# 17.4 Dynamic tables

- Let us now study the problem of dynamically expanding and contracting a table
- We show that the amortized cost of insertion/ deletion is only $O(1)$
- Though the actual cost of an operation is large when it triggers an expansion or a contraction
- Moreover, we see how to guarantee that the unused space in a dynamic table never exceeds a constant fraction of the total space



- Let the dynamic table support the operations TABLE-INSERT and TABLE-DELETE
- It is convenient to use the ***load factor*** $\alpha(T)$
  - $\alpha(T)$ of a nonempty table $T$ is the number of items stored in the table divided by the size (number of slots) of the table
- We assign an empty table size 0, and we define its load factor to be 1
- If $\alpha(T)$ is bounded below by a constant, the unused space in $T$ is never more than a constant fraction of the total amount of space

# 17.4.1 Table expansion

- Storage for a table is allocated as array of slots
- A table fills up when all slots have been used
  - equivalently, when its load factor is 1
- Upon inserting an item into a full table, we can expand the table by allocating a new table with more slots than the old table had
- We need the table to reside in contiguous memory, thus, we must allocate a new array and then copy items into the new table



- A common heuristic allocates a new table with twice as many slots as the old one
- If the only operations are insertions, then the load factor is always at least ½, and the amount of wasted space never exceeds half the space in the table
- The attribute $T.table$ contains a pointer to the block of storage representing the table
- $T.num$ contains the number of items in the table
- $T.size$ gives the total number of slots in the table
- Initially, the table is empty: $T.num = T.size = 0$

TABLE-INSERT$(T, x)$

**1. if** $T.size = 0$

2. allocate $T.table$ with 1 slot

3. $T.size \leftarrow 1$

**4. if** $T.num = T.size$

5. allocate *new–table* with $2 \cdot T.size$ slots

6. insert all items in $T.table$ into *new–table*

7. free $T.table$

8. $T.table \leftarrow$ *new–table*

9. $T.size \leftarrow 2 \cdot T.size$

10. insert $x$ into $T.table$

11. $T.num \leftarrow T.num + 1$



- A sequence of $n$ TABLE-INSERT operations on an initially empty table:
  - If the current table has room for the new item, then the cost $c_i$ of the $i$th operation is 1, since we only perform one elementary insertion
  - If the current table is full an expansion occurs, then $c_i = i$: cost of 1 for the elementary insertion plus $i - 1$ for the items that we copy from the old table to the new table
  - The worst-case cost of an operation is $O(n)$ $\Rightarrow$ upper bound of $O(n^2)$ on the total running time for $n$ operations

- This bound is not tight, we rarely expand the table in the course of $n$ TABLE-INSERT operations
- The $i$th operation causes an expansion only when $i - 1$ is an exact power of 2
- The amortized cost of an operation is in fact $O(1)$, as we can show using aggregate analysis
- The cost of the $i$th operation is

$$c_i = \begin{cases} i & \text{if } i-1 \text{ is an exact power of 2} \\ 1 & \text{otherwise} \end{cases}$$



- The total cost of $n$ TABLE-INSERT operations is therefore

$$\sum_{i=1}^{n} c_i \leq n + \sum_{j=0}^{\lfloor \lg n \rfloor} 2^j < n + 2n = 3n,$$

because at most $n$ operations cost 1 and the costs of the remaining operations form a geometric series

- Since the total cost of $n$ TABLE-INSERT operations is bounded by $3n$, the amortized cost of a single operation is at most 3

- By using the accounting method, we can gain some feeling for why the amortized cost of a TABLE-INSERT operation should be 3
- Intuitively, each item pays for 3 elementary insertions:
  - inserting itself into the current table,
  - moving itself when the table expands, and
  - moving another item that has already been moved once when the table expands
- For example, suppose that the size of the table is $m$ immediately after an expansion
- Then it holds $m/2$ items, and contains no credit



- We charge 3 units of cost for each insertion
  - The elementary insertion that occurs immediately costs 1 unit
  - We place another unit as credit on the item inserted
  - We place the third unit as credit on one of the $m/2$ items already in the table
- The table will not fill again until we have inserted another $m/2 - 1$ items, and thus, by the time the table contains $m$ items and is full, we will have placed a unit on each item to pay to reinsert it during the expansion

