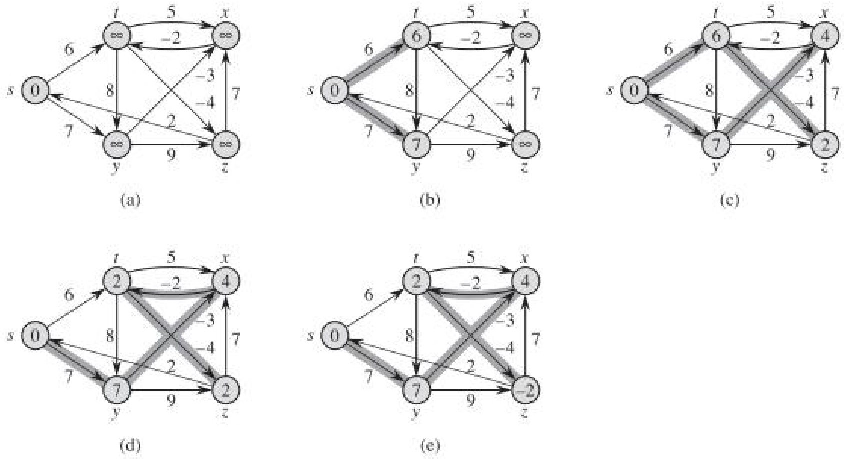
## 24.1 The Bellman-Ford algorithm

- Solves the single-source shortest-paths problem in the case in which weights may be negative
- Given a graph $G = (V, E)$ with source $s$ and weight function $w$, it returns a boolean value indicating whether or not there is a negative-weight cycle that is reachable from the source
- If there is such a cycle, the algorithm indicates that no solution exists
- If there is no such cycle, the algorithm produces the shortest paths and their weights

- Relax edges, progressively decreasing $v.d$ until we achieve the actual shortest-path weight $\delta(s, v)$

```
BELLMAN-FORD$(G, w, s)$
INITIALIZE-SINGLE-SOURCE$(G, s)$
for $i = 1$ to $|G.V| - 1$
    for each edge $(u, v) \in G.E$
        RELAX$(u, v, w)$
for each edge $(u, v) \in G.E$
    if $v.d > u.d + w(u, v)$
        return FALSE
return TRUE
```



- The algorithm makes $|V| - 1$ passes over the edges of the graph
- Each pass is one iteration of the **for** loop of lines 2–4 and consists of relaxing each edge of the graph once
- After making $|V| - 1$ passes, lines 5–8 check for a negative-weight cycle and return the appropriate boolean value
- The Bellman-Ford algorithm runs in time $O(VE)$, since the initialization in line 1 takes $\Theta(V)$ time, each of the $|V| - 1$ passes over the edges in lines 2–4 takes $\Theta(V)$ time, and the for loop of lines 5–7 takes $O(E)$ time

**Lemma 24.2:**

- *Let* $G = (V, E)$ *be a weighted, directed graph*
- *with source* $s$ *and*
- *weight function* $w: E \to \mathbb{R}$, *and*
- *assume that* $G$ *contains no negative-weight cycles that are reachable from* $s$.

*Then, after the* $|V| - 1$ *iterations of the* **for** *loop of lines 2–4 of* BELLMAN-FORD, *we have* $v.d = \delta(s, v)$ *for all vertices* $v$ *that are reachable from* $s$.

***Proof:*** We prove the lemma by appealing to the path-relaxation property.

Consider any vertex $v$ that is reachable from $s$, and let $p = \langle v_0, v_1, \dots, v_k \rangle$, where $v_0 = s$ and $v_k = v$, be any shortest path from $s$ to $v$.

Because shortest paths are simple, $p$ has at most $|V| - 1$ edges, and so $k \leq |V| - 1$ . Each of the $|V| - 1$ iterations of the **for** loop of lines 2–4 relaxes all $|E|$ edges.

Among the edges relaxed in the $i$th iteration, for $i = 1, 2, \dots, k$, is $(v_{i-1}, v_i)$. By the path-relaxation property, therefore,

$$v.d = v_k.d = \delta(s, v_k) = \delta(s, v).$$

■

**Corollary 24.3:**

- *Let $G = (V, E)$ be a weighted, directed graph*
- *with source $s$ and*
- *weight function $w: E \rightarrow \mathbb{R}$, and*
- *assume that $G$ contains no negative-weight cycles that are reachable from $s$.*

*Then, for each vertex $v \in V$, there is a path from $s$ to $v$ if and only if* BELLMAN-FORD *terminates with $v.d < \infty$ when it is run on $G$.*

