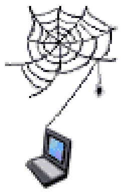
SGN-3057: General Information

Lectures given by Karén Egiazarian

(e-mail: firstname.lastname<at>tut.fi)

- Webpage: **www.cs.tut.fi/courses/SGN-3057/**
- Schedule lectures: **8** weeks, **4** hours/week,
- Time and place: **Mondays 12:15 – 14:00, TB219**

  period IV **Wednesdays, 14:15 – 16:00, TB 222**



Goals

- In-dept view of selected topics of image processing
- Practical tasks in image processing laboratory

General Information (cont.)

Pre-requisites and position

- Basic or introductory signal processing
- Digital Image Processing I or Digitaalinen kuvankäsittely I

Grade formation

- No exam.
- Final mark is computed based on four units as follows:
- Classroom exercises, 40% of the mark
- First laboratory work, 10% of the mark (5th period)
- Second laboratory work, 20% of the mark (5th period)
- Third laboratory work, 30% of the mark (5th period)
- All four units must be passed otherwise final mark is not given

Exercises

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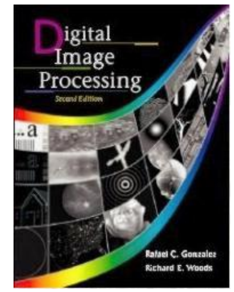
- A total of 7 exercises..
- 30% mandatory to pass the course.
- Two exercise groups
- -Group A: Tuesdays,
  10:15-11:45, TB215
- - Group B: Tuesdays,
  16:15-17:45, TB 215

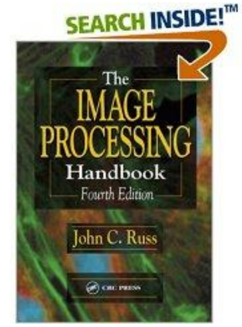
•Each class can take approx. 20-25 people – equal division into groups.

Students prepare the answers at home and present their solution at the whiteboard (with comments) if asked.

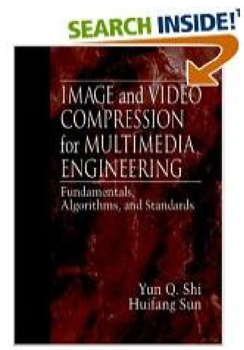
Texts

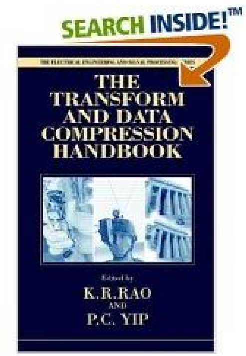
- **Lecture material (see webpage)**



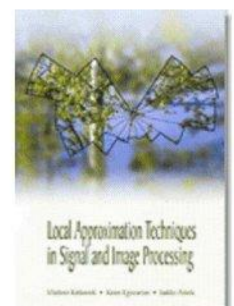


- **Textbook**
  [1] R. Gonzalez and R. Woods, *Digital Image Processing*, 2nd ed., Prentice-Hall, 2002.





- **Additional books**
  [2] John Russ, *The Image Processing Handbook*, Fourth Edition, CRC Press, 2002
  [3] Y. Shi, H. Sun, *Image and Video Compression for Multimedia Engineering: Fundamentals, Algorithms, and Standards*, CRC Press, 1999
  [4] K. Rao and P. Yip, *The Transform and Data Compression Handbook*, CRC Press, 2001
  [5] V.Katkovnik, K.Egiazarian, and J.Astola, *Local Approximations in Signal and Image Processing*, SPIE Press, vol. PM157, 2006.



Wavelets and Multiresolution Processing

- Preview
  - Good old Fourier transform
    - A transform where the basis functions are *sinusoids*, hence localized in frequency but not localized in time; in transform domain the temporal information is lost
  - Short-time Fourier transform
  - Wavelets
    - *Varying* frequency and *limited duration*; an attempt to reveal both the frequency and temporal information in the transform domain
  - Multiresolution analysis
    - Signal (image) representation at *multiple resolutions*

Background

- Images
  - Connected regions of similar texture combined to form different objects
    - Small sized or low in contrast objects: examined at high resolution
    - Large sized or high in contrast: examined at a coarse view
    - Several resolutions needed to distinguished between different objects
  - Mathematically images are 2-D arrays of intensity values of locally varying statistics (resulting from edges, homogenous regions, etc. – see Fig. 7.1)

Histograms computed in local squares

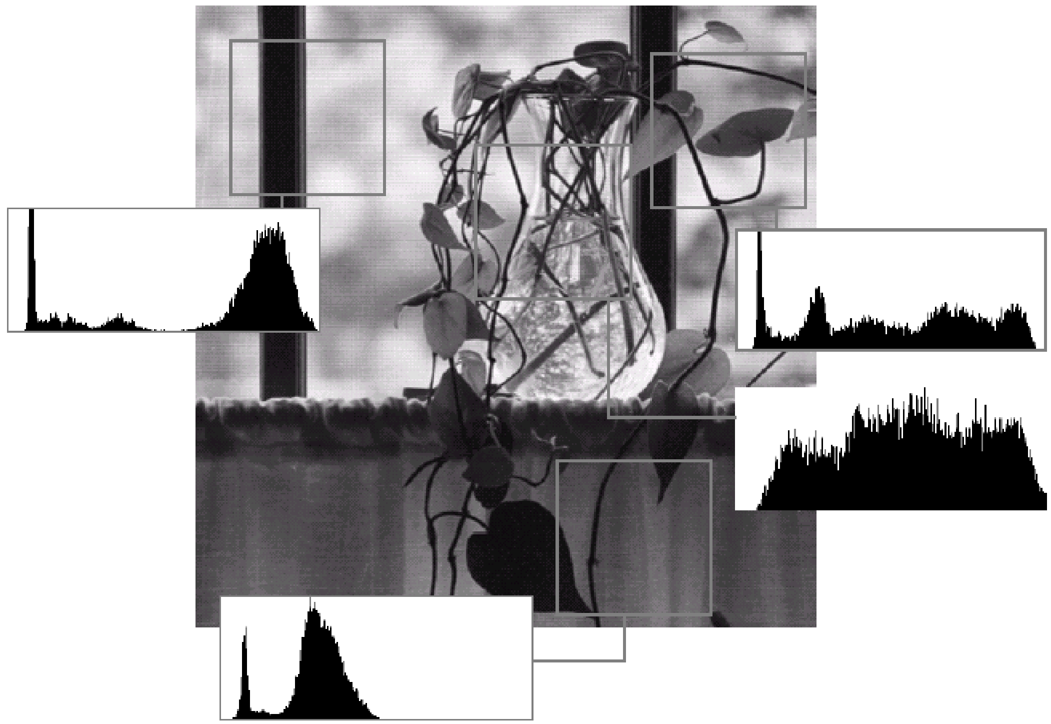


FIGURE 7.1 A natural image and its local histogram variations.

Basic ideas of linear transformation

- To change the coordinate system in which a signal (image/video) is represented in order to make it much better suited for processing (e.g.,compression).
- To represent all the useful signal features and important phenomena in as compact manner as possible.
- Important to compact the bulk of the signal energy into the fewest number of transform coefficients.

Four properties of a Good Transform:

Decorrelate the image pixels

Provide good energy compaction

Desirable to be orthogonal

efficient computational algorithm should exist

There is no universal good transform. Among block transforms – KLT is statistically optimal (first 3 properties fulfiled, but 4th – no). Instead, in image processing, fixed ”suboptimal” transforms are used, such as DFT, DCT, lapped transforms, wavelets, etc.

Introduction: Fourier, Short-time Fourier transforms and modulated filter banks

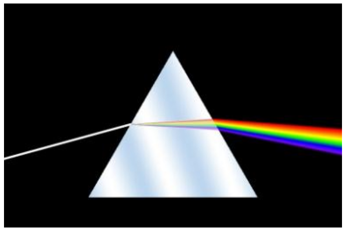
Classical Fourier transform:

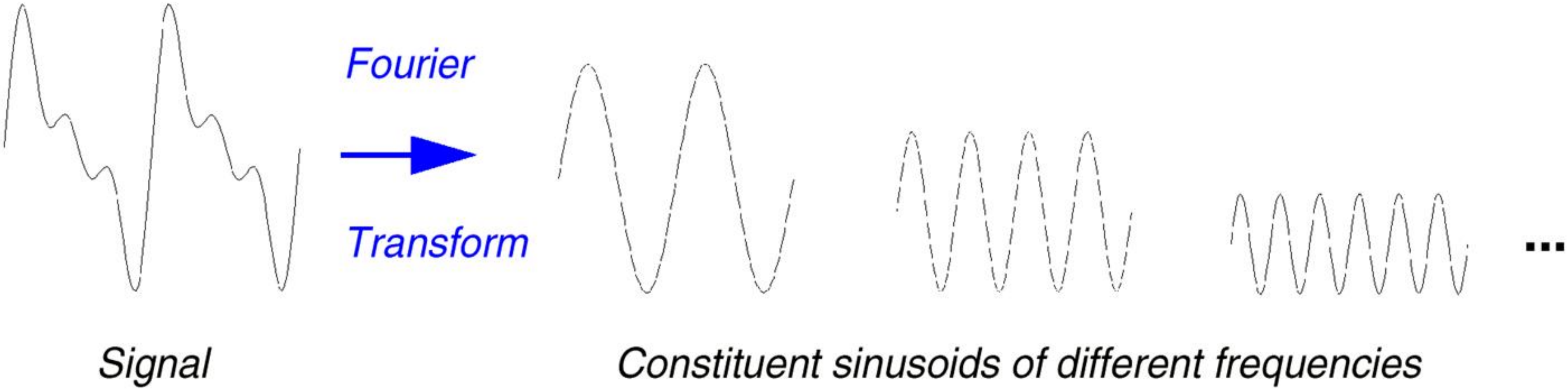
- 'uncertainty principle' *u[k]* has a narrow support (localized), then *U(.)* has a wide support (non-localized), and vice versa
- Notion of 'frequency that varies with time' not accommodated

$$U(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} u[k].e^{-j\omega k} \quad , \quad 0 \le \omega \le 2\pi \qquad\qquad u[k] = \frac{1}{2\pi}\int_0^{2\pi} U(e^{j\omega}).e^{j\omega k} d\omega$$

Fourier transform (Fourier spectrum)

Fourier transform acts as an optical prism

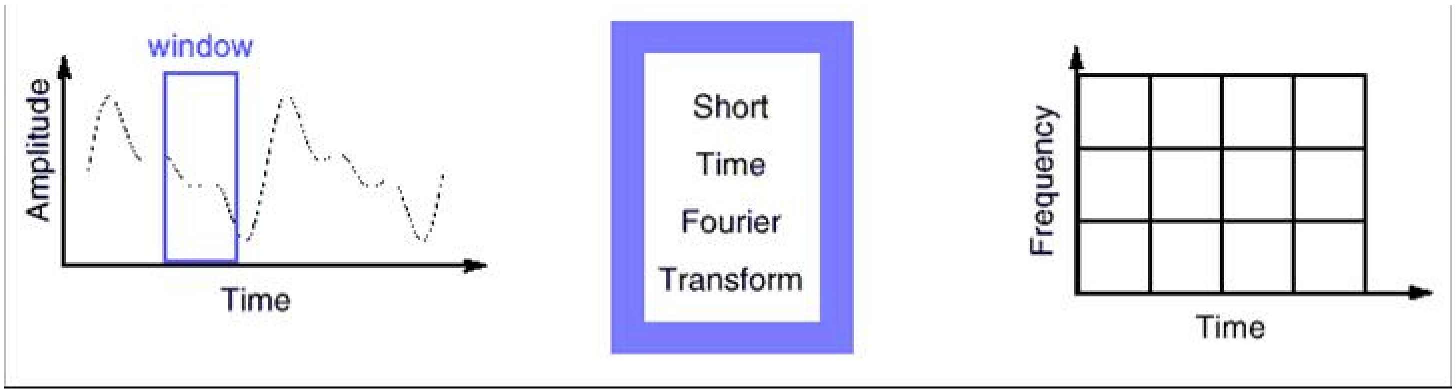




Short-time Fourier (STFT) transform

$$U_{STFT}(e^{j\omega}, n) = \sum_{k=-\infty}^{+\infty} u[k].w[k-n].e^{-j\omega k} \quad 0 \le \omega \le 2\pi \ , \ -\infty < n < +\infty$$

w[k] is a window function (Gabor transform *w[k]* is Gaussian)



STFT can be re-writen as

$$U_{STFT}(e^{j\omega}, n) = e^{-j\omega n}.\sum_{k=-\infty}^{+\infty} u[k].w[k-n].e^{-j\omega(k-n)}$$

This is a convolution with a filter $w[-n].e^{j\omega n}$

Computing it for discrete frequencies $\omega_k = k.\frac{2\pi}{N}$, $k = 0,\ldots, N-1$

make a set of filters $h_k[n] = h_0[n].e^{j2\pi k.n/N}$, $h_0[n] = w[-n]$

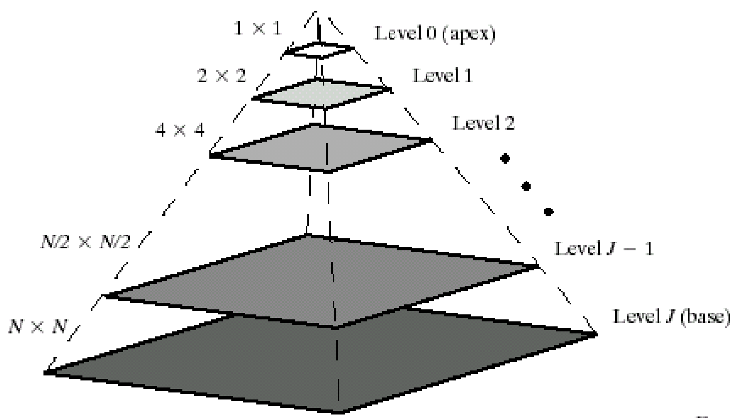
This leads to a DFT-modulated filter bank with a prototype filter – window function

