Problem set 4

Levinson- Durbin Algorithm and Lattice filters

Consider a signal $u(n)$ generated by an AR process of order 2:

$$u(n) = -a_1 u(n-1) - a_2 u(n-2) + v_1(n)$$

where $v_1(n)$ is a white noise sequence with variance σ_v^2.

6.1 Starting from AR parameters a_1, a_2 and from noise variance σ_v^2 compute $r(0), r(1), r(2)$.
Hint: use Yule-Walker equations.

We can write the Yule Walker equations:

$$\begin{aligned} r(0) + a_1 r_u(1) + a_2 r_u(2) &= \sigma_v^2 \\ r(1) + a_1 r_u(0) + a_2 r_u(1) &= 0 \\ r(2) + a_1 r_u(1) + a_2 r_u(0) &= 0 \end{aligned}$$

or

$$\begin{bmatrix} 1 & a_1 & a_2 \\ a_1 & 1 + a_2 & 0 \\ a_2 & a_1 & 1 \end{bmatrix} \begin{bmatrix} r(0) \\ r(1) \\ r(2) \end{bmatrix} = \begin{bmatrix} \sigma_v^2 \\ 0 \\ 0 \end{bmatrix}$$

6.2 Starting from autocorrelation sequence $r(0), r(1), r(2)$ compute the AR parameters a_1, a_2, and the noise variance σ_v^2.
Hint: use Yule-Walker equations.

From Yule Walker equations we can write

$$\begin{bmatrix} r(0) & r(1) & r(2) \\ r(1) & r(0) & r(1) \\ r(2) & r(1) & r(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sigma_v^2 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} r(0) & r(1) \\ r(1) & r(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -r(1) \\ -r(2) \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} &= \begin{bmatrix} r(0) & r(1) \\ r(1) & r(0) \end{bmatrix}^{-1} \begin{bmatrix} -r(1) \\ -r(2) \end{bmatrix} = \frac{1}{r(0)^2 - r(1)^2} \begin{bmatrix} r(0) & -r(1) \\ -r(1) & r(0) \end{bmatrix} \begin{bmatrix} -r(1) \\ -r(2) \end{bmatrix} \\ &= \frac{1}{r(0)^2 - r(1)^2} \begin{bmatrix} -r(0)r(1) + r(1)r(2) \\ r(1)^2 - r(0)r(2) \end{bmatrix} = \begin{bmatrix} \frac{-r(0)r(1)+r(1)r(2)}{r(0)^2-r(1)^2} \\ \frac{r(1)^2-r(0)r(2)}{r(0)^2-r(1)^2} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \sigma_v^2 &= r(0) + a_1 r(1) + a_2 r(2) = r(0) + r(1)\frac{-r(0)r(1) + r(1)r(2)}{r(0)^2 - r(1)^2} + r(2)\frac{r(1)^2 - r(0)r(2)}{r(0)^2 - r(1)^2} \\ &= \frac{r(0)^3 - 2r(0)r(1)^2 + 2r(1)^2 r(2) - r(0)r(2)^2}{r(0)^2 - r(1)^2} \end{aligned}$$

6.3 Write the iterations for the Levinson-Durbin algorithm.

Appying directly the Levinson Durbin algorithm we obtain

1. Initialize $\Delta_0 = r(1), \quad P_0 = r(0)$
2. $m = 1$
   - 2.1 $\Gamma_1 = -\frac{\Delta_0}{P_0} = -\frac{r(1)}{r(0)}$
   - 2.2 $a_{1,0} = 1$
   - 2.3 $a_{1,1} = a_{0,1} + \Gamma_1 a_{0,0} = -\frac{r(1)}{r(0)}$
   - 2.4 $\Delta_1 = r(2) + a_{1,1} r(1) = r(2) - \frac{r(1)^2}{r(0)} = \frac{r(2)r(0) - r(1)^2}{r(0)}$
   - 2.5 $P_1 = P_0(1 - \Gamma_1^2) = r(0)(1 - \frac{r(1)^2}{r(0)^2}) = \frac{r(0)^2 - r(1)^2}{r(0)}$
2. $m = 2$
   - 2.1 $\Gamma_2 = -\frac{\Delta_1}{P_1} = -\frac{r(2)r(0) - r(1)^2}{r(0)^2 - r(1)^2}$
   - 2.2 $a_{2,0} = 1$
   - 2.3 $a_{2,1} = a_{1,1} + \Gamma_2 a_{1,1} = -\frac{r(1)}{r(0)}(1 - \frac{r(2)r(0) - r(1)^2}{r(0)^2 - r(1)^2}) = \frac{-r(0)r(1) + r(1)r(2)}{r(0)^2 - r(1)^2}$
   - 2.3 $a_{2,2} = a_{1,2} + \Gamma_2 a_{1,0} = \Gamma_2 = -\frac{r(2)r(0) - r(1)^2}{r(0)^2 - r(1)^2}$
   - 2.4 Δ_2 Don't needed

$$2.5 \quad P_2 = P_1(1-\Gamma_2^2) = \frac{r(0)^2 - r(1)^2}{r(0)}(1 - (\frac{r(2)r(0) - r(1)^2}{r(0)^2 - r(1)^2})^2) =$$
$$= \frac{r(0)^3 - 2r(0)r(1)^2 + 2r(1)^2 r(2) - r(0)r(2)^2}{r(0)^2 - r(1)^2}$$