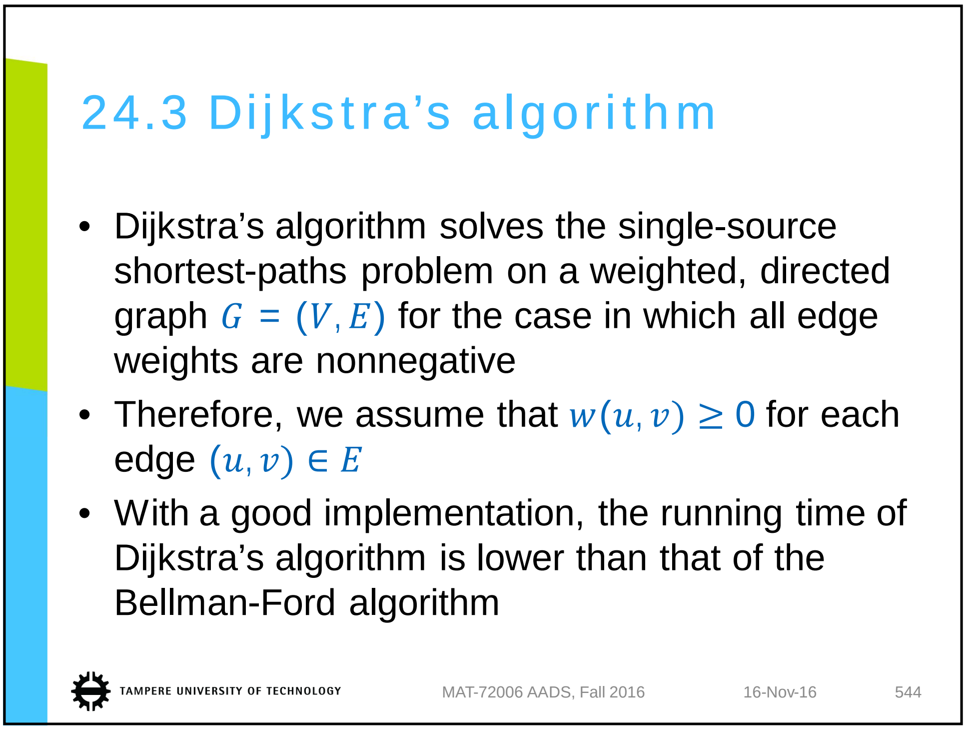
**Theorem 24.4** (Correctness of the B-F algorithm)

*Let* BELLMAN-FORD *be run on*

- *a weighted, directed graph $G = (V, E)$*
- *with source $s$ and*
- *weight function $w: E \rightarrow R$.*

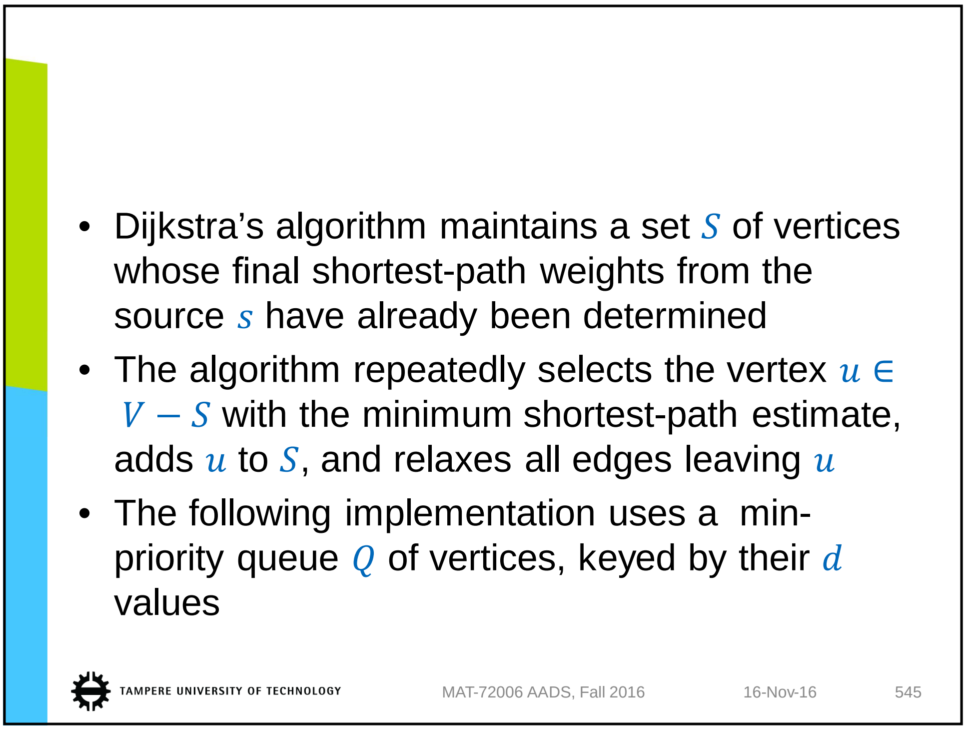
*If $G$ contains no negative-weight cycles that are reachable from $s$, then the algorithm returns* TRUE, *we have $v.d = \delta(s, v)$ for all vertices $v \in V$, and the predecessor subgraph $G_\pi$ is a shortest-paths tree rooted at $s$.*

*If $G$ does contain a negative-weight cycle reachable from $s$, then the algorithm returns* FALSE*.*



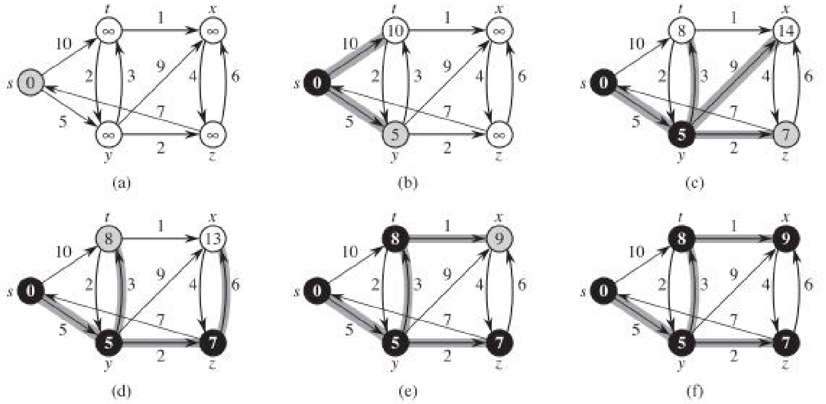
## 24.3 Dijkstra's algorithm

- Dijkstra's algorithm solves the single-source shortest-paths problem on a weighted, directed graph $G = (V, E)$ for the case in which all edge weights are nonnegative
- Therefore, we assume that $w(u, v) \geq 0$ for each edge $(u, v) \in E$
- With a good implementation, the running time of Dijkstra's algorithm is lower than that of the Bellman-Ford algorithm