- Use the potential method to analyze a sequence of $n$ TABLE-INSERT operations
- Potential function $\Phi$ is 0 after an expansion but builds to table size by the time the table is full
- $\Phi(T) = 2 \cdot T.num - T.size$ is one possibility
- Immediately after an expansion, we have $T.num = T.size/2$, and $\Phi(T) = 0$, as desired
- Before expansion, we have $T.num = T.size$, and $\Phi(T) = T.num$ as desired
- Table is always at least half full, $T.num \geq T.size/2$, which implies that $\Phi(T)$ is always nonnegative
- The sum of the amortized costs of $n$ TABLE-INSERTS upper bounds the sum of the actual costs



- Let, after the $i$th operation,
  – $num_i$ be the number of items stored in the table,
  – $size_i$ be the total size of the table, and
  – $\Phi_i$ be the potential after the operation
- Initially, $num_0 = 0$, $size_0 = 0$, and $\Phi_0 = 0$
- If the $i$th TABLE-INSERT operation does not trigger an expansion, then we have $size_i = size_{i-1}$ and the amortized cost of the operation is

$$\begin{aligned}\widehat{c_i} &= c_i + \Phi_i - \Phi_{i-1} \\ &= 1 + (2 \cdot num_i - size_i) - (2 \cdot num_{i-1} - size_{i-1}) \\ &= 1 + (2 \cdot num_i - size_i) - (2(num_i - 1) - size_i) \\ &= 3\end{aligned}$$

- If the $i$th operation **does** trigger an expansion,

$$size_i = 2 \cdot size_{i-1}$$

and

$$size_{i-1} = num_{i-1} = num_i - 1,$$

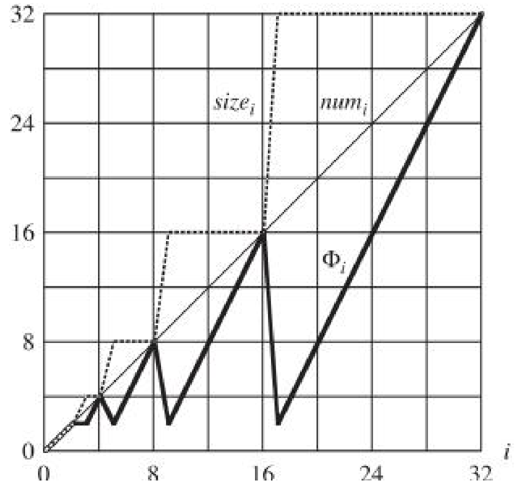
which implies that

$$size_i = 2(num_i - 1)$$

- Thus, the amortized cost of the operation is

$$\hat{c_i} = c_i + \Phi_i - \Phi_{i-1}$$
$$= num_i + (2 \cdot num_i - size_i) - (2 \cdot num_{i-1} - size_{i-1})$$
$$= num_i + (2 \cdot num_i - 2(num_i - 1)) - (2(num_i - 1) - (num_i - 1))$$
$$= num_i + 2 - (num_i - 1) = 3$$





## 17.4.2 Table expansion and contraction

- To implement TABLE-DELETE, it is enough to remove the specified item from the table
- To limit wasted space, we wish to ***contract*** the table when the load factor becomes too small
- Table contraction is analogous to expansion
- Ideally, we would like to preserve two properties:
  - the load factor of the dynamic table is bounded below by a positive constant, and
  - the amortized cost of a table operation is bounded above by a constant



- One might double the table size upon insertion into a full table and halve the size when a deletion would cause the table to become less than half full
- This would guarantee that the load factor is always above ½, but can cause quite large amortized cost
- Consider that we perform $n$ operations on a table $T$, where $n$ is an exact power of 2
- The first $n/2$ operations are insertions, which by our previous analysis cost a total of $\Theta(n)$
- At the end of this sequence, $T.num = T.size = n/2$
- For the second $n/2$ operations, we perform the following sequence: insert, delete, delete, insert, insert, delete, delete, insert, insert, …

