15.4 Longest common subsequence

- Biological applications often need to compare the DNA of two (or more) different organisms
- A strand of DNA consists of a string of molecules called ***bases***, where the possible bases are **A**denine, **G**uanine, **C**ytosine, and **T**hymine
- We express a strand of DNA as a string over the alphabet **{A,C,G,T}**
- E.g., the DNA of two organisms may be
  - S_1 = **ACCGGTCGAGTGCGCGGAAGCCGGCCGAA**
  - S_2 = **GTCGTTCGGAATGCCGTTGCTCTGTAAA**

- By comparing two strands of DNA we determine how “similar” they are, as some measure of how closely related the two organisms are
- We can define similarity in many different ways
- E.g., we can say that two DNA strands are similar if one is a substring of the other
  - Neither S_1 nor S_2 is a substring of the other
- Alternatively, we could say that two strands are similar if the number of changes needed to turn one into the other is small

- Yet another way to measure the similarity of S_1 and S_2 is by finding a third strand S_3 in which the bases in S_3 appear in each of S_1 and S_2
  - these bases must appear in the same order, but not necessarily consecutively
- The longer the strand S_3 we can find, the more similar S_1 and S_2 are
- In our example, the longest strand S_3 is
  - S_1 = ACCGGTCGAGTGCGCGGAAGCCGGCCGAA
  - S_2 = GTCGTTCGGAATGCCGTTGCTCTGTAAA
  - S_3 = GTCGTCGGAAGCCGGCCGAA

- Formalize this notion of similarity as the longest-common-subsequence problem
- A subsequence is just the given sequence with zero or more elements left out
- Formally, given a sequence $X = \langle x_1, x_2, \dots, x_m \rangle$, another sequence $Z = \langle z_1, z_2, \dots, z_k \rangle$ is a **subsequence** of X if there exists a strictly increasing sequence $\langle i_1, i_2, \dots, i_k \rangle$ of indices of X such that for all $j = 1, 2, \dots, k$, we have $x_{i_j} = z_j$
- For example, $Z = \langle B, C, D, B \rangle$ is a subsequence of $X = \langle A, B, C, B, D, A, B \rangle$ with corresponding index sequence $\langle 2,3,5,7 \rangle$

- A sequence Z is a ***common subsequence*** of X and Y if Z is a subsequence of both X and Y
- For example, if $X = \langle A, B, C, B, D, A, B \rangle$ and $Y = \langle B, D, C, A, B, A \rangle$, the sequence $\langle B, C, A \rangle$ is a common subsequence of both X and Y
- It is not a ***longest common subsequence*** (LCS) of X and Y
- The sequence $\langle B, C, B, A \rangle$ is also common to both X and Y and has length 4
- This sequence is an LCS of X and Y, as is $\langle B, D, A, B \rangle$; X and Y have no common subsequence of length 5 or greater

- In ***longest-common-subsequence problem***, we are given $X = \langle x_1, x_2, \dots, x_m \rangle$ and $Y = \langle y_1, y_2, \dots, y_n \rangle$ and wish to find a max-length common subsequence of X and Y

Step 1: Characterizing a longest common subsequence

- In a brute-force approach, we would enumerate all subsequences of X and check each of them to see whether it is also a subsequence of Y, keeping track of the longest subsequence we find
- Each subsequence of X corresponds to a subset of the indices $\langle 1, 2, \dots, m \rangle$ of X
- Because X has 2^m subsequences, this approach requires exponential time, making it impractical for long sequences

- The LCS problem has an optimal-substructure property, however, as the following theorem shows
- The natural classes of subproblems correspond to pairs of “prefixes” of the two input sequences
- Precisely, given a sequence $X = \langle x_1, x_2, \dots, x_m \rangle$, we define the ith prefix of X, for $i = 0, 1, \dots, m$, as $X_i = \langle x_1, x_2, \dots, x_i \rangle$
- For example, if $X = \langle A, B, C, B, D, A, B \rangle$, then $X_4 = \langle A, B, C, B \rangle$ and X_0 is the empty sequence

Theorem 15.1 (Optimal substructure of LCS)
Let $X = \langle x_1, x_2, \dots, x_m \rangle$ and $Y = \langle y_1, y_2, \dots, y_n \rangle$ be sequences, and let $Z = \langle z_1, z_2, \dots, z_k \rangle$ be any LCS of X and Y.