- Dijkstra's algorithm maintains a set $S$ of vertices whose final shortest-path weights from the source $s$ have already been determined
- The algorithm repeatedly selects the vertex $u \in V - S$ with the minimum shortest-path estimate, adds $u$ to $S$, and relaxes all edges leaving $u$
- The following implementation uses a min-priority queue $Q$ of vertices, keyed by their $d$ values

```
DIJKSTRA($(G, w, s)$)
INITIALIZE-SINGLE-SOURCE($(G, s)$)
$S \leftarrow \emptyset$
$Q \leftarrow G.V$
while $Q \neq \emptyset$
    $u \leftarrow$ EXTRACT-MIN($(Q)$)
    $S \leftarrow S \cup \{u\}$
    for each vertex $v \in G.Adj[u]$
        RELAX $(u, v, w)$
```



(a) (b) (c)

(d) (e) (f)

- Shaded edges indicate predecessor values
- Black vertices are in the set $S$

- Maintain the invariant that $Q = V - S$ at the start of each iteration of the **while** loop of lines 4–8
- Line 3 initializes the min-priority queue $Q$ to contain all the vertices in $V$; since $S = \emptyset$; at that time, the invariant is true after line 3
- Each time through the **while** loop, line 5 extracts a vertex $u$ from $Q = V - S$ and line 6 adds it to set $S$, thereby maintaining the invariant
- The first time through this loop, $u = s$
- Vertex $u$, therefore, has the smallest shortest-path estimate of any vertex in $V - S$

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- Then, lines 7–8 relax each edge $(u, v)$ leaving $u$, thus updating the estimate $v.d$ and the predecessor $v.\pi$ if we can improve the shortest path to $v$ found so far by going through $u$
- Observe that the algorithm never inserts vertices into $Q$ after line 3 and that each vertex is extracted from $Q$ and added to $S$ exactly once, so that the **while** loop iterates exactly $|V|$ times
- Because Dijkstra’s algorithm always chooses the “lightest” or “closest” vertex in $V - S$ to add to set $S$, we say that it uses a greedy strategy

**Theorem 24.6** (Correctness of Dijkstra's algorithm)

*Dijkstra's algorithm, run on a weighted, directed graph* $G = (V, E)$ *with non-negative weight function* $w$ *and source* $s$*, terminates with* $u.d = \delta(s, u)$ *for all vertices* $u \in V$*.*

***Proof:*** We use the following loop invariant:

- At the start of each iteration of the **while** loop of lines 4–8, $v.d = \delta(s, v)$ for each vertex $v \in S$.

It suffices to show for each vertex $u \in V$, we have $u.d = \delta(s, u)$ at the time when $u$ is added to set $S$. Once $u.d = \delta(s, u)$, we rely on the upper-bound property to show that it holds thereafter.

