularity and the frequency selectivity. Before considering these synthesis techniques, we start with a brief review of existing techniques for both orthogonal and biorthogonal octave filter banks in order to position the proposed technique among them. First, orthogonal filter banks are considered and, then, biorthogonal filter banks are studied.

Building blocks for conventional frequency-selective orthogonal FIR filter banks have been introduced by Mintzer [11] and Smith and Barnwell [12]. Using their design techniques, filter banks with subfilters exhibiting a minimax behavior in their stopbands are obtained. As shown by authors of this paper in [13], by applying an iterative technique, the subfilters can also be designed to exhibit a least-mean-square behavior in their stopbands.

Rioul and Duhamel have suggested in [14] a synthesis method for designing orthogonal filter banks with a minimax approximation in the filter stopbands and various regularities. In their synthesis scheme, the number of zeros of the lowpass analysis filter being located at \( z = -1 \) can be varied and the remaining parameters are optimized to make the filter amplitude response equiripple in the given stopband. In [15], Lu has proposed a parameterization method, whereas iterative methods have been described by Lang, Selesnick, Odegard, and Burrus in [16] and Blu in [17]. The synthesis scheme proposed by Blu deals with the design of rational (nonuniform) filter banks. The classical wavelet type filter banks can be considered as a special case of rational filter banks.

This paper introduce an efficient technique for synthesizing building-block orthogonal two-channel FIR filter banks in such a manner that the stopband energy of the analysis lowpass filter is minimized for a given number of fixed zeros at \( z = -1 \). Compared with the above-mentioned synthesis schemes, the proposed iterative method is more straightforward to implement, it is faster, and the convergence to the optimum solution is independent of the starting-point filter bank. Since the stopband energy of the analysis lowpass filter is minimized, for the same number of vanishing moments, more regular wavelet banks are obtained. Special cases of the proposed two-channel filter banks are banks used for generating the Daubechies wavelets [18] and the classical two-channel orthogonal filter banks without any regularity constraints [13].

Techniques for designing PR linear-phase biorthogonal FIR filter banks are based on optimization methods, like the one presented by Nguyen in [20], iterative method presented by Horng and Willson in [21], and an improved combination of the above two techniques described by the authors of this paper in [22]. All these design schemes result in frequency-selective filters of low regularity and a least-mean-square behavior in their stopbands.

Cooklev, Nishihara, and Sablataash in [23] as well as Zhao and Swamy in [24] have incorporated into the biorthogonal filter bank design procedure regularity constraints to obtain more regular filter banks. In this paper, like for the orthogonal case, an approach for designing biorthogonal filter banks is proposed. The stopband energies of the analysis and the synthesis lowpass filter are minimized after the numbers of vanishing moments for both filters are selected. The resulting overall filter bank (wavelet) has equal energies in all filter stopbands but independent regularities for the analysis and the synthesis parts. As for the orthogonal filter banks, two special cases of the proposed two-channel filter banks are banks used for generating the standard biorthogonal wavelets [25] and the classical two-channel biorthogonal filter banks without any regularity constraints [22].

An intermediate solution between orthogonal and biorthogonal filter banks, linear-phase nearly orthogonal wavelet banks, has been proposed by Saramäki and Egiazarian in [26]. They use nonlinear optimization to design such filter banks with additionally imposed regularity constraints. They show that such filter banks (wavelets) have a better performance.
in many applications than classical orthogonal filter banks even if these filter banks suffer from small alias and reconstruction errors.

The differences between the orthogonal and the biorthogonal filter banks is that for orthogonal filter banks, only one nonlinear-phase filter has to be designed and the regularity of the analysis and the synthesis filter bank is the same. In the biorthogonal case, different linear-phase filters can be used in the analysis and the synthesis part and the number of zeros at $z = -1$ may be different for the building-block lowpass analysis and synthesis filters. This can be utilized to achieve better filter bank properties. Many authors have noticed that it is beneficial to have different filter lengths for the analysis and the synthesis lowpass filters. Usually, filter banks with shorter synthesis lowpass filters result in a better overall filter bank performance.

The organization of this paper is as follows: Section 2 introduces the building-block two-channel filter bank. PR orthogonal FIR filter banks are considered in Section 3, whereas Section 5 considers PR biorthogonal FIR filter banks. The proposed iterative methods for orthogonal and biorthogonal filter banks are described in Sections 4 and 6, respectively. In Section 7, several design examples and comparisons with filter banks obtained using existing synthesis schemes are included. Finally, conclusions are given in Section 8.

2 Two-Channel Filter Banks

This section reviews some basic relations of alias-free two-channel FIR filter banks and introduces a transform that simplifies the filter bank synthesis to be described in the following sections.

2.1 Alias-Free Two-Channel Filter Banks

The block diagram for a two-channel filter bank providing the basic building-block for both the half tree-structured octave filter banks and discrete-time wavelet banks is shown in Figure 1. This system consists of an analysis bank containing a lowpass-highpass filter pair with transfer functions $H_0(z)$ and $H_1(z)$ and down-sampling their output signals by a factor of two and an synthesis bank containing up-sampling by a factor of two the two input signal followed by lowpass-highpass filter pair with transfer functions $F_0(z)$ and $F_1(z)$. The outputs of these two filters are added to form the overall output signal. In practice, there is a processing unit between down- and up-sampling operations for compressing and coding the two signals for transmission and storage purposes.

By omitting the processing unit, the input and the output of the overall system are related in the $z$-domain as [1]–[3]

$$Y(z) = T(z) X(z) + A(z) X(z),$$

where the first term

$$T(z) = \frac{1}{2} \left[ H_0(z) F_0(z) + H_1(z) F_1(z) \right]$$

is the distortion transfer function and the second term...
is the aliasing transfer function. The last term becomes zero by selecting the synthesis filter transfer functions as
\[ F_0(z) = 2H_1(-z) \quad \text{and} \quad F_1(z) = -2H_0(-z) \]
giving the following input-output relation:
\[ Y(z) = T(z)X(z) , \]
where
\[ T(z) = H_0(z)H_1(-z) - H_1(z)H_0(-z) . \]

### 2.2 Transforms Simplifying the Filter Bank Synthesis

In order to simplify the overall problem definition and the overall filter bank synthesis (especially for biorthogonal filter bank to be considered in Sections 5 and 6) we use, instead of \( H_0(z) \) and \( H_1(z) \), the following auxiliary transfer functions [13], [22]:
\[ G_0(z) = \sum_{n=0}^{N_0} g_0[n]z^{-n} = H_0(z) = \sum_{n=0}^{N_0} h_0[n]z^{-n} \quad (6a) \]
\[ G_1(z) = \sum_{n=0}^{N_1} g_1[n]z^{-n} = H_1(-z) = \sum_{n=0}^{N_1} (-1)^n h_1[n]z^{-n} . \quad (6b) \]

![Figure 2. Specifications for \( G_0(z) \) and \( G_1(z) \) and the relations between \( H_0(z) \) and \( G_0(z) \) and \( H_1(z) \) and \( G_1(z) \).](image)
Section 3 introduces the PR two-channel FIR filter bank under consideration and states the problem for optimizing this filter bank. An efficient algorithm for solving this problem will be described in Section 4.

