# Note on Connections between Active Contours and Rayleigh Quotients

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**ABSTRACT**

*A global optimization approach to active contours is necessary if images to be analyzed have low signal to noise ratio. In this setting, it is reasonable to study global properties of energy functions to be optimized. A simple connection between internal energy functions of active contour models of a certain type and Rayleigh quotients is derived in this paper. The importance of Rayleigh quotients lies in the fact that they are related to eigenvalues of real symmetric matrices. As a consequence, one can study the internal energy of an active contour model with numerical routines that are designed for eigenvalue computations of real symmetric matrices.*

## 1. INTRODUCTION

Deformable models [1] are widely used techniques in image analysis and processing. Particularly active contours [2], also termed snakes, have received a lot of attention. The idea behind snakes is to regularize edge-detection by imposing soft constraints on the shape of the contour to be extracted. This way it is possible to find a contour from a noisy image without knowing its exact shape or position. Active contours are frequently applied in medical image analysis [3], but also other applications exist [1].

To be more precise a snake is a curve with an associated energy function. A contour extraction from an image is formulated as the minimization of the energy function. The energy is divided into the internal energy and the external energy. The external energy is derived from image data. The internal energy depends only on the shape of the curve hence regularizing the often ill-posed problem.

The internal energy for the original snake-model [2] was not invariant to scaling of the curve in order to reduce sensitivity to initialization imposed by the applied local minimization technique. For most of the applications, this solution is not satisfactory, see for example [4], [5]. A possible solution is to minimize the energy globally and set hard constraints to ensure admissibility of the resulting curve. Normally, this requires the internal energy to be invariant to translation, rotation and scaling of the curve.

For implementation, it is convenient to approximate the curve by a polygon, which is completely described by its vertices. This simple representation is yet a powerful one. It permits one to incorporate detailed prior information about the expected shape of the target to be delineated in the internal energy of the snake [6], [7]. However, further analysis of the internal energy function is often omitted. The analysis of its global behaviour may prove to be important, especially as increased computation power and improved algorithms allow more efficient energy minimization. The intention here is to show that the internal energy of the snake can be interpreted as a Rayleigh quotient [8]. Rayleigh quotients relate to the eigenvalue problem for symmetric matrices for which there are a number of algorithms and software. Hence, the simple connection provides a fast way to obtain information about the specific snake model. Assumptions required are not prohibitive and many active contour models with little or no modification will satisfy them.

## 2. SNAKES AND RAYLEIGH QUOTIENTS

A snake is an ordered set of points $\mathbf{V} = \{\mathbf{v}_0, \ldots, \mathbf{v}_{N-1}\}$, where each *snaxel* $\mathbf{v}_i = [x_i, y_i]^T \in \mathbb{R}^2$. Only closed contours are considered and hence subscript arithmetic is modulo $N$. The energy of the snake is

$$E(\mathbf{V}) = \lambda E_{int}(\mathbf{V}) + (1-\lambda) E_{ext}(\mathbf{V}), \tag{1}$$

where $E_{int}$ is the internal energy, $E_{ext}$ is the external energy and $\lambda \in [0,1]$ is the regularization parameter. The internal energy is

$$\begin{aligned} &E_{int}(\mathbf{V}) \qquad (2)\\ &= \frac{\sum_{i=0}^{N-1} E_{int}(\mathbf{v}_i|\mathbf{v}_0,\ldots,\mathbf{v}_{i-1},\mathbf{v}_{i+1},\ldots,\mathbf{v}_{N-1})}{l(\mathbf{V})}\\ &= \frac{\sum_{i=0}^{N-1} ||\sum_{j=0}^{N-1} A_{i_j}\mathbf{v}_j - \mathbf{v}_i||^2}{\sum_{i=0}^{N-1} ||\mathbf{v}_{i+1}-\mathbf{v}_i||^2}, \end{aligned}$$

where all $A_{i_j}$ are $2\times 2$ matrices (with the convention that $A_{i_i}=0$). The purpose of the *normalization factor* $l(\mathbf{V})$ is to yield a scale invariant $E_{int}$. At least active contour models presented in [6] and [7] have internal energies, which can be written in a form (2).