- First the table expands to size $n$
- The two following deletions cause the table to contract back to size $n/2$
- Further insertions cause another expansion, …
- The cost of each expansion and contraction is $\Theta(n)$, and there are $\Theta(n)$ of them
- Thus, the total cost of the $n$ operations is $\Theta(n^2)$, making the amortized cost of an operation $\Theta(n)$
- Downside is that after expanding, we do not delete enough items to pay for contraction
- Likewise, for contracting the table



- Allow the load factor to drop below ½
- Still double the table size upon insertion into a full table, but halve the size when deletion causes the table to become less than ¼ full
- The load factor of the table is therefore bounded below by the constant ¼
- Intuitively, a load factor of ½ seems to be ideal, and the table's potential would then be 0
- As the load factor deviates from ½, the potential increases so that by the time we change the table, it has garnered sufficient potential to pay for copying all the items

- We need a potential function that has grown to $T.num$ by the time that the load factor has either increased to 1 or decreased to ¼
- After either expanding or contracting the table, the load factor goes back to ½ and the table's potential reduces back to 0
- Code for TABLE-DELETE is analogous to TABLE-INSERT
- We assume that whenever the number of items in the table drops to 0, we free the storage for the table
- That is, if $T.num = 0$, then $T.size = 0$



- Let us denote the load factor of a nonempty table $T$ by $\alpha(T) = T.num/T.size$
- Since for an empty table, $T.num = T.size = 0$ and $\alpha(T) = 1$, we always have $T.num = \alpha(T) \cdot T.size$, whether the table is empty or not
- We shall use as our potential function

$$\Phi(T) = \begin{cases} 2 \cdot T.num - T.size & \text{if } \alpha(T) \geq 1/2 \\ T.size/2 - T.num & \text{if } \alpha(T) < 1/2 \end{cases}$$

- The potential of an empty table is 0 and it is never negative
- The total amortized cost of a sequence w.r.t. $\Phi$ provides an upper bound on the actual cost

- When the load factor is $1/2$, the potential is $0$
- When $\alpha(T) = 1$, we have $T.size = T.num$, which implies $\Phi(T) = T.num$, and the potential can pay for an expansion if an item is inserted
- When $\alpha(T) = 1/4$, we have $T.size = 4 \cdot T.num$, which implies $\Phi(T) = T.num$, and the potential can pay for a contraction if an item is deleted
- When the $i$th operation is TABLE-INSERT the analysis is identical to the earlier one for table expansion if $\alpha_{i-1} \geq 1/2$
  – Whether the table expands or not, the amortized cost of the operation $\hat{c}_i \leq 3$



- If $\alpha_{i-1} < 1/2$, the table cannot expand, since it expands only when $\alpha_{i-1} = 1$
- If $\alpha_i < 1/2$ as well, then the amortized cost of the $i$th operation (INSERT) is

$$\begin{aligned}\hat{c}_i &= c_i + \Phi_i - \Phi_{i-1}\\ &= 1 + (size_i/2 - num_i) - (size_{i-1}/2 - num_{i-1})\\ &= 1 + (size_i/2 - num_i) - (size_i/2 - (num_i - 1))\\ &= 0\end{aligned}$$

- If $\alpha_{i-1} < 1/2$, but $\alpha_i \geq 1/2$, then

$$\hat{c}_i = c_i + \Phi_i - \Phi_{i-1}$$
$$= 1 + (2 \cdot num_i - size_i) - (size_{i-1}/2 - num_{i-1})$$
$$= 1 + (2(num_{i-1} + 1) - size_{i-1}) - (size_{i-1}/2 - num_{i-1})$$
$$= 3 \cdot num_{i-1} - \frac{3}{2} size_{i-1} + 3$$
$$= 3\alpha_{i-1} size_{i-1} - \frac{3}{2} size_{i-1} + 3$$
$$< \frac{3}{2} size_{i-1} - \frac{3}{2} size_{i-1} + 3$$
$$= 3$$



- Thus, the amortized cost of a TABLE-INSERT operation is at most 3
- When the $i$th operation is a TABLE-DELETE, the amortized cost is also bounded above by a constant
- In summary, since the amortized cost of each operation is bounded above by a constant, the actual time for any sequence of $n$ operations on a dynamic table is $O(n)$