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is an LCS of X_{m-1} and Y_{n-1}.
2. If $x_m \neq y_n$, then $z_k \neq x_m$ implies that Z is an LCS of X_{m-1} and Y.
3. If $x_m \neq y_n$, then $z_k \neq y_n$ implies that Z is an LCS of X and Y_{n-1}.

Proof (1) If $z_k \neq x_m$, then we could append $x_m = y_n$ to Z to obtain a common subsequence of X and Y of length $k + 1$, contradicting the supposition that Z is a LCS of X and Y. Thus, we must have $z_k = x_m = y_n$. Now, the prefix Z_{k-1} is a length-$(k-1)$ common subsequence of X_{m-1} and Y_{n-1}. We wish to show that it is an LCS. Suppose for the purpose of contradiction that there exists a common subsequence W of X_{m-1} and Y_{n-1} with length greater than $k-1$. Then, appending $x_m = y_n$ to W produces a common subsequence of X and Y whose length is greater than k, which is a contradiction.

(2) If $z_k \neq x_m$, then Z is a common subsequence of X_{m-1} and Y. If there were a common subsequence W with length greater than k, then W would also be a common subsequence of X_m and Y, contradicting the assumption that Z is an LCS of X and Y.

(3) The proof is symmetric to (2). ■

- Theorem 15.1 tells us that an LCS of two sequences contains within it an LCS of prefixes of the two sequences
- Thus, the LCS problem has an optimal-substructure property

Step 2: A recursive solution

- We examine either one or two subproblems when finding an LCS of X and Y
- If $x_m = y_n$, we find an LCS of X_{m-1} and Y_{n-1}
- Appending $x_m = y_n$ yields an LCS of X and Y
- If $x_m \neq y_n$, then we (1) find an LCS of X_{m-1} and Y and (2) find an LCS of X and Y_{n-1}
- Whichever of these two LCSs is longer is an LCS of X and Y
- These cases exhaust all possibilities, and we know that one of the optimal subproblem solutions must appear within an LCS of X and Y

- To find an LCS of X and Y, we may need to find the LCSs of X and Y_{n-1} and of X_{m-1} and Y
- Each subproblem has the subsubproblem of finding an LCS of X_{m-1} and Y_{n-1}
- Many other subproblems share subsubproblems
- As in the matrix-chain multiplication, recursive solution to the LCS problem involves a recurrence for the value of an optimal solution
- Let us define $c[i,j]$ to be the length of an LCS of the sequences X_i and Y_j
- If either $i = 0$ or $j = 0$, one of the sequences has length 0, and so the LCS has length 0

- The optimal substructure of the LCS problem gives

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1,j-1]+1 & \text{if } i,j > 0 \text{ and } x_i = y_j \\ \max(c[i,j-1], c[i-1,j]) & \text{if } i,j > 0 \text{ and } x_i \neq y_j \end{cases}$$

- Observe that a condition in the problem restricts which subproblems we may consider
- When $x_i = y_j$, we consider finding an LCS of X_{i-1} and Y_{j-1}
- Otherwise, we instead consider the two subproblems of finding an LCS of X_i and Y_{j-1} and of X_{i-1} and Y_j
- In the previous dynamic-programming algorithms — for rod cutting and matrix-chain multiplication — we ruled out no subproblems due to conditions in the problem