If we now assume that the internal energy (2) is translation invariant, we can interpret it as a Rayleigh quotient. For this, let $\mathbf{s} = [\mathbf{v}_0^T,\ldots,\mathbf{v}_{N-1}^T]^T = [x_0,y_0,x_1,y_1,\ldots,x_{N-1},y_{N-1}]^T$. Now (2) can be written as:

$$E_{int}(\mathbf{V}) = \frac{||B\mathbf{s}||^2}{l(\mathbf{V})}, \qquad (3)$$

where

$$B = \begin{bmatrix} -I & A_{0_1} & \cdots & & A_{0_{N-1}}\\ A_{1_0} & -I & A_{1_2} & \cdots & A_{1_{N-1}}\\ \vdots & & \ddots & & \vdots\\ A_{N-1_0} & \cdots & & & -I \end{bmatrix}$$

and $I$ is the $2\times 2$ identity matrix. However, this is still not what we are after; $l(\mathbf{V})$ can be zero even if $\mathbf{s}$ is not. Therefore, recalling that the internal energy is translation invariant, we assume $\mathbf{v}_0 = \mathbf{0}$ without any loss of generality. Let $\mathbf{z} = [x_1,y_1,\ldots,x_{N-1},y_{N-1}]^T$. Now

$$\begin{aligned} ||B\mathbf{s}|| &= ||B[0,0,\mathbf{z}^T]^T||\\ &= ||[\mathbf{b}_1,\ldots,\mathbf{b}_{2N}][0,0,\mathbf{z}^T]^T||\\ &= ||[\mathbf{b}_3,\ldots,\mathbf{b}_{2N}]\mathbf{z}|| = ||\hat{B}\mathbf{z}||. \end{aligned}$$

The normalization factor $l(\mathbf{V})$ is a quadratic form:

$$l(\mathbf{V}) = \sum_{i=2}^{N-2} ||\mathbf{v}_{i+1}-\mathbf{v}_i||^2 + ||\mathbf{v}_2||^2 + ||\mathbf{v}_{N-1}||^2 = \mathbf{z}^T L\mathbf{z},$$

where $L$ is $2N-2\times 2N-2$ non-singular matrix. The matrix $L$ is also positive definite and hence there is a positive definite matrix $\sqrt{L}$ such that $\sqrt{L}^2 = L$ [8, Thm. 2.14.2]. The introduction of a new variable $\mathbf{w} = \sqrt{L}\mathbf{z}$ gives the interpretation of (2) as a Rayleigh quotient

$$E_{int}(\mathbf{V}) = \frac{\mathbf{w}^T(\sqrt{L}^{-1})^T\hat{B}^T\hat{B}\sqrt{L}^{-1}\mathbf{w}}{\mathbf{w}^T\mathbf{w}}. \qquad (4)$$

## 3. PROPERTIES OF RAYLEIGH QUOTIENTS

The next theorem connects the Rayleigh quotient (4) and the internal energy (2) to the eigenvalues of the real symmetric matrix $(\sqrt{L}^{-1})^T\hat{B}^T\hat{B}\sqrt{L}^{-1}$, see [8] Theorems 3.2.1 and 3.3.1 and Exercise 1 at page 111.

**Theorem 1** *Let $R$ be real and symmetric square-matrix. The Rayleigh quotient $R(\mathbf{x}) = \frac{\mathbf{x}^T R\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$ is stationary at, and only at, the eigenvectors of the matrix $R$. At an eigenvector $\xi$, $R(\xi)=\mu$, where $\mu$ is the associated eigenvalue. Moreover $\mu_1 = \max R(\mathbf{x}), \mu_0 = \min R(\mathbf{x})$, where $\mu_1$ is the greatest eigenvalue and $\mu_0$ is the least eigenvalue of the matrix $R$.*

Define $\mathbf{V}+\mathbf{W} = \{\mathbf{v}_i+\mathbf{w}_i | i = 0,\ldots,N-1\}$, where $\mathbf{V},\mathbf{W}$ are snakes with $N$ snaxels. Note that when snakes are taken as vectors of $\mathbb{R}^{2N}$ their addition is simply vector addition. The scalar multiplication in $\mathbb{R}^{2N}$ corresponds to the scaling of snakes. If we now set $E_{int}(\mathbf{V}) = \mu_0$ if $l(\mathbf{V}) = 0$, where $\mu_0$ is the least eigenvalue of the related matrix $(\sqrt{L}^{-1})^T\hat{B}^T\hat{B}\sqrt{L}^{-1}$ we obtain a corollary to the Theorem 1.