# V Advanced Data Structures

Fibonacci Heaps

## 19 Fibonacci Heaps

1. The Fibonacci heap data structure supports a set of operations that constitutes what is known as a “mergeable heap”
2. Several Fibonacci-heap operations run in constant amortized time, which makes this data structure well suited for applications that invoke these operations frequently

## Mergeable heaps

- Support the following operations, each element has a key:
- MAKE-HEAP$()$ creates and returns a new empty heap
- INSERT$(H, x)$ inserts element $x$, whose key has already been filled in, into heap $H$
- MINIMUM$(H)$ returns a pointer to the element in heap $H$ whose key is minimum
- EXTRACT-MIN$(H)$ deletes the element from heap $H$ whose key is minimum, returning a pointer to the element



- UNION$(H_1, H_2)$ creates and returns a new heap that contains all the elements of heaps $H_1$ and $H_2$. Heaps $H_1$ and $H_2$ are “destroyed” by this operation
- Fibonacci heaps also support the following two operations:
- DECREASE-KEY$(H, x, k)$ assigns to element $x$ within heap $H$ the new key value $k$, which we assume to be no greater than its current key value
- DELETE$(H, x)$ deletes element $x$ from heap $H$

| Procedure | Binary Heap (worst-case) | Fibonacci Heap (amortized) |
|---|---|---|
| MAKE-HEAP | $\Theta(1)$ | $\Theta(1)$ |
| INSERT | $\Theta(\lg n)$ | $\Theta(1)$ |
| MINIMUM | $\Theta(1)$ | $\Theta(1)$ |
| EXTRACT-MIN | $\Theta(\lg n)$ | $\Theta(\lg n)$ |
| UNION | $\Theta(n)$ | $\Theta(1)$ |
| DECREASE-KEY | $\Theta(\lg n)$ | $\Theta(1)$ |
| DELETE | $\Theta(\lg n)$ | $\Theta(\lg n)$ |



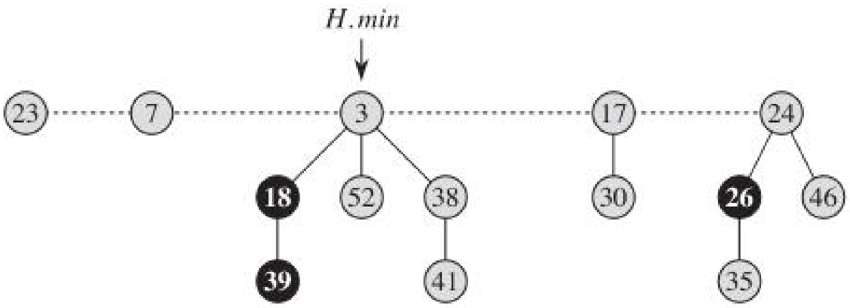
## Fibonacci heaps in theory and practice

- Fibonacci heaps are especially desirable when the number of EXTRACT-MIN and DELETE operations is small relative to the number of other operations performed
- E.g., some algorithms for graph problems may call DECREASE-KEY once per edge
- For dense graphs, with many edges, the $\Theta(1)$ amortized time of each call of DECREASE-KEY is a big improvement over the $\Theta(\lg n)$ worst-case time of binary heaps
- Fast algorithms for problems such as computing minimum spanning trees and finding single-source shortest paths make essential use of Fibonacci heaps

- The constant factors and programming complexity of Fibonacci heaps make them less desirable than ordinary binary (or $k$-ary) heaps for most applications, except for certain ones that manage large amounts of data
- Thus, Fibonacci heaps are predominantly of theoretical interest
- If a much simpler data structure with the same amortized time bounds as Fibonacci heaps were developed, it would be of practical use as well