### 3.1 General PR Two-Channel Orthogonal FIR Filter Banks

A system depicted in Figure 1 represents a PR orthogonal filter bank if in addition to the conditions of Equation (3), $G_0(z)$ and $G_1(z)$, as given by Equations (6a) and (6b), meet following three conditions:

1) $N_0$, the order of $G_0(z)$, is an odd integer.
2) $G_1(z) = z^{-N_0}G_0(z^{-1})$.
3) $T(z) = G_0(z)G_1(z) - G_0(-z)G_1(-z) = z^{-N_0}$.

In the above, Condition 1 is necessary due to the fact that a PR orthogonal system can be generated using only odd order filters [2], whereas Condition 2 defines the relation between the analysis filters. In terms of impulse-response coefficients, the coefficients of $G_1(z)$ are related to those of $G_0(z)$ via

$$g_1[n] = g_0[N_0 - n] \quad \text{for} \quad n = 0, 1, \cdots, N_0$$

making them time-reversed version of each other. Therefore, only $N_0+1$ coefficients of filter $G_0(z)$ are the unknowns. Condition 3 means that for the system of Figure 1 the output and input signal are related as $y[n] = x[n - N_0]$, thereby guaranteeing that the PR condition is satisfied. Equivalently, Condition 3 can be expressed, after some manipulations, in terms of the impulse-response coefficients of $G_0(z)$ as

$$\sum_{r=0}^{2l-1} g_0[N_0 + 1 + r - 2l]g_0[r] = \begin{cases} 0 & \text{for} \quad l = 1, 2, \cdots, (N_0-1)/2 \\ 1/2 & \text{for} \quad l = (N_0 + 1)/2. \end{cases}$$

This form will be exploited later on when solving the optimization problem to be stated for the proposed orthogonal PR filter bank in the following subsections.
3.2 Proposed Orthogonal Filter Bank

For the proposed filter bank with the given odd filter order \( N_0 \), it is assumed that \( G_0(z) \) possesses \( K_0 \) fixed zeros at \( z = -1 \). In this case \( G_0(z) \) is expressible as

\[
G_0(z) = \left[ \frac{1 + z^{-1}}{2} \right]^{K_0} \sum_{n=0}^{L_0} b_0[n] z^{-n}
\]

(10)

where \( L_0 = N_0 - K_0 \) is the number of remaining adjustable zeros.

3.3 Statement of the Problem

The optimization problem for the proposed two-channel orthogonal filter bank is stated as follows: Given \( N_0, K_0, \) and \( \omega_s \), find the \( L_0 + 1 \) (\( L_0 = N_0 - K_0 \)) adjustable coefficients of \( G_0(z) \), as given by Equation (10), to minimize

\[
\varepsilon_0 = \int_{-\pi}^{\pi} |G_0(e^{j\omega})|^2 d\omega
\]

(11)

subject to the condition that the coefficients of \( G_0(z) \) expressed in the direct form of Equation (6a) satisfy the conditions of Equation (9). Solving this problem results in a PR orthogonal filter bank where \( G_0(z) \) has at least \( K_0 \) zeros at \( z = -1 \) and its energy is minimized in the given stopband region.

4 Efficient Iterative Algorithm for Solving the Stated Problem for Orthogonal Filter Banks

This subsection describes an efficient algorithm for solving the optimization problem stated in Subsection 3.3.

4.1 Reformulation of the Stated Problem

According to Equation (10), there are only \( L_0 + 1 \) unknowns \( b_0[n] \) for \( G_0(z) \) in the stated problem. Using the vector and matrix notations, the optimization problem can be restated in the following modified form: Find the unknowns included in the vector

\[
b_0 = [b_0[0] \ b_0[1] \ \cdots \ b_0[L_0]]'
\]

(12a)

to minimize

\[
\varepsilon_0 = b_0^T \mathbf{S} b_0
\]

(12b)

subject to

\[
\mathbf{C} b_0 = \mathbf{m},
\]

(12c)

where \( \mathbf{S} \) is an \((L_0+1) \times (L_0+1)\) Toeplitz matrix with the elements given by

\[
s_{nl} = \frac{1}{2^{2K_0}} \sum_{p=0}^{2K_0} \left( \begin{array}{c} 2K_0 \\ p \end{array} \right) \int_{-\pi}^{\pi} \cos[(r - l + p - K_0)\omega] d\omega
\]

(12d)

with

\[
\left( \begin{array}{c} 2K_0 \\ p \end{array} \right) = \frac{(2K_0)!}{(2K_0 - p)! p!}
\]

(12e)

and

\[ \int_0^{\pi/\omega} \cos(\eta \omega) d\omega = \begin{cases} -\frac{\sin(\eta \omega)}{\eta} & \text{for } \eta \neq 0 \\ \pi - \omega & \text{for } \eta = 0. \end{cases} \quad (12f) \]

\( \mathbf{m} \) is an \((N_0+1)/2\) length vector given by

\[ \mathbf{m} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1/2 \end{bmatrix}^T \quad (12g) \]

and \( \mathbf{C} \) is an \((N_0+1)/2 \times (L_0+1)\) matrix with the elements given by

\[ c_{rl} = \frac{1}{2^{2K_0}} \sum_{p=0}^{2K_0} \left( \begin{bmatrix} 2K_0 \\ p \end{bmatrix} b_0[L_0 + p - 2r + l] \right) \quad (12h) \]

for \( r = 1, 2, \ldots, (N_0+1)/2 \) and \( l = 1, 2, \ldots, (L_0+1) \). In the above equation, \( b_0[n] = 0 \) for \( n < 0 \) and \( n > L_0 \).

Since the matrix \( \mathbf{C} \) depends on the unknowns included in \( \mathbf{b}_0 \), the restated optimization problem cannot be solved directly by applying the Lagrange multiplier method [13]. However, this problem can be avoided by determining \( \mathbf{C} \) separately. This can be accomplished by starting with properly determined initial values for the filter coefficients and then using them for calculating the matrix \( \mathbf{C} \). After fixing \( \mathbf{C} \), the overall problem becomes solvable with the aid of the Lagrange multiplier method. In the sequel, \( \mathbf{b}_0 \) is used for determining \( \mathbf{C} \), whereas \( \mathbf{b}_0^* \) is used as the solution of Lagrange multiplier method.

