A SYSTEMATIC TECHNIQUE FOR DESIGNING LINEAR-PHASE FIR PROTOTYPE FILTERS FOR PERFECT-RECONSTRUCTION COSINE-MODULATED AND MODIFIED DFT FILTERBANKS

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ABSTRACT

This paper describes a systematic technique for designing prototype filters for generating perfect-reconstruction (PR) orthogonal cosine-modulated and modified discrete Fourier transform filterbanks. In the proposed design scheme, the stopband energy of the prototype filter is minimized and the basic unknowns are the angles of a special lattice structure used for implementing the prototype filter so that the PR property is automatically satisfied independent of the angle values. This selection of the unknowns makes the overall optimization problem unconstrained.

Due to the fact that there are several local optima, the design is performed in multiple steps in order to arrive at least at a very good sub-optimal solution. First, for the given number of channels, the length of the channel filters, and the stopband edge of the prototype filter, the corresponding two-channel filterbank is designed based on the pre-optimized data. Then, after knowing the angles for the optimized two-channel case, the prototype filter for the desired filterbank is generated by gradually increasing the number of channels and by properly using the result of the previous step as a start-up solution for the present step. The main benefit of the proposed design technique is that it enables one to effectively design prototype filters for filterbanks with very high-order analysis and synthesis filters.

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I. INTRODUCTION

Among various classes of uniform multi-channel (M-channel) critically-sampled perfect-reconstruction (PR) filterbanks, cosine-modulated [1]–[16] and modified discrete Fourier transform [16]–[18] filterbanks have become very popular in many applications due to the following two main reasons. First, these banks can be generated by using a single prototype finite-impulse response (FIR) filter and by exploiting a proper transformation, thereby enabling one to simultaneously implement all the filters in the analysis and synthesis bank. This leads to extremely efficient implementation structures [1], [2], [4], [15] when compared to the filterbanks, where each filter in the analysis and synthesis banks is separately implemented. Second, the overall filterbank synthesis can concentrate on optimizing only the prototype filter, making the design significantly easier and more straightforward. Since the same prototype filter with only a scaling difference can be used for both above-mentioned types of filterbanks [18], this contribution concentrates on designing cosine-modulated filterbanks (CMFBs).

The CMFBs developed up to now can be divided into two basic types, namely, orthogonal [1]–[12] and biorthogonal (low-delay) [13]–[16] CMFBs. For the first type, the prototype filter is a linear-phase FIR filter and the filterbank delay is equal to the prototype filter order. For the second type, the prototype filter is a nonlinear-phase FIR filter and the filterbank delay is less than the prototype filter order. This paper concentrates on synthesizing orthogonal CMFBs.

For orthogonal CMFBs, the analysis and synthesis transfer functions $H_k(z)$ and $F_k(z)$ of order $N$ for $k=1, 2, \ldots, M$, as shown in Figure 1 [1], can be generated by properly modulating a linear-phase $N$th-order FIR transfer function of the following form:

$$H_p(z) = \sum_{n=0}^{N} h_p[n]z^{-n},$$

where $h_p[N-n] = h_p[n]$ for $n = 0, 1, \ldots, N$. The impulse response of the $k$th transfer function $H_k(z)$ can be obtained based on the above filter, referred to as the prototype filter, as follows [4] (for an alternative cosine-modulation technique, see [1], [2]):

$$h_k[n] = h_p[n]\frac{2}{M}\cos\left[k + \frac{1}{2}\right]\frac{\pi}{M}\left(n - \frac{N}{2}\right) + (-1)^{\frac{n}{2}}\frac{\pi}{4}$$

for $n = 0, 1, \ldots, N$, whereas the impulse response of the corresponding $F_k(z)$ satisfying $f_k[n] = h_k[N-n]$ for $n = 0, 1, \ldots, N$ is achieved by replacing the plus sign by the minus sign in the
front of \((-1)^k \pi/4\) in the above equation. Using the scaling constant of the value \(\sqrt{2/M}\) in the above equation, instead of the value 2 that has been used in [4], is due to the fact that this contribution concentrates on synthesizing the prototype filters based on the use of the lattice structures described in [1], [2], [4]. These structures are exploited because they automatically guarantee the PR property for the overall filterbank and enable one to use unconstrained optimization for designing the prototype filter in a manner to be described later in this paper.

For generating prototype filters for CMFBs, different strategies have been applied. The design can be performed using constrained minimization [7], [8], iterative methods [9], lattice factorizations [2], [4], [6], or by applying some other synthesis schemes [10], [11]. Some of these methods result in PR CMFBs and some in nearly PR CMFBs implying that in Figure 1 \(y[n] = x[n - N]\) and \(y[n] \approx x[n - N]\), respectively. The purpose of this paper is to introduce an efficient systematic technique for minimizing the stopband energy of the prototype filter used for generating PR CMFBs for the given number of channels and the given stopband edge of the prototype filter. In order to achieve this goal, first in order to guarantee the PR condition independent of the values of the design unknowns, these unknowns are selected to be the angles of the lattice structure used for implementing the prototype filter [1], [2], [4]. Second, in order to ensure a good channel selectivity, this proposed technique concentrates only on designing prototype filters with the order \(N\) being equal to \(2KM - 1\), where \(K\) is a positive integer [4]–[6].