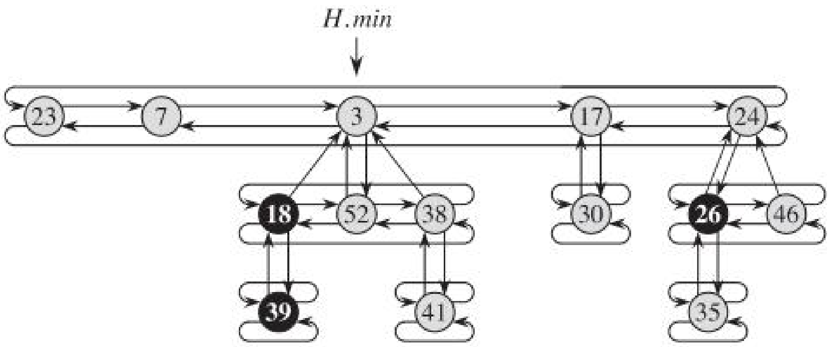


- Fibonacci heaps are based on rooted trees
  - We represent each element by a node within a tree, and each node has a key attribute
  - We use the term “node” instead of “element”
- We also ignore issues of allocating nodes prior to insertion and freeing nodes following deletion
- A Fibonacci heap is a collection of rooted trees that are min-heap ordered
- Each tree obeys the min-heap property:
  - the key of a node is greater than or equal to the key of its parent






- Each node $x$ contains a pointer $x.p$ to its parent and a pointer $x.child$ to any one of its children
- The children of $x$ are linked together in a circular, doubly linked list – the ***child list*** of $x$
- Each child $y$ in a child list has pointers $y.left$ and $y.right$ that point to $y$'s left and right siblings, respectively
- If $y$ is an only child, then $y.left = y.right = y$
- Siblings may appear in a child list in any order






- We store the number of children in the child list of node $x$ in $x.degree$
- The Boolean attribute $x.mark$ indicates whether node $x$ has lost a child since the last time $x$ was made the child of another node
- Newly created nodes are unmarked, and a node $x$ becomes unmarked whenever it is made the child of another node
- Until we look at the DECREASE-KEY operation we will just set all mark attributes to FALSE
- We access a given Fibonacci heap $H$ by a pointer $H.\mathrm{min}$ to the root of a tree containing the minimum key

- When a Fibonacci heap $H$ is empty, $H.\mathrm{min}$ is NIL
- The roots of all the trees in a heap are linked together using their left and right pointers into a circular, doubly linked list called the ***root list***
- The pointer $H.\mathrm{min}$ thus points to the node in the root list whose key is minimum
- Trees may appear in any order within a root list
- We rely on one other attribute for a Fibonacci heap $H$: $H.n$, the number of nodes currently in $H$



## Potential function

- We use the potential method to analyze the performance of Fibonacci heap operations
- Let $t(H)$ be the number of trees in the root list of Fibonacci heap $H$ and $m(H)$ the number of marked nodes in $H$
- We define the potential $\Phi(H)$ of heap $H$ by

$$\Phi(H) = t(H) + 2m(H)$$

- For example, the potential of the Fibonacci heap shown above is $5 + 2 \cdot 3 = 11$

- The potential of a set of Fibonacci heaps is the sum of the potentials of its constituent heaps
- We assume that a unit of potential can cover the cost of any of the specific constant-time pieces of work that we might encounter
- Fibonacci heap application begins with no heaps
- The initial potential, therefore, is 0, and the potential is nonnegative at all subsequent times
- An upper bound on the total amortized cost thus provides an upper bound on the total actual cost for the sequence of operations



## Maximum degree

- Amortized analyses we perform assume that we know an upper bound $D(n)$ on the maximum degree of any node in an $n$-node Fibonacci heap
- When only the mergeable-heap operations are supported

$$D(n) \leq \lfloor \lg n \rfloor$$

- We show that when we support DECREASE-KEY and DELETE as well, $D(n) = O(\lg n)$

## 19.2 Mergeable-heap operations

- The operations delay work as long as possible; various operations have performance trade-offs
- E.g., we insert a node by adding it to the root list, which takes just constant time
- If we insert $k$ nodes to an empty Fibonacci heap $H$, the heap consist of just a root list of $k$ nodes
- **Trade-off:** if we then perform EXTRACT-MIN on $H$, after removing the node that $H.\text{min}$ points to, we have to look through each of the remaining $k - 1$ nodes to find the new minimum node



- As long as we have to go through the entire root list during the EXTRACT-MIN operation,
  - we also consolidate nodes into min-heap-ordered trees to reduce the size of the root list
- We shall see that, no matter what the root list looks like before a EXTRACT-MIN operation,
  - afterward each node in the root list has a degree that is unique within the root list, which leads to a root list of size at most $D(n) + 1$