Introducing the Lagrange multiplier vector \( \lambda = [\lambda_1 \lambda_2 \ldots \lambda_{(N_0+1)/2}]^T \), the Lagrangian function for the linearized optimization problem can be expressed as

\[ \Lambda(\mathbf{b}_0, \lambda) = \mathbf{b}_0^T \mathbf{S} \mathbf{b}_0 - \lambda^T (\mathbf{C} \mathbf{b}_0 - \mathbf{m}) \quad (13) \]

Imposing the following necessary and sufficient conditions for the solution:

\[ \nabla_{\mathbf{b}_0} \Lambda(\mathbf{b}_0, \lambda) = 0 \quad (14a) \]

and

\[ \nabla_{\lambda} \Lambda(\mathbf{b}_0, \lambda) = 0 \quad , \quad (14b) \]

results in the following system of linear equations:

\[ \begin{bmatrix} -2\mathbf{S} & \mathbf{C}^T \\ \mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{b}_0^* \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{m} \end{bmatrix} \quad (15) \]

Solving this linear system of equations for \( \mathbf{b}_0^* \), a vector containing the new filter coefficient values is obtained. The next step involves updating \( \mathbf{b}_0 \) according to

\[ \mathbf{b}_0 := \tau \cdot \mathbf{b}_0 + (1 - \tau) \mathbf{b}_0^* \quad (16) \]

where \( \tau \) (\( 0 < \tau < 1 \)) is the convergence factor. Then, the resulting \( \mathbf{b}_0 \) is used for regenerating the fixed matrix \( \mathbf{C} \). \( \mathbf{b}_0 \) is determined from Equation (15), and \( \mathbf{b}_0 \) is updated according to Equation (16). This updating scheme is repeated until \( \mathbf{b}_0 \) and \( \mathbf{b}_0^* \) become equal according to the predetermined tolerance.
4.2 Proposed Algorithm for Orthogonal Filter Banks

The overall algorithm is carried out in the following steps:

Step 1) Select $b_0 = \left[ \frac{1}{(L_0 + 1)} ~ \frac{1}{(L_0 + 1)} ~ \cdots ~ \frac{1}{(L_0 + 1)} \right]^T$.

Step 2) Calculate the $(L_0+1) \times (L_0+1)$ matrix $S$ using Equations (12d), (12e), and (12f) and form the $(N_0+1)/2$ length vector $m$ according to Equation (12g).

Step 3) Use the old filter coefficient values $b_0$ to calculate the matrix $C$ with the aid of Equations (12h) and (12e).

Step 4) Calculate new values for the filter coefficients by solving the system of Equation (15) for $\tilde{b}$ and update the filter coefficients $b_0$ using Equation (16) with $\tau = 0.7$.

Step 5) If $\|b_0 - \tilde{b}_0\| \leq \delta$, where $\delta$ is a prescribed tolerance, then the filter coefficients included in the vector $b_0$ are the desired solution. Otherwise, go to Step 3.

It has turned out that very small number of iterations is required by the above algorithm to arrive at the optimum solution.

The above algorithm results in a filter bank where the amplitude response of $H_0(z) = G_0(z)$ achieves the value at unity at the zero frequency. When this bank is used as a building block for discrete-time wavelet banks, the corresponding value is desired to be $\sqrt{2}$. This is achieved by using the substitution

$$b_0[n] := \sqrt{2}b_0[n] \quad \text{for} \quad n = 0, 1, \ldots, L_0$$

and replacing the relations of Equation (3) by

$$F_0(z) = H_1(-z) \quad \text{and} \quad F_1(z) = -H_0(-z).$$

5 Statement of the Problem for Two-channel PR Linear-Phase Biorthogonal FIR Filter Banks

This section introduces the PR linear-phase biorthogonal two-channel FIR filter bank under consideration and states the problem for optimizing this filter bank. An efficient algorithm for solving this problem will be described in Section 6.

5.1 General PR Two-Channel Biorthogonal Linear-Phase FIR Filter Banks

A PR biorthogonal filter bank with linear-phase subfilters $G_0(z)$ and $G_1(z)$, as given by Equations (6a) and (6b), satisfies the following conditions [13]:

1) The impulse responses of $G_0(z)$ and $G_1(z)$ possess an even symmetry, that is, $g_0[n_0 - n] = g_0[n]$ for $n = 0, 1, \cdots, N_0$ and $g_1[N_1 - n] = g_1[n]$ for $n = 0, 1, \cdots, N_1$.

2) The sum of the filter orders $N_0$ and $N_1$ is two times an odd integer, that is, $N_0 + N_1 = 2K$ with $K$ being an odd integer.

3) $E(z) = G_0(z)G_1(z)$ is a linear-phase half-band FIR filter of order $N_0 + N_1$.

Here, Condition 3 implies that $E(z)$ is expressible as

$$E(z) = G_0(z)G_1(z) = \sum_{n=0}^{N_0+N_1} e[n]z^{-n},$$

where

$$e[(N_0 + N_1)/2] = 1/2$$

and

$$e[(N_0 + N_1)/2 + 2r] = 0 \quad \text{for} \quad r = \pm 1, \pm 2, \cdots, (N_0 + N_1)/2 - 1.$$
Only the following two filter classes can meet these conditions [21]:

**Type A**: \( N_0 \) and \( N_1 \) are odd integers and their sum is two times an odd integer \( K \).

**Type B**: \( N_0 \) and \( N_1 \) are even integers and their sum is two times an odd integer \( K \).

For both Type A and B, the overall filter bank delay is an odd integer given by \( K = (N_0 + N_1)/2 \). The frequency responses of \( G_k(z) \) for \( k = 0, 1 \) are expressible as

\[
G_k(e^{j\omega}) = e^{-jKN_k/2} \hat{G}_k(\omega)
\]

where the zero-phase frequency response \( \hat{G}_k(\omega) \) are given by

\[
\hat{G}_k(\omega) = \begin{cases} 
2 \sum_{n=0}^{(N_k-1)/2} g_k[(N_k - 1)/2 - n]\cos((n+1/2)\omega) & \text{for Type A} \\
g_k[N_k/2] + 2 \sum_{n=1}^{N_k/2} g_k[N_k/2 - n]\cos(n\omega) & \text{for Type B.}
\end{cases}
\]

As shown in [21], [13], Condition 3, guaranteeing the PR property, can be restated in terms of the impulse-response coefficients of \( G_0(z) \) and \( G_1(z) \) as

\[
\sum_{r=0}^{2l-1} g_l[2l-1-r]g_0[r] = \frac{1}{2} \delta(l - (K + 1)/2) \quad \text{for } l = 1, 2, \cdots, N_c
\]

or

\[
\sum_{r=0}^{2l-1} g_1[2l-1-r]g_1[r] = \frac{1}{2} \delta(l - (K + 1)/2) \quad \text{for } l = 1, 2, \cdots, N_c,
\]

where

\[
N_c = (N_0 + N_1 + 2)/4
\]

and \( \delta(r) = 1 \) for \( r = 0 \) and \( \delta(r) = 0 \) for \( r \neq 0 \).

### 5.2 Proposed Biorthogonal Filter Bank

For the proposed filter bank, it is assumed that \( G_0(z) \) (\( G_1(z) \)) has \( K_0 \) (\( K_1 \)) fixed zeros at \( z = -1 \). These transfer functions can be written as

\[
G_k(z) = \left[\frac{(1+z^{-1})}{2}\right]^{L_k} \sum_{n=0}^{L_k} b_k[n]z^{-n} \quad \text{for } k = 0, 1
\]

where \( L_k = N_k - K_k \), is the number of remaining adjustable zeros for \( G_k(z) \). Because \( G_0(z) \) and \( G_1(z) \) are linear-phase filters, the remaining coefficients satisfy \( b_k[L_k-n] = b_k[n] \) for \( n = 0, 1, \cdots, L_k \) and \( k = 0, 1 \).

