1. Let $f(n)$ and $g(n)$ be asymptotically nonnegative functions. Using the basic definition of $\Theta$-notation, prove that $\max(f(n), g(n)) = \Theta(f(n) + g(n))$.

2. Prove that the running time of an algorithm is $\Theta(g(n))$ if and only if its worst-case running time is $O(g(n))$ and its best-case running time is $\Omega(g(n))$.

3. Prove by induction that the $i$th Fibonacci number satisfies the equality

$$F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}},$$

where $\phi$ is the golden ratio and $\hat{\phi}$ its conjugate.

4. Let $f(n)$ and $g(n)$ be asymptotically positive functions. Prove or disprove each of the following conjectures.

(a) $f(n) = O(g(n))$ implies $g(n) = O(f(n))$.

(b) $f(n) + g(n) = \Theta(\min(f(n), g(n)))$.

(c) $f(n) = O(g(n))$ implies $2^{f(n)} = O\left(2^{g(n)}\right)$.

(d) $f(n) = O((f(n))^2)$.

(e) $f(n) = O(g(n))$ implies $g(n) = \Omega(f(n))$.

(f) $f(n) = \Theta(f(n/2))$.

(g) $f(n) = o(f(n)) = \Theta(f(n))$.

5. Show that the solution of $T(n) = T(n-1) + n$ is $O(n^2)$.

6. Show that the solution to $T(n) = 2T([n/2] + 17) + n$ is $O(n \log n)$.