TAMPERE UNIVERSITY OF TECHNOLOGY

Advanced Algorithms and Data Structures

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Course Prerequisites

- A seven credit unit course
- We take things a bit further than basic algorithms / data structures courses that you might have attended
- We will assume familiarity with
  - Necessary mathematics
  - Elementary data structures
  - Programming

Course Basics

- There will be 4 hours of lectures per week
- Weekly exercises start in a weeks time

- We will not have a programming exercise this year (unless you demand to have one)
- We might consider organizing a seminar with voluntary presentations (yielding extra points) at the end of the course

Organization & Timetable

- **Lectures**: Prof. Tapio Elomaa
  - Mon & Wed 12–14 PM in TB216
  - Aug. 29 – Dec. 7, 2016
    - Period break Oct. 17–23, 2016

- **Exercises**: M.Sc. Juho Lauri
  Thu 12–14 TB224, Start: Sept. 8
- **Exam**: Fri Dec. 16, 2016 @ 13—16

Course Grading

- **Exam**: Maximum of 30 points
- **Weekly exercises** yield extra points
  - 40% of questions answered: 1 point
  - 80% answered: 6 points
  - In between: linear scale (so that decimals are possible)
- Final grading depends on what we agree as course components

Material

- The textbook of the course is
  - Cormen, Leiserson, Rivest, Stein: *Introduction to Algorithms*, 3rd ed., MIT Press, 2009
- There is no prepared material, the slides appear in the web as the lectures proceed
  - http://www.cs.tut.fi/~elomaa/teach/72006.html
- The exam is based on the lectures (i.e., not on the slides only)

Content (Plan)

I Foundations

The Role of Algorithms in Computing
Getting Started
Growth of Functions
Recurrences
Probabilistic Analysis and Randomized Algorithms

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II (Sorting and) Order Statistics

~~Heapsort~~
~~Quicksort~~
~~Sorting in Linear Time~~
Medians and Order Statistics

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III Data Structures

~~Elementary Data Structures~~
Hash Tables
~~Binary Search Trees~~
~~Red-Black Trees~~

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IV Advanced Design and Analysis Techniques

Dynamic Programming

Greedy Algorithms

Amortized Analysis

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V Advanced Data Structures

B-Trees

~~Binomial Heaps~~

Fibonacci Heaps

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VI Graph Algorithms

Elementary Graph Algorithms
Minimum Spanning Trees
Single-Source Shortest Paths
All-Pairs Shortest Paths
Maximum Flow

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VII Selected Topics

Matrix Operations
Linear Programming
Number-Theoretic Algorithms
Approximation Algorithms

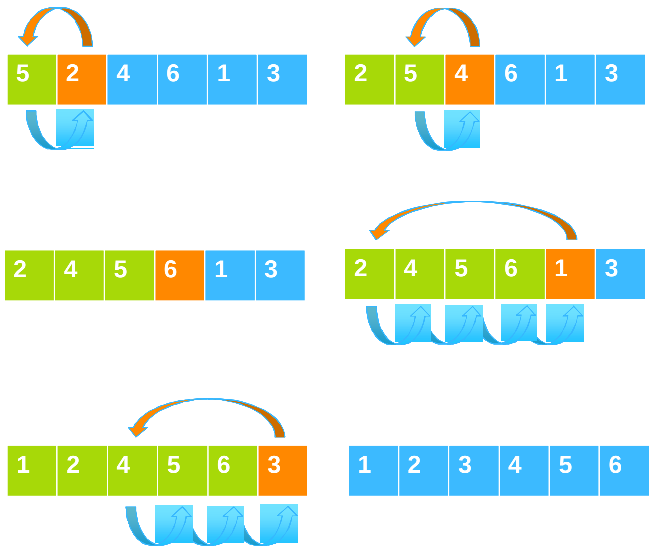
Part of these could also be
student presentation topics

The sorting problem

- **Input**: A sequence of n numbers
  $$\langle a_1, a_2, \ldots, a_n \rangle$$
- **Output**: A ***permutation*** (reordering)
  $$\langle a'_1, a'_2, \ldots, a'_n \rangle$$
  of the input sequence such that
  $$a'_1 \leq a'_2 \leq \cdots \leq a'_n$$
- The numbers that we wish to sort are also known as ***keys***

INSERTION-SORT(A)