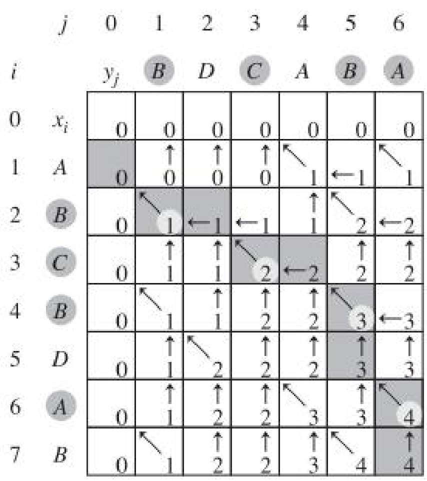
Step 3: Computing the length of an LCS

- Since the LCS problem has only $\Theta(mn)$ distinct subproblems, we can use dynamic programming to compute the solutions bottom up
- LCS-LENGTH stores the $c[i,j]$ values in $c[0..m, 0..n]$, and it computes the entries in ***row-major*** order
  - I.e., the procedure fills in the first row of c from left to right, then the second row, and so on
- The procedure also maintains the table $b[1..m, 1..n]$
- Intuitively, $b[i,j]$ points to the table entry corresponding to the optimal subproblem solution chosen when computing $c[i,j]$
- $c[m,n]$ contains the length of an LCS of X and Y

LCS-LENGTH(X, Y)

$m \leftarrow X.length$
$n \leftarrow Y.length$
let $b[1..m, 1..n]$ and $c[0..m, 0..n]$ be new tables
for $i \leftarrow 1$ **to** m
    $c[i, 0] \leftarrow 0$
for $j \leftarrow 0$ **to** n
    $c[0, j] \leftarrow 0$
for $i \leftarrow 1$ **to** m
    for $j \leftarrow 1$ **to** n
        if $x_i = y_j$
            $c[i, j] \leftarrow c[i-1, j-1] + 1$
            $b[i, j] \leftarrow$ "↖"
        elseif $c[i-1, j] \geq c[i, j-1]$
            $c[i, j] \leftarrow c[i-1, j]$
            $b[i, j] \leftarrow$ "↑"
        else $c[i, j] \leftarrow c[i, j-1]$
            $b[i, j] \leftarrow$ "←"
return c and b

Running time: $\Theta(mn)$



The c and b tables computed by LCS-LENGTH on $X = \langle A, B, C, B, D, A, B \rangle$ and $Y = \langle B, D, C, A, B, A \rangle$

Step 4: Constructing an LCS

- The b table returned by LCS-LENGTH enables us to quickly construct an LCS of X and Y
- We simply begin at $b[m, n]$ and trace through the table by following the arrows
- Whenever we encounter a "↖" in entry $b[i, j]$, it implies that $x_i = y_j$ is an element of the LCS that LCS-LENGTH found
- With this method, we encounter the elements of this LCS in reverse order
- A recursive procedure prints out an LCS of X and Y in the proper, forward order

- The square in row i and column j contains the value of $c[i, j]$ and the appropriate arrow for the value of $b[i, j]$
- The entry 4 in $c[7,6]$ — the lower right-hand corner of the table — is the length of an LCS $\langle B, C, B, A \rangle$
- For $i, j > 0$, entry $c[i, j]$ depends only on whether $x_i = y_j$ and the values in entries $c[i - 1, j]$, $c[i, j - 1]$, and $c[i - 1, j - 1]$, which are computed before $c[i, j]$
- To reconstruct the elements of an LCS, follow the $b[i, j]$ arrows from the lower right-hand corner
- Each "↖" on the shaded sequence corresponds to an entry (highlighted) for which $x_i = y_j$ is a member of an LCS

Improving the code

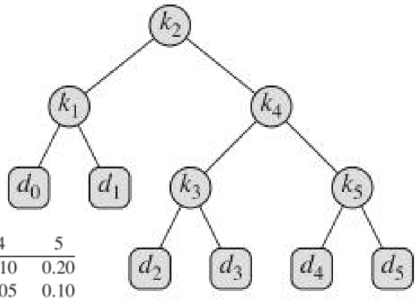
- Each $c[i,j]$ entry depends on only 3 other c table entries: $c[i-1,j]$, $c[i,j-1]$, and $c[i-1,j-1]$
- Given the value of $c[i,j]$, we can determine in $O(1)$ time which of these three values was used to compute $c[i,j]$, without inspecting table b
- We can reconstruct an LCS in $O(m+n)$ time
- The auxiliary space requirement for computing an LCS does not asymptotically decrease, since we need $\Theta(mn)$ space for the c table anyway