The synthesis of the proposed filter banks can be simplified by utilizing the fact that linear-phase FIR filter transfer functions having an impulse response with an even symmetry is characterized by the following fact. Filters of even order have an even number or no zeros at \( z = -1 \), whereas filters of odd order have an odd number of zeros at \( z = -1 \) [27]. Therefore, without lost of generality, the following conditions can be stated for Type A and Type B designs considered in the previous subsection:

**Type A Designs**: \( K_0 \) and \( K_1 \) are odd integers and \( L_0 = N_0 - K_0 \) and \( L_1 = N_1 - K_1 \) are even integers.

**Type B Designs**: \( K_0 \) as well as \( K_1 \) is either zero or an even integer and \( L_0 = N_0 - K_0 \) and \( L_1 = N_1 - K_1 \) are even integers.
These restrictions enable us to write for both design types the zero-phase frequency responses of the transfer functions $G_k(z)$ for $k = 0, 1$, given by Equation (20a), as

$$
\hat{G}_k(\omega) = \cos(\omega / 2)^{K_k} \cdot b_k[L_k / 2] + 2 \sum_{n=0}^{L_k/2-1} b_k[n] \cos\left((L_k / 2 - n)\omega\right).
$$

(23)

Hence, after fixing the filter type as well as the integers $N_0, N_1, K_0,$ and $K_1$, there are

$$
L_k / 2 + 1 = (N_k - K_k) / 2 + 1
$$

(24)

remaining unknowns for determining the transfer functions $G_k(z)$ for $k = 0, 1$.

5.3 Statement of the Problem

The optimization problem for the proposed two-channel linear-phase biorthogonal filter bank is stated as follows: Given the filter type as well as $N_0, N_1, K_0,$ and $K_1$, find the $L_0/2 + 1$ adjustable coefficients of $G_0(z)$ and the $L_1/2 + 1$ adjustable coefficients of $G_1(z)$ to minimize the following quantity:

$$
E = \alpha_{s_0} \int_{\omega=-\pi}^{\pi} \left| \hat{G}_0(\omega) \right|^2 d\omega + \alpha_{s_1} \int_{\omega=-\pi}^{\pi} \left| \hat{G}_1(\omega) \right|^2 d\omega
$$

$$
+ \alpha_{p_0} \int_{\omega=-\pi}^{\pi} \left[ \hat{G}_0(\omega) - 1 \right]^2 d\omega + \alpha_{p_1} \int_{\omega=-\pi}^{\pi} \left[ \hat{G}_1(\omega) - 1 \right]^2 d\omega,
$$

where

$$
\omega_{s}^{(k)} = \left(1 - \rho_{s}^{(k)}\right) \pi / 2 \quad \text{and} \quad \omega_{p}^{(k)} = \left(1 + \rho_{p}^{(k)}\right) \pi / 2 \quad \text{for} \quad k = 0, 1
$$

subject to the constraint that the overall two-channel filter banks satisfies the PR property. This means that the coefficients of $G_0(z)$ and $G_1(z)$ expressed in the direct form has to satisfy the conditions of Equations (21). Solving this problem results in a PR biorthogonal filter bank with filters with transfer functions $G_k(z)$ having minimized stopband energies and $K_k$ fixed zeros at $z = -1$.

6 Efficient Iterative Algorithm for Solving the Stated Problem for Biorthogonal Filter Banks

This section describes an efficient algorithm for solving the optimization problem stated in Subsection 5.3. First, the optimization problem is reformulated in Subsection 6.1 and, then, an efficient iterative procedure described in Subsection 6.2 for solving the stated problem.

6.1 Reformulation of the Stated Problem

Similar to the orthogonal case (Subsection 4.1), the unknown coefficients are included in the following two vectors:

$$
\mathbf{b}_k = [b_k[0] \quad b_k[1] \quad \cdots \quad b_k[L_k / 2]]^T, \quad \text{for} \quad k = 0, 1.
$$

(26a)

After some manipulations, Equation (25a) can be re-expressed using the matrix notations as

$$
E = E_0 + E_1
$$

(26b)

where for $k = 0, 1$
Here, the $d_k$'s for $k=0,1$ are vectors of the form
\[
d_k = [d_k[0] \ d_k[1] \ \cdots \ d_k[L_k/2-1] \ d_k[L_k/2]]^T
\] (26d)
with
\[
d_k[l] = -\frac{4}{2^{K_k}} \sum_{p=0}^{K_k} \left( \frac{K_k}{p} \right)^{\alpha_p} \int_0^{\alpha_p} \cos\left[ (L_k/2-l+p-K_k/2) \omega \right] d\omega \] for $l=0,1,\ldots,L_k/2$ (26e)
and
\[
\int_0^{\alpha_p} \cos(\eta \omega) d\omega = \begin{cases} \sin(\eta \omega_p) / \eta & \text{for } \eta \neq 0 \\ \omega_p & \text{for } \eta = 0. \end{cases} \] (26f)
Matrices $S^{(k)}$ and $P^{(k)}$ are $(L_k/2+1) \times (L_k/2+1)$ Toeplitz matrices with the elements given by for $r=0,1,\ldots,L_k/2$ and $l=0,1,\ldots,L_k/2$ as
\[
s_{rl} := \begin{cases} s_{rl}/2 & \text{for } r \neq l, r = L_k/2 \\ s_{rl}/2 & \text{for } r \neq l, l = L_k/2 \\ s_{rl}/4 & \text{for } r = l = L_k/2 \end{cases} \quad \text{and} \quad p_{rl} := \begin{cases} p_{rl}/2 & \text{for } r \neq l, r = L_k/2 \\ p_{rl}/2 & \text{for } r \neq l, l = L_k/2 \\ p_{rl}/4 & \text{for } r = l = L_k/2, \end{cases} \] (26g)
where
\[
p_{rl} = \frac{2}{2^{2K_k}} \sum_{p=0}^{2K_k} \left[ \left( \frac{2K_k}{p} \right)^{\alpha_p} \int_0^{\alpha_p} \cos\left[ (r-l+p-K_k) \omega \right] d\omega + \int_0^{\alpha_p} \cos\left[ (L_k-r-l+p-K_k) \omega \right] d\omega \right], \] (26h)
\[
s_{rl} = \frac{2}{2^{2K_k}} \sum_{p=0}^{2K_k} \left[ \left( \frac{2K_k}{p} \right)^{\alpha_p} \int_0^{\alpha_p} \cos\left[ (r-l+p-K_k) \omega \right] d\omega + \int_0^{\alpha_p} \cos\left[ (L_k-r-l+p-K_k) \omega \right] d\omega \right], \] (26i)
and
\[
\left( \frac{2K_k}{p} \right)^{\alpha_p} = \frac{(2K_k)!}{(2K_k-p)! p!}. \] (26j)
In the above, the integral terms can be calculated according to Equations (12f) and (26f).