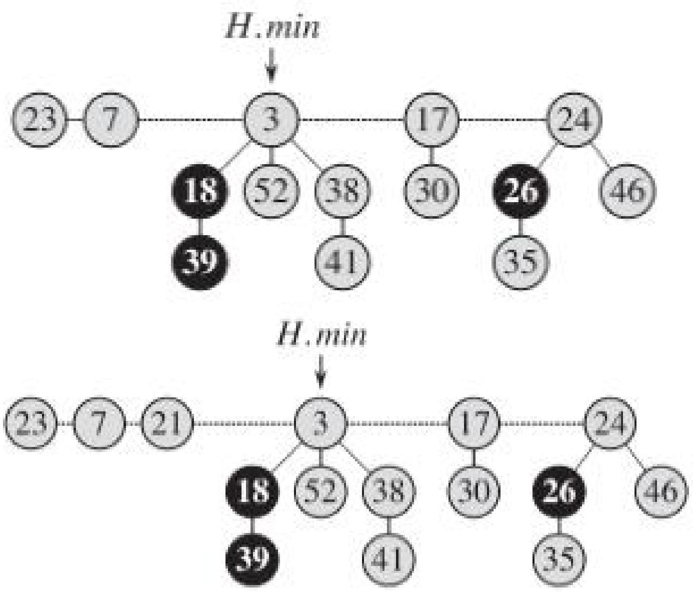
## Creating a new Fibonacci heap

- To make an empty Fibonacci heap, the MAKE-FIB-HEAP procedure allocates and returns the Fibonacci heap object $H$, where $H.n = 0$ and $H.min =$ NIL; there are no trees in $H$
- Because $t(H) = 0$ and $m(H) = 0$, the potential of the empty Fibonacci heap is $\Phi(H) = 0$
- The amortized cost of MAKE-FIB-HEAP is thus equal to its $O(1)$ actual cost



FIB-HEAP-INSERT$(H, x)$

1. $x.degree \leftarrow 0$
2. $x.p \leftarrow$ NIL
3. $x.child \leftarrow$ NIL
4. $x.mark \leftarrow$ FALSE
5. **if** $H.min =$ NIL
6. create a root list for $H$ containing just $x$
7. $H.\text{min} \leftarrow x$
8. **else** insert $x$ into $H$'s root list
9. **if** $x.key < H.\text{min}.key$
10. $H.\text{min} \leftarrow x$
11. $H.n \leftarrow H.n + 1$






- To determine the amortized cost of FIB-HEAP-INSERT, let $H$ be the input Fibonacci heap and $H'$ be the resulting Fibonacci heap
- Then, $t(H') = t(H) + 1$ and $m(H') = m(H)$, and the increase in potential is
  $\big((t(H) + 1) + 2m(H)\big) - \big(t(H) + 2m(H)\big) = 1$
- Since the actual cost is $O(1)$, the amortized cost is

$$O(1) + 1 = O(1)$$

FIB-HEAP-UNION$(H_1, H_2)$

1. $H \leftarrow$ MAKE–FIB–HEAP()
2. $H.\text{min} \leftarrow H_1.\text{min}$
3. concatenate the root list of $H_2$ with the root list of $H$
4. **if** $(H_1.\text{min} = \text{NIL})$ **or** $(H_2.\text{min} \neq \text{NIL}$ **and** $H_2.\text{min}.key < H_1.\text{min}.key)$
5. $\quad H.\text{min} \leftarrow H_2.\text{min}$
6. $H.n \leftarrow H_1.n + H_2.n$
7. **return** $H$



- The change in potential is

  $\Phi(H) - (\Phi(H_1) + \Phi(H_2))$

  $= (t(H) + 2m(H)) - \big((t(H_1) + 2m(H_1)) + (t(H_2) + 2m(H_2))\big)$

  $= 0$
- because $t(H) = t(H_1) + t(H_2)$ and $m(H) = m(H_1) + m(H_2)$
- The amortized cost of FIB-HEAP-UNION is therefore equal to its $O(1)$ actual cost