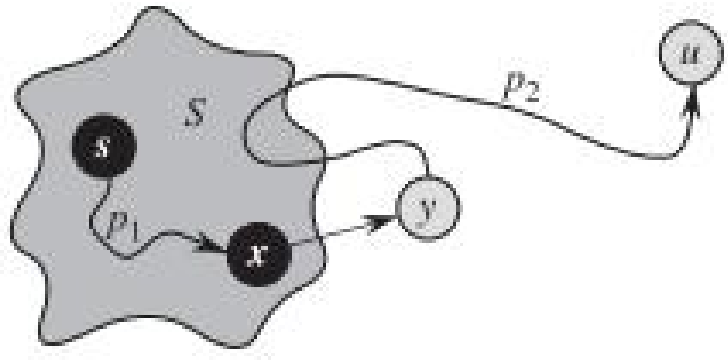
---

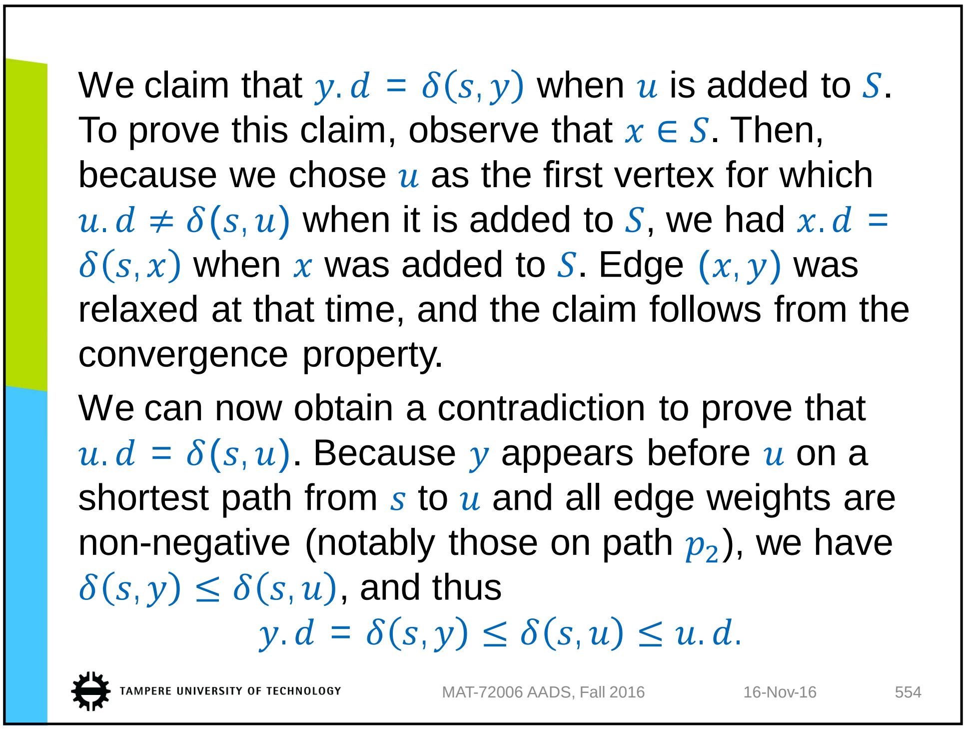
**Initialization:** Initially, $S = \emptyset$, and so the invariant is trivially true.

**Maintenance:** We wish to show that in each iteration, $u.d = \delta(s, u)$ for the vertex added to set $S$. For the purpose of contradiction, let $u$ be the first vertex for which $u.d \neq \delta(s, u)$ when it is added to set $S$. We shall focus our attention on the situation at the beginning of the iteration of the while loop in which $u$ is added to $S$ and derive the contradiction that $u.d = \delta(s, u)$ at that time by examining a shortest path from $s$ to $u$. We must have $u \neq s$ because $s$ is the first vertex added to set $S$ and $s.d = \delta(s, s) = 0$ at that time.

Because $u \neq s$, we also have that $S \neq \emptyset$ just before $u$ is added to $S$. There must be some path from $s$ to $u$, for otherwise $u.d = \delta(s,u) = \infty$ by the no-path property, which would violate our assumption that $u.d \neq \delta(s,u)$. Because there is at least one path, there is a shortest path $p$ from $s$ to $u$. Prior to adding $u$ to $S$, path $p$ connects a vertex in $S$, namely $s$, to a vertex in $V - S$, namely $u$. Let us consider the first vertex $y$ along $p$ such that $y \in V - S$, and let $x \in S$ be $y$'s predecessor along $p$.

Thus, we can decompose path $p$ into $s \overset{p_1}{\leadsto} x \to y \overset{p_2}{\leadsto} u$. (Either path $p_1$ or $p_2$ may have no edges.)

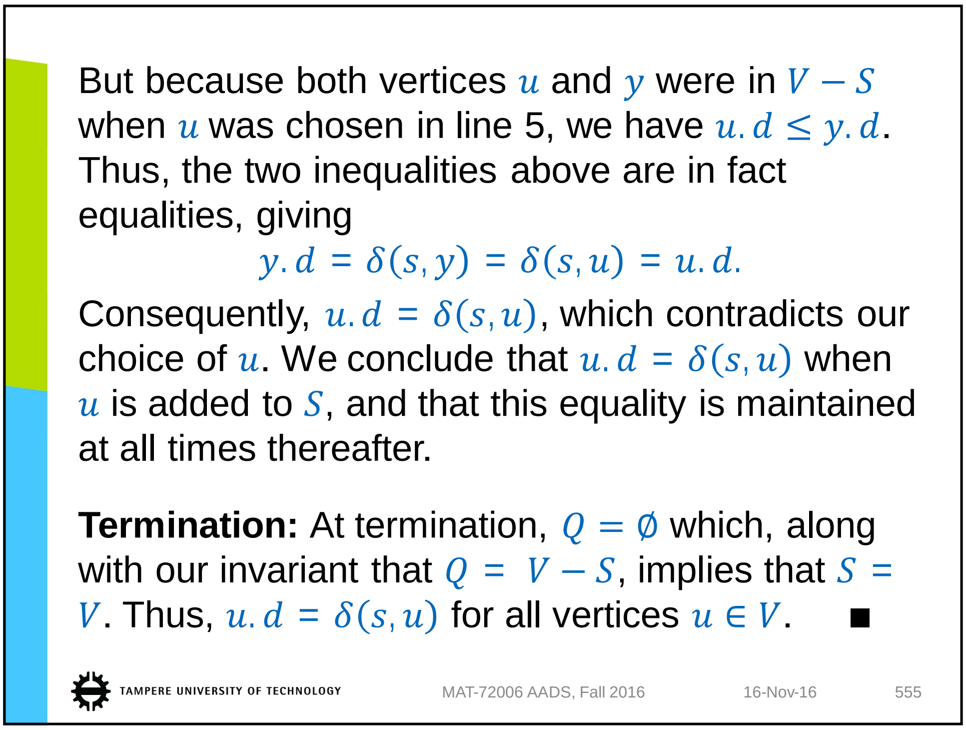




We claim that $y.d = \delta(s,y)$ when $u$ is added to $S$. To prove this claim, observe that $x \in S$. Then, because we chose $u$ as the first vertex for which $u.d \neq \delta(s,u)$ when it is added to $S$, we had $x.d = \delta(s,x)$ when $x$ was added to $S$. Edge $(x,y)$ was relaxed at that time, and the claim follows from the convergence property.