The PR property implies that the conditions of Equation (21) are satisfied. Like in the orthogonal case, they can be rewritten as
\[
C_0 b_0 = \frac{1}{2} m \] (27a)
and
\[
C_1 b_1 = \frac{1}{2} m \] (27b)
where $m$ is an $N_c = (N_0+N_1+2)/4$ length vector of the form
\[
m = [0 \ 0 \ 0 \ 1]^T, \] (27c)
and $C_0$ and $C_1$ are matrices defined for $k=0,1$ by
where the vector \( \mathbf{v}_{k,l}^{(k)} \) is the \( l \)th column of the following matrix:

\[
\mathbf{V}_k = \begin{bmatrix}
\mathbf{v}_0^{(k)} & \mathbf{v}_1^{(k)} & \cdots & \mathbf{v}_{L_k/2-1}^{(k)} & \mathbf{v}_{L_k/2+1}^{(k)}
\end{bmatrix}
\]  

(27e)

with

\[
\mathbf{v}_{nl}^{(k)} = \frac{1}{2^{2K_k}} \sum_{p=0}^{2K_k} \left( \frac{2K_k}{p} \right) b_0 \left[ L_k + p - 2r + l \right]
\]

(27f)

for \( r = 0, 1, \ldots, (N_c-1) \) and \( l = 0, 1, \ldots, \tilde{L}_k \). In the above equation, \( b_0[n] = 0 \) for \( n > L_k \) and \( n < 0 \) and the integer \( \tilde{L}_k \) is given by

\[
\tilde{L}_k = \begin{cases}
L_k & \text{for } k = 0 \\
L_0 & \text{for } k = 1.
\end{cases}
\]

(28)

In order to treat both transfer functions \( G_0(z) \) and \( G_1(z) \) in the same manner, Equations (27a) and (27b) are combined to give

\[
\mathbf{C}_1 \mathbf{b}_0 + \mathbf{C}_0 \mathbf{b}_1 = \mathbf{m}.
\]

(29)

In tree-structured filter banks, it is desired to have the filter gain equal to zero at the zero frequency \([26]\) (\( \sqrt{2} \) for wavelets). For orthogonal filter banks this was always satisfied due to the time-reverse relation between coefficients of the lowpass analysis and the lowpass synthesis filter. For biorthogonal filter banks, because there is no direct relation between the analysis and the synthesis filter, an extra constraint has to be imposed. The unity gain requirement at the zero frequency implies

\[
G_k(1) = 1 \quad \text{for} \quad k = 0, 1.
\]

(30)

The above equations can be rewritten using the unknown filter coefficients as

\[
\sum_{l=0}^{L_k} g_k[l] = 1 \quad \text{for} \quad k = 0, 1
\]

(31a)

and, after applying Equation (22), as

\[
\sum_{l=0}^{L_k} b_k[l] = 1 \quad \text{for} \quad k = 0, 1.
\]

(31b)

Taking into account the linear-phase property, Equation (31b) is expressible in terms of the vectors \( \mathbf{b}_k \) for \( k = 0, 1 \) as

\[
\mathbf{L}_{2k}^T\mathbf{b}_k = 1 \quad \text{for} \quad k = 0, 1,
\]

(32a)

where \( \mathbf{L}_{zk} \) is the following vector of length \( L_k/2 + 1 \):

\[
\mathbf{L}_{zk} = [2 \ 2 \ \cdots \ 2 \ 1]^T.
\]

(32b)

Due to the PR property of filter bank it is enough to impose only one out of the two constraints of Equation (32a), that is, the constraint for \( k = 0 \) or \( k = 1 \) is used. The second one is automatically satisfied due to the PR constraint. It has been observed that if both constraints are simultaneously applied, then the iterative procedure to be described in the next
subsection fails in some cases to give the optimized solution due to the too tight constraints. Experimental results have shown that it is better to apply the constraint is for the filter with the larger number of unknowns $L_k$.

Using the above-derived equations, the problem for the two-channel biorthogonal FIR linear-phase filter bank can be restated in the following modified form:

Minimize:

$$E = b^T Q b + d^T b + \omega_p^{(0)} + \omega_p^{(1)}$$  \hspace{1cm} (33a)

subject to

$$C b - m = 0$$  \hspace{1cm} (33b)

and

$$t^T b - 1 = 0$$  \hspace{1cm} (33c)

where

$$Q = \begin{bmatrix} Q_0 & 0 \\ 0 & Q_1 \end{bmatrix}, d = \begin{bmatrix} d_0 \\ d_1 \end{bmatrix}, b = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}, C = \begin{bmatrix} C_0 & C_1 \end{bmatrix}, \text{ and } t = \begin{bmatrix} L^{T} & 0_1 \\ 0_1 & L^{T}_{c1} \end{bmatrix}^T \text{ for } L_0 \geq L_i$$  \hspace{1cm} (33d)

with $Q_i = \alpha_{pk} S^{(k)} + \alpha_{pk} P^{(k)}$ and the $0_i$'s for $k = 0, 1$ being length $L_0/2 + 1$ zero vectors $0_i = [0 \cdots 0]$.

Since the matrix $C$ depends on the unknowns included in $b$, the overall problem cannot be solved directly by applying the Lagrange multiplier method. The above problem has similarities with the one described in Section 4 and can be solved using a similar iterative procedure. In the sequel, $b$ is used for determining $C$, whereas $\tilde{b}$ is used as the solution of the Lagrange multiplier method. Introducing the Lagrange multiplier vectors $\lambda = [\lambda_1, \lambda_2, ..., \lambda_k]^T$ and $\mu = [\mu_i]$, the Lagrangian function for the linearized optimization problem takes the following form:

$$\Lambda(\tilde{b}, \lambda, \mu) = \tilde{b}^T Q \tilde{b} + d^T \tilde{b} - \lambda^T (C \tilde{b} - m) - \mu^T (t^T \tilde{b} - 1) + \omega_p^{(0)} + \omega_p^{(1)}.$$  \hspace{1cm} (34)

Imposing the following necessary and sufficient conditions for the solution:

$$\nabla_\tilde{b} \Lambda(\tilde{b}, \lambda, \mu) = 0,$$  \hspace{1cm} (35a)

$$\nabla_\lambda \Lambda(\tilde{b}, \lambda, \mu) = 0,$$  \hspace{1cm} (35b)

and

$$\nabla_\mu \Lambda(\tilde{b}, \lambda, \mu) = 0,$$  \hspace{1cm} (35c)

results in the following system of linear equations:

$$\begin{bmatrix} -2Q & C^T \\ C & 0 & 0 \\ t & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{b} \\ \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} d \\ m \\ 1 \end{bmatrix}.$$  \hspace{1cm} (36)

Solving this linear system of equations for $\tilde{b}$, a vector containing the new filter coefficient values is obtained. Using these filter coefficient values, the initial filter coefficient values can be updated by using the following linear update formula:

$$b := (1 - \tau) b + \tau \tilde{b}.$$  \hspace{1cm} (37)
These updated filter coefficients are then used as the new initial values for determining $C$. Repeating this procedure several times until the difference between the initial and resulting filter coefficients are within a small tolerance, a very close solution to the original problem is found. In the above relation, the parameter $\tau$ is a convergence factor and it has been observed that for most cases $\tau=0.5$ is a good selection.