This paper is organized as follows: Section II gives the necessary and sufficient conditions for the prototype filter to be satisfied in order to arrive at an orthogonal PR CMFB in the case where the prototype filter is selected in the above-mentioned manner. These conditions are formed so that the least-mean-square problem to be stated for the prototype filter in Section III can be conveniently solved by using unconstrained optimization. The proposed multi-step synthesis scheme is described in Section IV first for the case where \(M\) is a power of two and then for other values of \(M\). In addition, the dependence of the optimized prototype filter performance on \(M\) and the

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1 When using these lattice structures, the amplitude value of the prototype filter is approximately equal to \(\sqrt{2/M}\) at the zero frequency. This implies, according to Equation (2), that the maximum amplitude values of the \(H_1(z)\)'s are approximately equal to \(\sqrt{M}\), whereas, from the signal point of view, those of the \(F_1(z)\)'s are very close to \(1/\sqrt{M}\) because the interpolation by a factor of \(M\) decreases the signal energy by this factor. The product of these two values is equal to unity, as is required for generating PR CMFBs.
prototype filter order is briefly studied. In order to arrive at least at good locally optimized solutions, Section V concentrates on optimizing the prototype filter in the two-channel case that is used as a start-up solution for the design scheme of Section IV. Finally, concluding remarks are drawn in Section VI.

II. PR CONDITIONS

This section gives the necessary and sufficient conditions for the prototype filter, with the transfer function as given by Equation (1), in order to result in a PR CMFB in the case, where this filter is implemented using the lattice structure [4] and its order is given by \( N = 2KM - 1 \) with \( K \) being a positive integer. These conditions are described in the forms that can be used in a straightforward manner for both stating the optimization problem in Section III and for solving it in Sections IV and V. The suggested conditions have been derived by slightly modifying those given in [4].

The conditions for the prototype filter can be expressed conveniently by rewriting its transfer function, as given by Equation (1), in the following form:

\[
H_p(z) = z^{-(KM - 1/2)} \sum_{n=1}^{KM} g[n] \left[ z^{-(n-1/2)} + z^{-(n+1/2)} \right],
\]

where \( g[n] = h_p[KM-n] \) for \( n = 1, 2, \ldots, KM \) are the coefficients representing the original coefficients \( h_p[n] \) being symmetrical around \( n = (KM-1)/2 \). Typical examples for these coefficients \( g[n] \) for both even and odd \( M \) are shown in Figures 2(a) and 2(b), respectively. Furthermore, these figures show how proper sets of the \( g[n] \)'s, being marked by different symbols, should be grouped together in order to conveniently state the criteria for the PR condition. For each of these sets, the required conditions are formed according to the following discussion.

First, if \( M \) is odd, then in order to arrive at a PR CMFB it is required that

\[
g[(M+1)/2] = \sqrt{1/2}
\]

and

\[
g[(M+1)/2 + kM] = 0 \quad \text{for} \quad k = 1, 2, \ldots, K-1.
\]

Selecting \( N \) in this manner ensures that the PR condition is achieved for \( M \) even without fixing any of the impulse-response values of the prototype filter to take on a predetermined value, as will be seen in Section II. For \( M \) odd, in turn, the number of these predetermined values is minimized, among the relations between \( N \) and \( M \), thereby resulting in the highest achievable selectivity for the filters in the analysis and synthesis banks.
In Figure 2(b), this set of the \( g[n] \)'s is marked by the symbol '•'. Second, for both \( M \) odd and even, there exist \( K \) constraints for each of the following \( \lfloor M/2 \rfloor \) \( g[n] \)'s:

\[
\Theta_r = \{ g[r], g[M+1-r], g[M+r], g[2M+1-r], g[2M+r], \ldots, g[(K-1)M+r], g[KM+1-r] \}
\]  

for \( r = 1, 2, \ldots, \lfloor M/2 \rfloor \) as shown in Figures 2(a) and 2(b).

For the \( r \)th set \( \Theta_r \), the above-mentioned \( K \) constraints can be conveniently expressed in terms of

\[
c_{r,k} = \cos(\phi_{r,k}) \quad \text{and} \quad s_{r,k} = \sin(\phi_{r,k}) \quad \text{for} \quad k = 1, 2, \ldots, K,
\]

where the \( \phi_{r,k} \)'s for \( k = 1, 2, \ldots, K \) are the basic adjustable parameters. To generate these constraints, first, the following transfer functions:

\[
P^{(K)}_r(z) = \sum_{n=0}^{K-1} p_r[n] z^{-n} \quad \text{and} \quad Q^{(K)}_r(z) = \sum_{n=0}^{K-1} q_r[n] z^{-n}
\]

are generated recursively as

\[
P^{(k)}_r(z) = s_{r,k} P^{(k-1)}_r(z) + c_{r,k} z^{-1} Q^{(k-1)}_r(z) \quad \text{and} \quad Q^{(k)}_r(z) = c_{r,k} P^{(k-1)}_r(z) - s_{r,k} z^{-1} Q^{(k-1)}_r(z)
\]

by starting with

\[
P^{(1)}_r(z) = s_{r,1} \quad \text{and} \quad Q^{(1)}_r(z) = c_{r,1}.
\]