Image Pyramids

- Creation by iterated approximations and interpolations (predictions)
- The approximation output is taken as the input for the next resolution level
- For a number of levels P, two pyramids: approximation (Gaussian) and prediction residual (Laplacian), are created
- approximation

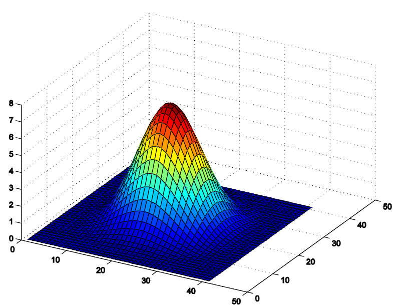
filters could be:

- averaging
- low-pass (Gaussian)

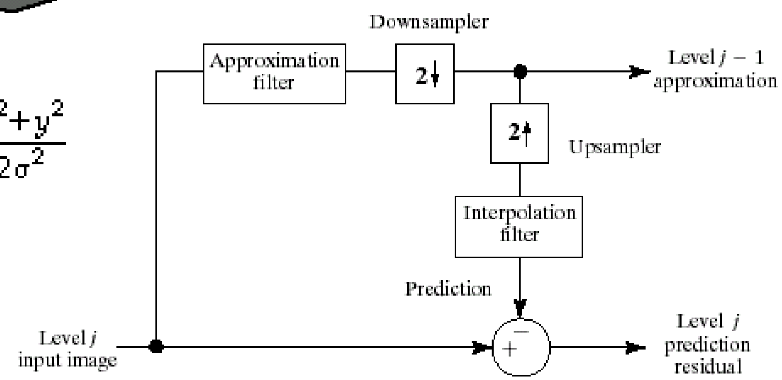


a b

FIGURE 7.2 (a) A pyramidal image structure and (b) system block diagram for creating it.



$$G(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$



Gaussian and Laplacian Pyramids

Starting image is 512×512; convolution kernel (approximation kernel) is 5×5 Gaussian

Different resolutions are appropriate for different image objects (window, vase, flower, etc.)

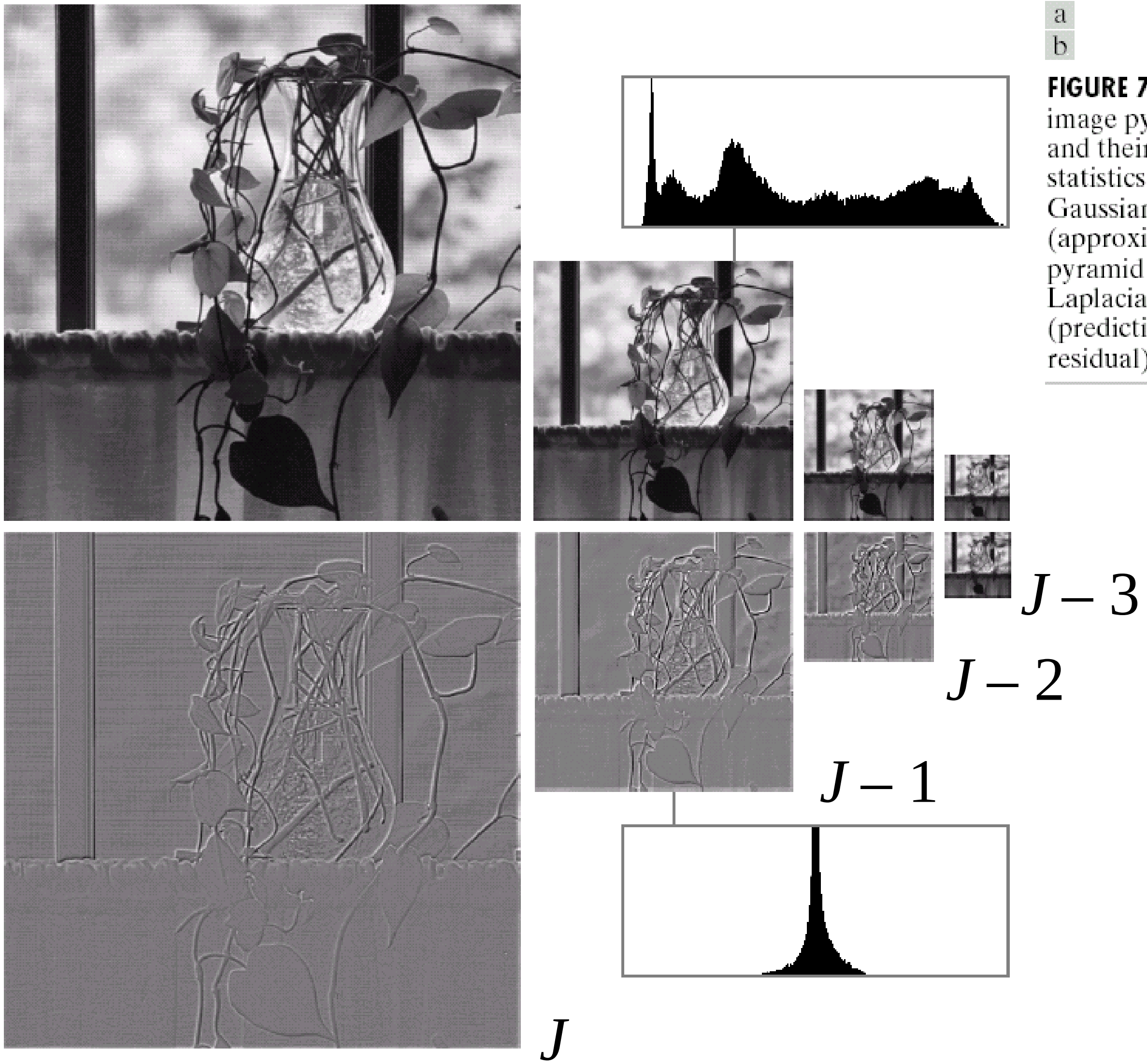


FIGURE 7.3 Two image pyramids and their statistics: (a) a Gaussian (approximation) pyramid and (b) a Laplacian (prediction residual) pyramid.

Subband Coding

- The image is decomposed into a set of band-limited components (subbands).
- Two-channel, perfect reconstruction system; two sets of half-band filters
- <u>Analysis</u>: h_0 – low-pass and h_1 – high-pass and <u>Synthesis</u>: g_0 – low-pass and g_1 – high-pass

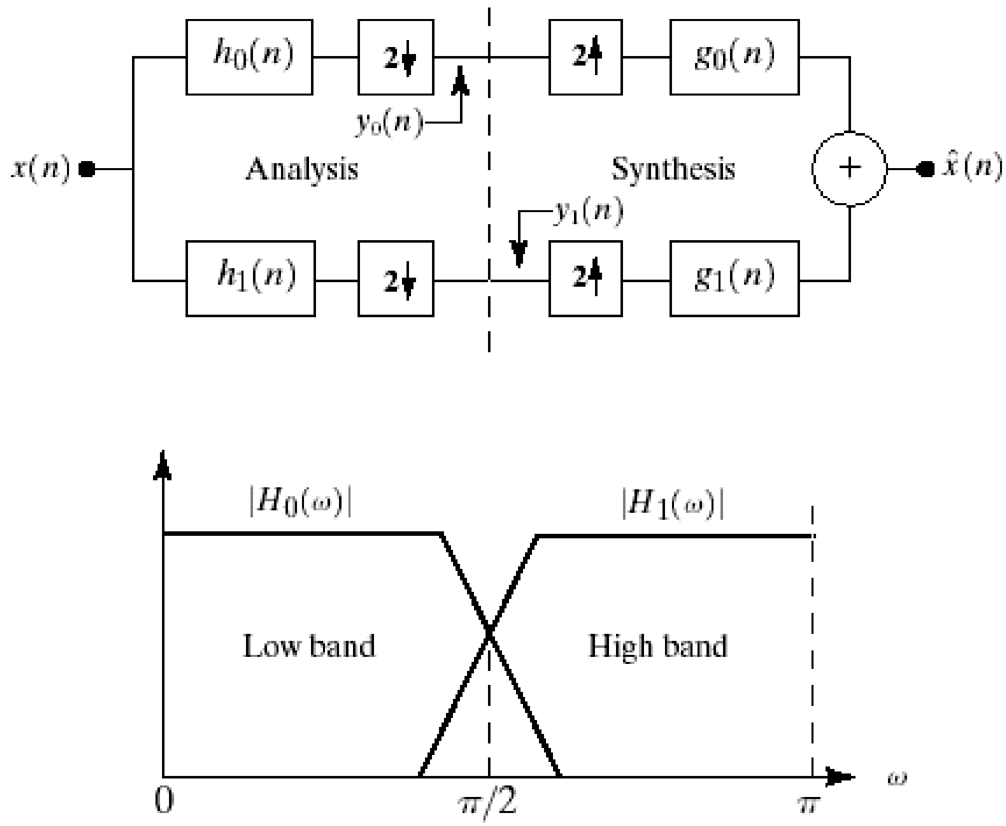


FIGURE 7.4 (a) A two-band filter bank for one-dimensional subband coding and decoding, and (b) its spectrum splitting properties.

Brief reminder of z-transform properties

Downsampled and upsampled by a factor of two sequences in z-domain

$$[2\downarrow]x(n) = x_{down}(n) = x(2n) \Leftrightarrow X_{down}(z) = \frac{1}{2}\left[X(z^{1/2}) + X(-z^{1/2})\right]$$

$$[2\uparrow]x(n) = x^{up}(x) = \begin{cases} x(n/2) & n = 0,2,4,\ldots \\ 0 & n = 1,3,5,\ldots. \end{cases} \Leftrightarrow X^{up}(z) = X(z^2)$$

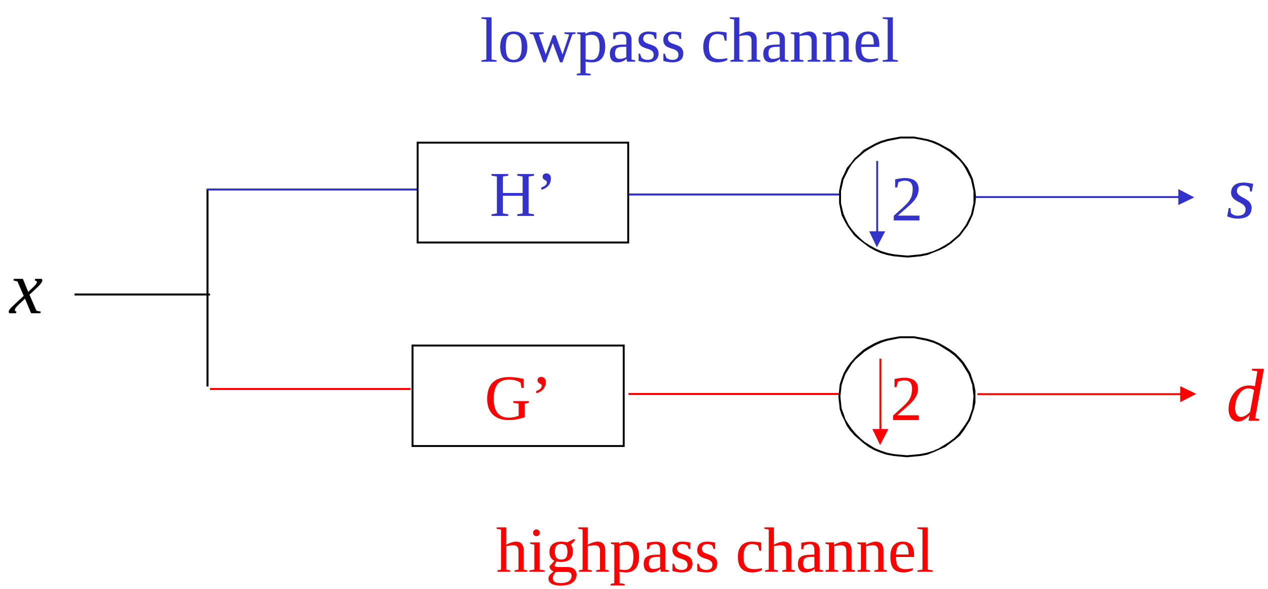
Downsampling followed by upsampling

$$\hat{x}(n) = [2\uparrow][2\downarrow]x(n) \Leftrightarrow \hat{X}(z) = \frac{1}{2}\left[X(z) + X(-z)\right]$$

$$Z^{-1}\left(X(-z)\right) = (-1)^n x(n)$$

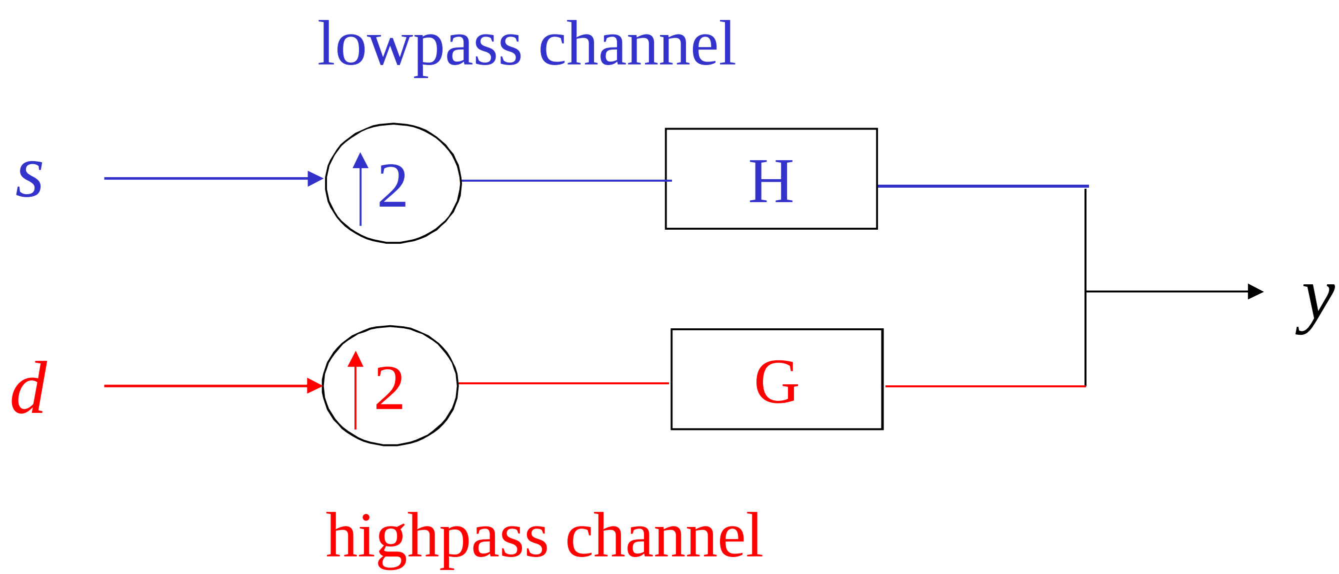
2-channel filter bank: Analysis bank

- H' is the lowpass filter and G' is the highpass filter.
- $\downarrow$2 is the downsampling operator: (1 3 4 6 5) $\longrightarrow$ (1 4 5).