### 6.2 Proposed Algorithm for Biorthogonal Filter Banks

The overall algorithm can be carried out in the following steps:

**Step 1)** Use a conventional method to design two linear-phase lowpass filters of orders $N_0$ and $N_1$ with the given band edges and prescribed numbers of zeros at $z=-1$.

**Step 2)** Calculate the matrix $Q$ and the vector $d$ using Equations (26d-j) and (33d) and form the $(N_0+N_1+2)/4$ length vector $m$ according to Equation ((27c).

**Step 3)** Use the old filter coefficient values $b$ to calculate the matrix $C$ with the aid of Equations (27d-f) and (33d).

**Step 4)** Calculate new values for the filter coefficients by solving the system of Equation (36) for $\tilde{b}$ and update the filter coefficients using Equation (37).

**Step 5)** If $\| \tilde{b} - b \| < \delta$, where $\delta$ is a prescribed tolerance, then the filter coefficients included in the vector $b$ are the desired solution. Otherwise, go to Step 3.

Very few iterations are required by the above algorithm to arrive at the optimum solution. Due to the higher non-linearity of the design problem compared with the orthogonal case, it is necessary to repeat the above design procedure a several times for different weighting factors and, then, to choose the most appropriate solution. Another approach would be to utilize a proper non-linear optimization technique to directly arrive at the desired solution.

Like in the orthogonal case, the above algorithm results in a filter bank with normalized filters, that is, $G_k(1)=1$ for $k=0,1$. If these filters are used as building block for discrete-time wavelets, then the corresponding values are desired to be $\sqrt{2}$. This is achieved by first using the substitution

$$b_k[n] = \sqrt{2}b_k[n], \quad \text{for } n = 0,1,...,L_k/2$$

for $k=0,1$ and then, by replacing, for the synthesis filters, the relations of Equation (3) by the relations of Equation (18).

### 7 Numerical examples

This section illustrates, by means of examples, the properties of the proposed orthogonal and biorthogonal two-channel filter banks and their efficiency in providing tradeoffs between the regularity and selectivity of the multistage filter banks. In addition, a de-noising example is included and the proposed orthogonal filter banks are compared with the corresponding minimax solutions of Rioul and Duhamel [14].

**Example 1.** Orthogonal filter banks with $N_0=11$ have been designed for $\omega_s=0.64\pi$ and for different values of $K_0$, the number of vanishing moments. Figure 3 shows the amplitude characteristics of $G_0(z)$ for the resulting filter banks. For $K_0=0$, a classical orthogonal irregular filter bank is obtained. As $K_0$ increases, the stopband attenuation decreases, but simultaneously the regularity of the corresponding wavelet bank increases due to the larger number of vanishing moments. For $K_0=(N_0+1)/2=6$, the lowpass analysis filter gives rise to the Daubechies wavelet [18]. In all the cases $\delta=10^{-3}$ has been used as the convergence criterion in the algorithm described in Subsection 4.2.

Figure 3. Amplitude responses for $G_0(z)$ of order $N_0 = 11$ for different numbers of vanishing moments: $K_0 = 0$ (dot-dashed line), $K_0 = 2$ (solid line), $K_0 = 4$ (dashed line), and $K_0 = 6$ (dotted line).

**Example 2.** In order to illustrate the tradeoffs between the frequency selectivity of the filter banks and the regularity of corresponding wavelet banks, several filter banks have been designed in the $\omega_s = 0.64\pi$ case for various values of $N_0$, the order of the lowpass analysis filter and $K_0$, the number of vanishing moments. The regularity of the corresponding wavelet bank has been estimated by the lower bound of the Hölder regularity. This lower bound has been determined using the method described by Rioul in [6]. The real regularity is more or less larger than this lower bound.

Figure 4 shows the attenuation of $G_0(z)$ at the stopband edge and the regularity of the resulting wavelet bank for various proposed two-channel filter banks, whereas in Figure 5 as a measure of selectivity the stopband energy of $G_0(z)$ has been used. These figures provide information on how to choose the minimum filter order and the number of vanishing moments in order to simultaneously achieve the required regularity and filter selectivity. The appropriate numerical data for some values of $N_0$ and $K_0$ are given in Table 1. The quantities included in the table are: $A_{sb}$, the attenuation of the first peak and $A_e$, the attenuation at the stopband edge, both given in decibels, $A_{pb}$, the passband ripple, $E_{sb}$, the stopband energy, $r$, the regularity, and $t$, the design time in seconds (Pentium II, 333MHz, WIN98).

In order to illustrate the efficiency of the filter banks resulting when using the proposed design scheme, these filter banks have been compared in the above case with the corresponding banks of Rioul and Duhamel [14]. The main difference of the Rioul-Duhamel designs compared with the present solutions is that, instead of the least-mean-square error criterion, the amplitude response $G_0(z)$ is minimized in the given stopband region for the given values of $N_0$ and $K_0$. Table 2 shows for the Rioul-Duhamel designs the same results as presented in Table 1 for the proposed filter banks. The quantities in the table are again, $A_{pb}$, the passband ripple, $E_{sb}$, the stopband energy, $r$, the regularity, and $t$, the design time in seconds as well as $A_m$, the minimum stopband attenuation for the lowpass analysis filter given in decibels.

When comparing Tables 1 and 2, it is observed that the proposed banks provide better regularities for the corresponding wavelet banks, and, as can be expected, the stopband energy is lower for the proposed banks.
Figure 4. Comparison between the regularity of the wavelet bank and the attenuation of $G_0(z)$ at the stopband edge for the proposed two-channel filter banks for various values of $N_0$ and $K_0$.

Figure 5. Comparison between the regularity of the wavelet bank and the stopband energy of $G_0(z)$ for the proposed two-channel filter banks for various values of $N_0$ and $K_0$. 
Table 1. Properties of filter banks designed using the proposed least-mean-square method.

