DESIGN OF CAUSAL STABLE PERFECT RECONSTRUCTION TWO-CHANNEL IIR FILTER BANKS

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Abstract

An efficient optimization algorithm is proposed for designing causal stable perfect-reconstruction two-channel infinite impulse response filter banks originally introduced by Basu, Chiang, and Choi. The problem is stated as a constrained optimization problem where the maximum of the stopband energies of the analysis filters is minimized subject to the given constraints. First, the maximum allowable passband ripple of both filters is fixed and the maximum value in the transition bands is the maximum allowable passband value. Second, the poles of these filters are forced to lie inside the unit circle. It is shown that the second constraint can be included in the proposed algorithm by using Rouché’s theorem. Several examples are included illustrating the efficiency of the proposed design scheme as well as the superiority of the resulting banks compared to those ones obtainable by using other existing design techniques.

1. Introduction

Perfect-reconstruction (PR) or nearly PR (NPR) two-channel filter banks can be constructed using either finite impulse response (FIR) or infinite impulse response (IIR) filters. IIR filter banks are very attractive compared to their FIR counterparts due to their lower complexity, shorter filter bank delay, and better frequency selectivity. However, in most applications FIR filter banks are used. The main reason for this is that IIR filter banks are more complicated to design. Therefore, in the past, most of the research work has been concentrated on synthesizing FIR filter banks, whereas the design of IIR filter banks has not received so much attention.

There exist various ways of designing two-channel IIR filter banks [1]. The simplest way is to use the allpass filters of the so-called lowpass half-band IIR filters [2], [3] as basic building blocks. These filters are special cases of lattice wave digital filters or parallel connections of two allpass filters having all the poles on the imaginary axis. If only the phase distortion is allowed between the input and output of the overall bank, then a frequency-selective two-channel filter bank with a few number of multipliers can be constructed with the aid of the above-mentioned building blocks, as shown, e.g., in [4]. Another alternative, leading to a bank where the output is identically equal to the input, is to construct the synthesis filters using the noncausal counterparts of the allpass filter building blocks as proposed in [6], [7] for processing finite-length signals and in [8] for processing infinite-length signals. However, in practice, in both cases there is a need to form a time-reversed input for the corresponding causal filters and at the filter outputs again to produce the time-reversed output, generating a long extra delay. Also, additional information regarding the state variables of analysis filters must be transferred to the synthesis filters, making the overall implementation complicated.

For designing causal PR IIR filter banks, two approaches have been proposed in the literature. The first one is based on the use of a special filter bank structure considered in [9]–[12], where the PR property is structurally guaranteed. The drawback of this approach is that the overall filter bank delay is considerably longer than in the second approach. In this second technique, the analysis filters are constructed by using IIR filters of a special type as proposed in [13]–[16] and the alias-free property is achieved by relating the synthesis filter to them in a proper manner. These PR banks have been originally introduced by Basu, Chiang, and Choi in [13], but no practical design method has been given. In [14], an iterative method has been proposed without imposing any stability constraints on the IIR filters. Therefore, most of the resulting banks are unstable. In [15], a technique based on nonlinear optimization has been applied for designing stable PR banks such that the resulting analysis filters exhibit minimax stopband responses. The stability criterion is included in
the overall procedure by separating the denominators into second-order terms and applying the stability constraint to each term. Separating the denominators makes the overall problem complicated.

This paper describes an efficient approach for designing PR Basu-Chiang-Choi IIR filter banks. The problem is stated as a constrained optimization problem being solvable by using a modified version of the Dutta-Vidyasagar algorithm [17, 18] similar to that described in [19] for designing FIR filter banks. In the problem, the maximum of the stopband energies of the analysis filters is minimized subject to the following constraints. First, the maximum allowable passband ripple of both filters is fixed and the maximum value in the transition bands is the maximum allowable passband value. Second, the poles of these filters should stay inside the unit circle. The proposed algorithm is simplified by using Rouche’s theorem [20] for satisfying the second constraints like in the iterative algorithm used for designing IIR filters in [21]. This enables us to consider the denominators in the direct form. Several examples are included illustrating the superiority of the resulting filter banks compared to those ones obtainable by using other existing design techniques.