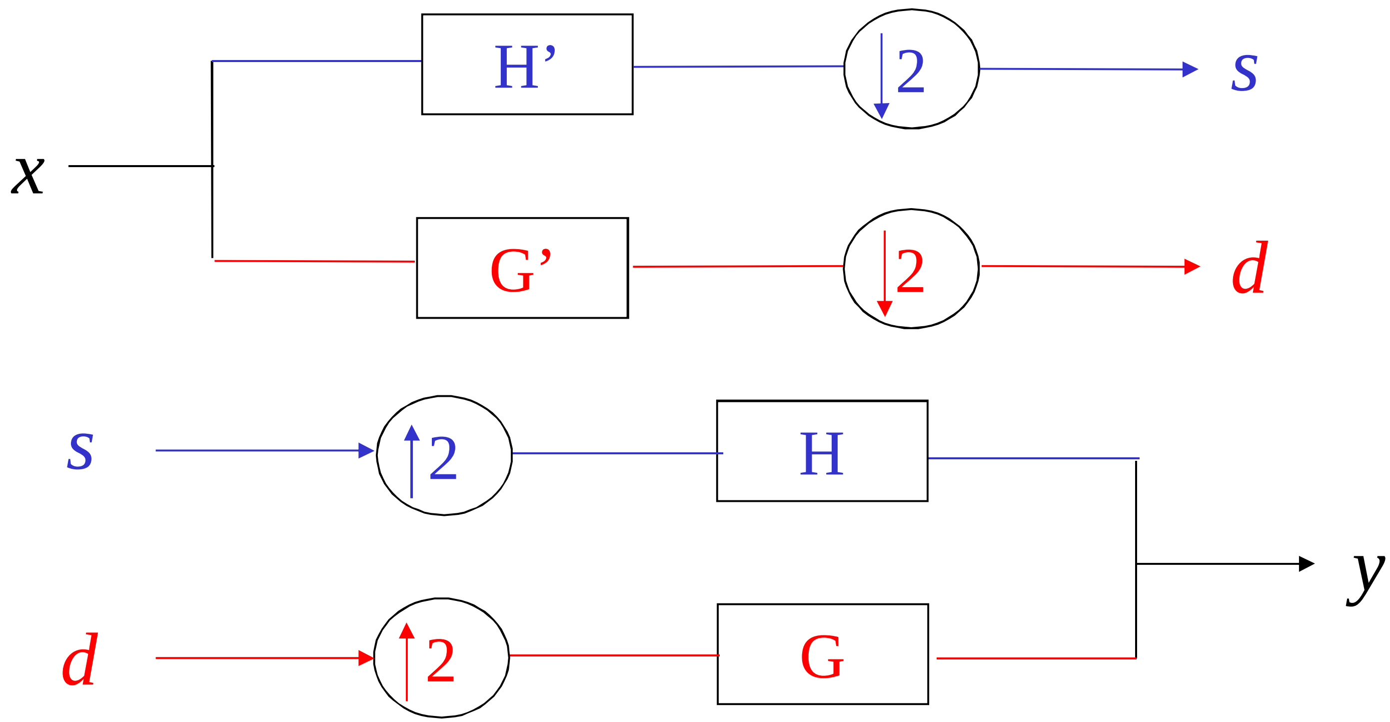


2-channel filter bank: Synthesis bank

- H is the lowpass filter and G is the highpass filter.
- $\uparrow 2$ is the upsampling operator: (1 4 5) $\longrightarrow$ (1 0 4 0 5).

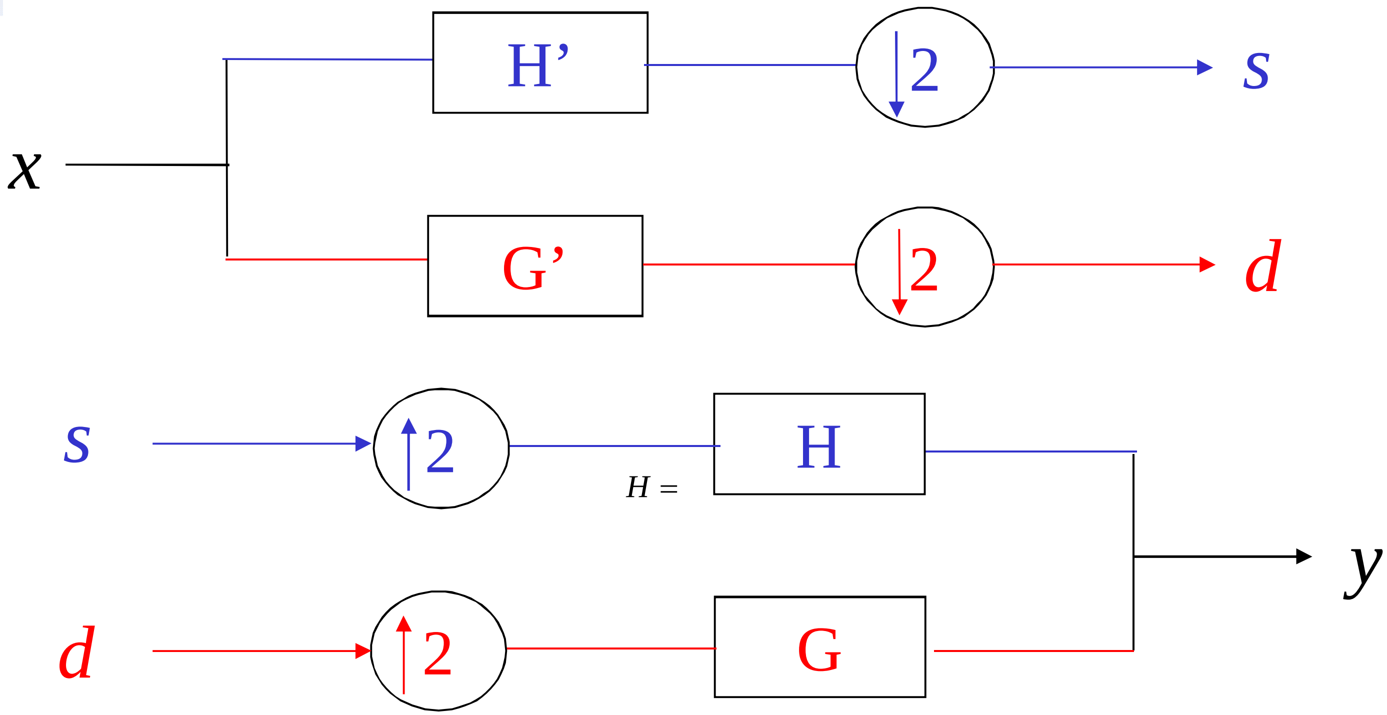


A *biorthogonal* filter bank



Biorthogonal (or perfect) filter bank: if $y=x$ for all inputs x .

An *orthogonal* filter bank



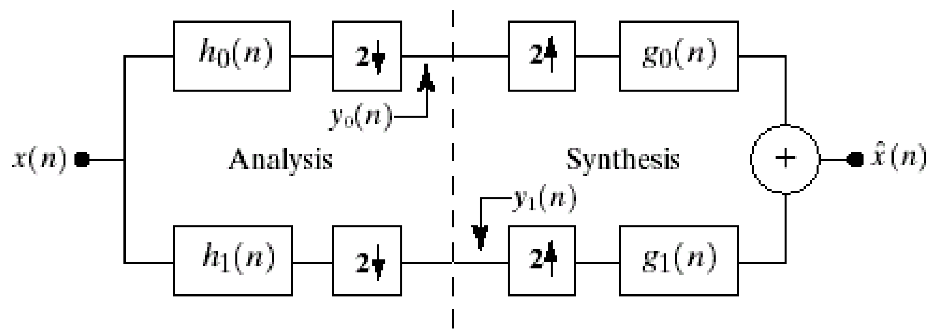
Orthogonal filter bank: if it is biorthogonal, and both *analysis* filters H' and G' are the *time reversals* of the *synthesis* filters H & G: H=(1, 2, 3) $\longrightarrow$ H'=(3, 2, 1).

Subband Coding

$$h_0(n) * x(n) \Leftrightarrow H_0(z)X(z)$$

Analysis branch:
$$Y_0(z) = \frac{1}{2}\left[H_0(z^{1/2})X(z^{1/2}) + H_0(-z^{1/2})X(-z^{1/2})\right]$$
$$Y_1(z) = \frac{1}{2}\left[H_1(z^{1/2})X(z^{1/2}) + H_1(-z^{1/2})X(-z^{1/2})\right]$$

Synthesis branch:
$$\hat{X}(z) = \frac{1}{2}G_0(z)\left[H_0(z)X(z) + H_0(-z)X(-z)\right] + $$
$$+ \frac{1}{2}G_1(z)\left[H_1(z)X(z) + H_1(-z)X(-z)\right]$$



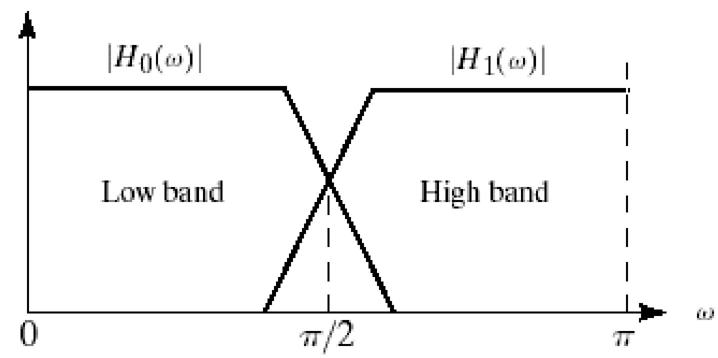
Rearanged terms:
$$\hat{X}(z) = \frac{1}{2}\left[H_0(z)G_0(z) + H_1(z)G_1(z)\right]X(z) + $$
$$+ \frac{1}{2}\left[H_0(-z)G_0(z) + H_1(-z)G_1(z)\right]X(-z)$$

For error-free (perfect reconstruction - PR): $\hat{X}(z) = X(z)$

Two PR conditions:
1) $H_0(z)G_0(z) + H_1(z)G_1(z) = 2$
2) $H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0$

$$\left[G_0(z)\ G_1(z)\right]\begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} = [2\ 0], \text{ where } \mathbf{H}_m(z) = \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix}$$

$$\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \frac{2}{\det(\mathbf{H}_m(z))}\begin{bmatrix} H_1(-z) \\ -H_0(-z) \end{bmatrix}$$



Subband Coding

For FIR filters $\det(\mathbf{H}_m(z))$ is a pure delay, i.e. $\det(\mathbf{H}_m(z)) = az^{-(2k+1)}$

Let omit the delay term $z^{-(2k+1)}$ and let $a = 2$

Take inverse z - trasform of $\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \begin{bmatrix} H_1(-z) \\ -H_0(-z) \end{bmatrix} \Leftrightarrow \begin{matrix} g_0(n) = (-1)^n h_1(n) \\ g_1(n) = (-1)^{n+1} h_0(n) \end{matrix}$ or for a = -2: $\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = -\begin{bmatrix} H_1(-z) \\ -H_0(-z) \end{bmatrix} \Leftrightarrow \begin{matrix} g_0(n) = (-1)^{n+1} h_1(n) \\ g_1(n) = (-1)^n h_0(n) \end{matrix}$

Biorthogonality : Take the product of the low - pas analysis and synthesis filters $P(z) = H_0(z)G_0(z) = \frac{2}{\det(\mathbf{H}_m(z))} H_0(z)H_1(-z)$

Since $\det(\mathbf{H}_m(z)) = -\det(\mathbf{H}_m(-z))$: $H_!(z)G_1(z) = \frac{-2}{\det(\mathbf{H}_m(z))} H_0(-z)H_1(z) = P(-z)$

The first PR condition transforms to: $H_0(z)G_0(z) + H_0(-z)G_0(-z) = 2$

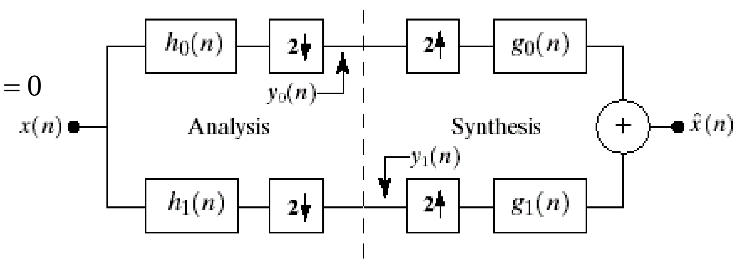
By inverse z - transform: $\sum_k g_0(k)h_0(n-k) + (-1)^n \sum_k g_0(k)h_0(n-k) = 2\delta(n)$