For \( K \) even, the required constraints for the \( g[n] \)'s belonging to the \( r \)th set \( \Theta_r \), as given by Equation (5), can be written in terms of the coefficients of \( P^{(K)}_r(z) \) and \( Q^{(K)}_r(z) \) as follows:

\[
g[2kM+r] = p_r[K/2+k] \quad \text{for} \quad k = 0,1,\ldots,K/2-1,
\]

\[
g[(2k+1)M+r] = q_r[K/2+k] \quad \text{for} \quad k = 0,1,\ldots,K/2-1,
\]

\[
g[2kM+1-r] = p_r[K/2-k] \quad \text{for} \quad k = 1,2,\ldots,K/2,
\]

and

\[
g[(2k-1)M+1-r] = q_r[K/2-k] \quad \text{for} \quad k = 1,2,\ldots,K/2,
\]

whereas for \( K \) odd, the corresponding requirements are expressible as

\[\text{[x]} \] stands for the integer part of \( x \).
\[
g[2kM+r] = q_r[(K-1)/2 + k] \quad \text{for} \quad k = 0, 1, \ldots, (K-1)/2, \\
g[(2k-1)M+r] = p_r[(K-1)/2 + k] \quad \text{for} \quad k = 1, 2, \ldots, (K-1)/2, \\
g[2kM+1-r] = q_r[(K-1)/2 - k] \quad \text{for} \quad k = 1, 2, \ldots, (K-1)/2, \\
g[(2k+1)M+1-r] = p_r[(K-1)/2 - k] \quad \text{for} \quad k = 0, 1, \ldots, (K-1)/2.
\]

There are the following close connections between the proposed adjustable angles \( \phi_{r,k} \) to the corresponding angles used in [2] and [4]. For the angles used in [2], denoted by \( \theta_{r,k} \) for \( r = 1, 2, \ldots, \lfloor M/2 \rfloor \) and \( k = 0, 1, \ldots, K-1 \), these relations are given by
\[
\theta_{r,2k} = \pi/2 - \phi_{r+1,K-2k} \quad \text{for} \quad k = 0, 1, \ldots, \lfloor (K-1)/2 \rfloor
\]
and
\[
\theta_{r,2k+1} = \pi/2 - \phi_{r+1,K-2k+1} \quad \text{for} \quad k = 0, 1, \ldots, \lfloor K/2 \rfloor - 1,
\]
whereas for the corresponding angles used in [4], the relations are expressible as
\[
\theta_{r,k} = \pi/2 - \phi_{r+1,k+1} \quad \text{for} \quad k = 0, 1, \ldots, K-1.
\]

The main benefit of using the angles \( \phi_{r,k} \), as defined earlier in this section, is that if the \( \phi_{r,k} \)'s for \( k=k_1, k_1+1, \ldots, K \) and \( r = 1, 2, \ldots, \lfloor M/2 \rfloor \) are made zero, then the resulting \( g[n] \)'s become non-zero only for \( n \leq k_1M-1 \) and are zero-valued otherwise, thereby corresponding to the \( K=k_1 \) case. This makes the construction of the optimization algorithm, to be described in the following two sections, more straightforward. This is because of the fact that by selecting both the values of the proper angles and the partial derivatives of the objective function with respect to these angles to be zero-valued does not increase the prototype filter order during the optimization, thereby automatically resulting in a prototype filter with the desired reduced overall order.

III. STATEMENT OF THE PROBLEM

This section states the optimization problem for minimizing the stopband energy of the prototype filter resulting in a PR CMFB in the case considered in the previous section.

The resulting optimization problem can be stated as follows:
**Optimization Problem:** Given $M$, the number of channels and $K$, determining the prototype filter order to be equal to $2KM - 1$, as well as the roll-off factor $\rho > 0$ typically used for defining the stopband edge angle as [6]

$$\omega_s = (1 + \rho) \pi / (2M),$$  

(13)

find the unknowns $g[n]$ of the prototype filter transfer function $H_p(z)$, as given by Equation (3), to minimize the stopband energy of the prototype filter as given by

$$E_2 = \int_\omega_s^{\pi} |H_p(e^{j\omega})|^2 d\omega$$  

(14)

subject to the conditions stated in the previous section.

Solving the above optimization problem can be significantly simplified by properly taking into account the constraints stated on the $g[n]$’s in the previous section. To do this, the first step is to start with the zero-phase frequency response of $H_p(z)$, as given by Equation (3), that is, the response resulting when omitting the phase term $e^{-j(2KM - 1/2)\omega}$ in the overall frequency response [23]. For the optimization purposes, it is beneficial to express this response as

$$G_p(\omega, \Phi) = 2 \sum_{n=1}^{KM} g(n, \Phi) \cos[(n - 1/2)\omega]$$  

(15a)

where

$$\Phi = [\phi_{1,1}, \phi_{1,2}, \ldots, \phi_{1,K}, \phi_{2,1}, \phi_{2,2}, \ldots, \phi_{2,K}, \ldots, \phi_{[M/2],1}, \phi_{[M/2],2}, \ldots, \phi_{[M/2],K}]$$  