This paper is organized as follows. In Section 2, the basic relations for two-channel IIR filter banks under consideration and the conditions required to be met by the denominator and numerator orders are given. In Section 3, the design problem is stated, whereas the proposed synthesis method is described in Section 4. Special emphasis is laid in Section 4 on how to keep the filters stable when finding the optimized solution. Section 5 shows, by means of several examples, the efficiency of the proposed method.

2. Two-channel IIR filter banks

This section reviews some basic properties of alias-free two-channel filter banks and specifies the transfer functions of IIR filters used for building these banks. The constraints required to be met by the numerator and denominator orders of the filters are also discussed.

2.1. Basic relations of two-channel IIR filter banks

Figure 1 shows the block diagram for a two-channel filter bank [1]. This filter bank consists of analysis filters \( H_d(z) \) and \( H_s(z) \) followed by down-samplers (↓2), the processing unit, and up-samplers (↑2) followed by synthesis filters \( F_d(z) \) and \( F_s(z) \) the outputs of which are added to generate the overall output.

\[ x[n] \xrightarrow{H_d(z)} \downarrow 2 \xrightarrow{H_s(z)} \downarrow 2 \xrightarrow{\text{Processing unit}} \uparrow 2 \xrightarrow{F_d(z)} \uparrow 2 \xrightarrow{F_s(z)} y[n] \]

Figure 1. Two-channel filter bank.

It is assumed that there are no coding, quantization, or channel degradations in the processing unit. Furthermore, to achieve an alias-free system, it is assumed that \( F_d(z) = 2H_d(-z) \) and \( F_s(z) = -2H_s(-z) \), yielding the following input-output relation in the \( z \)-domain:

\[ Y(z) = T(z)X(z), \quad (1) \]

where

\[ T(z) = H_0(z)H_1(-z) - H_1(z)H_0(-z). \quad (2) \]

To simplify the overall synthesis scheme to be described in the next sections, we use instead of \( H_d(z) \) and \( H_s(z) \) the following transfer functions [19]:

\[ G_0(z) = H_0(z) \quad \text{and} \quad G_1(z) = H_1(-z). \quad (3) \]

Here, \( G_0(z) \) and \( H_0(z) \) are identical lowpass filters. Generating \( G_1(z) \) from \( H_1(z) \) by replacing \( z \) by \(-z\) corresponds to mapping a highpass transfer function \( H_1(z) \) to a lowpass transfer function \( G_1(z) \) such that \( [G_1(e^{j\omega})] = H_1(e^{-j\omega}) \). Hence, the amplitude response of \( G_1(z) \) is obtained from that of \( H_1(z) \) using the substitution \( \omega \rightarrow -\omega \) and vice versa. If \( \tilde{\omega}_p^{(1)} \) and \( \tilde{\omega}_s^{(1)} \) are the passband and stopband edges of \( H_1(z) \), then the corresponding edges for \( G_1(z) \) are \( \tilde{\omega}_p^{(1)} = \pi - \tilde{\omega}_p^{(1)} \) and \( \tilde{\omega}_s^{(1)} = \pi - \tilde{\omega}_s^{(1)} \), respectively.

The advantage of using \( G_0(z) \) and \( G_1(z) \) as primary transfer functions lies in the fact that both of them are transfer function of lowpass filters. This makes their design more straightforward as they can be treated in the same manner and all design formulas take simplified forms. In terms of \( G_0(z) \) and \( G_1(z) \), \( T(z) \), as given by Eq. (2), becomes

\[ T(z) = G_0(z)G_1(z) - G_0(-z)G_1(-z). \quad (4) \]

This paper concentrates on the case where \( G_0(z) \) and \( G_1(z) \) have the following IIR filter transfer functions [13]:

\[ G_k(z) = \frac{A_k(z)}{B_k(z^2)} = \frac{\sum_{m=0}^{M_k} a_k[m]z^{-m}}{1 + \sum_{n=1}^{N_k} b_k[n]z^{-n}} \quad \text{for} \quad k=0, 1 \quad (5) \]

yielding

\[ T(z) = \frac{A_0(z)A_1(z) - A_0(-z)A_1(-z)}{B_0(z^2)B_1(z^2)}. \quad (6) \]

It can be shown that for a PR filter bank it is required that \( T(z) = z^{-(2K+1)} \), where \( K \) is an integer. In the case of Figure 1, this means that \( y[n] = x[n-(2K+1)] \).
It should be pointed out that once $G_k(z)$ has been determined, the corresponding $H_k(z) = G_k(-z)$ is obtained by simply changing the numerator coefficients to be $(-1)^m a_k[m]$ for $m = 0, 1, \ldots, M_k$.