- We can, however, reduce the asymptotic space requirements for LCS-LENGTH, since it needs only two rows of table c at a time
  - the row being computed and the previous row
- This improvement works if we need only the length of an LCS
  - if we need to reconstruct the elements of an LCS, the smaller table does not keep enough information to retrace our steps in $O(m+n)$ time

15.5 Optimal binary search trees

- We are designing a program to translate text
- Perform lookup operations by building a BST with n words as keys and their equivalents as satellite data
- We can ensure an $O(\lg n)$ search time per occurrence by using a RBT or any other balanced BST
- A frequently used word may appear far from the root while a rarely used word appears near the root
- We want frequent words to be placed nearer the root
- How do we organize a BST so as to minimize the number of nodes visited in all searches, given that we know how often each word occurs?

- What we need is an ***optimal binary search tree***
- Formally, given a sequence $K = \langle k_1, k_2, \ldots, k_n \rangle$ of n distinct sorted keys ($k_1 < k_2 < \cdots < k_n$), we wish to build a BST from these keys
- For each key k_i, we have a probability p_i that a search will be for k_i
- Some searches may be for values not in K, so we also have $n + 1$ "dummy keys" $d_0, d_1, \ldots, d_n$ representing values not in K
- In particular, d_0 represents all values less than k_1, d_n represents all values greater than k_n



i	0	1	2	3	4	5
p_i		0.15	0.10	0.05	0.10	0.20
q_i	0.05	0.10	0.05	0.05	0.05	0.10

- For $i = 1, 2, \dots, n-1$, the dummy key d_i represents all values between k_i and k_{i+1}
- For each dummy key d_i, we have a probability q_i that a search will correspond to d_i

- Each key k_i is an internal node, and each dummy key d_i is a leaf
- Every search is either successful (finds a key k_i) or unsuccessful (finds a dummy key d_i), and so we have

$$\sum_{i=1}^{n} p_i + \sum_{i=0}^{n} q_i = 1$$

- Because we have probabilities of searches for each key and each dummy key, we can determine the expected cost of a search in a given BST T

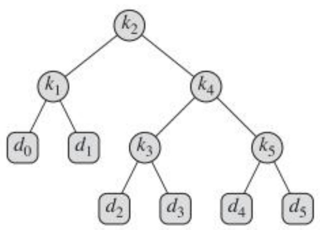
- Let us assume that the actual cost of a search equals the number of nodes examined, i.e., the depth of the node found by the search in $T + 1$
- Then the expected cost of a search in T, $\mathrm{E}[\text{search cost in } T] =$

$$\sum_{i=1}^{n}(\mathrm{depth}_T(k_i) + 1) \cdot p_i + \sum_{i=0}^{n}(\mathrm{depth}_T(d_i) + 1) \cdot q_i$$

$$= 1 + \sum_{i=1}^{n}\mathrm{depth}_T(k_i) \cdot p_i + \sum_{i=0}^{n}\mathrm{depth}_T(d_i) \cdot q_i \, ,$$

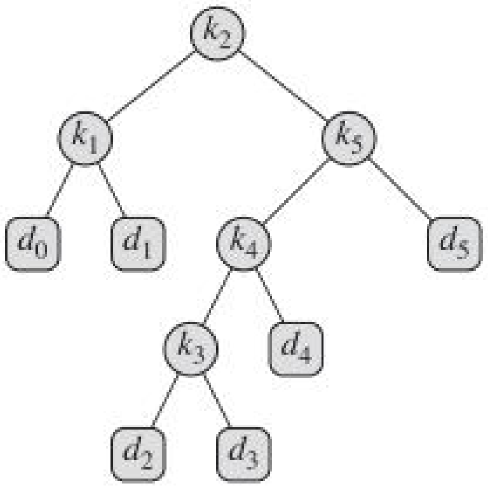
where depth_T denotes a node's depth in tree T