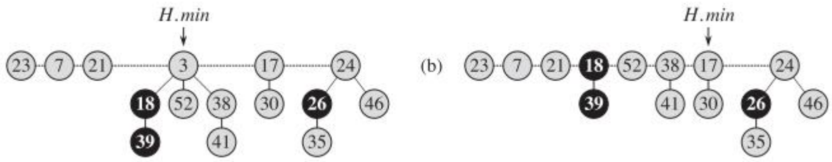
## Extracting the minimum node

- The process of extracting the minimum node is the most complicated of the operations so far
- It is also where the delayed work of consolidating trees in the root list finally occurs
- The following code assumes that when a node is removed, pointers remaining in the linked list are updated, but pointers in the extracted node are left unchanged
- It also calls the auxiliary procedure CONSOLIDATE



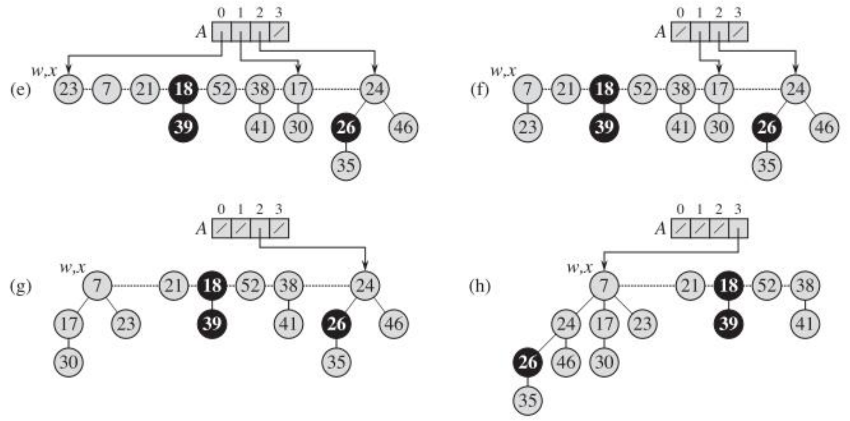
FIB-HEAP-EXTRACT-MIN($H$)

1. $z \leftarrow H.\text{min}$
2. **if** $z \neq$ NIL
3.     **for** each child $x$ of $z$
4.         add $x$ to the root list of $H$
5.         $x.p \leftarrow$ NIL
6.     remove $z$ from the root list of $H$
7.     **if** $z = z.right$
8.         $H.\text{min} \leftarrow$ NIL
9.     **else** $H.\text{min} \leftarrow z.right$
10.         CONSOLIDATE($H$)
11.     $H.n \leftarrow H.n - 1$
12. **return** $z$






- CONSOLIDATE$(H)$ reduces the number of trees in the Fibonacci heap
- Consolidating the root list consists of repeatedly executing the following steps until every root in the root list has a distinct degree value:
  1. Find two roots $x$ and $y$ in the root list with the same degree. Without loss of generality, let $x.key \leq y.key$
  2. Remove $y$ from the root list, and make $y$ a child of $x$ by calling the FIB-HEAP-LINK procedure. This procedure increments the attribute $x.degree$ and clears the mark on $y$



(e)
(f)
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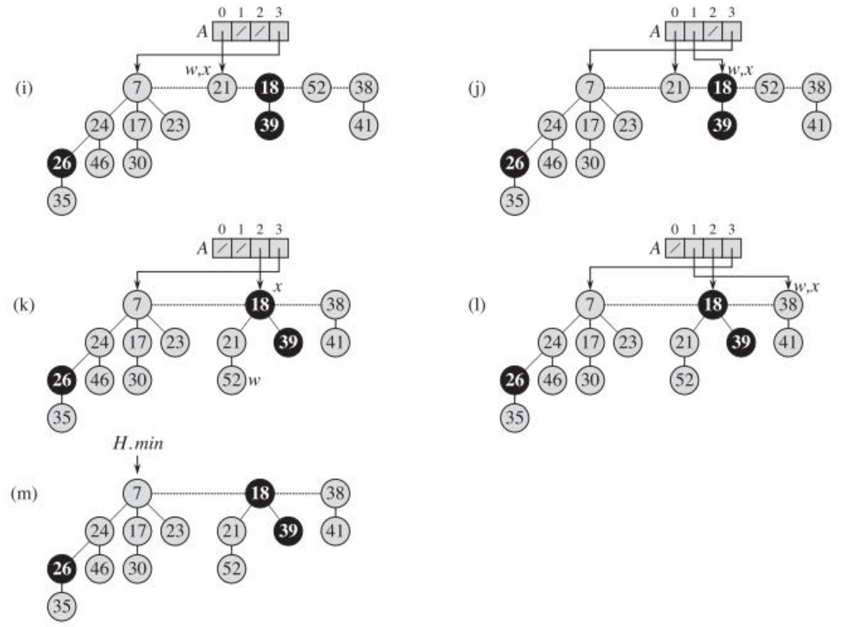
A
w,x