(15b)

is an $K[M/2]$-length adjustable parameter vector containing as unknown parameters the angles $\phi_{r,k}$, for $r = 1, 2, \ldots, [M/2]$ and for $k = 1, 2, \ldots, K$, as stated in the previous section in order to arrive at a PR CMFB. Recall that for $M$ odd, in addition to the $(M-1)/2$ angle sets specifying the $K(M-1)$ $g[n]$’s according to the discussion of the previous section, there are also $K$ fixed $g[n]$’s, as given by Equations (4a) and (4b), that should be taken into account when determining the above $G_p(\omega, \Phi)$ even though, due to their fixed values, they do not provide any contribution to the adjustable parameter vector $\Phi$. The notations $g(n, \Phi)$ and $G(\omega, \Phi)$ are used both for emphasizing their dependence on the elements of the adjustable parameter vector $\Phi$ and for making the actual optimization more efficient and more straightforward, as will be seen later in this section.

After some manipulations, the objective function to be minimized, that is, the stopband energy of the prototype filter, as given by Equation (14), is expressible with the aid of the above $G(\omega, \Phi)$
and the stopband edge angle $\omega_s$, as defined in terms of $\rho > 0$ by Equation (13), in the following form:

$$E_2 = E_2(\Phi) = \int_{\omega_s}^{\pi} [G_p(\omega, \Phi)]^2 d\omega = \sum_{\mu=1}^{KM} \sum_{\eta=1}^{KM} \Re[g(\mu, \Phi) g(\eta, \Phi)] \Psi(\mu, \eta),$$ (16a)

where

$$\Psi(\mu, \eta) = \begin{cases} 2\pi - 2\omega_s - \frac{2\sin[(2\mu - 1)\omega_s]}{2\mu - 1}, & \mu = \eta \\ -\frac{2 \sin[(\mu + \eta - 1)\omega_s]}{\mu + \eta - 1} - \frac{2 \sin[(\mu - \eta)\omega_s]}{\mu - \eta}, & \mu \neq \eta. \end{cases}$$ (16b)

The partial derivatives of the above objective function with respect to $\phi_{r,n}$'s for $r = 1, 2, \ldots, \lfloor M/2 \rfloor$ and $n = 1, 2, \ldots, K$ are expressible as

$$\frac{\partial E_2(\Phi)}{\partial \phi_{r,n}} = 2\sum_{\mu=1}^{KM} \sum_{\eta=1}^{KM} \hat{g}_{r,n}(\mu, \Phi) g(\eta, \Phi) \Psi(\mu, \eta),$$ (17a)

where

$$\hat{g}_{r,n}(\mu, \Phi) = \frac{\partial g(\mu, \Phi)}{\partial \phi_{r,n}}.$$ (17b)

It can be shown that in the $n \neq k$ case the following relations are valid for $k = 1, 2, \ldots, K$:

$$\hat{g}_{r,n}((k - 1)M + r, \Phi) = g((k - 1)M + r, \Phi)$$ (18a)

and

$$\hat{g}_{r,n}(kM + 1 - r, \Phi) = g(kM + 1 - r, \Phi),$$ (18b)

whereas the remaining elements are zero-valued. Hence, the values of the $\hat{g}_{r,n}(n, \Phi)$'s being non-zero can be determined as the corresponding values of the $g[n]$'s belonging to the $r$th set $\Theta_r$, as given by Equation (5). This can be performed according to the conditions stated in the previous section, that is, by using Equations (6), (7), (8a), (8b), (9), and (10). Also, in the $n = k$ case, only the elements $\hat{g}_{r,n}((k - 1)M + r, \Phi)$ and $\hat{g}_{r,n}(kM + 1 - r, \Phi)$ for $k = 1, 2, \ldots, K$ are nonzero. Their values can be determined by simply replacing in Equation (6) $c_{r,k} = \cos(\phi_{r,k})$ and $s_{r,k} = \sin(\phi_{r,k})$ by $c_{r,k} = -\sin(\phi_{r,k})$ and $s_{r,k} = \cos(\phi_{r,k})$, respectively.

Expressing the objective function in the form of Equations (15a), (15b), (16a), and (16b) is very attractive for the following reasons. First, the objective function and its partial derivatives
with respect to the unknowns have easy-to-evaluate closed form expressions. Second, most importantly, since the conditions required by the prototype filter leading to a PR CMFB are included in the objective function, there are no constraints on minimizing this objective function. These facts enable one to use any effective unconstrained optimization algorithm exploiting the existence of the closed-form expressions for both the objective function and its partial derivatives with respect to the unknowns. However, due to the fact that for the problem under consideration, there are several local optima, the convergence at least to a good local optimum solution implies that a good enough initial solution exists. Therefore, the proposed overall design scheme to be described in the following two sections is generated so that such an initial solution is available for each step.

IV. PROPOSED MULTI-STEP SYNTHESIS SCHEME AND PROPERTIES OF THE RESULTING FILTERBANKS

This section shows how to effectively design the prototype filter leading to a PR CMFB for large values of $K$ and $M$ in the case where the prototype filter order is given by $N=2KM−1$. First, the case with $M$ being a power of two is considered. Then, the synthesis of CMFBs for an arbitrary value of $M$ is discussed. Finally, examples are included for indicating the dependence of the prototype filter performance on $M$ and $K$ (or equivalently $N=2KM−1$).