Node	Depth	Probability	Contribution
k_1	1	0.15	0.30
k_2	0	0.10	0.10
k_3	2	0.05	0.15
k_4	1	0.10	0.20
k_5	2	0.20	0.60
d_0	2	0.05	0.15
d_1	2	0.10	0.30
d_2	3	0.05	0.20
d_3	3	0.05	0.20
d_4	3	0.05	0.20
d_5	3	0.10	0.40
Total			2.80



i	0	1	2	3	4	5
p_i		0.15	0.10	0.05	0.10	0.20
q_i	0.05	0.10	0.05	0.05	0.05	0.10

- For a given set of probabilities, we wish to construct a BST whose expected search cost is smallest
- We call such a tree an ***optimal binary search tree***
- An optimal BST for the probabilities given has expected cost 2.75
- An optimal BST is not necessarily a tree whose overall height is smallest
- Nor can we necessarily construct an optimal BST by always putting the key with the greatest probability at the root
- The lowest expected cost of any BST with k_5 at the root is 2.85



Step 1: The structure of an optimal BST

- Consider any subtree of a BST
- It must contain keys in a contiguous range $k_i, \ldots, k_j$, for some $1 \leq i \leq j \leq n$
- In addition, a subtree that contains keys $k_i, \ldots, k_j$ must also have as its leaves the dummy keys $d_{i-1}, \ldots, d_j$
- If an optimal BST T has a subtree T'containing keys $k_i, \ldots, k_j$, then this subtree T' must be optimal as well for the subproblem with keys $k_i, \ldots, k_j$ and dummy keys $d_{i-1}, \ldots, d_j$

- Given keys $k_i, \ldots, k_j$, one of them, say k_r, is the root of an optimal subtree containing these keys
- The left subtree of the root k_r contains the keys $k_i, \ldots, k_{r-1}$ (and dummy keys $d_{i-1}, \ldots, d_{r-1}$)
- The right subtree contains the keys $k_{r+1}, \ldots, k_j$ (and dummy keys $d_r, \ldots, d_j$)
- As long as we
  - examine all candidate roots k_r, where $i \leq r \leq j$,
  - and determine all optimal BSTs containing $k_i, \ldots, k_{r-1}$ and those containing $k_{r+1}, \ldots, k_j$,

  we are guaranteed to find an optimal BST

- Suppose that in a subtree with keys $k_i, \dots, k_j$, we select k_i as the root
- k_i's left subtree contains the keys $k_i, \dots, k_{i-1}$
- Interpret this sequence as containing no keys
- Subtrees, however, also contain dummy keys
- Adopt the convention that a subtree containing keys $k_i, \dots, k_{i-1}$ has no actual keys but does contain the single dummy key d_{i-1}
- Symmetrically, if we select k_j as the root, then k_j's right subtree contains no actual keys, but it does contain the dummy key d_j

Step 2: A recursive solution

- We pick our subproblem domain as finding an optimal BST containing the keys $k_i, \dots, k_j$, where $i \geq 1$, $j \leq n$, and $j \geq i - 1$
- Let us define $e[i, j]$ as the expected cost of searching an optimal BST containing the keys $k_i, \dots, k_j$
- Ultimately, we wish to compute $e[1, n]$
- The easy case occurs when $j = i - 1$
- Then we have just the dummy key d_{i-1}
- The expected search cost is $e[i, i-1] = q_{i-1}$

- When $j > i$, we need to select a root k_r from among $k_i, \dots, k_j$ and make an optimal BST with keys $k_i, \dots, k_{r-1}$ as its left subtree and an optimal BST with keys $k_{r+1}, \dots, k_j$ as its right subtree
- What happens to the expected search cost of a subtree when it becomes a subtree of a node?
  - Depth of each node increases by 1
  - Expected search cost of this subtree increases by the sum of all the probabilities in it
- For a subtree with keys $k_i, \dots, k_j$, let us denote this sum of probabilities as

$$w(i,j) = \sum_{l=i}^{j} p_l + \sum_{l=i-1}^{j} q_l$$

- Thus, if k_r is the root of an optimal subtree containing keys $k_i, \dots, k_j$, we have

$$e[i,j] = p_r + \big(e[i,r-1] + w(i,r-1)\big) + \big(e[r+1,j] + w(r+1,j)\big)$$

- Noting that

$$w(i,j) = w(i,r-1) + p_r + w(r+1,j)$$

we rewrite

$$e[i,j] = e[i,r-1] + e[r+1,j] + w(i,j)$$

- We choose the root k_r that gives the lowest expected search cost: $e[i,j] =$

$$\begin{cases} q_{i-1} & \text{if } j = i-1 \\ \min_{i \le r \le j} e[i,j] = e[i,r-1] + e[r+1,j] + w(i,j) & \text{if } i \le j \end{cases}$$