1. **for** $j \leftarrow 2$ **to** $A.length$
2. $\quad key \leftarrow A[j]$
3. // Insert $A[j]$ into the sorted sequence $A[1..j-1]$
4. $\quad i \leftarrow j - 1$
5. $\quad$ **while** $i > 0$ **and** $A[i] > key$
6. $\quad\quad A[i + 1] \leftarrow A[i]$
7. $\quad\quad i \leftarrow i - 1$
8. $\quad A[i + 1] \leftarrow key$



Correctness of the Algorithm

- The following ***loop invariant*** helps us understand why the algorithm is correct:

  At the start of each iteration of the **for** loop of lines 1–8, the subarray $A[1\,..\,j-1]$ consists of the elements originally in $A[1\,..\,j-1]$ but in sorted order

Initialization

- The loop invariant holds before the first loop iteration, when $j = 2$:
  - The subarray, therefore, consists of just the single element $A[1]$
  - It is the original element in $A[1]$
  - This subarray is trivially sorted
  - Therefore, the loop invariant holds prior to the first iteration of the loop

Maintenance

- Each iteration maintains the loop invariant:
  - The body of the **for** loop works by moving $A[j-1]$, $A[j-2]$, $A[j-3]$, … by one position to the right until the proper position for $A[j]$ is found (lines 4–7)
  - At this point the value of $A[j]$ is inserted (line 8)
  - The subarray $A[1..j]$ then consists of the elements originally in $A[1..j]$, but in sorted order

Termination

- The condition causing the for loop to terminate is that $j > A.length = n$
- Because each loop iteration increases j by 1, we must have $j = n + 1$ at that time
- Substituting $n + 1$ for j in the wording of loop invariant, we have that the subarray $A[1..n]$ consists of the elements originally in $A[1..n]$, but in sorted order
- $A[1..n]$ is the entire array

Analysis of insertion sort

- The time taken by the INSERTION-SORT depends on the input:
  - sorting a thousand numbers takes longer than sorting three numbers
- Moreover, the procedure can take different amounts of time to sort two input sequences of the same size
  - depending on how nearly sorted they already are

Input size

- The time taken by an algorithm grows with the size of the input
- Traditional to describe the ***running time*** of a program as a function of the ***size*** of its ***input***
- For many problems, such as sorting most natural measure for input size is the number of items in the input—i.e., the array size n

- For, e.g., multiplying two integers, the best measure is the total number of bits needed to represent the input in binary notation
- Sometimes, more appropriate to describe the size with two numbers rather than one
- E.g., if the input to an algorithm is a graph, the input size can be described by the numbers of vertices and edges in it

Running time

- Running time of an algorithm on an input:
  - The number of primitive operations ("steps") executed
- Step as machine-independent as possible
- For the moment:
  - Constant amount of time to execute each line of pseudocode
  - We assume that each execution of the ith line takes time c_i, where c_i is a constant

	INSERTION-SORT(A)	cost	times
1	**for** $j \leftarrow 2$ **to** $A.length$	c_1	n
2	$key \leftarrow A[j]$	c_2	$n-1$
3	// Insert $A[j]$ into the sorted sequence $A[1..j]$	0	$n-1$
4	$i \leftarrow j-1$	c_4	$n-1$
5	**while** $i > 0$ **and** $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
6	$A[i+1] \leftarrow A[i]$	c_6	$\sum_{j=2}^{n} (t_j - 1)$
7	$i \leftarrow i-1$	c_7	$\sum_{j=2}^{n} (t_j - 1)$
8	$A[i+1] \leftarrow key$	c_8	$n-1$

- t_j denotes the number of times the **while** loop test in line 5 is executed for that value of j
- When a **for** or **while** loop exits in the usual way, the test is executed one time more than the loop body
- Comments are not executable statements, so they take no time
- Running time of the algorithm is the sum of those for each statement executed

- To compute $T(n)$, the running time of INSERTION-SORT on an input of n values,
  - we sum the products of the cost and times columns, obtaining

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j$$

$$+ c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1)$$

Best case

- The best case occurs if the array is already sorted
- For each $j = 2, 3, \dots, n$, we then find that $A[i] < key$ in line 5 when i has its initial value of $j - 1$
- Thus $t_j = 1$ for $j = 2, 3, \dots, n$, and the best-case running time is