2.2. Constraints on numerator and denominator orders

For the PR IIR filter banks under consideration there are constraints on numerator and denominator orders of the analysis IIR transfer functions. It can be shown that Eq. (6) implies that for a PR filter bank with the overall delay equal to $2K+1$ following relation has to be satisfied:

$$N_0 + N_1 \leq \frac{M_0 + M_1 - 2K - 1}{2}, \quad (7)$$

where $M_0$ and $N_1$ are the orders of the numerator and denominator of the analysis filters [cf. Eq. (5)]. If the sum of $N_0$ and $N_1$ is larger than the upper limit as given by Eq. (7), then for the optimized filter bank some of the poles move to the origin, thereby decreasing the effective denominator orders to satisfy the condition of Eq. (7).

3. Statement of the Problem

Before stating the optimization problem for the filter bank under consideration, $G_k(z)$ and $G_k(z)$, as given by Eq. (5), are denoted by $G_0(a_0, b_0, z)$ and $G_1(a_1, b_1, z)$, where

$$a_k = [a_k[0 \ a_k[1] \cdots a_k[M_k]]^T \quad (8)$$

are the vectors containing the adjustable numerator and denominator coefficients of the analysis filters.

This paper concentrates on the following optimization problem: Given the numerator and denominator filter orders, $N_0$, $M_0$, $N_1$, and $M_1$, $\rho_p^{(k)}$ and $\rho_s^{(k)}$ for $k = 0, 1$, $\delta_0$, and $\delta_1$ as well as $K$, find $a_0$, $b_0$, $a_1$, and $b_1$ to minimize

$$\epsilon = \max \left\{ \varepsilon_p(a_0, b_0), \varepsilon_1(a_1, b_1) \right\}, \quad (9a)$$

where

$$\varepsilon_p(a_k, b_k) = \sum_{\omega_i \in [0, \pi]} \left| T(a_0, b_1, a_1, b_1, e^{j\omega_i}) e^{-j2(2K+1)\omega_i} \right| \quad (9b)$$

subject to the condition that the poles of $G_0(a_0, b_0, z)$ and $G_1(a_1, b_1, z)$ stay within the unit circle,

$$\max_{\omega_i \in [0, \pi]} \left| T(a_0, b_0, a_1, b_1, e^{j\omega_i}) - e^{-j2(2K+1)\omega_i} \right| \leq \delta_0 \quad (9c)$$

and

$$\max_{\omega_i \in [0, \pi]} \left| G_k(a_k, b_k, e^{j\omega_i}) - 1 \right| \leq \delta_1 \quad (9d)$$

subject to the condition that the poles of $G_0(a_0, b_0, z)$ and $G_1(a_1, b_1, z)$ stay within the unit circle,

$$\max_{\omega_i \in [0, \pi]} \left| G_k(a_k, b_k, e^{j\omega_i}) - 1 \right| \leq \delta_1 \quad (9e)$$

where

$$T(a_0, b_1, a_1, b_1, e^{j\omega_i}) = A_1(a_0, b_0, e^{j\omega_i}) - A_0(a_0, b_0, e^{j(\omega_i + \pi/2)}) A_1(a_1, b_1, e^{j(\omega_i + \pi/2)})$$

and

$$\varepsilon_1(k) = \pi \sum_{\omega_i \in [0, \pi]} \left| G_k(a_k, b_k, e^{j\omega_i}) \right|^2 \quad (9g)$$

for $k = 0, 1$. In the above, the passband and stopband regions of $G_0(a_0, b_0, z)$ and $G_1(a_1, b_1, z)$, denoted by $\omega_p(a_0, b_0)$ and $\omega_s(a_1, b_1)$, subject to following constraints. First, the poles of $G_0(a_0, b_0, z)$ and $G_1(a_1, b_1, z)$ should stay inside the unit circle. Second, the overall frequency response $T(a_0, b_0, a_1, b_1, e^{j\omega})$ should approximate on $[0, \pi]$ the delay term $e^{j(2K+1)\omega}$ with the given tolerance $\delta_0$. For the PR filter banks, $\delta_0$ is zero. Third, the passband amplitude responses of $G_0(a_0, b_0, z)$ and $G_1(a_1, b_1, z)$ have to stay within $1 \pm \delta_1$. Fourth, in the transition bands maximum values are restricted to be less than or equal to $1 + \delta_0$.