We can now obtain a contradiction to prove that $u.d = \delta(s,u)$. Because $y$ appears before $u$ on a shortest path from $s$ to $u$ and all edge weights are non-negative (notably those on path $p_2$), we have $\delta(s,y) \leq \delta(s,u)$, and thus

$$y.d = \delta(s,y) \leq \delta(s,u) \leq u.d.$$



But because both vertices $u$ and $y$ were in $V - S$ when $u$ was chosen in line 5, we have $u.d \leq y.d$. Thus, the two inequalities above are in fact equalities, giving

$$y.d = \delta(s,y) = \delta(s,u) = u.d.$$

Consequently, $u.d = \delta(s,u)$, which contradicts our choice of $u$. We conclude that $u.d = \delta(s,u)$ when $u$ is added to $S$, and that this equality is maintained at all times thereafter.

**Termination:** At termination, $Q = \emptyset$ which, along with our invariant that $Q = V - S$, implies that $S = V$. Thus, $u.d = \delta(s,u)$ for all vertices $u \in V$. ■

# 25 All-Pairs Shortest Paths

- Consider the problem of finding shortest paths between all pairs of vertices in a graph
- Given a weighted, directed graph $G = (V, E)$ with a weight function $w: E \to \mathbb{R}$ that maps edges to real-valued weights
- We wish to find, for every pair of vertices $u, v \in V$, a shortest (least-weight) path from $u$ to $v$, where the weight of a path is the sum of the weights of its constituent edges

- We can solve the problem by running a single-source shortest-paths algorithm $|V|$ times
- If all edge weights are nonnegative, we can use Dijkstra's algorithm
- If we use the linear-array implementation of the min-priority queue, the running time is
$$O(V^3 + VE) = O(V^3)$$
- Binary min-heap implementation of min-priority queue yields a running time of $O(VE \lg V)$, which is an improvement if the graph is sparse
- Alternatively, we can implement the min-priority queue with a Fibonacci heap, yielding a running time of $O(V^2 \lg V + VE)$

- If the graph has negative-weight edges, we cannot use Dijkstra’s algorithm
- Instead, we must run the slower Bellman-Ford algorithm once from each vertex
- The resulting running time is $O(V^2E)$, which on a dense graph is $O(V^4)$
- We shall see how to do better

- APSP = all-pairs shortest-paths (APSP) problem

- Unlike the single-source algorithms, most of the following algorithms use an adjacency-matrix representation
- For convenience, we assume that the vertices are numbered $1,2,\ldots|V|$, so that the input is an $n \times n$ matrix $W$ representing the edge weights of an $n$-vertex directed graph $G = (V, E)$
- I.e., $W = (w_{ij})$, where

$$w_{ij} = \begin{cases} 0 & \text{if } i = j \\ \text{the weight of directed edge } (i,j) & \text{if } i \neq j \text{ and } (i,j) \in E \\ \infty & \text{if } i \neq j \text{ and } (i,j) \notin E \end{cases}$$

- We allow negative-weight edges, but assume that the input graph contains no negative-weight cycles
- The tabular output of the APSP algorithms is an $n \times n$ matrix $D = (d_{ij})$, where entry $d_{ij}$ contains the weight of a shortest path from vertex $i$ to vertex $j$
- That is, if we let $\delta(i, j)$ denote the shortest-path weight from vertex $i$ to vertex $j$, then $d_{ij} = \delta(i, j)$ at termination

- To solve the APSP problem on an input adjacency matrix, we need to compute also a predecessor matrix $\Pi = (\pi_{ij})$,
  - where $\pi_{ij}$ is NIL if either $i = j$ or there is no path from $i$ to $j$, and otherwise $\pi_{ij}$ is the predecessor of $j$ on some shortest path from $i$
- Just as the predecessor subgraph $G_\pi$ is a shortest-paths tree for a given source vertex, the subgraph induced by the $i$th row of the $\Pi$ matrix should be a shortest-paths tree with root $i$

- For each vertex $i \in V$, define the predecessor subgraph of $G$ for $i$ as $G_{\pi,i} = (V_{\pi,i}, E_{\pi,i})$, where

$$V_{\pi,i} = \{j \in V : \pi_{ij} \neq \text{NIL}\} \cup \{i\}$$

and

$$E_{\pi,i} = \left\{(\pi_{ij}, j) : j \in V_{\pi,i} - \{i\}\right\}$$

- If $G_{\pi,i}$ is a shortest-paths tree, then the following procedure prints a shortest path from vertex $i$ to vertex $j$