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5(n-1) + c_8(n-1)$$
$$= (c_1 + c_2 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8)$$

Worst case

- We can express this as $an + b$ for constants a and b that depend on the statement costs c_i
- It is a ***linear function*** of n
- The worst case results when the array is in reverse sorted order — in decreasing order
- We must compare each element $A[j]$ with each element in the entire sorted subarray $A[1..j-1]$, and so $t_j = j$ for $j = 2, 3, \dots, n$

- Note that

$$\sum_{j=2}^{n} j = \frac{n(n+1)}{2} - 1$$

and

$$\sum_{j=2}^{n} (j-1) = \frac{n(n-1)}{2}$$

by the summation of an arithmetic series

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

- The worst-case running time of INSERTION-SORT is

$$\begin{aligned} T(n) &= c_1 n + c_2(n-1) + c_4(n-1) \\ &+ c_5\left(\frac{n(n+1)}{2} - 1\right) + c_6\left(\frac{n(n-1)}{2}\right) \\ &+ c_7\left(\frac{n(n-1)}{2}\right) + c_8(n-1) \\ &= \left(\frac{c_5}{2} + \frac{c_6}{2} + \frac{c_7}{2}\right) n^2 + (c_1 + \cdots + c_8)n \\ &- (c_2 + \cdots + c_8) \end{aligned}$$

- We can express this worst-case running time as $an^2 + bn + c$ for constants a, b, and c that depend on the statement costs c_i
- It is a ***quadratic function*** of n
- The ***rate of growth***, or ***order of growth***, of the running time really interests us
- We consider only the leading term of a formula (an^2); the lower-order terms are relatively insignificant for large values of n

- We also ignore the leading term's coefficient, constant factors are less significant than the rate of growth in determining computational efficiency for large inputs
- For insertion sort, we are left with the factor of n^2 from the leading term
- We write that insertion sort has a worst-case running time of $\Theta(n^2)$

  ("theta of n-squared")

2.3 Designing algorithms

- Insertion sort is an incremental approach: having sorted $A[1..j-1]$, we insert $A[j]$ into its proper place, yielding sorted subarray $A[1..j]$
- Let us examine an alternative design approach, known as "divide-and-conquer"
- We design a sorting algorithm whose worst-case running time is much lower
- The running times of divide-and-conquer algorithms are often easily determined

The divide-and-conquer approach

- Many useful algorithms are ***recursive***:
  - to solve a problem, they call themselves to deal with closely related subproblems
- These algorithms typically follow a divide-and-conquer approach:
  - Break the problem into subproblems that resemble the original problem but are smaller,
  - Solve the subproblems recursively,
  - Combine these solutions to create a solution to the original problem

- The paradigm involves three steps at each level of the recursion:
  1. **Divide** the problem into a number of subproblems that are smaller instances of the same problem
  2. **Conquer** the subproblems by solving them recursively
     - If the sizes are small enough, just solve the subproblems in a straightforward manner
  3. **Combine** the solutions to the subproblems into the solution for the original problem

The merge sort algorithm

- **Divide:** Divide the n-element sequence into two subsequences of $n/2$ elements each
- **Conquer:** Sort the two subsequences recursively using merge sort
- **Combine:** Merge the two sorted subsequences to produce the sorted answer
  – Recursion "bottoms out" when the sequence to be sorted has length 1: a sequence of length 1 is already in sorted order

- The key operation is the merging of two sorted sequences in the "combine" step
- We call auxiliary procedure MERGE(A, p, q, r), where A is an array and p, q, and r are indices such that $p \leq q < r$
- The procedure assumes that the subarrays $A[p..q]$ and $A[q+1..r]$ are in sorted order and
- merges them to form a single sorted subarray that replaces the current subarray $A[p..r]$

MERGE(A, p, q, r)