**Corollary 1** *Let $\mu_0$ be the least eigenvalue of the matrix $(\sqrt{L}^{-1})^T\hat{B}^T\hat{B}\sqrt{L}^{-1}$ related to $E_{int}(\mathbf{V})$. Let the multiplicity of $\mu_0$ be $K$. Then the set of snakes of minimum internal energy $\mathcal{V} = \{\mathbf{V} : E_{int}(\mathbf{V}) = \mu_0\}$ is a vector space. Moreover, if $\mathbf{V}_j, j = 1,\ldots,K$, are $K$ snakes corresponding to $K$ linearly independent eigenvectors associated with $\mu_0$, a basis for $\mathcal{V}$ is*

$$\{\mathbf{V}_j | j = 1\ldots K\} \cup \{\mathbf{X},\mathbf{Y}\},$$

*where* $\mathbf{X} = \{[1,0]^T,\ldots,[1,0]^T\}$ *and* $\mathbf{Y} = \{[0,1]^T,\ldots,[0,1]^T\}$.

The proof of the Corollary is given in the Appendix. Of course, while performing actual computations, one normally does not want to find a contour whose length is zero. However, the above Corollary is still a useful one. For example, it is applied in the Section 4.

## 4. EXPERIMENTS

As an example two particular internal energy functions are analyzed by computing eigenvalues and eigenvectors of the related matrices. The internal energy functions are

$$E^1_{int}(\mathbf{V}) = \frac{\sum_{i=0}^{N-1} ||\mathbf{v}_i - \frac{1}{2}(\mathbf{v}_{i-1}+\mathbf{v}_i)||^2}{l(\mathbf{V})},$$

$$E^2_{int}(\mathbf{V}) = \frac{\sum_{i=0}^{N-1} ||\mathbf{v}_i - \hat{\mathbf{v}}_i||^2}{l(\mathbf{V})},$$

where

$$\hat{\mathbf{v}}_i = \frac{1}{2}(\mathbf{v}_{i-1}+\mathbf{v}_i+\tan\frac{\pi}{N}R_{90}(\mathbf{v}_{i-1}-\mathbf{v}_{i+1}))$$

and $R_{90}$ is 90 degrees rotation matrix. The function $E^1_{int}$ is the discretized version of the curvature term of the internal energy of the original snake model [2]. It has been normalized by $l(\mathbf{V})$ for scale invariance. The function $E^2_{int}$ is from [6]. Symbols $M_i, i = 1,2$, are used when referring to the matrix $(\sqrt{L}^{-1})^T\hat{B}^T\hat{B}\sqrt{L}^{-1}$ corresponding the function $E^i_{int}$.

**Table 1:** Minima and maxima of the two energy functions when the number of snaxels is varied. Minima of $E^2_{int}$ are always zero.

| $N$ | $\min E^1_{int}$ | $\max E^1_{int}$ | $\max E^2_{int}$ |
|---|---|---|---|
| 20 | 0.0245 | 1 | 1.0251 |
| 21 | 0.0222 | 0.9944 | 1.0170 |
| 30 | 0.0109 | 1 | 1.0110 |
| 31 | 0.0102 | 0.9974 | 1.0077 |
| 50 | 0.0039 | 1 | 1.0040 |
| 51 | 0.0038 | 0.9991 | 1.0029 |
| 100 | 0.0010 | 1 | 1.0010 |
| 101 | 0.0010 | 0.9998 | 1.0007 |

Numerical computations were performed by Matlab 5.3 (Mathworks, Natick, MA, U.S.). It uses the EISPACK routines [9] for eigenvalue calculations. Square roots of matrices $L$ were also computed by Matlab. For this, it applies the Parlett-algorithm described in [10, p.384]. The properties of the two internal energy functions that will be presented are based on numerical simulations. Some of these ought to be taken with caution. For example, it is possible to make an error when stating results concerning multiplicities of eigenvalues. We may not notice that two eigenvalues are not equal if they are very close to each other.

Minima and maxima of the both energy functions for several values of $N$ are listed in Table 1. As can be seen from Table 1, their ranges tended to $[0, 1]$ as $N$ increased. The least eigenvalue of $M_1$ had multiplicity 4. Snakes $\mathbf{V}^j, j = 1, \ldots 4$, corresponding some four linearly independent eigenvectors were related by a linear transformation, i.e. $\mathbf{V}^i = T\mathbf{V}^j = \{T\mathbf{v}^j_k | k = 0, \ldots, N-1\}$, where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. Now, noting that the curves $\mathbf{V}^j, j = 1, 2, 3, 4$, all had a shape of an ellipse, by Corollary 1 it follows that all minimum energy curves of $E^1_{int}$ are ellipses. Curves corresponding to all the other eigenvalues of $M_1$ were self-intersecting and hence classified as inadmissible solutions to the problem. Also from the shape of these curves it was clear that all linear combinations of them were also self-intersecting.
For the function $E^2_{int}$ the curve of minimal energy is, by the construction, circle. Our simulation verified the fact. Moreover, since the multiplicity of the least eigenvalue of $M_2$ was 2, we can conclude that circle is the only minimum energy curve of $E^2_{int}$. However, $E^2_{int}$ had also other admissible curves as stationary points. Some of these are shown in Fig. 1.