<table>
<thead>
<tr>
<th>(N_0)</th>
<th>(K_0)</th>
<th>(A_{lb} \text{ [dB]})</th>
<th>(A_e \text{ [dB]})</th>
<th>(A_{pb} )</th>
<th>(E_{lb} )</th>
<th>(r)</th>
<th>(t \text{ [s]})</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1</td>
<td>19.5</td>
<td>11.2</td>
<td>3.9(10^{-2})</td>
<td>1.2(10^{-2})</td>
<td>0.48</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-</td>
<td>8.4</td>
<td>7.6(10^{-2})</td>
<td>2.9(10^{-2})</td>
<td>1.08</td>
<td>0.05</td>
</tr>
<tr>
<td>11</td>
<td>0</td>
<td>29.9</td>
<td>21.5</td>
<td>3.6(10^{-3})</td>
<td>6.2(10^{-4})</td>
<td>-0.07</td>
<td>0.08</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>29.3</td>
<td>21.0</td>
<td>4.0(10^{-3})</td>
<td>7.0 (10^{-4})</td>
<td>1.22</td>
<td>0.11</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>26.1</td>
<td>18.1</td>
<td>7.7(10^{-3})</td>
<td>1.5(10^{-3})</td>
<td>1.82</td>
<td>0.11</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>-</td>
<td>11.9</td>
<td>3.3(10^{-2})</td>
<td>7.6(10^{-3})</td>
<td>2.16</td>
<td>0.11</td>
</tr>
<tr>
<td>17</td>
<td>3</td>
<td>39.6</td>
<td>31.4</td>
<td>3.6(10^{-4})</td>
<td>4.5(10^{-5})</td>
<td>1.86</td>
<td>0.16</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>31.3</td>
<td>23.7</td>
<td>2.1(10^{-3})</td>
<td>3.0(10^{-4})</td>
<td>2.91</td>
<td>0.20</td>
</tr>
<tr>
<td>19</td>
<td>4</td>
<td>42.3</td>
<td>34.2</td>
<td>1.9(10^{-4})</td>
<td>2.2(10^{-5})</td>
<td>2.24</td>
<td>0.17</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>38.9</td>
<td>30.8</td>
<td>4.1(10^{-4})</td>
<td>4.9(10^{-5})</td>
<td>2.77</td>
<td>0.20</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>33.0</td>
<td>25.4</td>
<td>1.5(10^{-4})</td>
<td>1.9(10^{-4})</td>
<td>3.23</td>
<td>0.22</td>
</tr>
</tbody>
</table>

Table 2. Properties of filter banks designed using the minimax method [14].

<table>
<thead>
<tr>
<th>(N_0)</th>
<th>(K_0)</th>
<th>(A_{m} \text{ [dB]})</th>
<th>(A_{pb} \text{ [dB]})</th>
<th>(E_{lb} \text{ [dB]})</th>
<th>(r)</th>
<th>(t \text{ [s]})</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1</td>
<td>13.8</td>
<td>2.1(10^{-2})</td>
<td>2.2(10^{-2})</td>
<td>0.31</td>
<td>0.30</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>8.4</td>
<td>7.6(10^{-2})</td>
<td>2.9(10^{-2})</td>
<td>1.08</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>0</td>
<td>26.8</td>
<td>1.0(10^{-3})</td>
<td>1.2(10^{-3})</td>
<td>-0.09</td>
<td>1.17</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>26.0</td>
<td>1.3(10^{-3})</td>
<td>1.3(10^{-3})</td>
<td>1.04</td>
<td>0.71</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>21.8</td>
<td>3.3(10^{-3})</td>
<td>2.3(10^{-3})</td>
<td>1.66</td>
<td>0.36</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>11.9</td>
<td>3.3(10^{-2})</td>
<td>7.6(10^{-3})</td>
<td>2.16</td>
<td>0</td>
</tr>
<tr>
<td>17</td>
<td>3</td>
<td>37.9</td>
<td>8.7(10^{-5})</td>
<td>7.7(10^{-5})</td>
<td>1.67</td>
<td>1.38</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>27.7</td>
<td>8.4(10^{-4})</td>
<td>4.6(10^{-4})</td>
<td>2.75</td>
<td>0.44</td>
</tr>
<tr>
<td>19</td>
<td>4</td>
<td>40.9</td>
<td>4.3(10^{-5})</td>
<td>3.6(10^{-5})</td>
<td>2.04</td>
<td>1.43</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>36.6</td>
<td>1.2(10^{-4})</td>
<td>7.8(10^{-5})</td>
<td>2.59</td>
<td>0.99</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>29.5</td>
<td>5.7(10^{-4})</td>
<td>2.8(10^{-4})</td>
<td>3.08</td>
<td>0.49</td>
</tr>
</tbody>
</table>

**Example 3.** A typical application of wavelet banks in digital signal processing is de-noising. The method used in this example is the one proposed by Donoho and Johnstone in [28]. It consists of three steps: First, the noisy data is transformed into wavelet coefficients by applying a wavelet transform. Second, a “soft” or “hard” thresholding is applied to these coefficients resulting in a suppression of the coefficients of lower energies. Third, the results are transformed back in to the original domain using the inverse wavelet transform. To avoid some artifacts near singularities (pseudo-Gibbs phenomenon) an undecimated (stationary, shift invariant) wavelet transform has been used as suggested by Coifman and Donoho in [29]. All the necessary calculations have been performed using the MATLAB Wavelet toolbox [30] and the Standford WaveLab toolbox for MATLAB [31]. In this example, 5 level wavelet decomposition has been used followed by “hard” thresholding. The threshold level was selected as proposed in [28].

A set of four test signals was generated, 2048 samples each, as shown in Figure 6. To these signals Gaussian white noise (noise \(\sim \mathcal{N}(0,1)\) [29]) has been added. Figure 7 shows the resulting noisy signals.

![Graphs of test signals for de-noising.](image)

Figure 6. Test signals for de-noising.

![Graphs of test signals with added Gaussian white noise.](image)

Figure 7. Test signals with added Gaussian white noise.

For the noisy signals, de-noising was performed using different well-known wavelets as well as some filter banks obtained by using the proposed methods. The following wavelets have been considered: ‘sym9’ (symlet wavelet, $N_0 = 17$), ‘coif3’ (Coiflet wavelet, $N_0 = 17$), ‘db9’ (Daubechies wavelet, $N_0 = 17$) and Haar wavelet ($N_0 = 1$) as representatives of orthogonal wavelets, and some standard biorthogonal, namely, ‘bior6.8’, ‘bior2.8’, and ‘rbio2.8’ [30]. These wavelets have been compared in the above de-noising application with two filter banks (wavelets) designed with the proposed methods. For the first proposed wavelet, denoted by ‘1708’, the building block two-channel filter bank is the orthogonal one designed for $N_0 = 11$, $K_0 = 8$, $L_0 = 9$, and $\rho = 0$ using the algorithm of Subsection 4.2. For the second wavelet, denoted by ‘1206’, a biorthogonal two-channel filter bank is used. It was generated using $N_0 = 12$, $K_0 = 6$, $L_0 = 6$, $\rho_0 = 0.1$, $N_1 = 6$, $K_1 = 2$, $L_1 = 4$, $\rho_1 = 0.2$, $\alpha_{0\theta} = 5$, $\alpha_{1\theta} = 2$, $\alpha_{0p} = 1$, and $\alpha_{1p} = 1$ in the algorithm of Subsection 6.2. Figures 8 and 9 show the
amplitude responses of the resulting building-block two-channel filter banks ‘1708’ and ‘1206’, respectively, whereas Figures 10 and 11 show the corresponding zero locations.