A. SYNTHESIS SCHEME FOR $M$ BEING A POWER OF TWO

In order to make the description of the proposed multi-step algorithm more self-explanatory, the following illustrative example is first used in this subsection: It is desired to design a CMFB for $M=32=2^5$, $K=8$, and $\rho=1$. A very fast procedure giving at least a good local optimum can be carried out as follows (see Figures 3 and 4):

**Step 1.** Optimize the prototype filter for $M=2$, $K=8$, and $\rho=1$ by selecting the initial angle values of the lattice structure according to the discussion of Section V.

**Step 2.** Optimize the prototype filter for $M=4$, $K=8$, and $\rho=1$ using the initial values $\phi_{r,k}^{(2)} = \phi_{r,k}^{(1)}$ for $r=1, 2$ and for $k=1, 2, \ldots, 8$, where the $\phi_{r,k}^{(1)}$’s are the optimized parameters of Step 1.

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Note that the values of $K$ and $\rho$ are the same as for the final prototype filter implying that the stopband edge is given by Equation (13) with $M=2$. This is also true for Steps 2, 3, 4, and 5, where the stopband edge is determined in the same manner by using $M=4$, $M=8$, $M=16$, and $M=32$, respectively.
Step 3. Optimize the prototype filter for \(M = 8\), \(K = 8\), and \(\rho = 1\) using the initial values \(\phi_{r,k}^{(3)} = \phi_{r,k}^{(2)}\) for \(r = 1, 2\) and \(\phi_{r,k}^{(3)} = \phi_{r,k}^{(2)}\) for \(r = 3, 4\), where the \(\phi_{r,k}^{(2)}\)’s and \(\phi_{r,k}^{(2)}\)’s are the optimized values of Step 2.

Step 4. Optimize the prototype filter for \(M = 16\), \(K = 8\), and \(\rho = 1\) using the initial values \(\phi_{r,k}^{(4)} = \phi_{r(2),k}^{(3)}\) for \(r = 1, 2, \ldots, 8\), where the \(\phi_{r(2),k}^{(3)}\)’s are the optimized parameters of Step 3.

Step 5. Optimize the prototype filter for \(M = 32\), \(K = 8\), and \(\rho = 1\) using the initial values of the previous step in the manner similar to Steps 2, 3, and 4.

In both Figures 3 and 4, the scaled prototype filter with transfer function \(H_s(z)\) being related to the prototype filter transfer function, as given by Equation (1), via

\[
H_s(z) = H_p(z)/\sqrt{2M},
\]

is considered since in this case the resulting amplitude value at the zero frequency is approximately equal to unity.

Using the prototype filter transfer function after performing Step 5 and utilizing the cosine-modulation, as given by Equation (2), results in a CMFB with 32 channels. The scaled amplitude responses for the analysis filters in this bank are shown in Figure 5.

There are the following main differences between the above illustrative design procedure compared with the general case, where \(M = 2^L\) and \(K\) is an arbitrary integer. First, the number of steps in the procedure is the given value of \(L\), instead of \(L = 5\). Second, the number of sets of angles under consideration is the given value of \(K\), instead of \(K = 8\).

The key idea in generating the initial solutions for Steps 2, 3, 4, and 5 in the above illustrative example, by means of repeating all sets of the angles of the previous steps as described, is based on the following fact. If the optimized scaled transfer function of the previous step is given by

\[
\tilde{H}_s(z) = \sum_{n=0}^{\tilde{N}_s} \tilde{h}_s(n)z^{-n},
\]

then the resulting start-up scaled transfer function for the present step becomes

\[
H_s(z) = \sum_{n=0}^{2\tilde{N}_s+1} h_s(n)z^{-n},
\]

where

\[
h_s(2n) = h_s(2n + 1) = \tilde{h}_s(n)/2 \quad \text{for} \quad n = 0, 1, \ldots, \tilde{N}_s.
\]
Alternatively, \( H_s(z) \) is expressible in terms of \( \tilde{H}_s(z) \) as
\[
H_s(z) = \tilde{H}_s(z^2)(1 + z^{-1})/2
\] (23)

As seen in Figures 3 and 6, this means that the amplitude response of \( \tilde{H}_s(z^2) \) is a frequency-axis compressed version of that of \( \tilde{H}_s(z) \) so that the interval \([0, 2\pi]\) is shrunk onto \([0, \pi]\). The amplitude response \( \tilde{H}_s(z^2) \) is shown by the solid line in Figure 6 when performing Step 2 in the above design scheme. As seen from this figure, it has an extra passband at \( \omega = \pi \). This is partially attenuated by the term \( (1 + z^{-1})/2 \) with the amplitude response shown by the dashed line in Figure 6. The resulting overall response with an additional peak around \( \omega = \pi \) is shown by the dashed line in the second figure of Figure 3. After the optimization, the solid line in the second figure of Figure 3 shows that this additional peak is attenuated, as is desired. It is interesting to observe from Figure 3 that when \( M \) is increased, the term \((1 + z^{-1})/2\) provides a better attenuation for the extra passband of \( \tilde{H}_s(z^2) \) and the initial filter becomes closer to the optimized one.

