TAMPERE UNIVERSITY OF TECHNOLOGY

# VI Graph Algorithms

Elementary Graph Algorithms
Minimum Spanning Trees
Single-Source Shortest Paths
All-Pairs Shortest Paths

## 22 Elementary Graph Algorithms

- There are two standard ways to represent a graph $G = (V, E)$:
  - as a collection of adjacency lists or
  - as an adjacency matrix
- Either way applies to both directed and undirected graphs
- The adjacency-list representation provides a compact way to represent sparse graphs — those for which $|E| \ll |V|^2$

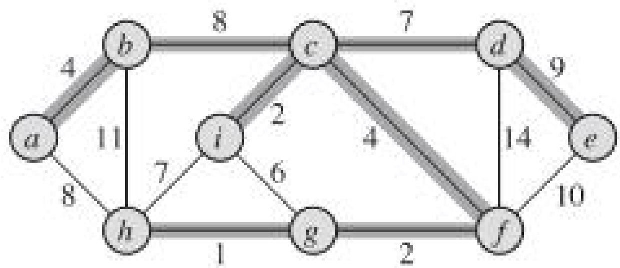
# 23 Minimum Spanning Trees

- Electronic circuit designs often need to make the pins of several components electrically equivalent by wiring them together
- To interconnect a set of $n$ pins, we can use an arrangement of $n - 1$ wires, each connecting two pins
- Of all such arrangements, the one that uses the least amount of wire is usually the most desirable



- Model this wiring problem with a connected, undirected graph $G = (V, E)$, where
  - $V$ is the set of pins,
  - $E$ is the set of possible interconnections between pairs of pins, and
  - for each edge $(u, v) \in E$, we have a weight $w(u, v)$ specifying the cost (amount of wire) to connect $u$ and $v$
- Find an acyclic subset $T \subseteq E$ that connects all of the vertices and whose total weight is minimized

$$w(T) = \sum_{(u,v)\in T} w(u, v)$$




- Since $T$ is acyclic and connects all of the vertices, it must form a tree, which we call a spanning tree since it "spans" the graph $G$
- We call the problem of determining the tree $T$ the ***minimum-spanning-tree problem*** (MST)



- We examine two algorithms for solving the MST problem: Kruskal's and Prim's algorithms
- We can easily make each of them run in time $O(E \lg V)$ using ordinary binary heaps
- By using Fibonacci heaps, Prim's algorithm runs in time $O(E + V \lg V)$, which improves over the binary-heap implementation if $|V| \ll |E|$
- The two algorithms are greedy algorithms
- Greedy strategy does not generally guarantee finding globally optimal solutions to problems
- For the MST problem we can prove that greedy strategies do yield a tree with minimum weight

## 23.1 Growing a minimum spanning tree

- Assume that we have a connected, undirected graph $G = (V, E)$ with a weight function $w\colon E \to \mathbb{R}$, and we wish to find a MST for $G$
- The following generic method grows the MST one edge at a time
- The generic method manages a set of edges $A$, maintaining the following loop invariant:

Prior to each iteration, $A$ is a subset of some minimum spanning tree



- At each step, we determine an edge $(u, v)$ that we can add to $A$ without violating this invariant, in the sense that $A \cup \{(u, v)\}$ is also a subset of a MST
- We call such an edge a ***safe edge*** for $A$

GENERIC-MST$(G, w)$

1. $A \leftarrow \emptyset$;
2. **while** $A$ does not form a spanning tree
3. find an edge $(u, v)$ that is safe for $A$
4. $A \leftarrow A \cup \{(u, v)\}$
5. **return** $A$