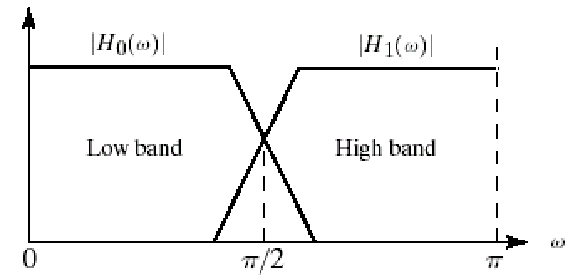
Odd indexed terms cancel: $\sum_k g_0(k)h_0(2n-k) = \langle g_0(k), h_0(2n-k) \rangle = \delta(n)$

It can be also shown that: $\langle g_1(k), h_1(2n-k) \rangle = \delta(n); \langle g_0(k), h_1(2n-k) \rangle = 0;\ \langle g_1(k), h_0(2n-k) \rangle = 0$

Biorthogonal conditions: $\langle g_j(k), h_i(2n-k) \rangle = \delta(j-i)\delta(n),\ i, j = \{0,1\}$

More restrictive **orthogonal** conditions: $\langle g_j(k), g_i(2m+k) \rangle = \delta(j-i)\delta(m),\ i, j = \{0,1\}$





Families of half-band PR filters

Filter	QMF	CQF	Orthonormal
$H_0(z)$	$H_0^2(z) - H_0^2(-z) = 2$	$H_0(z)H_0(z^{-1}) + H_0^2(-z)H_0(-z^{-1}) = 2$	$G_0(z^{-1})$
$H_1(z)$	$H_0(-z)$	$z^{-1}H_0(-z^{-1})$	$G_1(z^{-1})$
$G_0(z)$	$H_0(z)$	$H_0(z^{-1})$	$G_0(z)G_0(z^{-1}) + G_0(-z)G_0(-z^{-1}) = 2$
$G_1(z)$	$-H_0(-z)$	$zH_0(-z)$	$-z^{-2K+1}G_0(-z^{-1})$

TABLE 7.1 Perfect reconstruction filter families.

• For each family of filters, a prototype filter is designed and the other filters are computed from the prototype.

• QMF – quadrature-mirror filters

• CQF – conjugate-quadrature filters

• $2K$ is the length (number of coefficients, filter taps)

$$g_1(n) = (-1)^n g_0(2K-1-n)$$
$$h_i(n) = g_i(2K-1-n),\ i = \{0,1\}$$

Separable filtering in 2-D (images)

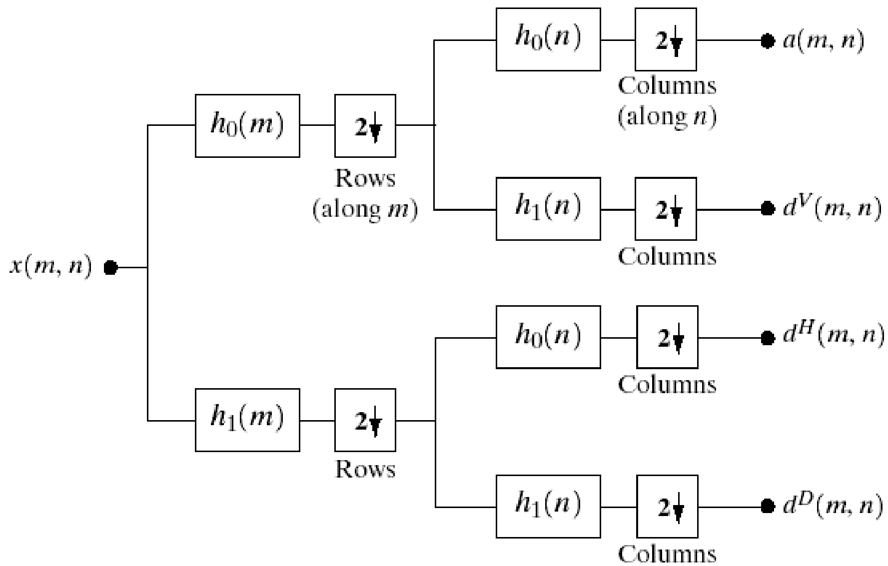
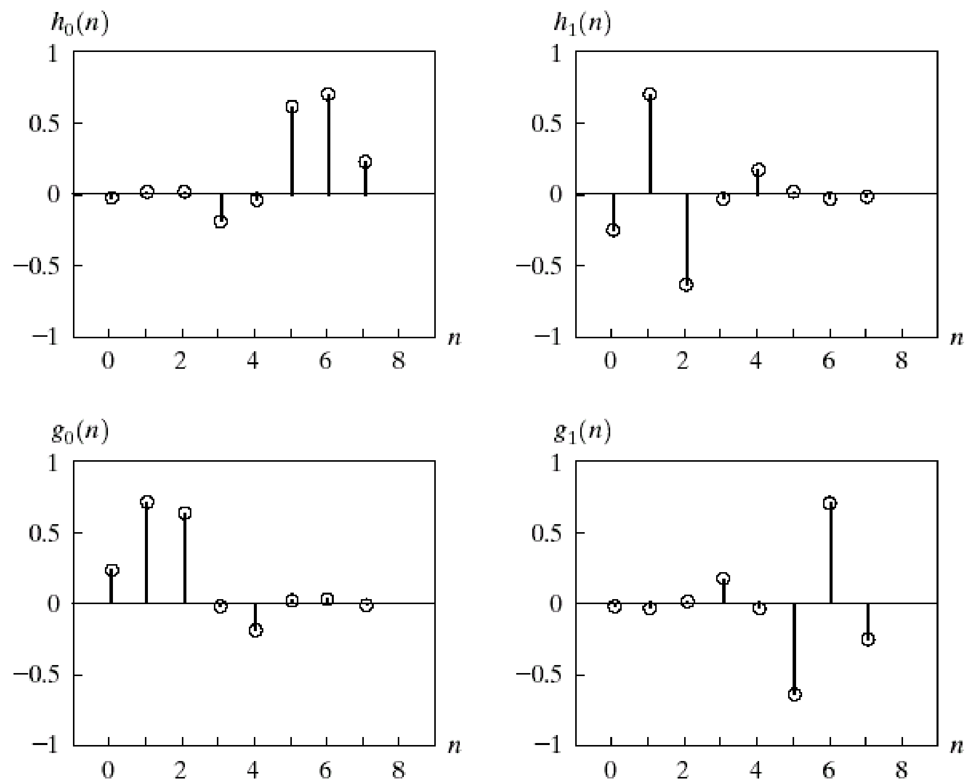


FIGURE 7.5 A two-dimensional, four-band filter bank for subband image coding.

- Vertical followed by horizontal filtering
- Four output subbands:
  - approximation, and vertical, horizontal, and diagonal detail subbands

Example: 8-tap orthonormal filter designed by Daubechies

FIGURE 7.6 The impulse responses of four 8-tap Daubechies orthonormal filters.



Example: four band subband filtering of an image

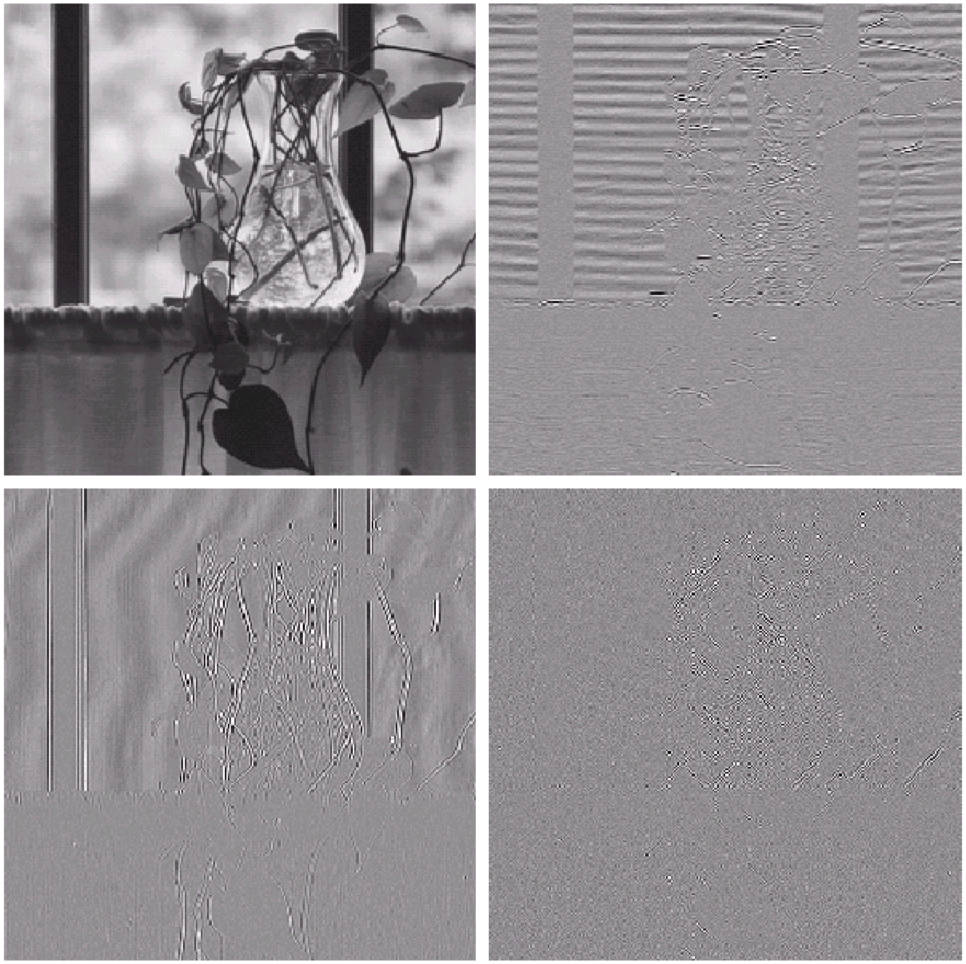


FIGURE 7.7 A four-band split of the vase in Fig. 7.1 using the subband coding system of Fig. 7.5.

Aliasing is presented in the vertical and horizontal detail subbands. It is due to the down-sampling and will be canceled during the reconstruction stage.

Continuous Wavelet transform

- Wavelet transform decomposes a signal into a set of basis functions.
- These basis functions are called *wavelets*
- Transform a continuous function into a highly redundant function of two variables – translation (*t*) and scale (*s*).

$$W_\psi(s,\tau) = \int_{-\infty}^{\infty} f(x)\psi_{s,\tau}(x)$$

$$\text{where } \psi_{s,\tau}(x) = \frac{1}{\sqrt{s}}\psi\left(\frac{x-\tau}{s}\right)$$

$$f(x) = \frac{1}{C_\psi}\int_0^{\infty}\int_{-\infty}^{\infty} W_\psi(s,\tau)\frac{\psi_{s,\tau}(x)}{s^2}\,d\tau\, ds$$

$$\text{where } C_\psi = \int_{-\infty}^{\infty}\frac{|\Psi(\omega)|^2}{|\omega|}\,d\omega$$

Comparison between Fourier Transform and Continuous WT

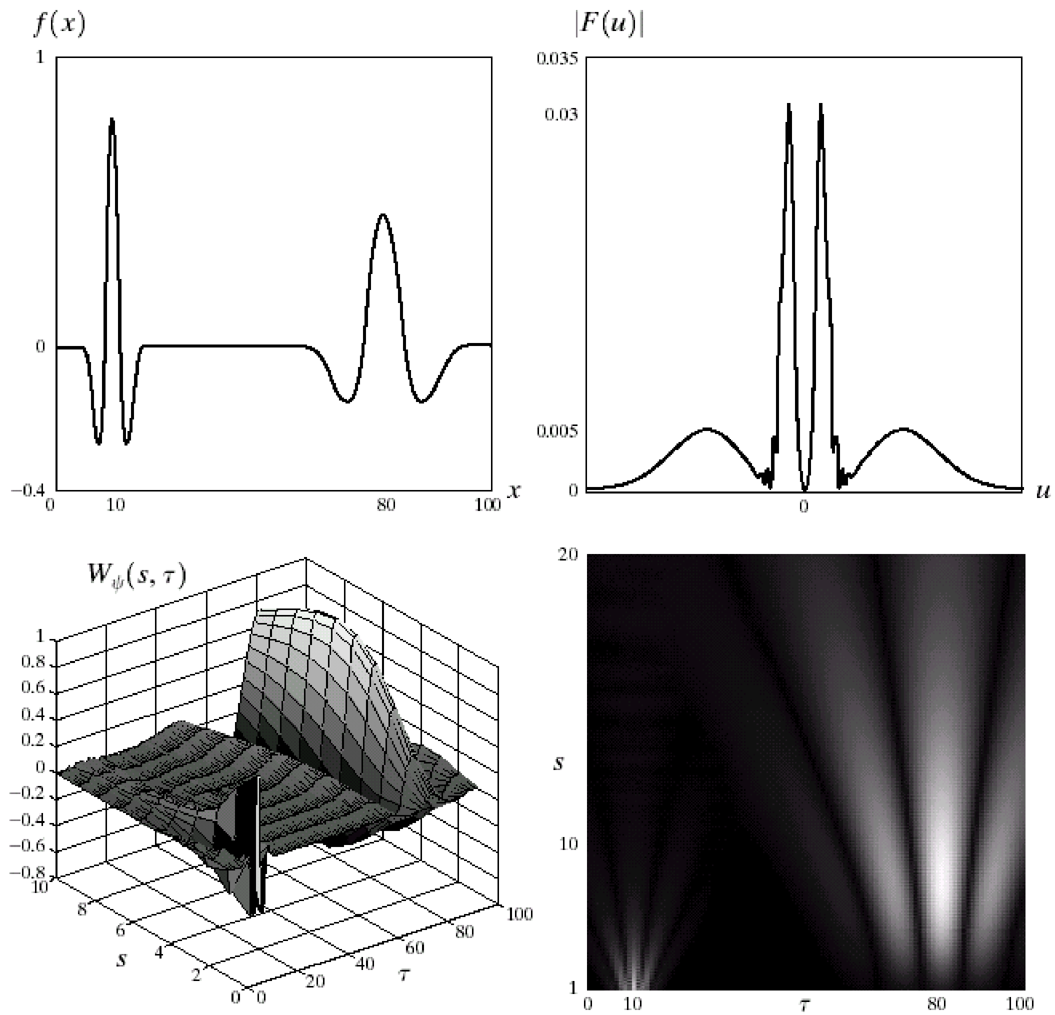


FIGURE 7.14 The continuous wavelet transform (c and d) and Fourier spectrum (b) of a continuous one-dimensional function (a).