$n_1 \leftarrow q - p + 1$

$n_2 \leftarrow r - q$

Let $L[1..n_1+1]$ and $R[1..n_2+1]$ be new arrays

for $i \leftarrow 1$ **to** n_1

    $L[i] \leftarrow A[p+i-1]$

for $j \leftarrow 1$ **to** n_2

    $R[j] \leftarrow A[q+j]$

$L[n_1+1] \leftarrow \infty$

$R[n_2+1] \leftarrow \infty$

$i \leftarrow 1$

$j \leftarrow 1$

for $k \leftarrow p$ **to** r

    if $L[i] \leq R[j]$

        $A[k] \leftarrow L[i]$

        $i \leftarrow i + 1$

    else $A[k] \leftarrow R[j]$

        $j \leftarrow j + 1$

- Line 1 computes the length n_1 of the subarray $A[p..q]$; similarly for n_2 and $A[q+1..r]$ on line 2
- Line 3 creates arrays L (left) and R (right), of lengths $n_1 + 1$ and $n_2 + 1$, respectively
  - the extra position will hold the sentinel ∞
- The **for** loop of lines 4–5 copies $A[p..q]$ into $L[1..n_1]$;
- Lines 6–7 copy $A[q+1..r]$ into $R[1..n_2]$
- Lines 8–9 put the sentinels at the ends of L and R

- Lines 10–17 perform the $r - p + 1$ basic steps by maintaining the following loop invariant:
  - At the start of each iteration of the **for** loop of lines 12–17, $A[p..k-1]$ contains the $k-p$ smallest elements of $L[1..n_1+1]$ and $R[1..n_2+1]$, in sorted order
  - Moreover, $L[i]$ and $R[j]$ are the smallest elements of their arrays that have not been copied back into A