- The $e[i,j]$ values give the expected search costs in optimal BSTs
- To help us keep track of the structure of optimal BSTs, we define $root[i,j]$, for $1 \le i \le j \le n$, to be the index r for which k_r is the root of an optimal BST containing keys $k_i, \ldots, k_j$
- Although we will see how to compute the values of $root[i,j]$, we leave the construction of an optimal binary search tree from these values as an exercise

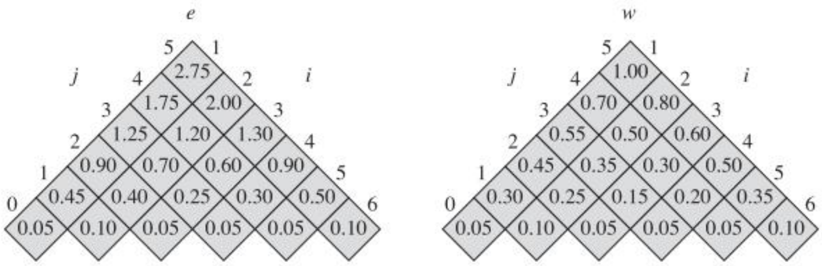
Step 3: Computing the expected search cost of an optimal BST

- We store $e[i,j]$ values in a table $e[1..n+1, 0..n]$
- The first index needs to run to $n+1$ because to have a subtree containing only the dummy key d_n, we need to compute and store $e[n+1, n]$
- The second index needs to start from 0 because to have a subtree containing only the dummy key d_0, we need to compute and store $e[1,0]$

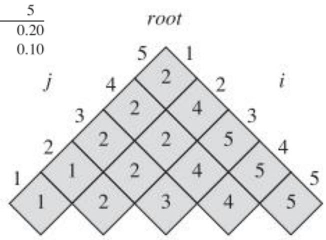
- We use only the entries $e[i,j]$ for which $j \geq i-1$
- We also use a table *root*$[i,j]$, for recording the root of the subtree containing keys $k_i, \dots, k_j$
- This table uses only the entries $1 \leq i \leq j \leq n$
- We also store the $w(i,j)$ values in a table $w[1..n+1, 0..n]$
- For the base case, we compute $w[i,i-1] = q_i$
- For $j \geq i$, we compute
$$w[i,j] = w[i,j-1] + p_j + q_j$$
- Thus, we can compute the $\Theta(n^2)$ values of $w[i,j]$ in $\Theta(1)$ time each

OPTIMAL-BST(p,q,n)

let $e[1..n+1, 0..n]$, $w[1..n+1, 0..n]$, *root*$[1..n, 1..n]$ be new tables
for $i = 1$ **to** $n+1$
  $e[i,i-1] = q_{i-1}$
  $w[i,i-1] = q_{i-1}$
for $l = 1$ **to** n
  for $i = 1$ **to** $n-l+1$
    $j = i+l-1$
    $e[i,j] = \infty$
    $w[i,j] = w[i,j-1] + p_j + q_j$
    for $r = i$ **to** j
      $t = e[i,r-1] + e[r+1,j] + w[i,j]$
      if $t < e[i,j]$
        $e[i,j] = t$
        root$[i,j] = r$
return e and *root*



i	0	1	2	3	4	5
p_i		0.15	0.10	0.05	0.10	0.20
q_i	0.05	0.10	0.05	0.05	0.05	0.10



- The OPTIMAL-BST procedure takes $\Theta(n^3)$ time, just like MATRIX-CHAIN-ORDER
- Its running time is $O(n^3)$, since its **for** loops are nested three deep and each loop index takes on at most n values
- The loop indices in OPTIMAL-BST do not have exactly the same bounds as those in MATRIX-CHAIN-ORDER, but they are within ≤ 1 in all directions
- Thus, like MATRIX-CHAIN-ORDER, the OPTIMAL-BST procedure takes $\Omega(n^3)$ time