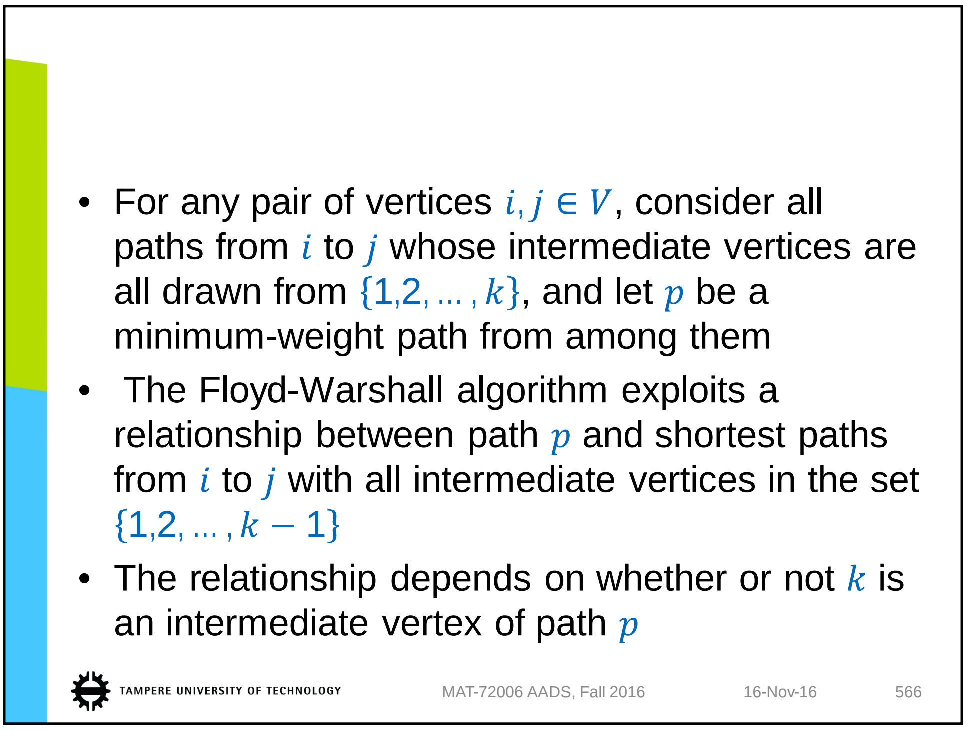
```
PRINT-ALL-PAIRS-SHORTEST-PATH$(\Pi, i, j)$
if $i = j$
    print $i$
elseif $\pi_{ij}$ = NIL
    print "no path from" $i$ "to" $j$ "exists"
else PRINT-ALL-PAIRS-SHORTEST-PATH$(\Pi, i, \pi_{ij})$
    print $j$
```

## 25.2 The Floyd-Warshall algorithm

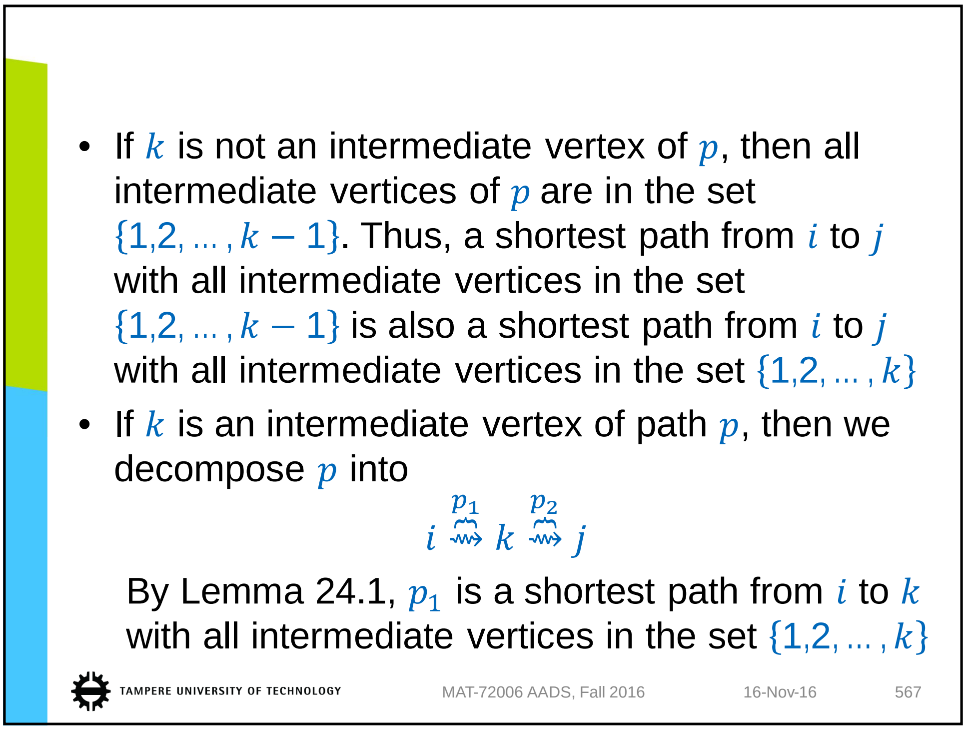
- We use a dynamic-programming formulation to solve the all-pairs shortest-paths problem on a directed graph $G = (V, E)$
- The resulting algorithm runs in $\Theta(V^3)$ time
- As before, negative-weight edges may be present, but we assume that there are no negative-weight cycles
- After studying the resulting algorithm, we look at a similar method for finding the transitive closure of a directed graph

## The structure of a shortest path

- The Floyd-Warshall algorithm considers the intermediate vertices of a shortest path
- An intermediate vertex of a simple path $p = \{v_1, v_2, \dots, v_l\}$ is any vertex of $p$ other than $v_1$ or $v_l$
- I.e., any vertex in the set $\{v_2, v_3, \dots, v_{l-1}\}$
- Under our assumption that the vertices of $G$ are $V = \{1,2, \dots, n\}$, let us consider a subset $\{1,2, \dots, k\}$ of vertices for some $k$