MERGE$(A, 9, 12, 16)$

	8	9	10	11	12	13	14	15	16	17
A	...	2	4	5	7	1	2	3	6	...
		k								

	1	2	3	4	5
L	2	4	5	7	∞
	i				

	1	2	3	4	5
R	1	2	3	6	∞
	j				

	8	9	10	11	12	13	14	15	16	17
A	...	1	4	5	7	1	2	3	6	...
			k							

	1	2	3	4	5
L	2	4	5	7	∞
	i				

	1	2	3	4	5
R	1	2	3	6	∞
		j			

	8	9	10	11	12	13	14	15	16	17
A	...	1	2	5	7	1	2	3	6	...
				k						

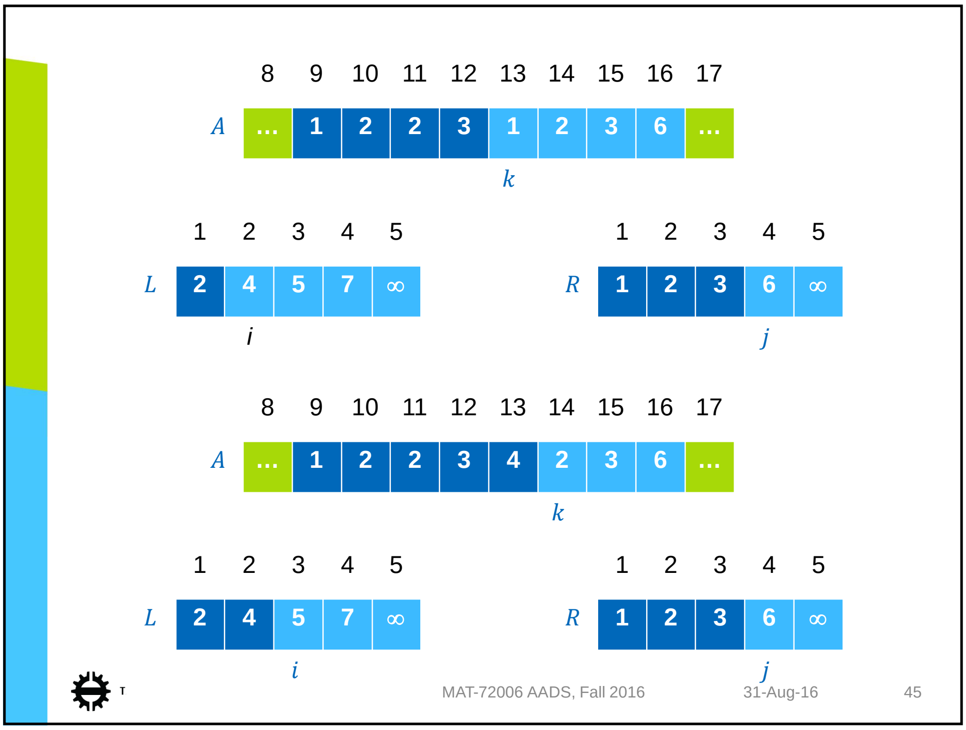
	1	2	3	4	5
L	2	4	5	7	∞
		i			

	1	2	3	4	5
R	1	2	3	6	∞
		j			

	8	9	10	11	12	13	14	15	16	17
A	...	1	2	2	7	1	2	3	6	...
					k					

	1	2	3	4	5
L	2	4	5	7	∞
		i			

	1	2	3	4	5
R	1	2	3	6	∞
			j		


8 9 10 11 12 13 14 15 16 17
A … 1 2 2 3 1 2 3 6 …
k
1 2 3 4 5
L 2 4 5 7 ∞
i
1 2 3 4 5
R 1 2 3 6 ∞
j
8 9 10 11 12 13 14 15 16 17
A … 1 2 2 3 4 2 3 6 …
k
1 2 3 4 5
L 2 4 5 7 ∞
i
1 2 3 4 5
R 1 2 3 6 ∞
j
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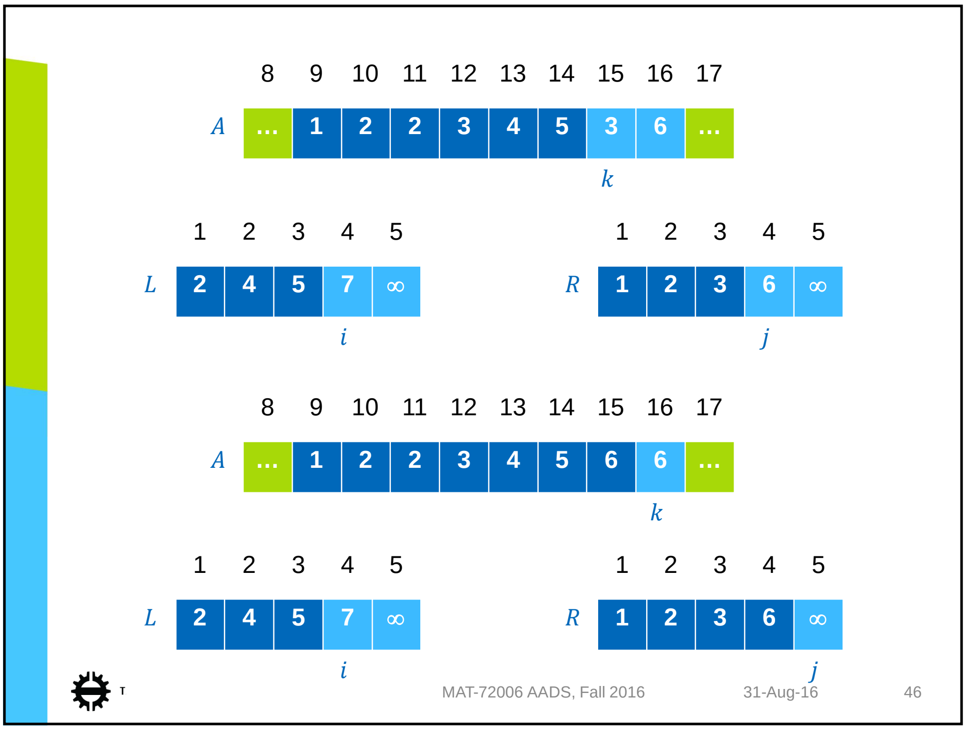
8 9 10 11 12 13 14 15 16 17
A … 1 2 2 3 4 5 3 6 …
k
1 2 3 4 5
L 2 4 5 7 ∞
i
1 2 3 4 5
R 1 2 3 6 ∞
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R 1 2 3 6 ∞
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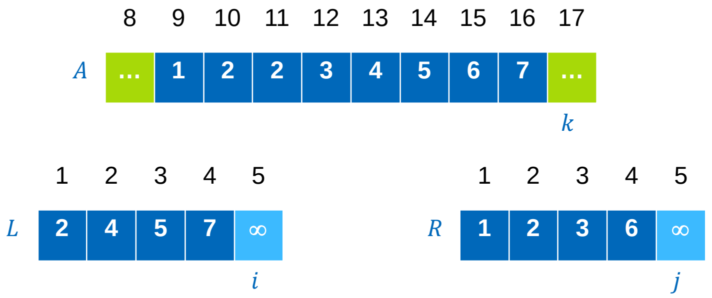