16 Greedy Algorithms

- Optimization algorithms typically go through a sequence of steps, with a set of choices at each
- For many optimization problems, using dynamic programming to determine the best choices is overkill; simpler, more efficient algorithms will do
- A ***greedy algorithm*** always makes the choice that looks best at the moment
- That is, it makes a locally optimal choice in the hope that this choice will lead to a globally optimal solution

16.1 An activity-selection problem

- Suppose we have a set $S = \{a_1, a_2, \dots, a_n\}$ of n proposed activities that wish to use a resource (e.g., a lecture hall), which can serve only one activity at a time
- Each activity a_i has a start time s_i and a finish time f_i, where $0 \leq s_i < f_i < \infty$
- If selected, activity a_i takes place during the half-open time interval $[s_i, f_i)$

- Activities a_i and a_j are ***compatible*** if the intervals $[s_i, f_i)$ and $[s_j, f_j)$ do not overlap
- I.e., a_i and a_j are compatible if $s_i \geq f_j$ or $s_j \geq f_i$
- We wish to select a maximum-size subset of mutually compatible activities
- We assume that the activities are sorted in monotonically increasing order of finish time:

$$f_1 \leq f_2 \leq f_3 \leq \cdots \leq f_{n-1} \leq f_n$$

- Consider, e.g., the following set S of activities:

i	1	2	3	4	5	6	7	8	9	10	11
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16

- For this example, the subset $\{a_3, a_9, a_{11}\}$ consists of mutually compatible activities
- It is not a maximum subset, however, since the subset $\{a_1, a_4, a_8, a_{11}\}$ is larger
- In fact, it is a largest subset of mutually compatible activities; another largest subset is $\{a_2, a_4, a_9, a_{11}\}$

The optimal substructure of the activity-selection problem

- Let S_{ij} be the set of activities that start after a_i finishes and that finish before a_j starts
- We wish to find a maximum set of mutually compatible activities in S_{ij}
- Suppose that such a maximum set is A_{ij}, which includes some activity a_k
- By including a_k in an optimal solution, we are left with two subproblems: finding mutually compatible activities in the set S_{ik} and finding mutually compatible activities in the set S_{kj}

- Let $A_{ik} = A_{ij} \cap S_{ik}$ and $A_{kj} = A_{ij} \cap S_{kj}$, so that
  - A_{ik} contains the activities in A_{ij} that finish before a_k starts and
  - A_{kj} contains the activities in A_{ij} that start after a_k finishes
- Thus, $A_{ij} = A_{ik} \cup \{a_k\} \cup A_{kj}$, and so the maximum-size set A_{ij} in S_{ij} consists of $|A_{ij}| = |A_{ik}| + |A_{kj}| + 1$ activities
- The usual cut-and-paste argument shows that the optimal solution A_{ij} must also include optimal solutions for S_{ik} and S_{kj}

- This suggests that we might solve the activity-selection problem by dynamic programming
- If we denote the size of an optimal solution for the set S_{ij} by $c[i,j]$, then we would have the recurrence

$$c[i,j] = c[i,k] + c[k,j] + 1$$

- Of course, if we did not know that an optimal solution for the set S_{ij} includes activity a_k, we would have to examine all activities in S_{ij} to find which one to choose, so that

$$c[i,j] = \begin{cases} 0 & \text{if } S_{ij} = \emptyset \\ \max\limits_{a_k \in S_{ij}} \{c[i,j] = c[i,k] + c[k,j] + 1\} & \text{if} S_{ij} \neq \emptyset \end{cases}$$

Making the greedy choice

- For the activity-selection problem, we need consider only the greedy choice
- We choose an activity that leaves the resource available for as many other activities as possible
- Now, of the activities we end up choosing, one of them must be the first one to finish
- Choose the activity in S with the earliest finish time, since that leaves the resource available for as many of the activities that follow it as possible
- Activities are sorted in monotonically increasing order by finish time; greedy choice is activity a_1