6.4 Starting from the AR parameters a_1, a_2 find the reflection coefficients Γ_1, Γ_2.
Hint: apply the inverse Levinson-Durbin algorithm.

Applying the inverse Levinson Durbin algorithm

$$a_{m-1,k} = \frac{a_{m,k} - a_{m,m}a_{m,m-k}}{1 - (a_{m,m})^2} \qquad k = 1, \ldots, m-1$$

we obtain

$$m = 2$$
$$\Gamma_2 = a_{2,2} = a_2$$
$$a_{1,1} = \frac{a_{2,1} - a_{2,2}a_{2,1}}{1 - (a_{2,2})^2}$$
$$\Gamma_1 = a_{1,1} = \frac{a_1 - a_2 a_1}{1 - (a_2)^2} = \frac{a_1}{1 + a_2}$$

6.5 Find the transfer function of the Lattice predictor of order 2, with parameters (reflection coefficients) Γ_1, Γ_2.

Using the recursions

$$\begin{bmatrix} f_m(n) \\ b_m(n) \end{bmatrix} = \begin{bmatrix} 1 & \Gamma_m q^{-1} \\ \Gamma_m & q^{-1} \end{bmatrix} \begin{bmatrix} f_{m-1}(n) \\ b_{m-1}(n) \end{bmatrix}$$

we can write

$$\begin{aligned}
\begin{bmatrix} f_2(n) \\ b_2(n) \end{bmatrix} &= \begin{bmatrix} 1 & \Gamma_2 q^{-1} \\ \Gamma_2 & q^{-1} \end{bmatrix} \begin{bmatrix} f_1(n) \\ b_1(n) \end{bmatrix} = \begin{bmatrix} 1 & \Gamma_2 q^{-1} \\ \Gamma_2 & q^{-1} \end{bmatrix} \begin{bmatrix} 1 & \Gamma_1 q^{-1} \\ \Gamma_1 & q^{-1} \end{bmatrix} \begin{bmatrix} f_0(n) \\ b_0(n) \end{bmatrix} \\
&= \begin{bmatrix} 1 & \Gamma_2 q^{-1} \\ \Gamma_2 & q^{-1} \end{bmatrix} \begin{bmatrix} 1 & \Gamma_1 q^{-1} \\ \Gamma_1 & q^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(n) = \begin{bmatrix} 1+\Gamma_1\Gamma_2 q^{-1} & (\Gamma_1 q^{-1} + \Gamma_2 q^{-2}) \\ \Gamma_2 + \Gamma_1 q^{-1} & \Gamma_1\Gamma_2 q^{-1} + q^{-2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(n) \\
&= \begin{bmatrix} 1 + (\Gamma_1 + \Gamma_2\Gamma_1) q^{-1} + \Gamma_2 q^{-2} \\ \Gamma_2 + (\Gamma_1 + \Gamma_2\Gamma_1) q^{-1} + q^{-2} \end{bmatrix} u(n) = \begin{bmatrix} A(q) \\ A^B(q) \end{bmatrix} u(n)
\end{aligned}$$

6.6 Find the transfer function of the Lattice Inverse filter of order 2, with parameters (reflection coefficients) Γ_1, Γ_2.

$$\begin{aligned}
u(n) &= -\Gamma_1 q^{-1} u(n) + v(n) - \Gamma_2 q^{-1} g(n) \\
g(n) &= \Gamma_1 u(n) + q^{-1} u(n) = (\Gamma_1 + q^{-1}) u(n)
\end{aligned}$$

$$\begin{aligned}
&u(n) = -\Gamma_1 q^{-1} u(n) + v(n) - \Gamma_2 q^{-1} (\Gamma_1 + q^{-1}) u(n) \\
&(1 + \Gamma_1 q^{-1} + \Gamma_1\Gamma_2 q^{-1} + \Gamma_2 q^{-2}) u(n) = v(n) \\
&(1 + (\Gamma_1 + \Gamma_1\Gamma_2) q^{-1} + \Gamma_2 q^{-2}) u(n) = v(n) \\
&u(n) = \frac{1}{1 + (\Gamma_1 + \Gamma_1\Gamma_2) q^{-1} + \Gamma_2 q^{-2}} v(n) = \frac{1}{A(q)} v(n)
\end{aligned}$$