ANSWER QUESTIONS 1 AND 2

1. Let \( AB \) be the language

\[
AB = \{ a^n b^n \mid n \geq 0 \}.
\]

Give well-grounded answers to the following questions.

(a) Is \( AB \) a regular language?
(b) Can language \( AB \) be recognized with a pushdown automaton?
(c) Can language \( AB \) be generated by a context-free grammar?

2. Define the universal language \( U \) over the binary alphabet. Show that \( U \) is not decidable. Take advantage of the knowledge that the "diagonal language"

\[
D = \{ \langle M \rangle \in \{ 0, 1 \}^* \mid \langle M \rangle \notin L(M) \}
\]

is not Turing-recognizable. Is \( U \) Turing-recognizable?
3. Let $A, B \subseteq \Sigma^*$ be decidable languages. Prove that then also languages

$$\overline{A} = \Sigma^* \setminus A, \ A \cup B, \text{ and } A \cap B$$

are decidable.

4. Show that incompressible strings of every length exist.

5. Two numbers are relatively prime if 1 is the largest integer that evenly divides them both. For instance, 10 and 21 are relatively prime. Is recognizing pairs of numbers that are relatively prime a problem that can be solved efficiently? Justify your answer carefully.

6. Prove that if $B$ is NP-complete and $B \leq^p_m C$ for some $C \in \text{NP}$, then also $C$ is NP-complete.