**Initialization:** After line 1, the set $A$ trivially satisfies the loop invariant

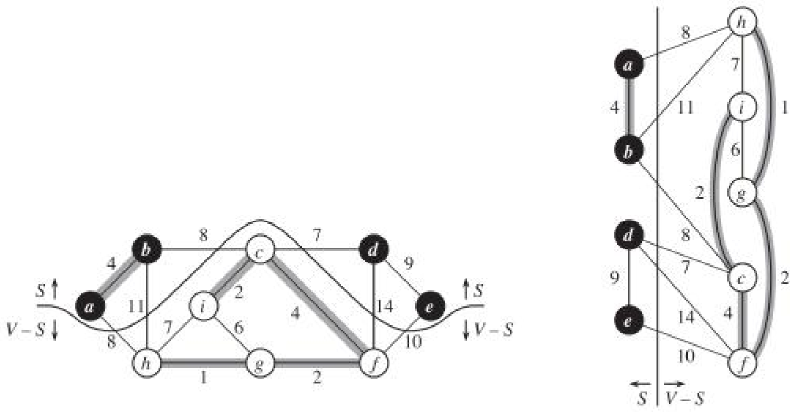
**Maintenance:** The loop in lines 2–4 maintains the invariant by adding only safe edges

**Termination:** All edges added to $A$ are in a MST, and so the set $A$ returned in line 5 must be a MST

- The tricky part is finding a safe edge in line 3
- One must exist, since the invariant dictates that there is a spanning tree $T$ such that $A \subseteq T$
- Within the while loop body, $A$ must be a proper subset of $T$, and there must be an edge $(u, v) \in T$ s.t. $(u, v) \notin A$ and $(u, v)$ is safe for $A$



- A ***cut*** $(S, V - S)$ of an undirected graph $G = (V, E)$ is a partition of $V$
- An edge $(u, v) \in E$ ***crosses*** the cut if one of its endpoints is in $S$ and the other is in $V - S$
- We say that a cut ***respects*** a set $A$ of edges if no edge in $A$ crosses the cut
- A ***light edge*** crossing a cut has the minimum weight of any edge crossing the cut
- Note that there can be ties
- More generally, an edge is a light edge satisfying a given property if its weight is the minimum of any edge satisfying the property




(a) (b)



**Theorem 23.1:** *Let*

- $G = (V, E)$ *be a connected, undirected graph*
- *with a real-valued weight function* $w$ *on* $E$.
- *Let* $A$ *be a subset of* $E$ *that is included in some MST for* $G$,
- *let* $(S, V - S)$ *be any cut of* $G$ *that respects* $A$, *and*
- *let* $(u, v)$ *be a light edge crossing* $(S, V - S)$.

*Then, edge* $(u, v)$ *is safe for* $A$.

- Theorem 23.1 gives us a better understanding of the workings of the GENERIC-MST method on a connected graph $G = (V, E)$
- As the method proceeds, the set $A$ is always acyclic; otherwise, a MST including $A$ would contain a cycle, which is a contradiction
- At any point in the execution, the graph $G_A = (V, A)$ is a forest, and each of the connected components of $G_A$ is a tree
- Some of the trees may contain just one vertex, as is the case, e.g., when the method begins: $A$ is empty and the forest contains $|V|$ trees, one for each vertex



- Moreover, any safe edge $(u, v)$ for $A$ connects distinct components of $G_A$, since $A \cup \{(u, v)\}$ must be acyclic
- The while loop in lines 2–4 of GENERIC-MST executes $|V| - 1$ times because it finds one of the $|V| - 1$ edges of a minimum spanning tree in each iteration
- Initially, when $A = \emptyset$, there are $|V|$ trees in $G_A$, and each iteration reduces that number by 1
- When the forest contains only a single tree, the method terminates

**Corollary 23.2:** *Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function $w$ defined on $E$.*

*Let $A$ be a subset of $E$ that is included in some MST for $G$, and let $C = (V_C, E_c)$* be a connected component (tree) in the forest $G_A = (V, A)$. If $(u, v)$ is a light edge connecting $C$ to some other component in $G_A$, then $(u, v)$ is safe for $A$.