Wavelet transforms vs. Fourier transform

- The standard Fourier Transform (FT) decomposes the signal into individual frequency components.
- The Fourier basis functions are infinite in extent.
- FT can never tell when or where a frequency occurs.
- Any abrupt changes in time in the input signal $f(t)$ are spread out over the whole frequency axis in the transform output $F(\omega)$ and vice versa.
- WT uses short window at high frequencies and long window at low frequencies (recall a and b in previous formula). It can localize abrupt changes in both time and frequency domains.

Discrete Wavelet Transform (DWT)

Transform pair (forward and inverse transform) applicable for discrete signals (or sampled signals)

$$W_\varphi(j_0,k) = \frac{1}{\sqrt{M}} \sum_x f(x)\varphi_{j_0,k}(x)$$

$$W_\psi(j,k) = \frac{1}{\sqrt{M}} \sum_x f(x)\psi_{j,k}(x)$$

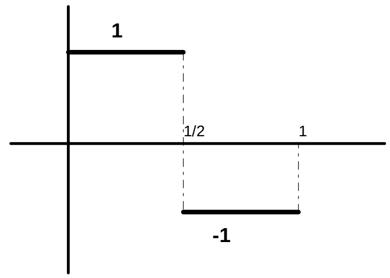
$$f(x) = \frac{1}{\sqrt{M}} \sum_k W_\varphi(j_0,k)\varphi_{j_0,k}(x) + \frac{1}{\sqrt{M}} \sum_{j=j_0}^{\infty} \sum_k W_\psi(j,k)\psi_{j,k}(x)$$

Here x is a *discrete* variable: $x = 0,1,2,\ldots M - 1$
Normally, $M = 2^J$; $j_0 = 0$; $j = 0,1,2,\ldots J$-1 and $k = 0,1,2,..2^{j}$-1
W_φ and W_ψ are referred as approximation and detail coefficients respectively.

Haar Transform
(oldest and simplest wavelets)

- The Haar "aperture" function is

$$\psi^{\text{harr}}(x) = 1_{0 \le x < 1/2}(x) - 1_{1/2 \le x < 1}(x).$$



- Haar's theorem (1905):

All Haar wavelets $\psi^{\text{haar}}_{j,k}$ *, together with the constant function 1, consist into an* ***orthonormal basis*** *for the Hilbert space of all square integrable functions on* [*0, 1*].

Haar Transform (example)

Let's consider a 1D 4-pixel Image [9 7 3 5]

[9 7 3 5]

Average (smoothing)

Detail coefficients (edge detection)

(9 + 7)/2 (3 + 5)/2

[8 4]

(9 - 7)/2 (3 - 5)/2

[1 -1]

WT

(8 + 4)/2

6

(8 – 4)/2

2

[6 2 1 -1]

Haar Transform (cont.)

$\mathbf{T} = \mathbf{HFH}$ where $\mathbf{F}$ is an $N \times N$ image matrix; $\mathbf{H}$ is an $N \times N$ transform matrix, and $\mathbf{T}$ is the resulting transform coefficients matrix.

$\mathbf{H}$ contains the Haar basis functions $h_k(z)$. They are defined over the continuous, closed interval $z \in [0,1]$ for $k = 0,1,\ldots N-1$, where $N = 2^n$.

To generate $\mathbf{H}$, we define the integer k such that $k = 2^p + q - 1$, where $0 \le p \le n-1$, $q = 0$ or 1 for $p = 0$ and $1 \le q \le 2^p$ for $p \ne 0$:

$$h_0(z) = h_{00}(z) = \frac{1}{\sqrt{N}}, \quad z \in [0,1]$$

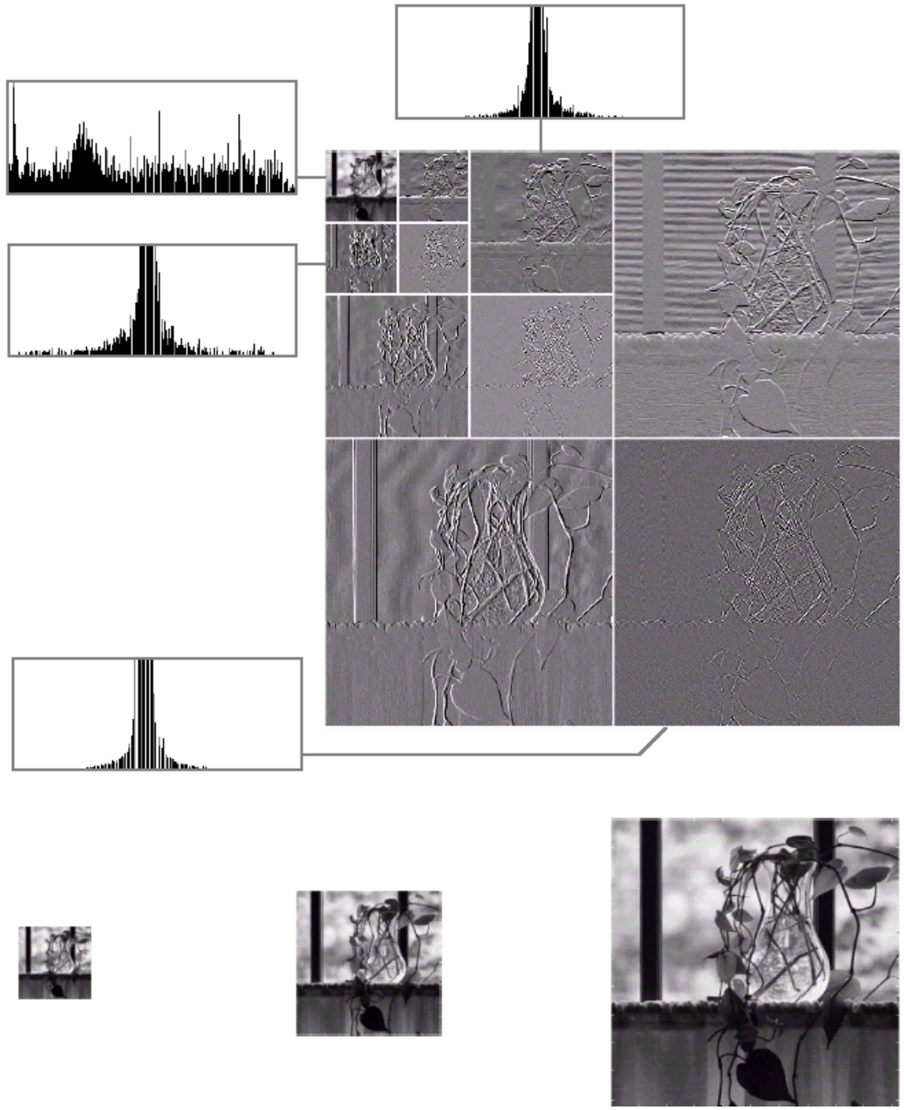
$$h_k(z) = h_{pq}(z) = \frac{1}{\sqrt{N}} \begin{cases} 2^{p/2} & (q-1)/2^p \le z < (q-0.5)/2^p \\ -2^{p/2} & (q-0.5)/2^p \le z < q/2^p \\ 0 & \text{otherwize } z \in [0,1] \end{cases}$$

The i-th row of the Haar matrix contains the elements of $h_i(z)$ for $z = 0/N, 1/N, 2/N, \ldots, (N-1)/N$.

$$\mathbf{H}_4 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}; \qquad \mathbf{H}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The Haar basis functions define two-tap FIR filter bank. The coefficients of corresponding QMF analysis filters are in the rows of $\mathbf{H}_2$

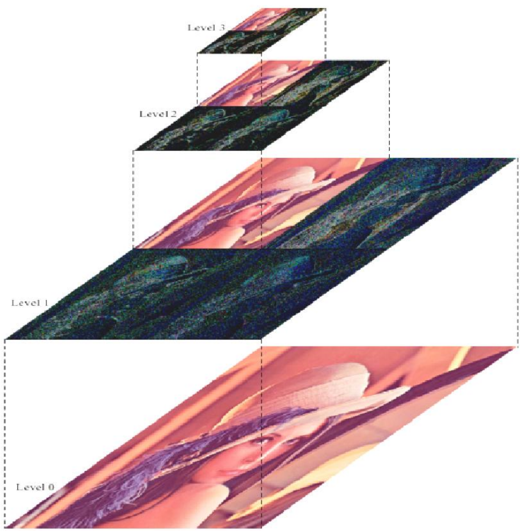
Haar Functions in a Discrete Wavelet Transform

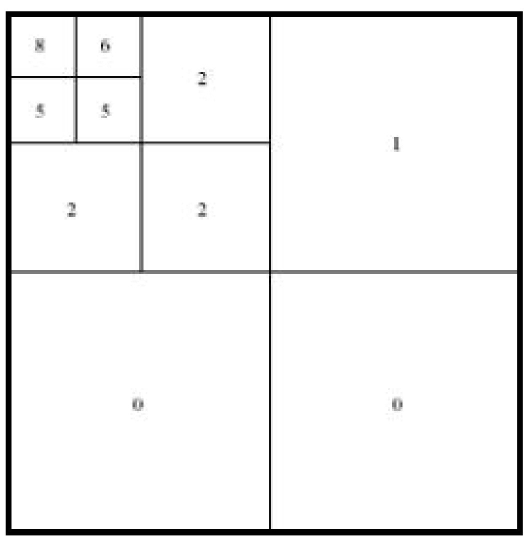


a
b c d

FIGURE 7.8 (a) A discrete wavelet transform using Haar basis functions. Its local histogram variations are also shown; (b)–(d) Several different approximations (64×64, 128×128, and 256×256) that can be obtained from (a).

Wavelet transform of images





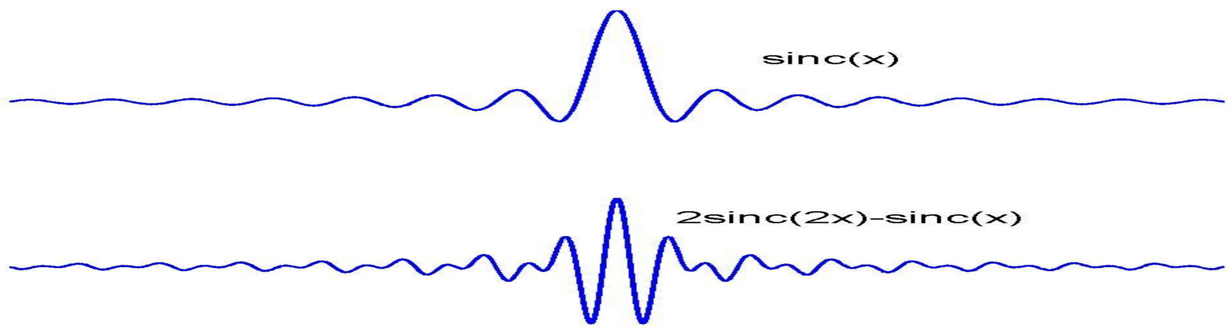
Wavelets work for decomposing signals (such as images) into hierarchy of increasing resolutions: as we consider more layers, we get more and more detailed look at the image.

From http://www.maths.bris.ac.uk/~wavethresh/

Another Example: The Shannon wavelets

- The Shannon's "aperture" function is:

$$\psi^{\text{Shannon}}(x) = 2\,\text{sinc}\,(2x) - \text{sinc}\,(x).$$



- Theorem:

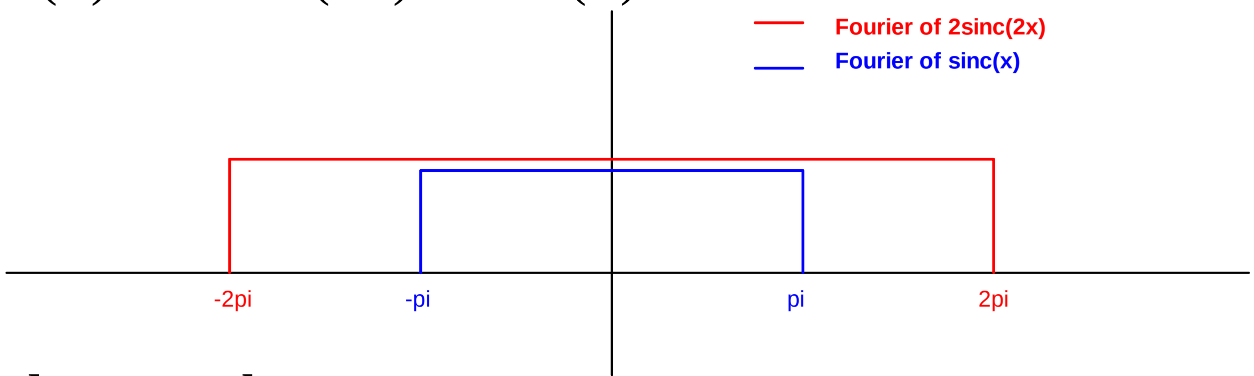
$\{\psi_{j,k}^{\text{Shannon}}(x) : j, k \in Z\}$ is an orthonormal basis of $L_2(\mathrm{R})$.