In the time domain, the relation between the impulse responses of \( H_s(z) \) and \( \tilde{H}_s(z) \) are given by Equation (22). As seen in Figures 4(a)–4(c) for Step 2 and Figures 4(c)–4(g) for Steps 3, 4, and 5, the impulse-response coefficients of the initial filter are obtained from the optimized ones of the previous steps by dividing the impulse-response coefficients by two and then repeating them two times, thereby increasing the length of the resulting filter by a factor of two. After the optimization, the resulting impulse response becomes smooth, as seen when comparing Figures 4(b) and 4(c) for Step 2 or Figures 4(d) and 4(e) for Step 3. The same procedure is then repeated for Steps 4 and 5 in this design example.6

As discussed in Section II, the angles of the corresponding lattice implementation are used as the unknowns in the proposed design. The neat property in the above technique is that the start-up solutions are generated by simply repeating all sets of angles as shown at Steps 2–5 in the illustrative example. Therefore, at each step, the initial solution satisfies the following two attractive properties. First, as seen from Figure 3, the initial frequency response is close to the optimized one except for an interval around \( \pi \), and second, the angles of the initial solution used for generating the next-step prototype filter guarantee that the PR condition is automatically satisfied.

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5 For more details about the validity of this formula and its interpretation, see, for instance, [23].
6 It is worth pointing out that an idea similar to the proposed one has also been used earlier for designing NPR \( M \)-channel filter banks (pseudo QMF banks) by Cox [19].
The above procedure enables one to quickly design at least sub-optimum filters for very high values of $K$ and $M$ compared to the case where the overall filter is designed using only a one-step procedure. In order to arrive at a good overall solution, it is very crucial to find a good solution for $M=2$. This problem will be considered in Section V.

**B. SYNTHESIS SCHEME FOR AN ARBITRARY NUMBER OF CHANNELS**

If $M$, the number of channels, is even but not a power of two, then at least a good local optimum solution can be achieved by using the following procedure:

1) Find nearest power of two, denoted by $\tilde{M}$, and synthesize the filter using the procedure described in the previous subsection.

2) If $M < \tilde{M}$, then gradually decrease $\tilde{M}$ by two until $\tilde{M} = M$.

3) If $M > \tilde{M}$, then gradually increase $\tilde{M}$ by two until $\tilde{M} = M$.

When increasing the number of channels, one angle set is used twice following each other. The decrease in the number of channels is performed, in turn, by neglecting one angle set. It should be emphasized that when increasing or decreasing the number of channels, the stopband edge should be changed according to Equation (13), where $\rho$ has the fixed predetermined value.

If $M$, the number of channels is odd, then the best overall design is obtained by first designing the prototype filter for $M+1$ and then neglecting the second last angle set. This observation has turned out to be valid up to 63 channels.

**C. PROPERTIES OF THE OPTIMIZED FILTERBANKS**

In order to investigate how the stopband energy of the optimized prototype filter depends on $M$, the number of channels, and $K$ being related to the prototype filter order via $N=2KM-1$, four sets of prototype filters have been optimized for $\rho=1$ and $12 \leq M \leq 20$. For these four sets, the values of $K$ are 2, 4, 6 and 8. For each set, the dependence of the energy on the number of channels is shown in Figure 7. As can be seen from this figure, except for the value$^7$ of $K=2$, the use of filterbanks with odd number of channels is questionable due to their poor stopband attenuation when compared with filterbanks having an even number of channels. This is a direct consequence of the fact that some impulse-response values for an odd value of $M$ are forced to take on prede-

$^7$ The same is true for $K=1$, but not for $K=3$. 
terminated values as discussed in Section II, whereas for an even value of $M$ there are no such restrictions.

V. INITIAL SOLUTIONS FOR TWO-CHANNEL CASES

As discussed earlier, the convergence of the proposed multi-step design scheme to at least a good local optimum solution relies in finding a good solution for the two-channel case ($M = 2$). The purpose of this section is twofold. First, the characteristics of the angle values for the lattice structure of the two-channel filterbank being achieved by optimizing them with the aid of numerous optimization techniques for various values of $K$ and $\rho$ [cf. Equation (13)] are considered. Second, the resulting optimized angle values are expressed in simple polynomial forms such that when using these expressions, the optimized angle values can be found very quickly through optimization.

It has been experimentally observed that in many cases a good filterbank performance for the given values of $K$ and $\rho$ is obtained by starting the optimization with all adjustable angle values being equal to zero. However, there are several local optima. Therefore, additional random search optimization algorithms have been applied for finding an improved solution. The stopband energies of the resulting optimized prototype filters are shown in Figure 8 for various values of $K$ and $\rho$.

As can be expected, the stopband energy of the prototype filter for a fixed value of $\rho$ decreases when $K$ is made larger. Moreover, it is interesting to observe from Figure 8 that the energy decreases in a very regular manner as a function of the stopband edge frequency. The consequence of this behavior is that the corresponding optimized angle values also change very smoothly as a function of $\rho$. This is illustrated in Figure 9 that shows how the value of the first optimized angle in the $K = 5$ case depends on $\rho$. In this figure as well as later on, the notation $\phi_k^{(K)}$ is used for the $k$th angle for the given value of $K$.