- For any pair of vertices $i, j \in V$, consider all paths from $i$ to $j$ whose intermediate vertices are all drawn from $\{1,2,\dots,k\}$, and let $p$ be a minimum-weight path from among them
- The Floyd-Warshall algorithm exploits a relationship between path $p$ and shortest paths from $i$ to $j$ with all intermediate vertices in the set $\{1,2,\dots,k-1\}$
- The relationship depends on whether or not $k$ is an intermediate vertex of path $p$

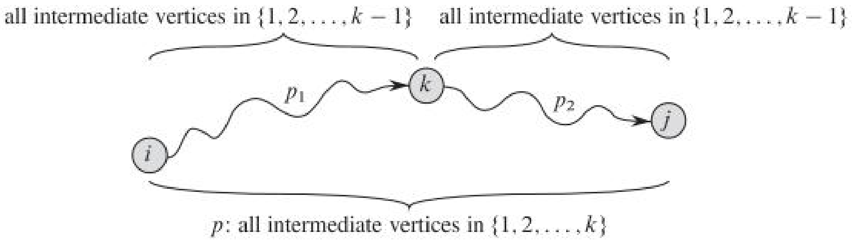


- If $k$ is not an intermediate vertex of $p$, then all intermediate vertices of $p$ are in the set $\{1,2,\dots,k-1\}$. Thus, a shortest path from $i$ to $j$ with all intermediate vertices in the set $\{1,2,\dots,k-1\}$ is also a shortest path from $i$ to $j$ with all intermediate vertices in the set $\{1,2,\dots,k\}$
- If $k$ is an intermediate vertex of path $p$, then we decompose $p$ into

$$i \overset{p_1}{\leadsto} k \overset{p_2}{\leadsto} j$$

By Lemma 24.1, $p_1$ is a shortest path from $i$ to $k$ with all intermediate vertices in the set $\{1,2,\dots,k\}$

- In fact, we can make a slightly stronger statement
- Because vertex $k$ is not an intermediate vertex of path $p_1$, all intermediate vertices of $p_1$ are in the set $\{1,2,\ldots,k-1\}$
- Therefore, $p_1$ is a shortest path from $i$ to $k$ with all intermediate vertices in the set $\{1,2,\ldots,k-1\}$
- Similarly, $p_2$ is a shortest path from vertex $k$ to vertex $j$ with all intermediate vertices in the set $\{1,2,\ldots,k-1\}$



## A recursive solution to the all-pairs shortest-paths problem

- Let $d_{ij}^{(k)}$ be the weight of a shortest path from vertex $i$ to vertex $j$ for which all intermediate vertices are in the set $\{1,2,\ldots,k\}$
- When $k = 0$, a path from $i$ to $j$ with no intermediate vertex numbered higher than $0$ has no intermediate vertices at all
- Such a path has at most one edge, and hence $d_{ij}^{(0)} = w_{ij}$

---

- Let us define recursively

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0 \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1 \end{cases}$$

- Because for any path, all intermediate vertices are in the set $\{1,2,\ldots,n\}$, the matrix $D^{(n)} = \left(d_{ij}^{(n)}\right)$ gives the final answer:

$$d_{ij}^{(n)} = \delta(i,j)$$

for all $i,j \in V$

## Computing the shortest-path weights bottom up

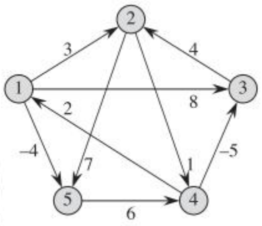
- Based on the prev. recurrence, we can use the following bottom-up procedure to compute the values $d_{ij}^{(k)}$ in order of increasing values of $k$
- Its input is an $n \times n$ matrix $W = (w_{ij})$, where

$$w_{ij} = \begin{cases} 0 & \text{if } i = j \\ \text{the weight of directed edge } (i,j) & \text{if } i \neq j \text{ and } (i,j) \in E \\ \infty & \text{if } i \neq j \text{ and } (i,j) \notin E \end{cases}$$

- The procedure returns the matrix $D^{(n)}$ of shortest-path weights

---

```
FLOYD-WARSHALL $(W)$
$n \leftarrow W.rows$
$D^{(0)} \leftarrow W$
for $k \leftarrow 1$ to $n$
    let $D^{(k)} = \left(d_{ij}^{(k)}\right)$ be a new $n \times n$ matrix
    for $i \leftarrow 1$ to $n$
        for $j \leftarrow 1$ to $n$
            $d_{ij}^{(k)} \leftarrow \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$
return $D^{(n)}$
```



$$D^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad D^{(4)} = \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

$$D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad D^{(5)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

- The running time of the Floyd-Warshall algorithm is determined by the triply nested **for** loops of lines 3–7
- Because each execution of line 7 takes $O(1)$ time, the algorithm runs in time $\Theta(n^3)$
- The code is tight, with no elaborate data structures, and so the constant hidden in the $\Theta$-notation is small
- Thus, the Floyd-Warshall algorithm is quite practical for even moderate-sized input graphs

## Transitive closure of a directed graph

- Given a directed $G = (V, E)$, $V = \{1,2, \dots, n\}$, we wish to determine whether $G$ contains a path from $i$ to $j$ for all vertex pairs $i, j \in V$
- We define the ***transitive closure*** of $G$ as the graph $G^* = (V, E^*)$, where
  $E^* = \{\ (i, j)\colon \text{there is a path from } i \text{ to } j \text{ in } G\ \}$
- One way to compute it is to assign a weight of $1$ to each edge of $E$ and run the F-W algorithm
- If there is a path from $i$ to $j$, we get $d_{ij} < n$. Otherwise, $d_{ij} = \infty$.