Shannon wavelets (cont'd)

- How to visualize the orthonormal basis ?

  Answer: go to the Fourier domain !

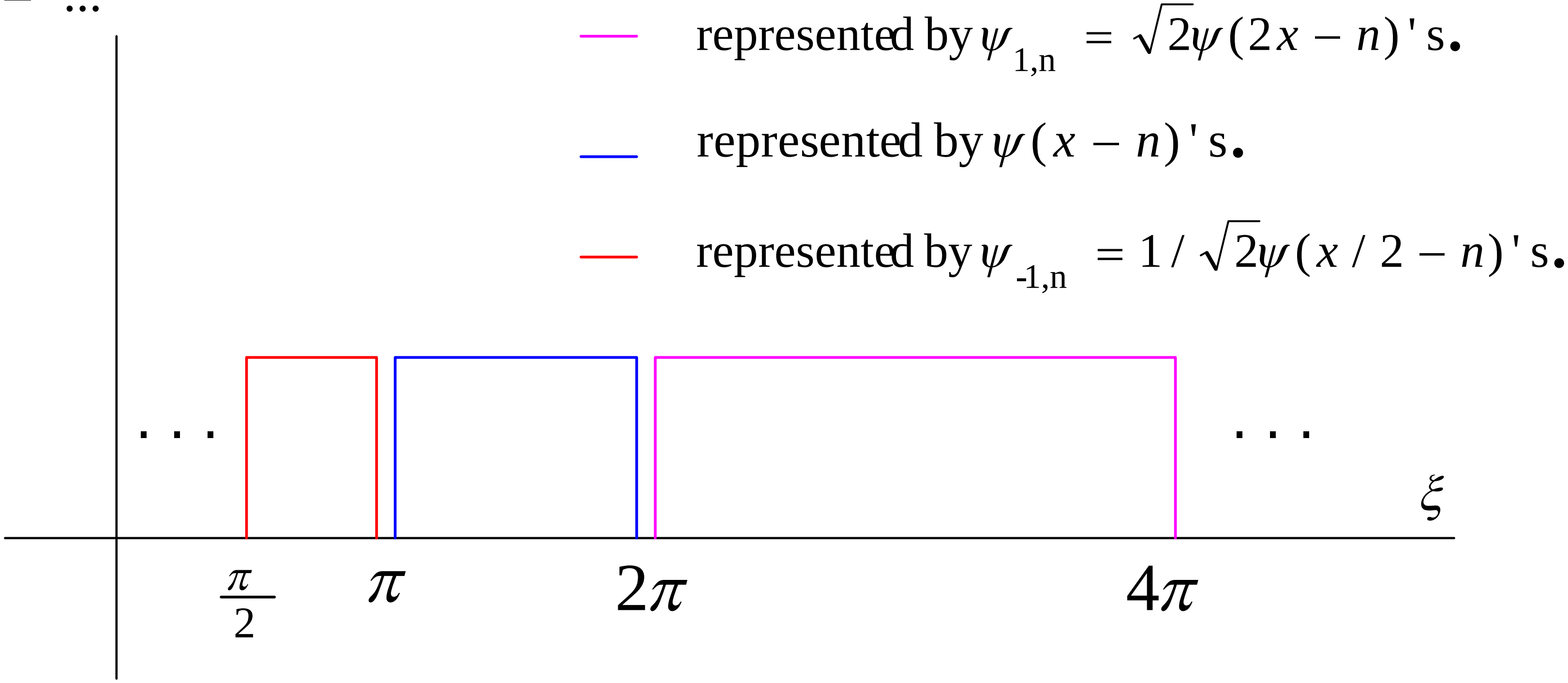
$$\psi^{\text{shannon}}(x) = 2\,\mathrm{sinc}\,(2x) - \mathrm{sinc}\,(x).$$



- According to Shannon:
  - All signals bandlimited to (–π, π) can be represented by sinc(***x-n***)…
  - those bandlimited to (–2π, π) U (π, 2π), by $\psi(x-n)$.
  - those bandlimited to (–4π, 2π) U (2π, 4π), by $\psi_{1,n} = \sqrt{2}\psi(2x-n)$.
  - ...

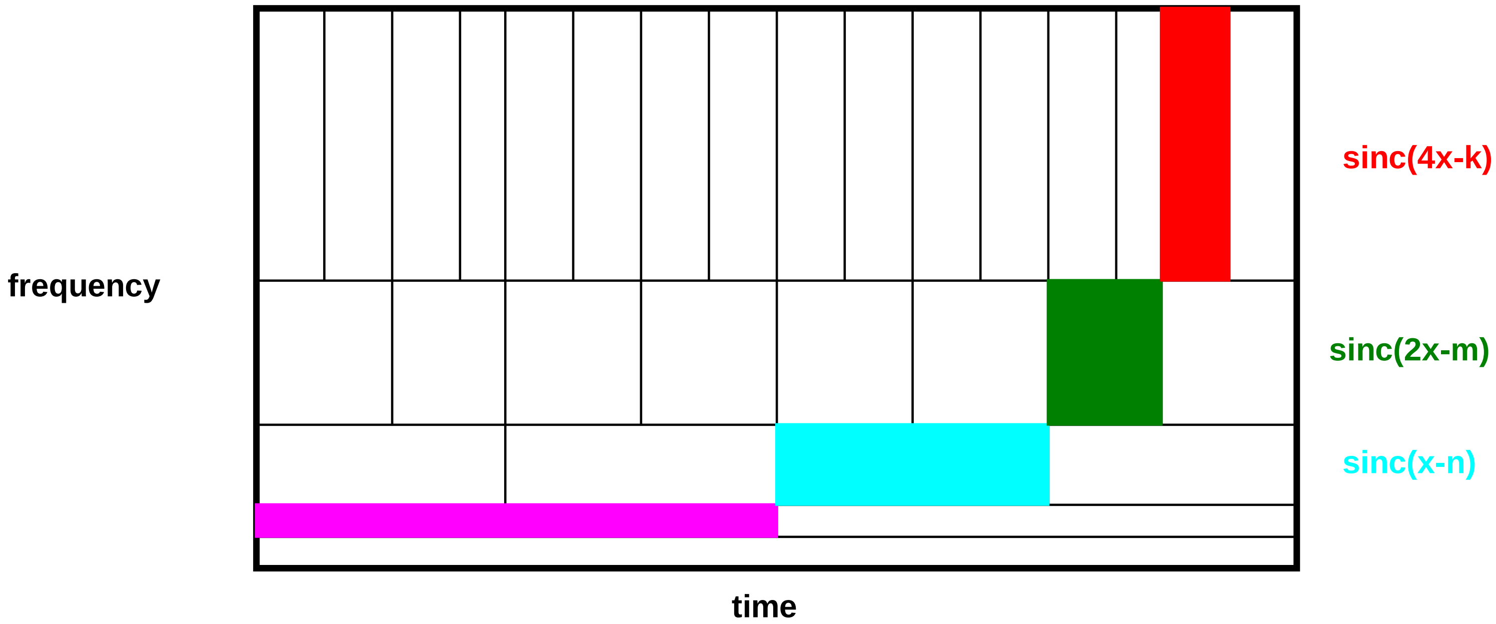
Shannon wavelets (cont'd)

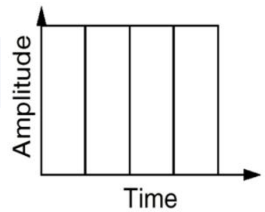
- According to Shannon:
  - All signals bandlimited to (- π, π) can be represented by sinc(***x-n***)…
  - those bandlimited to (-2 π, - π) U (π, 2 π), by $\psi(x-n)$.
  - those bandlimited to (-4 π,-2 π)U(2 π, 4 π), by $\psi_{1,n} = \sqrt{2}\psi(2x-n)$.
  - ...



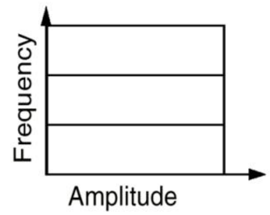
Partition of the time-frequency plane

- Heisenberg's uncertainty principle requires that each TF atom must have: $\Delta t \cdot \Delta x \geq 2\pi$.
- Thus, for an *optimal* localization, the "life time" of an atom must influence its scale or frequency content.

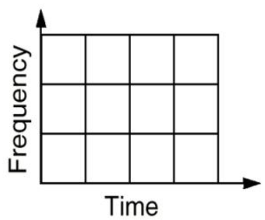




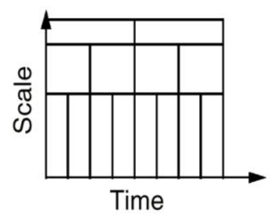
Time Domain (Shannon)



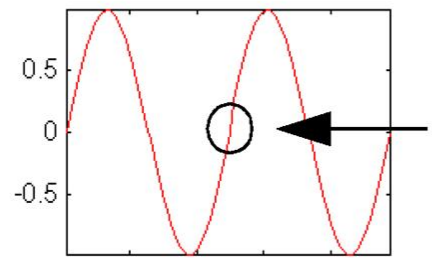
Frequency Domain (Fourier)



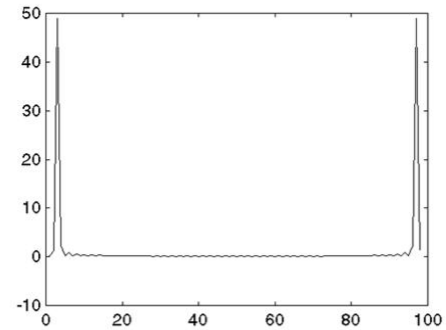
STFT (Gabor)



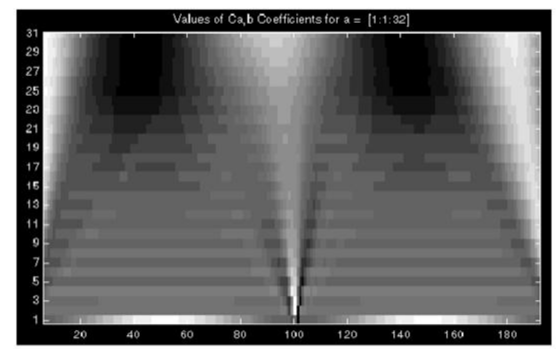
Wavelet Analysis



Sinusoid with a small discontinuity



Fourier Coefficients



Wavelet Coefficients

Multiresolution Expansions

In *Multiresolution Analysis* (MRA) *a scaling function* is used to create a series of approximations of a function (image), each differing by a factor of two. Additional functions, *wavelets*, are used to encode the difference between adjacent approximations.

- Series Expansions
  - Linear combination of expansion functions: $f(x) = \sum_k \alpha_k \varphi_k(x)$
  - Basis functions and function space: $V = \operatorname{span}_k \{\varphi_k(x)\}$
  - Dual functions: $\{\tilde{\varphi}(x)\}$ such that $\alpha_k = \langle \tilde{\varphi}_k(x), f(x) \rangle = \int \tilde{\varphi}_k^*(x) f(x) dx$
  - Cases: orthonormal basis, biorthogonal basis, overcomplete expansions

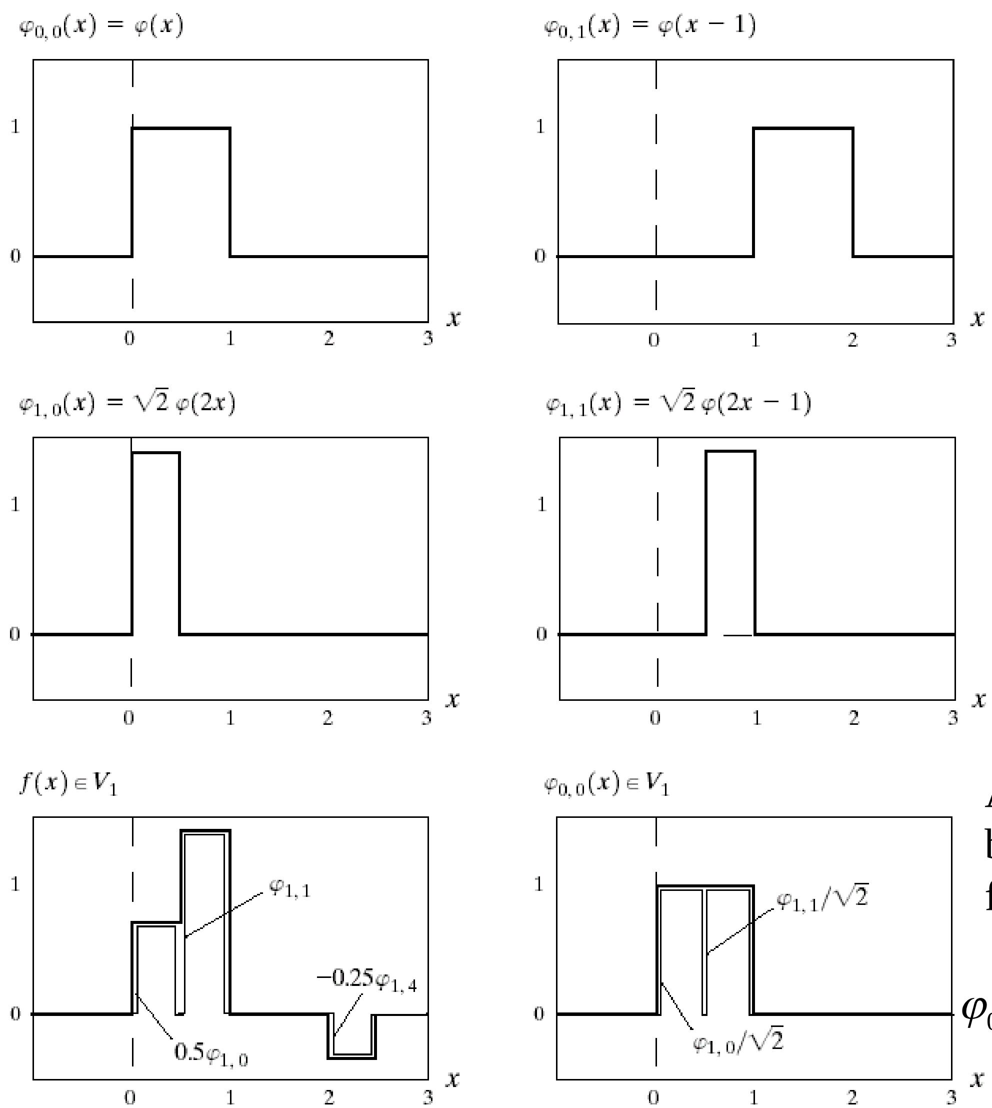
Multiresolution Expansions

- Scaling Functions
  - Set of functions $\{\varphi_{j,k}(x)\}$ formed by integer translations and binary scaling of a prototype function $\varphi(x)$: $\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k)$
  - translation (position) parameter k; scale parameter j
  - scaling functions can be made to span the space of all measurable, square-integrable functions, denoted by $L^2(\mathbf{R})$
  - by fixing the scale to $j=j_0$, the resulting set spans a subspace of $L^2(\mathbf{R})$

$$V_{j_0} = \underset{k}{\text{span}}\left\{\varphi_{j_0,k}(x)\right\}$$

  - if $f(x) \in V_{j0}$ then $f(x) = \sum_k \alpha_k \varphi_{j_0,k}(x)$

The Haar Scaling Function



a b
c d
e f

FIGURE 7.9 Haar scaling functions in V_0 in V_1.