4. Proposed Design Technique

This section shows how to effectively solve the problem stated in the previous section. We start by converting the problem into its discretized equivalent. Then, special emphasis is laid on how to find the optimized solution such that the analysis filters remain stable.

4.1. Discretized Problem

To solve the above problem, the region $[0, \pi]$ is discretized into a very dense grid of angular frequencies $\omega_i \in [0, \pi]$ for $i = 1, 2, \ldots, L$. The resulting discretized problem is to find $a_0$, $b_0$, $a_1$, and $b_1$ to minimize

$$\epsilon = \max \left\{ \varepsilon_p(a_0, b_0), \varepsilon_1(a_1, b_1) \right\}, \quad (10a)$$

where for $k = 0, 1$

$$\varepsilon_p(a_k, b_k) = \frac{\pi}{N_k} \sum_{\omega_i \in [0, \pi]} \left| G_k(a_k, b_k, e^{j\omega_i}) \right|^2 \quad (10b)$$

subject to the condition that the poles of $G_0(a_0, b_0, z)$ and $G_1(a_1, b_1, z)$ stay within the unit circle,

$$\left| T(a_0, b_0, a_1, b_1, e^{j\omega_i}) - e^{j2(2K+1)\omega_i} \right| \leq \delta_0 \quad (10c)$$

for $\omega_i \in [0, \pi/2]$ and for $k = 0, 1$
\[ G_k^{(a)}(\omega) = \frac{1}{1 - \delta} G_k^{(a)}(\omega) \text{ for } \omega \in [0, \omega_k] \]

and

\[ G_k^{(b)}(\omega) = (1 + \delta) G_k^{(b)}(\omega) \text{ for } \omega \in [\omega_k, \omega_0) \] (10e)

In Eq. (10b), \( N_k^{(b)} \) is the number of grid points in the interval \([\omega_k, \pi]\).

4.2. Description of the algorithm

For solving this constrained problem the Dutta-Vidyasagar algorithm [17], [18] is applied in a manner similar to that described in [19] for designing FIR filter banks. The main difference is that now after starting with a stable solution the poles of the analysis filters are not allowed to go outside the unit circle.

Inside the modified version of the Dutta-Vidyasagar algorithm the Quasi-Newton method [22] is used as a subalgorithm. At each step of the Quasi-Newton method, the direction in the parameter space is determined along which a better solution is determined by starting with the solution of the previous step. For the denominator polynomials of the analysis transfer functions, as given by Eq. (5), this means that the new denominators for \( k = 0, 1 \) are formed as follows:

\[ B_k^{(new)}(z^2) = B_k^{(old)}(z^2) + \alpha \Delta_k(z^2), \] (11)

where

\[ B_k^{(old)}(z^2) = 1 + \sum a^{(old)}[n]z^{-2n}, \] (12)

is the starting-point solution,

\[ \Delta_k(z^2) = \Delta_k^{(1)} z^2 + \Delta_k^{(2)} z^4 + \ldots + \Delta_k^{(N_k)} z^{-2N_k} \] (13)

is the direction along which a better new solution is searched for (for simplicity, only the parameters involved in determining the new denominators are included), and \( \alpha \) is the step size. In normal cases, \( \alpha \) is determined to minimize the objective function under consideration. For the problem stated in Subsection 4.1, there is a need to find an upper limit for \( \alpha \) to still guarantee the stability.

Among various alternatives for ensuring the stability at each step, the use of the following Rouche’s theorem [20], earlier utilized in [21] for designing IIR filters, has turned out to be the most convenient:

**Rouche’s Theorem**: If functions \( f(z) \) and \( g(z) \) are analytic inside and on a closed contour \( C \), and \( |g(z)| < |f(z)| \) on \( C \), then \( f(z) \) and \( f(z) + g(z) \) have the same number of zeros inside \( C \).