***Proof:*** The cut $(V_C, V - V_C)$ respects $A$, and $(u, v)$ is a light edge for this cut. Therefore, $(u, v)$ is safe for $A$. ■



## 23.2 The algorithms of Kruskal and Prim

- These algorithms use a specific rule to determine a safe edge in line 3 of GENERIC-MST
- In Kruskal's algorithm, the set $A$ is a forest whose vertices are all those of the given graph
- The safe edge added to $A$ is always a least-weight edge in the graph that connects two distinct components
- In Prim's algorithm, the set $A$ forms a single tree
- The safe edge added to $A$ is always a least-weight edge connecting the tree to a vertex not in the tree

## Kruskal's algorithm

- Find a safe edge to add to the growing forest by finding, of all the edges that connect any two trees in the forest, an edge $(u, v)$ of least weight
- Let $C_1$ and $C_2$ denote the two trees that are connected by $(u, v)$
- Since $(u, v)$ must be a light edge connecting $C_1$ to some other tree, Corollary 23.2 implies that $(u, v)$ is a safe edge for $C_1$
- This is a greedy algorithm because at each step it adds an edge of least possible weight



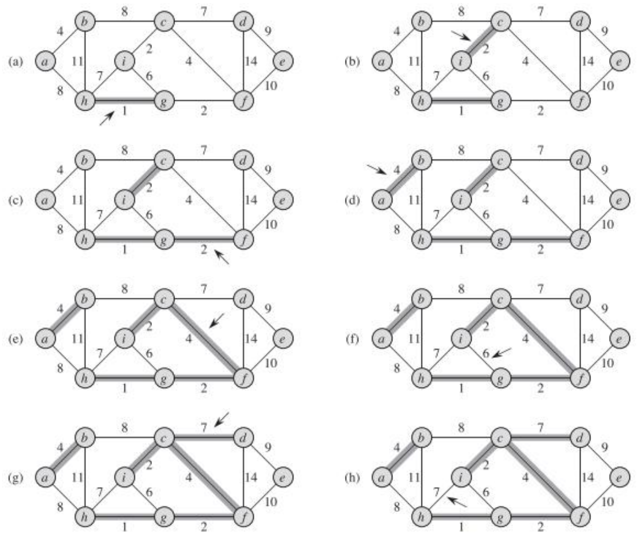
MST-KRUSKAL$(G, w)$

1. $A \leftarrow \emptyset$
2. **for** each vertex $v \in G.V$
3. MAKE-SET$(v)$
4. sort the edges of $G.E$ into nondecreasing order by weight $w$
5. **for** each edge $(u, v) \in G.E$, taken in nondecreasing order by weight
6. **if** FIND-SET$(u) \neq$ FIND-SET$(v)$
7. $A \leftarrow A \cup \{(u, v)\}$
8. UNION$(u, v)$
9. **return** $A$

- FIND-SET$(u)$ returns a representative element from the set that contains $u$
- Determine whether $u$ and $v$ belong to the same tree by testing FIND-SET$(u)$ = FIND-SET$(v)$
- UNION procedure combines trees
- Lines 1–3 initialize the set $A$ to the empty set and create $|V|$ trees, one containing each vertex
- The for loop in lines 5–8 examines edges in order of weight, from lowest to highest