Because of this very smooth dependence of the optimized angle values on $\rho$, it is beneficial to fit a polynomial in the least-mean-square sense through the optimized data. After knowing the polynomial coefficients, denoted by $c_{l_k}^{(K)}(l)$, and the corresponding order, denoted by $L_k^{(K)}$, the angle values for $k = 1, 2, \ldots, K$ can be estimated by the following expression:
The main idea in the polynomial fitting is that given the value of $\rho$, very accurate initial angle values are available in the storage and can be used for the optimization routine giving very quickly the real optimized angles. Due to the slow changes of angle values as functions of $\rho$, rather low-order polynomials are needed for the polynomial fitting. In order to keep the polynomial orders as low as possible and at the same time to achieve good approximations to the optimized values, it has turned out that for high values of $K$ it is beneficial to generate different low-order polynomials in two or three intervals of $\rho$. Using these polynomials in generating initial angle values, the optimization algorithm very quickly finds the optimized angle values. Moreover, when using polynomials of orders given in Table I, the corresponding solutions will result in prototype filters with their stopband energies differing from the optimized ones by less than one percent.

As an example, two sets of coefficients in the $K=5$ case are given in Table II and for the subintervals $0.5 < \rho < 1.05$ and $1.05 < \rho < 1.5$, respectively. Table IV, in turn, compares for some values of $\rho$ the energies of the prototype filters obtained by using the polynomial fitting approach with the optimized ones. In this table, $E_{an}$, $E_{diff}$, and $E_{%}$ denote the stopband energy of the prototype filter obtained by the polynomial fitting, the absolute error compared to the optimized solution, and the same error in percents, respectively. As seen from this table, the difference is in all the cases negligible.

VI. CONCLUDING REMARKS

Due to the highly non-linear optimization problem, it is not worth claiming that the solution resulting when applying the multi-step procedure proposed in this paper is the global optimum in all cases. However, based on extensive simulation studies, the proposed technique has always been successful in finding at least a very good sub-optimal solution up to $K=10$.

The proposed procedure can be used without difficulties for designing filterbanks up to $M=256$ channels. Furthermore, it provides good start-up solutions for cases where it is desired to optimize the prototype filter using either the minimax or the peak-constrained least-squared error criterion. In addition, this procedure can be applied for designing filterbanks with an odd or an
even number of channels. However, filterbanks with an odd number of channels are of a practical value only for $K \leq 2$, as was shown by means of an example.

It is worth mentioning that based on the design method introduced in [20] by the authors of this contribution, another technique has been proposed in [22]. In this design scheme, the angles of the lattice implementation, as given by Equation (15b), are approximated by polynomial functions in order to further reduce the number of unknowns. This simplifies the design problem, but the global optimum can not be achieved. It is worth studying in the future whether these two design schemes can be combined in order to achieve better solutions.

VII. REFERENCES


Figure 1. $M$-channel critically sampled filterbank.
Figure 2. Grouping the coefficients $g[n]$, as defined by Equation (3) and representing the original coefficients of the prototype filter, into proper sets in order to conveniently state the criteria for the PR property. For each set being marked by their own symbols, the PR condition implies the specific relations between the coefficients in this set according to the discussion in the text. (a) $M=8$, $K=4$, and $N=2KM−1=63$. (b) $M=9$, $K=4$, and $N=2KM−1=71$. 
Figure 3. Amplitude responses of the prototype filters for Steps 1–5 in the proposed synthesis scheme in the illustrative example with $M=32$, $K=8$, $N=511$, and $\rho=1$. The dashed and solid lines show the initial and the optimized responses for the scaled prototype filters, respectively.
Figure 4. Scaled impulse-response coefficients of prototype filters for Steps 1–5 in the proposed synthesis scheme in the illustrative example with $M=32$, $K=8$, $N=511$, and $\rho=1$. (a) Optimized filter for Step 1. (b) Initial filter for Step 2. (c) Optimized filter for Step 2. (d) Initial filter for Step 3. (e) Optimized filter for Step 3. (f) Optimized filter for Step 4. (g) Optimized filter for Step 5.
Figure 5. Scaled amplitude characteristics of the analysis filters achieved by using the prototype filter obtained at Step 5 of the illustrative example.

Figure 6. Generating the amplitude response of $H_i(z)$, as given by Equation (23), based on the cascade of $\tilde{H}_i(z^2)$ and $(1+z^{-1})/2$. The solid and dashed lines show the amplitude responses of $\tilde{H}_i(z^2)$ and $(1+z^{-1})/2$, respectively, for Step 2 in the illustrative example.
Figure 7. Stopband energies of the optimized prototype filters for $\rho=1$ and $12 \leq M \leq 20$ for four values of $K$, namely, $K=2$, $K=4$, $K=6$, and $K=8$.