Figure 8. Amplitude responses of the analysis filters for the orthogonal filter bank ‘1708’.

Figure 9. Amplitude responses of the analysis filters for the biorthogonal filter bank ‘1206’.

Figure 10. Zero locations for the lowpass analysis filter of the orthogonal filter bank ‘1708’.

![Figure 11. Zero locations for the analysis filters of the biorthogonal filter bank ‘1206’.](image)

(a) Lowpass analysis filter $H_0(z)$. (b) Highpass analysis filter $H_1(z)$.

In Table 3 quantitative results are given that were obtained by applying the denoising method on the test signals using the above mentioned wavelets (filter banks). For every numerical entry in the table, 400 experiments were made and then an averaging over the obtained results was performed.

<table>
<thead>
<tr>
<th>Filter type</th>
<th>Signal</th>
<th>RMSE / MAE</th>
<th>$N_0$</th>
<th>$K_0$</th>
<th>$L_0$</th>
<th>$N_1$</th>
<th>$K_1$</th>
<th>$L_1$</th>
<th>Regularity</th>
<th>$r_a$</th>
<th>$r_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Noisy signal</td>
<td>Doppler</td>
<td>45.23/0.80</td>
<td>45.28/0.80</td>
<td>45.25/0.80</td>
<td>45.23/0.80</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>sym9</td>
<td>Blocks</td>
<td>10.24/0.14</td>
<td>17.43/0.24</td>
<td>17.91/0.26</td>
<td>9.16/0.12</td>
<td>17</td>
<td>9</td>
<td>8</td>
<td>-</td>
<td>-</td>
<td>3.00</td>
</tr>
<tr>
<td>db9</td>
<td>Bumps</td>
<td>10.66/0.15</td>
<td>15.93/0.23</td>
<td>16.62/0.24</td>
<td>8.60/0.12</td>
<td>17</td>
<td>6</td>
<td>11</td>
<td>-</td>
<td>-</td>
<td>2.19</td>
</tr>
<tr>
<td>Haar</td>
<td>HeaviSin</td>
<td>11.23/0.16</td>
<td>21.63/0.29</td>
<td>19.90/0.28</td>
<td>10.78/0.14</td>
<td>17</td>
<td>9</td>
<td>8</td>
<td>-</td>
<td>-</td>
<td>3.04</td>
</tr>
<tr>
<td>1708</td>
<td>-</td>
<td>18.22/0.27</td>
<td>7.95/0.12</td>
<td>18.32/0.22</td>
<td>9.18/0.15</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-</td>
<td>-</td>
<td>1.00</td>
</tr>
<tr>
<td>Bior.</td>
<td>-</td>
<td>9.50/0.14</td>
<td>21.87/0.29</td>
<td>19.91/0.28</td>
<td>10.84/0.14</td>
<td>17</td>
<td>8</td>
<td>9</td>
<td>-</td>
<td>-</td>
<td>3.05</td>
</tr>
<tr>
<td>bior6.8</td>
<td>-</td>
<td>10.58/0.15</td>
<td>16.10/0.23</td>
<td>16.73/0.24</td>
<td>8.66/0.12</td>
<td>16</td>
<td>8</td>
<td>8</td>
<td>10</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>bior2.8</td>
<td>-</td>
<td>11.91/0.17</td>
<td>14.03/0.19</td>
<td>13.83/0.20</td>
<td>8.11/0.12</td>
<td>16</td>
<td>8</td>
<td>8</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>rbio2.8</td>
<td>-</td>
<td>11.74/0.16</td>
<td>17.04/0.25</td>
<td>19.27/0.29</td>
<td>9.46/0.13</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>16</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>1206</td>
<td>-</td>
<td>12.26/0.17</td>
<td>12.63/0.17</td>
<td>13.70/0.19</td>
<td>7.95/0.11</td>
<td>12</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

The numerical performances of the wavelets have been estimated by evaluating the root mean square error (RMSE) and the maximum absolute error (MAE). These values are given for every signal and every wavelet (filter bank) under consideration. The quantities, denoted by $r_a$ and $r_s$ in the Table 3, show the regularities of the analysis and the synthesis part, respectively. For orthogonal filter banks, the regularities of the synthesis and the analysis parts are the same. The regularities were calculated using the method described by Rioul in [6] using a 19-level decomposition.

It is seen from Table 3 that in order to obtain good results with orthogonal filter banks, the lowpass analysis filter must be a mixed-phase filter. It is observed that the ‘db9’ filter results in a very poor performance for Blocks and Bumps types of signals, whereas ‘sym9’, that is a mixed phase design with the most symmetrical version out of the ‘db9’ filters, gives a significantly better result. Releasing some constraints on regularity, even better results can be obtained, like in the case of the filter ‘1708’ (a mixed phase filter, see Figure 10) for the doppler signal. Releasing constraints too much, for achieving a better filter symmetry like the case of the ‘coif3’ wavelet, has an opposite effect, namely, the results become worse.

When comparing in Table 3 the biorthogonal wavelets ‘bior2.8’ and ‘rbio2.8’ that are built using the same two filters, but in different arrangement (the lowpass analysis filter and the lowpass synthesis filters are swapped), it is seen that better results are obtained by
‘bior2.8’. The corresponding building-block two-channel filter bank has for this wavelet a short lowpass synthesis filter and a long lowpass analysis filter. Furthermore, it is seen that ‘bior2.8’ has even fewer coefficients then ‘bior6.8’, but still gives better results. It is interesting to observe from Table 3 that the filter bank ‘1206’ has the worst regularity (especially for the synthesis part), but still gives very good results for all test signals except for the doppler one. This clearly shows that the direct use of the simple measures of ‘goodness’ such as the maximal regularity, the maximum number of vanishing moments, the phase linearity, and the maximal frequency selectivity, do not result in a solution having the best possible properties for the given application. In practice, proper compromises must be done between these measures of ‘goodness’ in order to arrive at a good solution. Another problem of interest is to study whether it is beneficial to slightly release the perfect-reconstruction condition in order to even improve the performance of the overall system.

8 Conclusions

In this paper iterative methods for designing orthogonal and biorthogonal two-channel PR FIR filter banks were presented. The filters in the filter bank are designed in such a way that, first, the number of zeros for the analysis and the synthesis lowpass filter at $z = -1$ are fixed and, then, the energies in the given stop band regions are minimized. The attractive features of the proposed methods can be summarized as follows. First, it is easy to control the number of fixed zeros at $z = -1$. Second, the design methods enable us to make compromises between the regularity and the frequency selectivity of the resulting filter bank. Third, the convergence of the proposed method is fast. Fourth, the problem is defined in a form that gives directly the lowpass and highpass analysis filters. There is no need to factorize any half-band filter into two components like in several other synthesis schemes. Finally, using different weightings for designing biorthogonal filter banks, various designs can be obtained. The supplementary examples show that this is very important due to the fact that it is not known in advance which type of filter bank will give best results for a given problem.

References