(i)
(j)
(k)
(l)
(m)
A
w,x
x
w
H.min

## Decreasing a key

FIB-HEAP-DECREASE-KEY$(H, x, k)$

1. **if** $k > x.key$
2. **error** "new key is greater than current key"
3. $x.key \leftarrow k$
4. $y \leftarrow x.p$
5. **if** $y \neq$ NIL **and** $x.key < y.key$
6. CUT$(H, x, y)$
7. CASCADING-CUT$(H, y)$
8. **if** $x.key < H.\text{min}.key$
9. $H.\text{min} \leftarrow x$

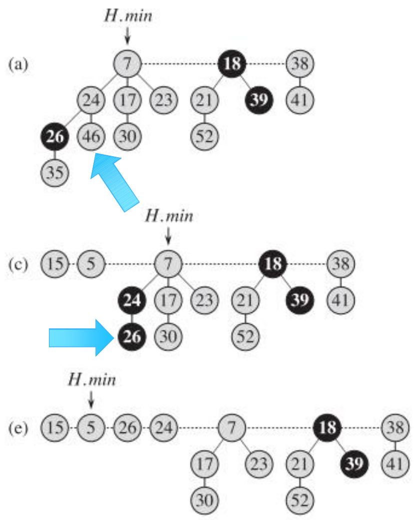


CUT$(H, x, y)$

1. remove $x$ from the child list of $y$, decrementing $y.degree$
2. add $x$ to the root list of $H$
3. $x.p \leftarrow$ NIL
4. $x.mark \leftarrow$ FALSE

CASCADING-CUT$(H, y)$

1. $z \leftarrow y.p$
2. **if** $z \neq$ NIL
3. **if** $y.mark =$ FALSE
4. $y.mark \leftarrow$ TRUE
5. **else** CUT$(H, y, z)$
6. CASCADING-CUT$(H, z)$






- Fib-Heap-Decrease-Key creates a new tree rooted at node $x$ and clears $x$'s mark bit
- Each of the $c$ calls of Cascading-Cut, except the last one, cuts a marked node and clears the mark bit
- Afterward, the heap contains $t(H) + c$ trees
  - the original $t(H)$ trees, $c - 1$ trees produced by cascading cuts, and the tree rooted at $x$

  and at most $m(H) - c + 2$ marked nodes
  - $c - 1$ were unmarked by cascading cuts and the last call of Cascading-Cut may have marked a node

- The change in potential is therefore at most $\big((t(H)+c)+2(m(H)-c+2)\big)-\big(t(H)+2m(H)\big)=4-c$
- Thus, the amortized cost of FIB-HEAP-DECREASE-KEY is at most $O(c)+4\ -c=O(1)$, since we can scale up the units of potential to dominate the constant hidden in $O(c)$
- When a marked node $y$ is cut by a cascading cut, its mark bit is cleared, which reduces the potential by 2
- One unit of potential pays for the cut and the clearing of the mark bit, and the other unit compensates for the unit increase in potential due to node $y$ becoming a root



## Deleting a node

- We assume that there is no key value of $-\infty$ currently in the Fibonacci heap

FIB-HEAP-DELETE$(H, x)$

1. FIB-HEAP-DECREASE-KEY$(H, x, -\infty)$
2. FIB-HEAP-EXTRACT-MIN$(H)$

- The amortized time of FIB-HEAP-DELETE is the sum of the $O(1)$ amortized time of FIB-HEAP-DECREASE-KEY and the $O(D(n))$ amortized time of FIB-HEAP-EXTRACT-MIN