According to this theorem, the roots of the new denominators stay inside a circle with the origin being located at zero in the \( z \)-plane and the radius being \( \lambda \) \((\lambda < 1)\) provided that the roots of the starting point denominators lie inside this circle and the step size is determined as

\[ \alpha < \frac{B_k^{(old)}(z^2)}{\Delta_k(z^2)} \text{ for } |z| = \lambda \] (14)

for \( k = 0, 1 \). In practice, due to the computational inaccuracies and the fact that the radius of the outermost poles for the optimum solution are not close to unity, it is beneficial to use a value of \( \lambda \) being slightly less than unity. This parameter determines how close to the unit circle the poles are allowed to be located. By using a step size \( \alpha \) satisfying above constraints at every step of the optimization algorithm, the stability of the resulting filters is guaranteed provided that the start-up filters for the overall algorithm are stable.

The maximum allowable value for \( \alpha \) can be found conveniently as follows. First, the quantities on the right side of Eq. (14) are calculated for \( k = 0, 1 \) using a sparse grid along \( z = \lambda e^{j\omega}, 0 \leq \omega \leq \pi \) in order to roughly locate the minimum values. Then the golden section search is applied around each minimum point to find the minimum values accurately. Finally, the smallest value among the minimum values is the desired maximum allowable value for the step size \( \alpha \). It should be pointed out that at most steps of the Quasi-Newton method, the actual step size \( \alpha \) used is less than the maximum allowable value.

Due to the fact that the optimization algorithm is nonlinear, it is very crucial to find a good enough start-up solution for the algorithm. As suggested in [15], such a solution can be generated by first designing the corresponding FIR filter bank with the denominator orders equal to zero. The initial solution is then obtained from this bank by selecting the denominator polynomials to have all their roots at the origin. The drawback of this approach is the long computation time required to arrive at the desired IIR filter bank. Moreover, there is no guarantee that a good enough IIR filter bank will be obtained. Therefore, in most cases it is faster to start with simple FIR filters designed in the least-mean-square sense to roughly satisfy the prescribed specifications. In the case where the resulting filter bank after optimization is not satisfactory, there is a need to restart the optimization with a different initial solution.

As mentioned earlier, for PR filter banks, \( \delta \), the allowable reconstruction error, has to be zero. However, it has turned out to be beneficial to allow a small value for \( \delta \) in the very beginning of the optimization. As the optimization continues, this value is then gradually decreased to zero. This approach simplifies the optimization problem and reduces the overall computation time to arrive at an optimized solution.

5. Examples

This section illustrates, by means of some examples, the efficiency of the proposed design method as well as the
superiority of the resulting filter banks to those obtained by using other existing design schemes. In all examples to be considered \( \rho_0^{(0)} = \rho_0^{(1)} = \rho_s^{(0)} = \rho_s^{(1)} = \rho \).

**Example 1.** The specifications for a PR filter bank are: \( M_0 = 7, \, N_0 = 4, \, M_1 = 11, \, N_1 = 4, \, \rho = 0.1, \) and \( \delta_p = 0.01 \). These filter bank parameters are similar to those in Example 2 in [14]. Figure 2 shows the amplitude responses for the optimized analysis filters, whereas some characteristics for this solution are given in Table I. These characteristics include for both analysis filters \( A_{se} \), the attenuation at the stopband edge, \( A_{sb} \), the attenuation of the first peak in the stopband, and \( \varepsilon \), the filter stopband energy. It should be pointed out that the proposed optimization scheme gives an overall solution where the stopband energies of both analysis filters are the same. When comparing this design with the one given in [14], it is seen that the new design provides both better attenuations and smaller passband ripples.

**Example 2.** The specifications for a PR filter bank are: \( M_0 = M_1 = 9, \, N_0 = N_1 = 4, \, \rho = 0.2 \) and \( \delta_p = 0.04 \). The filter bank parameters are similar to the two-channel filter bank example in [15]. Figure 3 shows the amplitude responses for the optimized analysis filters whereas some characteristics of this design are given in Table I. The resulting filter bank outperforms the one given in [15].
6. Conclusion

In this paper an efficient optimization algorithm was proposed for designing causal stable perfect-reconstruction two-channel IIR filter banks. In this synthesis scheme, the maximum of the stopband energies of the analysis filters is minimized subject to the given passband and transition band constraints and the constraint that the poles of the filters are forced to lie inside the unit circle. Due to the nonlinearity of the design problem, to obtain a good result, the proposed method has to be restarted with different initial solutions. Future work is devoted to finding good startup solutions such that the optimum filter bank can be directly obtained.

7. References