- The loop checks, for each edge $(u, v)$, whether the endpoints $u$ and $v$ belong to the same tree
- If they do, then the edge $(u, v)$ cannot be added to the forest without creating a cycle, and the edge is discarded
- Otherwise, the two vertices belong to different trees
- In this case, line 7 adds the edge $(u, v)$ to $A$, and line 8 merges the vertices in the two trees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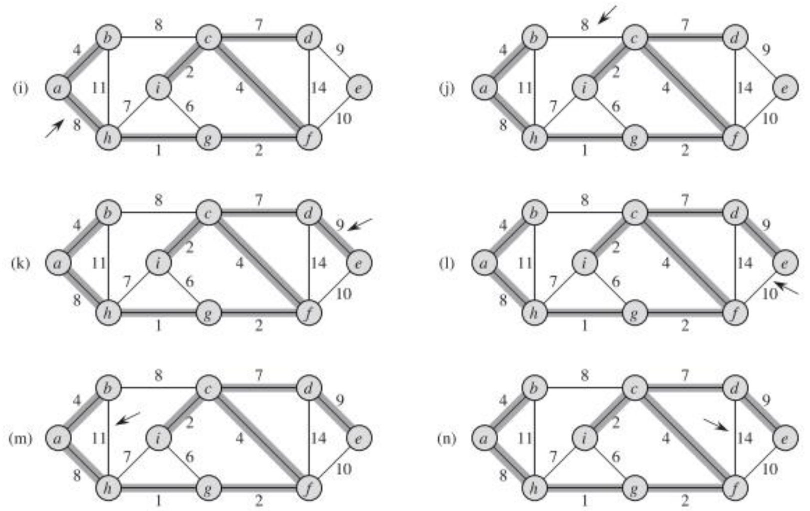
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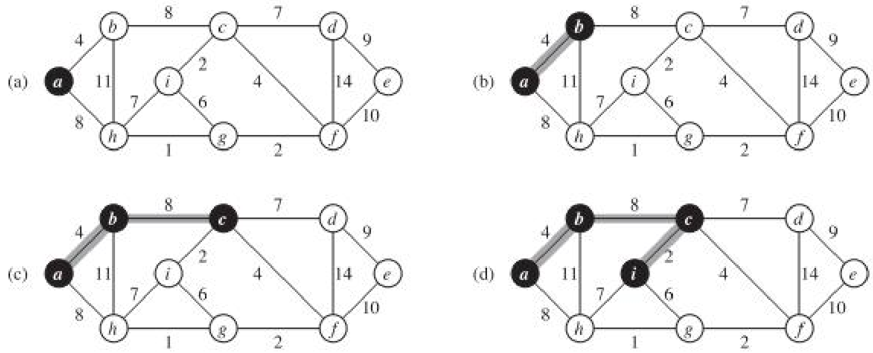
- The running time depends on how we implement the disjoint-set data structure
- Initializing the set $A$ (line 1) takes $O(1)$ time, and the time to sort the edges (line 4) is $O(E \lg E)$
- The **for** loop (lines 5–8) performs $O(E)$ FIND-SET and UNION operations on the disjoint-set forest
- With the $|V|$ MAKE-SET operations, these take a total of $O((V + E)\alpha(V))$ time $\alpha$ is the very slowly growing funct
- We assume that $G$ is connected, so have $|E| \geq |V| - 1$, and so the disjoint-set operations take $O(E\alpha(V))$ time
- Moreover, since $\alpha(|V|) = O(\lg V) = O(\lg E)$, the total running time of Kruskal's algorithm is $O(E \lg E)$
- Observing that $|E| < |V|^2$, we have $\lg|E| = O(\lg V)$, and the running time of Kruskal's algorithm is $O(E \lg V)$



## Prim's algorithm

- Prim's algorithm has the property that the edges in the set $A$ always form a single tree
- We start from an arbitrary root vertex $r$ and grow until the tree spans all the vertices in $V$
- Each step adds to $A$ a light edge that connects $A$ to an isolated vertex (no edge of $A$ is incident)
- By Corollary 23.2, this rule adds only edges that are safe for $A$ and eventually $A$ forms a MST
- Greedy: each step adds to the tree an edge that contributes the min amount to the tree's weight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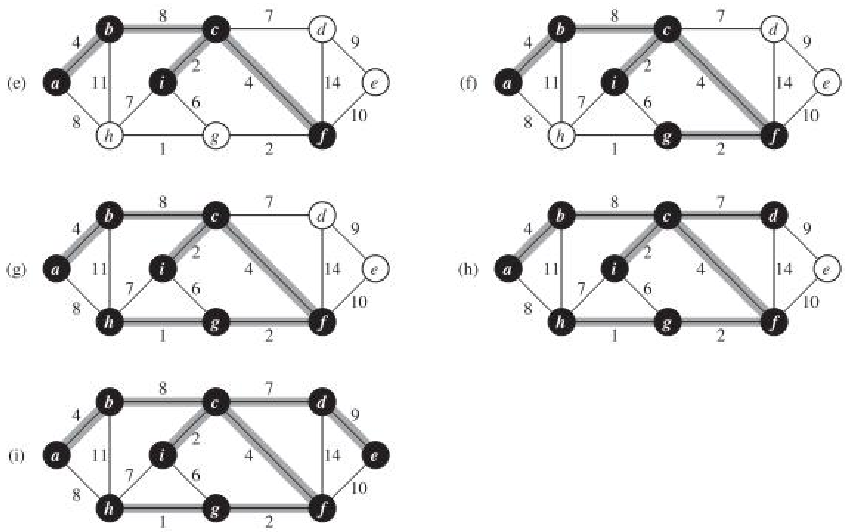
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- In order to implement Prim's algorithm efficiently, we need a fast way to select a new edge to add to the tree formed by the edges in $A$
- In the pseudocode below, the connected graph $G$ and the root $r$ of the MST to be grown are inputs to the algorithm
- During execution of the algorithm, all vertices that are not in the tree reside in a min-priority queue $Q$ based on a *key* attribute
- For each vertex $v$, the attribute $v.key$ is the minimum weight of any edge connecting $v$ to a vertex in the tree; by convention $v.key = \infty$ if there is no such edge