## 5. DISCUSSION

We have shown how to interpret a scale and translation invariant internal energy of a snake as a Rayleigh quotient. The approach is quite general. For example, the snake models from [6] and [7] can be seen to satisfy our assumptions. The only real restriction of our approach is the choice of normalization factor. Also normalization factors

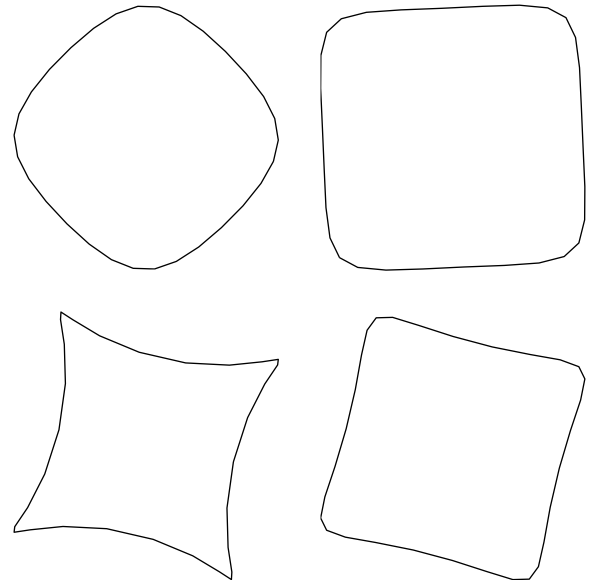

**Figure 1:** Few curves for which $E^2_{int}$ is stationary when $N = 30$.

that do not permit the Rayleigh quotient interpretation can of course be used. However, further studies and discussions about the meaning of the form of the normalization factor are beyond the scope of this paper.

Rayleigh quotients relate to eigenvalues of the real symmetric matrices. Because the symmetric eigenvalue problem is well-studied, the connection allows one to analyze global properties of the internal energy functions of the snake models. Here minima, maxima and stationary points of two internal energy functions were found by using the derived connection. Another function had also admissible, i.e. non-intersecting, curves as stationary points in addition to the ones of minimal energy. This is an interesting result, because it clearly demonstrates a disadvantage of gradient descent techniques for the optimization in the framework of active contours.

## APPENDIX

The proof of Corollary 1 is presented. Snakes $\mathbf{V}_j, j = 1, \ldots, K$, belong to $\mathcal{V}$ by Theorem 1. By assumption that if $l(\mathbf{V}) = 0$ then $E_{int}(\mathbf{V}) = \mu_0$, also $\mathbf{X}, \mathbf{Y} \in \mathcal{V}$. Since the (algebraic) multiplicity and the geometric multiplicity of an eigenvalue of a real symmetric matrix are equal [8], $l(\mathbf{V}_j) \neq 0$ and the first snaxel of $\mathbf{V}_j$ is zero for each $j$, the set $\mathcal{B} = \{\mathbf{V}_j | j = 1 \ldots K\} \cup \{\mathbf{X}, \mathbf{Y}\}$ is linearly independent.

Now let $\mathbf{W} \in \mathcal{V}$ be arbitrary. Then also $\mathbf{V} = \mathbf{W} - x_0\mathbf{X} - y_0\mathbf{Y} \in \mathcal{V}$, where $x_0$ (resp. $y_0$) is the $x$-coordinate ($y$-coordinate) of the first snaxel of $\mathbf{W}$. Furthermore $\mathbf{v}_0 = \mathbf{0}$. Since Rayleigh quotients are differentiable where defined and $\mu_0$ is the minimum of the Rayleigh quotient corresponding to $E_{int}$, from Theorem 1 it follows that there is an eigenvector associated with $\mu_0$ that corresponds to $\mathbf{V}$. Hence, $\mathbf{W}$ belongs to a space spanned by $\mathcal{B}$. Assume now that $\mathbf{W}$ is an arbitrary element of the space spanned by $\mathcal{B}$. Then $\mathbf{W}$ is obtained by a translation from some linear combination of $\mathbf{V}_j, j = 1, \ldots, K$. It follows that $E_{int}(\mathbf{W}) = \mu_0$ and the proof is completed.

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