- If we make the greedy choice, we only have to find activities that start after a_1 finishes
- $s_1 < f_1$ and f_1 is the earliest finish time of any activity $\Rightarrow$ no activity can have a finish time $\leq s_1$
- Thus, all activities that are compatible with activity a_1 must start after a_1 finishes
- Let $S_k = \{a_i \in S: s_i \geq f_k\}$ be the set of activities that start after a_k finishes
- Optimal substructure: if a_1 is in the optimal solution, then an optimal solution to the original problem consists of a_1 and all the activities in an optimal solution to the subproblem S_1

Theorem 16.1 *Consider any nonempty subproblem S_k, and let a_m be an activity in S_k with the earliest finish time. Then a_m is included in some maximum-size subset of mutually compatible activities of S_k.*

Proof Let A_k be a max-size subset of mutually compatible activities in S_k, and let a_j be the activity in A_k with the earliest finish time. If $a_j = a_m$, we are done, since a_m is in a max-size subset of mutually compatible activities of S_k.

If $a_j \neq a_m$, let the set $A'_k = A_k - \{a_j\} \cup \{a_m\}$. The activities in A'_k are disjoint because the activities in A_k are disjoint, a_j is the first activity in A_k to finish, and $f_m \leq f_j$. Since $|A'_k| = |A_k|$, we conclude that A'_k is a maximum-size subset of mutually compatible activities of S_k and includes a_m. ■

- We can repeatedly choose the activity that finishes first, keep only the activities compatible with this activity, and repeat until no activities remain
- Moreover, because we always choose the activity with the earliest finish time, the finish times of the activities we choose must strictly increase
- We can consider each activity just once overall, in monotonically increasing order of finish times

A recursive greedy algorithm

RECURSIVE-ACTIVITY-SELECTOR(s, f, k, n)

1. $m \leftarrow k + 1$

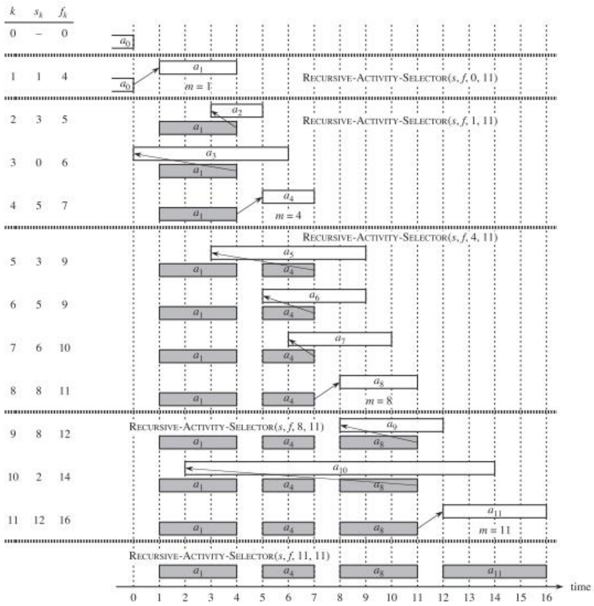
2. **while** $m \leq n$ **and** $s[m] < f[k]$ // find the first
// activity in S_k to finish

3. $m \leftarrow m + 1$

4. **if** $m \leq n$

5. **return** $\{a_m\} \cup$ RECURSIVE-ACTIVITY-SELECTOR(s, f, m, n)

6. **else return** $\emptyset$



An iterative greedy algorithm

GREEDY-ACTIVITY-SELECTOR(s, f)

$n \leftarrow s.length$
$A \leftarrow \{a_1\}$
$k \leftarrow 1$
for $m \leftarrow 2$ **to** n
    if $s[m] \geq f[k]$
        $A \leftarrow A \cup \{a_m\}$
        $k \leftarrow m$
return A

- The set A returned by the call

  GREEDY-ACTIVITY-SELECTOR(s, f)

  is precisely the set returned by the call

  RECURSIVE-ACTIVITY-SELECTOR(s, f, k, n)

- Both the recursive version and the iterative algorithm schedule a set of n activities in $\Theta(n)$ time, assuming that the activities were already sorted initially by their finish times