- Attribute $v.\pi$ names the parent of $v$ in the tree
- The algorithm implicitly maintains the set $A$ from GENERIC-MST as

$$A = \{(v, v.\pi) | v \in V - \{r\} - Q\}$$

- When the algorithm terminates, the min-priority queue $Q$ is empty; the minimum spanning tree $A$ for $G$ is thus

$$A = \{(v, v.\pi) | v \in V - \{r\}\}$$

MST-PRIM$(G, w, r)$

1. **for** each $u \in G.V$
2. $\quad u.key \leftarrow \infty$
3. $\quad u.\pi \leftarrow$ NIL
4. $r.key \leftarrow 0$
5. $Q \leftarrow G.V$
6. **while** $Q \neq \emptyset$
7. $\quad u \leftarrow$ EXTRACT-MIN$(Q)$
8. $\quad$ **for** each $v \in G.Adj[u]$
9. $\quad\quad$ **if** $v \in Q$ **and** $w(u, v) < v.key$
10. $\quad\quad\quad v.\pi \leftarrow u$
11. $\quad\quad\quad v.key \leftarrow w(u, v)$



- The algorithm maintains the following three-part loop invariant:
- Prior to each iteration of the **while** loop of lines 6–11,
  1. $A = \{(v, v.\pi) | v \in V - \{r\} - Q\}$
  2. The vertices already placed into the minimum spanning tree are those in $V - Q$
  3. For all vertices $v \in Q$, if $v.\pi \neq$ NIL, then $v.key < \infty$ and $v.key$ is the weight of a light edge $(v, v.\pi)$ connecting $v$ to some vertex already placed into the MST

- Line 7 identifies a vertex $u \in Q$ incident on a light edge that crosses the cut $(V - Q, Q)$
- Removing $u$ from the set $Q$ adds it to the set $V - Q$ of vertices in the tree, thus adding $(u, u.\pi)$ to $A$
- The **for** loop of lines 8–11 updates the $key$ and $\pi$ attributes of every vertex $v$ adjacent to $u$ but not in the tree, thereby maintaining the third part of the loop invariant



- The total time for Prim's algorithm is $O(V \lg V + E \lg V) = O(E \lg V)$, which is asympt. the same as for Kruskal's algorithm
- We can improve the asymptotic running time of Prim's algorithm by using Fibonacci heaps
- If a Fibonacci heap holds $|V|$ elements, an EXTRACT-MIN operation takes $O(\lg V)$ amortized time and a DECREASE-KEY operation (to implement line 11) takes $O(1)$ amortized time
- Therefore, by using a Fibonacci heap for the min-priority queue $Q$, the running time of Prim's algorithm improves to $O(E + V \lg V)$

## 24 Single-Source Shortest Paths

- In a ***shortest-paths problem***, we are given a weighted, directed graph $G = (V, E)$, with weight function $w: E \rightarrow \mathbb{R}$ mapping edges to real-valued weights
- The weight $w(p)$ of path $p = \langle v_0, v_1, \dots, v_k \rangle$ is the sum of the weights of its constituent edges:

$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$$



- We define the shortest-path weight $\delta(u, v)$ from $u$ to $v$ by $\delta(u, v) =$

$$\begin{cases} \min\left\{w(p) : u \overset{p}{\rightsquigarrow} v\right\} & \text{if there is a path from } u \text{ to } v \\ \infty & \text{otherwise} \end{cases}$$