- The needed $r - p + 1$ iterations of the last **for** loop have been executed:
  - $A[9..16]$ is sorted, and
  - the two sentinels in L and R are the only two elements in these arrays that have not been copied into A

- MERGE procedure runs in $\Theta(n)$ time, where $n = r - p + 1$
  - each of lines 1–3 and 8–11 takes constant time
  - the **for** loops of lines 4–7 take $\Theta(n_1 + n_2) = \Theta(n)$ time
  - there are n iterations of the **for** loop of lines 12–17, each of which takes constant time

Merge sort

- The procedure MERGE-SORT(A, p, r) sorts the elements in $A[p..r]$
- If $p \geq r$, the subarray has at most one element and is therefore already sorted
- Otherwise, the divide step computes an index q that partitions $A[p..r]$ into two subarrays:
  - $A[p..q]$, containing $\lceil n/2 \rceil$ elements
  - $A[q+1..r]$, containing $\lfloor n/2 \rfloor$ elements

MERGE-SORT(A, p, r)

1. **if** $p < r$
2. $\quad q \leftarrow \lfloor (p+r)/2 \rfloor$
3. $\quad$ MERGE-SORT(A, p, q)
4. $\quad$ MERGE-SORT$(A, q+1, r)$
5. $\quad$ MERGE(A, p, q, r)

Analysis of merge sort

- Our analysis assumes that the original problem size is a power of 2
- Each divide step then yields two subsequences of size exactly $n/2$
- We set up the recurrence for $T(n)$, the worst-case running time of merge sort on n numbers
- Merge sort on just one element takes constant time

- When we have $n > 1$ elements, we break down the running time as follows:
  - **Divide**: The step just computes the middle of the subarray, which takes constant time: $D(n) = \Theta(1)$
  - **Conquer**: We recursively solve two subproblems, each of size $n/2$, which contributes $2T(n/2)$ to the running time
  - **Combine**: the MERGE procedure on an n-element array takes time $\Theta(n)$, and so $C(n) = \Theta(n)$

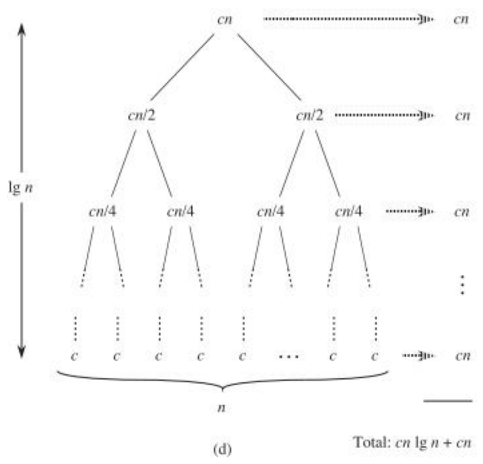
- When we add the $D(n)$ and $C(n)$, we are adding functions that are $\Theta(n)$ and $\Theta(1)$
- This sum is a linear function of n
- Adding it to the $2T(n/2)$ term from the "conquer" step gives the recurrence for $T(n)$:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 2T(n/2) + \Theta(n) & \text{if } n > 1 \end{cases}$$

- To intuitively see that the solution to the recurrence is $T(n) = \Theta(n \lg n)$, where $\lg n$ stands for $\log_2 n$, let us rewrite it as

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ 2T(n/2) + cn & \text{if } n > 1 \end{cases}$$

where constant c represents time required to solve problems of size 1 and that per array element of the divide and combine steps



- Inductive argument shows that total number of levels of the recursion tree is $\lg n + 1$
  - base case $n = 1$: tree has only one level; $\lg 1 = 0 \Rightarrow \lg n + 1$ is the correct number of levels
  - Inductive hypothesis: number of levels of a tree with 2^i leaves is $\lg 2^i + 1 = i + 1$
  - We assume that the input size is a power of 2, the next input size to consider is 2^{i+1}
  - A tree with $n = 2^{i+1}$ leaves has one more level than a tree with 2^i leaves, and so the total number of levels is $(i + 1) + 1 = \lg 2^{i+1} + 1$

- To compute the total cost represented by the recurrence, we simply add up the costs of all the levels:
  - The recursion tree has $\lg n + 1$ levels, each costing cn, for a total cost of

$$cn(\lg n + 1) = cn \lg n + cn$$

  - Ignoring the low-order term and the constant c gives the desired result of $\Theta(n \lg n)$