Figure 8. Stopband energy of the prototype filter for various values of $K$ and $\rho$. 
Figure 9. Value of the first angle in the $K=5$ case as a function of $\rho$. 
### TABLE I

**REQUIRED POLYNOMIAL ORDERS FOR** $0.5 < \rho < 1.5$ **TO KEEP THE ENERGY ERROR LESS THAN 1%**

<table>
<thead>
<tr>
<th>$K$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L^{(K)}$</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>4+6</td>
<td>6+8</td>
</tr>
</tbody>
</table>

### TABLE II

**POLYNOMIAL COEFFICIENTS FOR $K=5$ AND $0.5 < \rho < 1.05$ WITH $L^{(5)} = 4$**

<table>
<thead>
<tr>
<th>$k$</th>
<th>$c_k^{(5)}(0)$</th>
<th>$c_k^{(5)}(1)$</th>
<th>$c_k^{(5)}(2)$</th>
<th>$c_k^{(5)}(3)$</th>
<th>$c_k^{(5)}(4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.80099</td>
<td>−2.87892</td>
<td>6.51772</td>
<td>−6.51983</td>
<td>2.53331</td>
</tr>
<tr>
<td>2</td>
<td>0.14353</td>
<td>−0.27843</td>
<td>0.25244</td>
<td>−0.16907</td>
<td>0.05933</td>
</tr>
<tr>
<td>3</td>
<td>−0.66512</td>
<td>2.84760</td>
<td>−6.51593</td>
<td>6.43453</td>
<td>−2.48184</td>
</tr>
<tr>
<td>4</td>
<td>−0.20461</td>
<td>0.27802</td>
<td>−0.28523</td>
<td>0.22100</td>
<td>−0.07755</td>
</tr>
<tr>
<td>5</td>
<td>0.04808</td>
<td>0.04233</td>
<td>0.02447</td>
<td>0.02811</td>
<td>−0.02934</td>
</tr>
</tbody>
</table>
TABLE III

POLYNOMIAL COEFFICIENTS FOR \( K = 5 \) AND \( 1.05 < \rho < 1.5 \) WITH \( L^{(5)} = 6 \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>( c_k^{(5)}(0) )</th>
<th>( c_k^{(5)}(1) )</th>
<th>( c_k^{(5)}(2) )</th>
<th>( c_k^{(5)}(3) )</th>
<th>( c_k^{(5)}(4) )</th>
<th>( c_k^{(5)}(5) )</th>
<th>( c_k^{(5)}(6) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-533.6072</td>
<td>2783.822</td>
<td>-5959.943</td>
<td>6712.219</td>
<td>-4198.006</td>
<td>1384.239</td>
<td>-188.2563</td>
</tr>
<tr>
<td>2</td>
<td>-21.16929</td>
<td>102.2146</td>
<td>-203.6576</td>
<td>214.5891</td>
<td>-126.2182</td>
<td>39.32101</td>
<td>-5.072057</td>
</tr>
<tr>
<td>3</td>
<td>523.7613</td>
<td>-27.42570</td>
<td>5891.328</td>
<td>-6654.325</td>
<td>4172.305</td>
<td>-1378.759</td>
<td>187.8630</td>
</tr>
<tr>
<td>4</td>
<td>20.75647</td>
<td>-100.2287</td>
<td>199.0703</td>
<td>-209.098</td>
<td>122.6300</td>
<td>-38.10026</td>
<td>4.902311</td>
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<tr>
<td>5</td>
<td>10.36380</td>
<td>-43.1651</td>
<td>73.12838</td>
<td>-63.5914</td>
<td>29.34241</td>
<td>-6.732564</td>
<td>0.568735</td>
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</tbody>
</table>

TABLE IV

ENERGIES OF THE PROTOTYPE FILTERS ACHIEVED BY THE PROPOSED POLYNOMIAL FITTING, ABSOLUTE ERRORS BETWEEN THE OPTIMIZED SOLUTIONS AND THE POLYNOMIAL SOLUTIONS, AND THE CORRESPONDING RELATIVE ERRORS FOR SOME VALUES OF \( \rho \) IN THE \( K = 5 \) CASE

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( E_{an} )</th>
<th>( E_{diff} )</th>
<th>( E_{%} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.566</td>
<td>1.57 \cdot 10^{-3}</td>
<td>9.06 \cdot 10^{-8}</td>
<td>5.8 \cdot 10^{-3}</td>
</tr>
<tr>
<td>0.699</td>
<td>4.31 \cdot 10^{-4}</td>
<td>3.12 \cdot 10^{-8}</td>
<td>7.2 \cdot 10^{-3}</td>
</tr>
<tr>
<td>0.792</td>
<td>1.77 \cdot 10^{-4}</td>
<td>1.66 \cdot 10^{-9}</td>
<td>9.4 \cdot 10^{-4}</td>
</tr>
<tr>
<td>0.942</td>
<td>3.84 \cdot 10^{-5}</td>
<td>2.32 \cdot 10^{-9}</td>
<td>6.0 \cdot 10^{-3}</td>
</tr>
<tr>
<td>1.021</td>
<td>1.56 \cdot 10^{-5}</td>
<td>1.11 \cdot 10^{-9}</td>
<td>7.1 \cdot 10^{-3}</td>
</tr>
<tr>
<td>1.171</td>
<td>2.10 \cdot 10^{-6}</td>
<td>6.14 \cdot 10^{-10}</td>
<td>2.9 \cdot 10^{-2}</td>
</tr>
<tr>
<td>1.328</td>
<td>2.08 \cdot 10^{-7}</td>
<td>1.08 \cdot 10^{-10}</td>
<td>5.2 \cdot 10^{-2}</td>
</tr>
</tbody>
</table>