- A shortest path from vertex $u$ to vertex $v$ is then defined as any path $p$ with weight $w(p) = \delta(u, v)$

## Optimal substructure of a shortest path

- Shortest-paths algorithms typically rely on the property that a shortest path between two vertices contains other shortest paths within it
- Recall that optimal substructure is one of the key indicators that dynamic programming and the greedy method might apply
- The following lemma states the optimal-substructure property of shortest paths more precisely



**Lemma 24.1** (Subpaths of shortest paths are shortest paths)
*Given a weighted, directed graph $G = (V, E)$, with weight function $w: E \to \mathbb{R}$, let $p = \langle v_0, v_1, \dots, v_k \rangle$ be a shortest path from vertex $v_0$ to vertex $v_k$ and, for any $i$ and $j$ such that $0 \le i \le j \le k$, let $p_{ij} = \langle v_i, v_{i+1}, \dots, v_j \rangle$ be the subpath from vertex $v_i$ to $v_j$. Then, $p_{ij}$ is a shortest path from $v_i$ to $v_j$.*

***Proof:*** If we decompose path $p$ into $v_0 \overset{p_{0i}}{\leadsto} v_i \overset{p_{ij}}{\leadsto} v_j \overset{p_{jk}}{\leadsto} v_k$, then we have that $w(p) = w(p_{0i}) + w(p_{ij}) + w(p_{jk})$. Now, assume that there is a path $p'_{ij}$ from $v_i$ *to* $v_j$ with weight $w(p'_{ij}) < w(p_{ij})$.

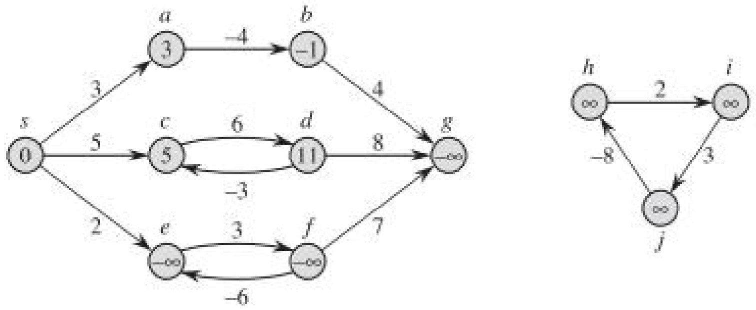
Then, $v_0 \overset{p_{0i}}{\leadsto} v_i \overset{p'_{ij}}{\leadsto} v_j \overset{p_{jk}}{\leadsto} v_k$ is a path from $v_0$ *to* $v_k$ whose weight $w(p_{0i}) + w(p'_{ij}) + w(p_{jk})$ is less than $w(p)$, which contradicts the assumption that $p$ is a shortest path from $v_0$ *to* $v_k$. ■

## Negative-weight edges

- If the graph $G = (V, E)$ contains no negative-weight cycles reachable from the source $s$, then for all $v \in V$, the shortest-path weight $\delta(s, v)$ is well defined, even if it has a negative value
- If $G$ contains a negative-weight cycle reachable from $s$, shortest-path weights aren't well defined
- No path from $s$ to a vertex on the cycle can be a shortest path—we always find a path with lower weight by following the proposed "shortest" path and then traversing the negative-weight cycle



- If there is a negative-weight cycle on some path from $s$ to $v$, we define $\delta(s, v) = -\infty$



## Cycles

- A shortest path cannot either contain a positive-weight cycle, since removing the cycle from the path produces a path with the same source and destination vertices and a lower path weight
- If $p = \langle v_0, v_1, \dots, v_k \rangle$ is a path and $c = \langle v_i, v_{i+1}, \dots, v_j \rangle$ is a positive-weight cycle on this path ($v_i = v_j$ and $w(c) > 0$), then the path $p' = \langle v_0, v_1, \dots, v_i, v_{j+1}, \dots, v_k \rangle$ has weight $w(p') = w(p) - w(c) < w(p)$, and so $p$ cannot be a shortest path from $v_0$ *to* $v_k$