---

- Another, similar way to compute the transitive closure of $G$ in $\Theta(n^3)$ time that can save time and space in practice
- This method substitutes the logical operations $\vee$ and $\wedge$ for the arithmetic operations $\min$ and $+$ in the Floyd-Warshall algorithm
- For $i, j, k = 1,2; , \dots, n$, we define $t_{ij}^{(k)}$ to be $1$ if there exists a path in graph $G$ from vertex $i$ to vertex $j$ with all intermediate vertices in the set $\{1,2, \dots, k\}$, and $0$ otherwise
- We construct the transitive closure $G^* = (V, E^*)$ by putting edge $(i, j)$ into $E^*$ if and only if $t_{ij}^{(n)} = 1$

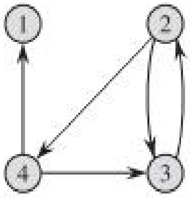
- A recursive definition of $t_{ij}^{(k)}$ is

$$t_{ij}^{(0)} = \begin{cases} 0 & \text{if } i \neq j \text{ and } (i,j) \notin E \\ 1 & \text{if } i = j \text{ or } (i,j) \in E \end{cases}$$

- and for $k \geq 1$,

$$t_{ij}^{(k)} = t_{ij}^{(k-1)} \vee \left(t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)}\right)$$

- As in the Floyd-Warshall algorithm, we compute the matrices $T^{(k)} = \left(t_{ij}^{(k)}\right)$ in order of increasing $k$

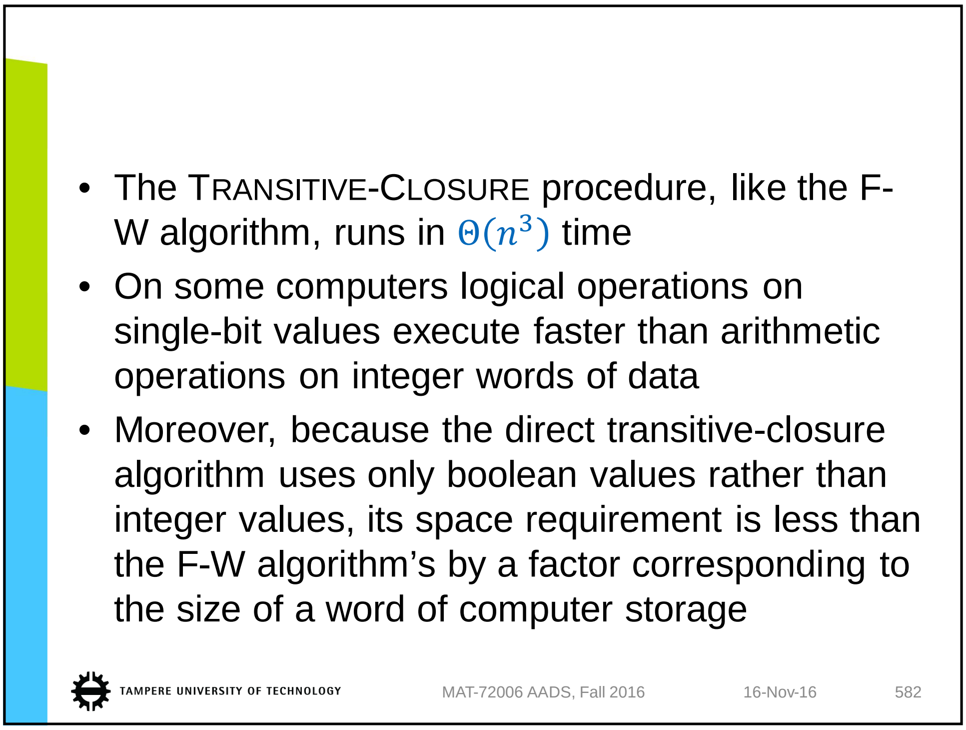


$$T^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad T^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad T^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

$$T^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad T^{(4)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

```
TRANSITIVE-CLOSURE$(G)$
$n \leftarrow |G.V|$
let $T^{(0)} = \left(t_{ij}^{(0)}\right)$ be a new $n \times n$ matrix
for $i \leftarrow 1$ to $n$
    for $j \leftarrow 1$ to $n$
        if $i = j$ or $(i, j) \in G.E$
            $t_{ij}^{(0)} \leftarrow 1$
        else $t_{ij}^{(0)} \leftarrow 0$
```

```
for $k \leftarrow 1$ to $n$
    let $T^{(k)} = \left(t_{ij}^{(k)}\right)$ be a new $n \times n$ matrix
    for $i \leftarrow 1$ to $n$
        for $j \leftarrow 1$ to $n$
            $t_{ij}^{(k)} = t_{ij}^{(k-1)} \vee \left(t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)}\right)$
return $T^{(n)}$
```



- The TRANSITIVE-CLOSURE procedure, like the F-W algorithm, runs in $\Theta(n^3)$ time
- On some computers logical operations on single-bit values execute faster than arithmetic operations on integer words of data
- Moreover, because the direct transitive-closure algorithm uses only boolean values rather than integer values, its space requirement is less than the F-W algorithm’s by a factor corresponding to the size of a word of computer storage