- If there is a shortest path from a source vertex $s$ to a destination vertex $v$ that contains a $0$-weight cycle, then there is another shortest path from $s$ to $v$ without this cycle
- We can repeatedly remove $0$-weight cycles from the path until the shortest path is cycle-free
- Therefore, we can assume that when we are finding shortest paths, they have no cycles, i.e., they are *simple paths*
- Any acyclic path in $G = (V, E)$ contains at most $|V|$ distinct vertices, and at most $|V| - 1$ edges
- Thus, we can restrict our attention to shortest paths of at most $|V| - 1$ edges

## Relaxation

- The attribute $v.d$ is an upper bound on the weight of a shortest path from source $s$ to $v$
- We call $v.d$ a ***shortest-path estimate***
- The following $\Theta(V)$-time procedure initializes the shortest-path estimates and predecessors ($\pi$):

INITIALIZE-SINGLE-SOURCE$(G, s)$

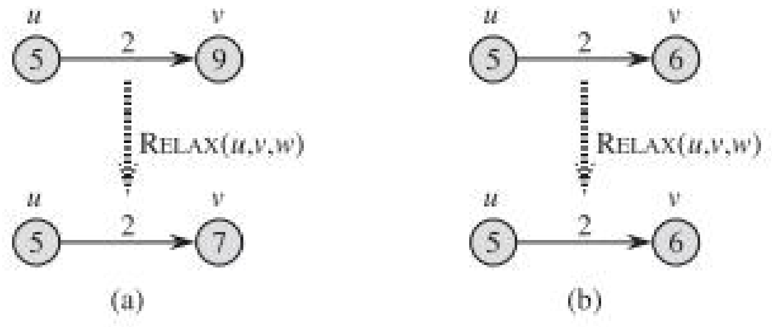
1. **for** each vertex $v \in G.V$
2. $\quad v.d \leftarrow \infty$
3. $\quad v.\pi \leftarrow$ NIL
4. $s.d \leftarrow 0$



- The process of relaxing an edge $(u, v)$ consists of testing whether we can improve the shortest path to $v$ found so far by going through $u$ and, if so, updating $v.d$ and $v.\pi$
- A relaxation step may decrease the value of the shortest-path estimate $v.d$ and update $v$'s predecessor attribute $v.\pi$
- The following code performs a relaxation step on edge $(u, v)$ in $O(1)$ time:

RELAX$(u, v, w)$

1. **if** $v.d > u.d + w(u, v)$
2. $\quad v.d \leftarrow u.d + w(u, v)$
3. $\quad v.\pi \leftarrow u$






## Properties of shortest paths and relaxation

- To prove the following algorithms correct, we appeal to several properties of shortest paths and relaxation:

**Triangle inequality** (Lemma 24.10)

*For any edge* $(u, v) \in E$, *we have* $\delta(s, v) \leq \delta(s, u) + w(u, v)$

**Upper-bound property** (Lemma 24.11)

*We always have* $v.d \geq \delta(s, v)$ *for all vertices* $v \in V$, *and once* $v.d$ *achieves the value* $\delta(s, v)$, *it never changes*

**No-path property** (Corollary 24.12)

*If there is no path from $s$ to $v$, then we always have $v.d = \delta(s, v) = \infty$*

**Convergence property** (Lemma 24.14)

*If $s \rightsquigarrow u \to v$ is a shortest path in $G$ for some $u, v \in V$, and if $u.d = \delta(s, u)$ at any time prior to relaxing edge $(u, v)$, then $v.d = \delta(s, v)$ at all times afterward*



**Path-relaxation property** (Lemma 24.15)

*If $p = \langle v_0, v_1, \ldots, v_k \rangle$ is a shortest path from $s = v_0$ to $v_k$, and we relax the edges of $p$ in the order $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$. This property holds regardless of any other relaxation steps that occur, even if they are intermixed with relaxations of the edges of $p$*

**Predecessor-subgraph property** (Lemma 24.17)

*Once $v.d = \delta(s, v)$ for all $v \in V$, the predecessor subgraph is a shortest-paths tree rooted at $s$*