Any V_0 expansion function can be represented by V_1 expansion functions, i.e. $V_0 \subset V_1$.

$$\varphi_{0,k}(x) = \frac{1}{\sqrt{2}}\varphi_{1,2k}(x) + \frac{1}{\sqrt{2}}\varphi_{1,2k+1}(x)$$

Basis for Vector space V^j

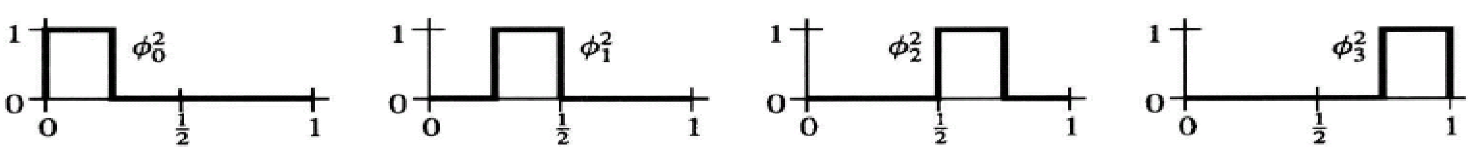
- Basis functions for $\mathbf{V^j}$ are called **scaling functions** and are denoted by φ.

❖A simple basis for $\mathbf{V^j}$ is given by the set of scaled and translated **box functions**:

$$\varphi^{ij}(x) := \varphi\,(2^j x - i) \qquad i = 0,\ldots, 2^j - 1$$

where
$$\varphi(x) := \begin{cases} 1, \text{ for } 0 \leq x < 1 \\ 0, \text{ otherwise} \end{cases}$$

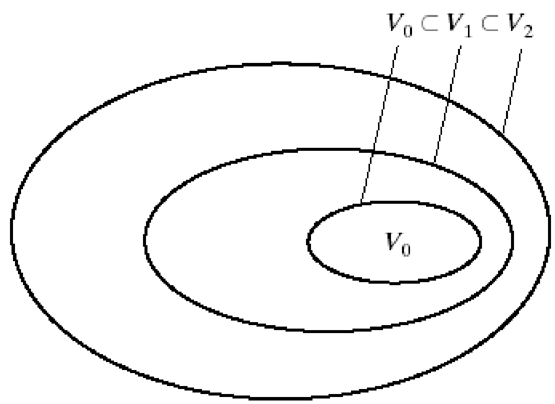
❖Example basis for $\mathbf{V^2}$:



MRA Requirements

MRA requirements 1:

FIGURE 7.10 The nested function spaces spanned by a scaling function.



The scaling function is orthogonal to its integer translates

MRA requirement 2: The subspaces spanned by the scaling function at low scales are nested within those spanned at higher scales

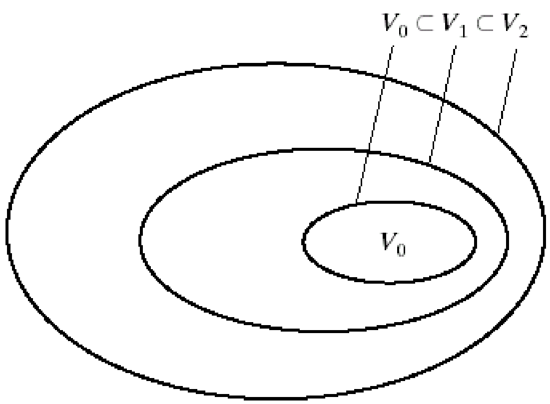
$$V_{-\infty} \subset \ldots V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots V_{\infty}$$

MRA requirement 3: The only functions that is common to all V_j is

$f(x) = 0$

MRA requirement 4: Any function can be represented with arbitrary precision

Dilatation Equation



Under the MRA requirements, the expansion functions of subspace V_j can be expressed as a weighted sum of the expansion functions of subspace V_{j+1}.

$$\varphi_{j,k}(x) = \sum_n \alpha_n \varphi_{j+1,n}(x)$$

$$\varphi_{j,k}(x) = \sum_n h_\varphi(n) 2^{(j+1)/2} \varphi(2^{j+1}x - n)$$

setting both j and k to 0 we obtain the so-called dilatation equation

$$\varphi(x) = \sum_n h_\varphi(n) \sqrt{2} \varphi(2x - n)$$

Wavelet Functions

Define a wavelet function $\psi(x)$ that, together with its integer translates and binary scalings, spans the difference between any two adjacent scaling subspaces

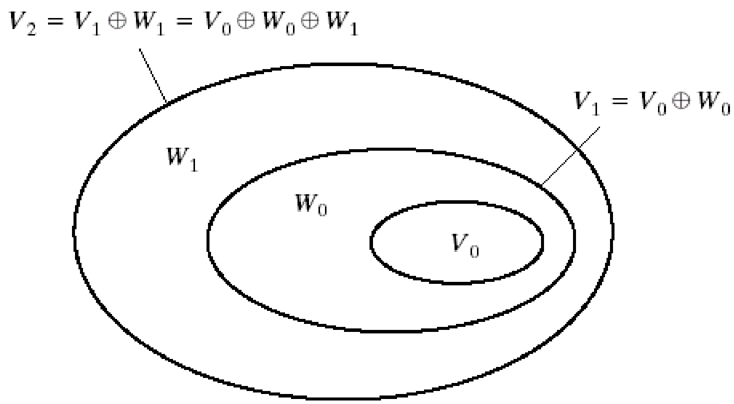


FIGURE 7.11 The relationship between scaling and wavelet function spaces.

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$$

$$W_j = \operatorname*{span}_k \{\psi_{j,k}(x)\}$$

$$\text{if } f(x) \in W_j \Rightarrow f(x) = \sum_k \alpha_k \psi_{j,k}$$

$$V_{j+1} = V_j \oplus W_j$$

We call W_j orthogonal complement of V_j in V_{j+1}. $\langle \varphi_{j,k}(x), \psi_{j,l}(x) \rangle = 0$

Wavelet Functions

The space of measurable, square-integrable functions can be expressed as

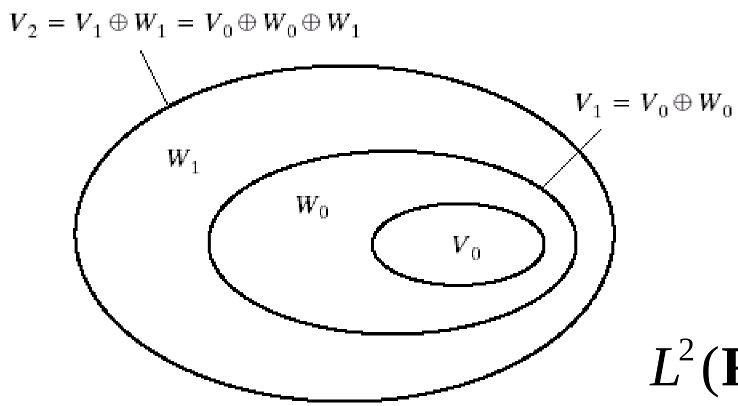


FIGURE 7.11 The relationship between scaling and wavelet function spaces.

$$L^2(\mathbf{R}) = V_0 \oplus W_0 \oplus W_1 \oplus \ldots$$

$$L^2(\mathbf{R}) = V_1 \oplus W_1 \oplus W_2 \oplus \ldots$$

$$L^2(\mathbf{R}) = \ldots W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \ldots$$

Any *wavelet* function can be expressed as a weighted sum of shifted, double-resolution *scaling* functions

$$\psi(x) = \sum_n h_\psi(n)\sqrt{2}\varphi(2x-n)$$

where $h_\psi(n) = (-1)^n h_\varphi(1-n)$

Definition of Wavelets

- Define $\mathbf{W}^{\mathbf{j}}$ as the orthogonal complement of $\mathbf{V}^{\mathbf{j}}$ in $\mathbf{V}^{\mathbf{j+1}}$, i.e. $\mathbf{W}^{\mathbf{j}} \oplus \mathbf{V}^{\mathbf{j}} = \mathbf{V}^{\mathbf{j+1}}$
- A collection of linearly independent functions $\psi^{ij}(x)$ spanning $\mathbf{W}^{\mathbf{j}}$ are called ***wavelets***:
  - The basis functions $\psi^{ij}(x)$ and $\varphi^{ij}(x)$ form a basis in $\mathbf{V}^{\mathbf{j+1}}$
  - For each j $\psi^{ij}(x)$ orthogonal to $\varphi^{ij}(x)$
  - Wavelets are orthogonal to each other

Haar Wavelets revisited

- The wavelets corresponding to the box basis are known as Haar Wavelets:

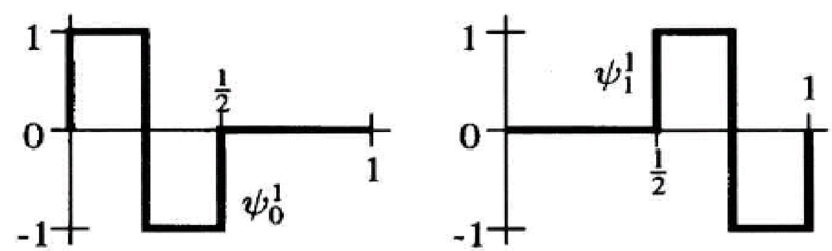
$$\psi^{ij}(x) := \psi(2^j x - i) \quad i = 0,\dots, 2^j - 1$$

where

$$\psi(x) := \begin{cases} 1, \text{ for } 0 \le x < 1/2 \\ -1, \text{ for } 1/2 \le x < 1 \\ 0, \text{ otherwise} \end{cases}$$

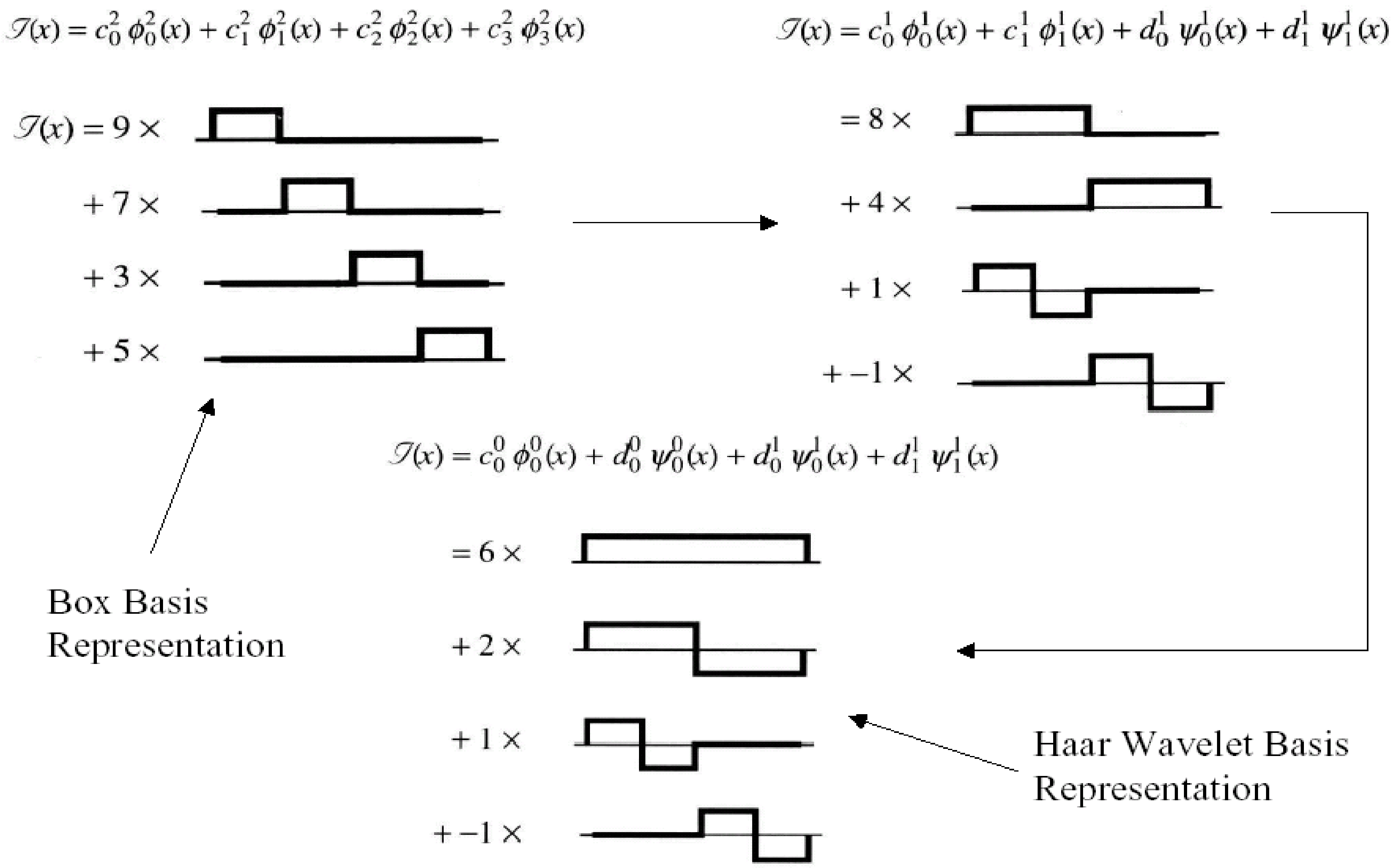
❖Example:

Haar Wavelets for $\mathbf{W}^1$

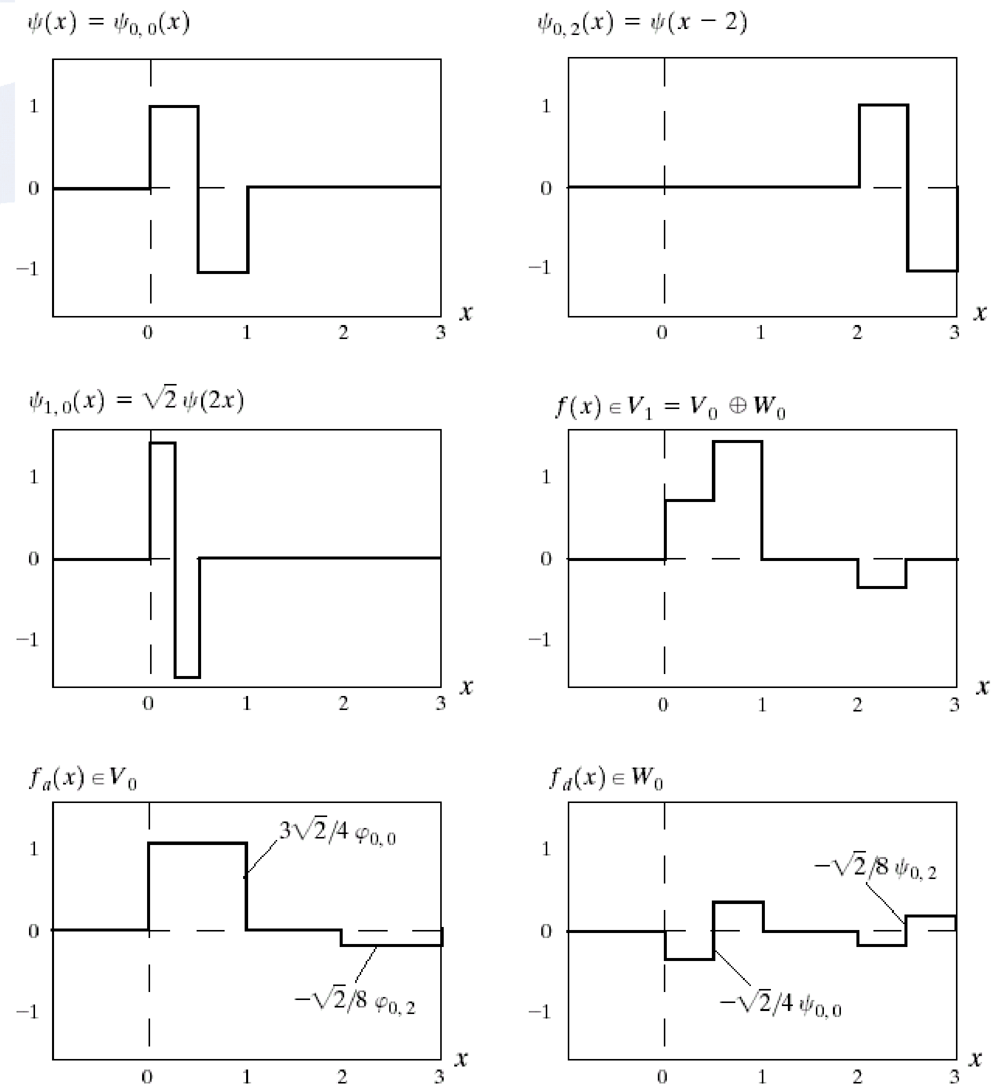


Haar Wavelets example revisited

Our Image [9 7 3 5] Expressed Using Various Bases



Haar Wavelet Function Coefficients



a b
c d
e f

FIGURE 7.12 Haar wavelet functions in W_0 and W_1.

Haar scaling filter: $h_\varphi(0) = h_\varphi(1) = 1/\sqrt{2}$

Haar wavelet filter: $h_\psi(0) = 1/\sqrt{2}; h_\psi(1) = -1/\sqrt{2}$

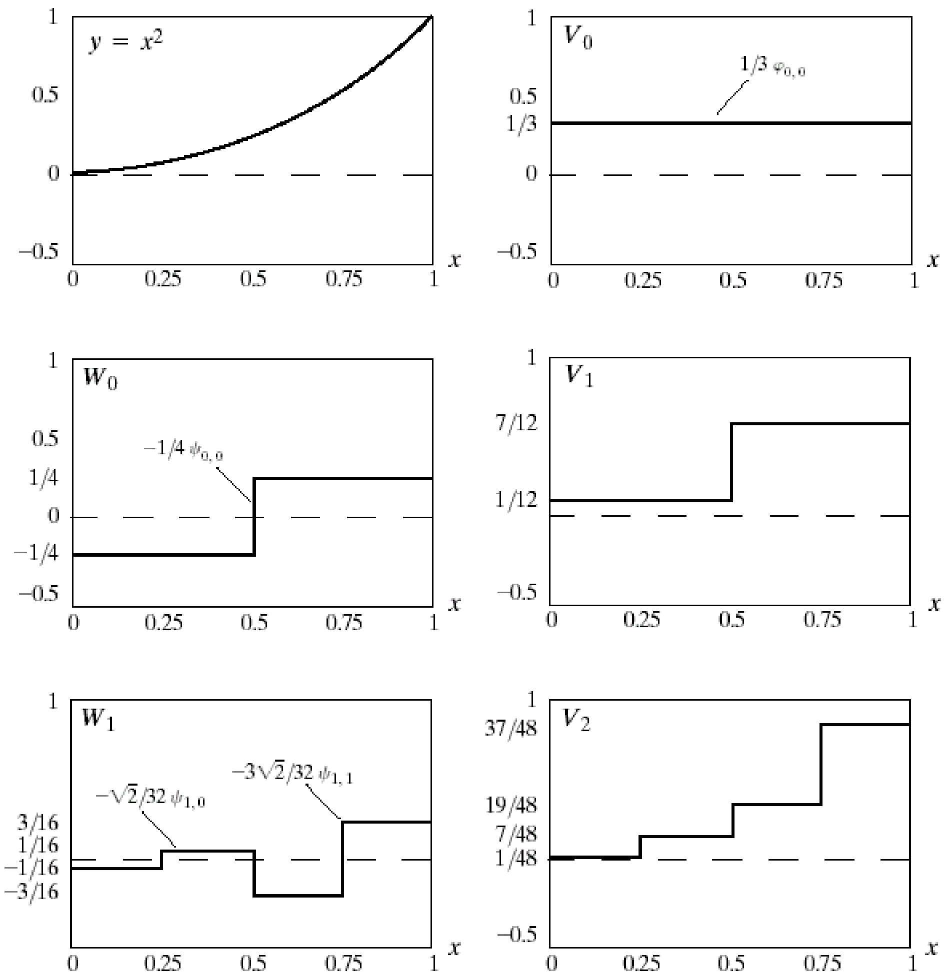
$$\text{Haar wavelet function}: \psi(x) = \begin{cases} 1 & 0 \le x < 0.5 \\ -1 & 0.5 \le x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$f(x) = f_a(x) + f_d(x)$$

$$f_a(x) = \frac{3\sqrt{2}}{4}\varphi_{0,0}(x) - \frac{\sqrt{2}}{8}\varphi_{0,2}(x)$$

$$f_d(x) = \frac{-\sqrt{2}}{4}\psi_{0,0}(x) - \frac{\sqrt{2}}{8}\psi_{0,2}(x)$$

Wavelet Series Expansions



Consider the orthogonal case

$$f(x) = \sum_k c_{j_0}(k)\varphi_{j_0,k}(x) + \sum_{j=j_0}^{\infty} \sum_k d_j(k)\psi_{j,k}(x)$$

$$c_{j_0}(k) = \langle f(x), \varphi_{j_0,k}(x) \rangle$$

$$d_j(k) = \langle f(x), \psi_{j,k}(x) \rangle$$

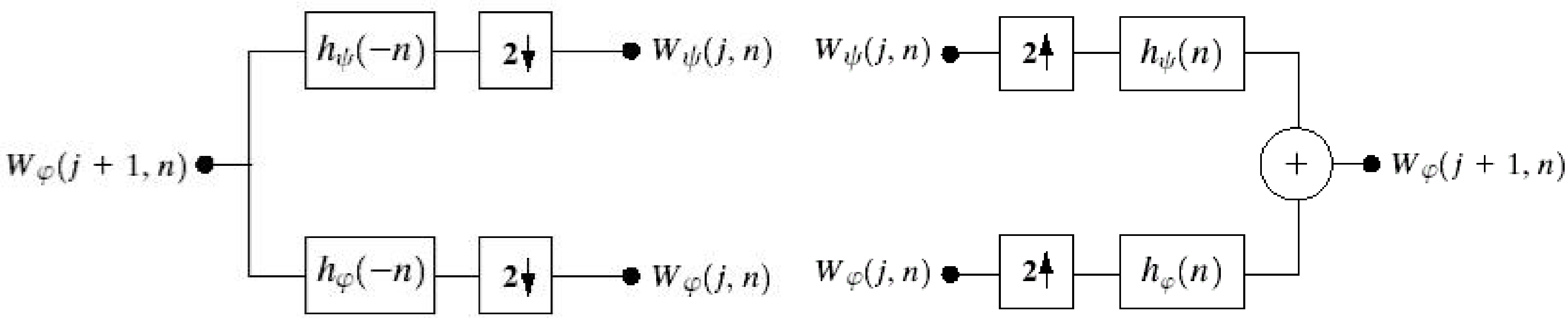
$c(k)$ are referred as approximation coefficients and $d(k)$ are referred as detail coefficients.

Example: $f(x) = \begin{cases} x^2 & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$

a	b
c	d
e	f

FIGURE 7.13 A wavelet series expansion of $y = x^2$ using Haar wavelets.

Fast Wavelet Transform (FWT)
In brief



Analysis filter bank　　　　Synthesis filter bank

A Two-stage (Two-scale) FWT: Analysis

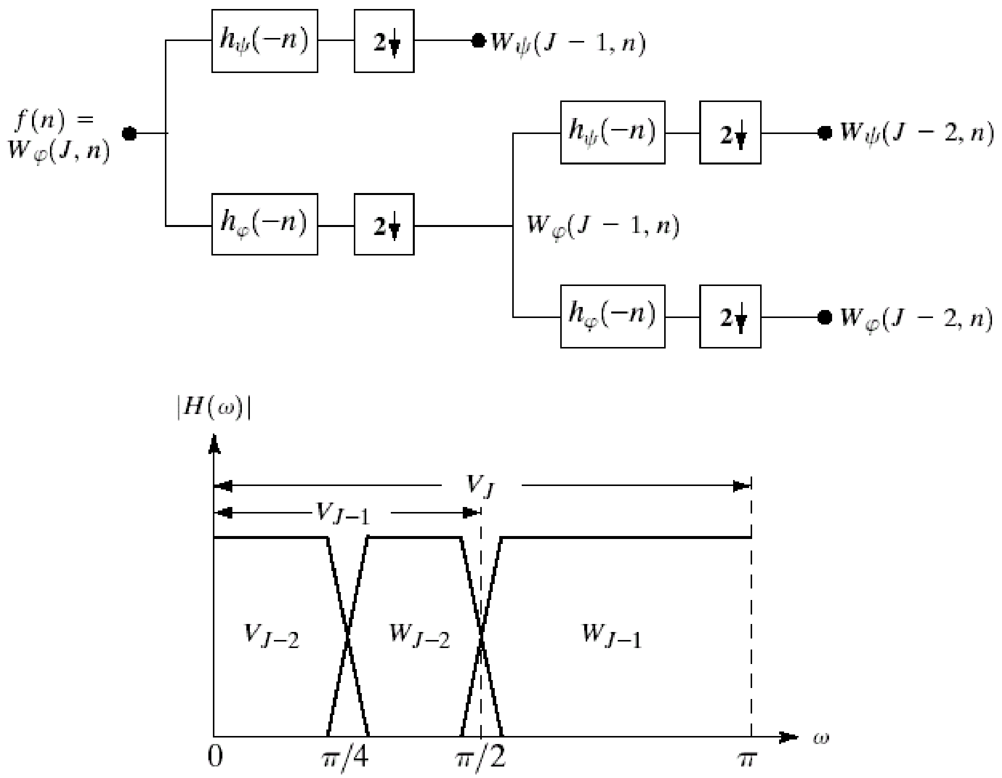
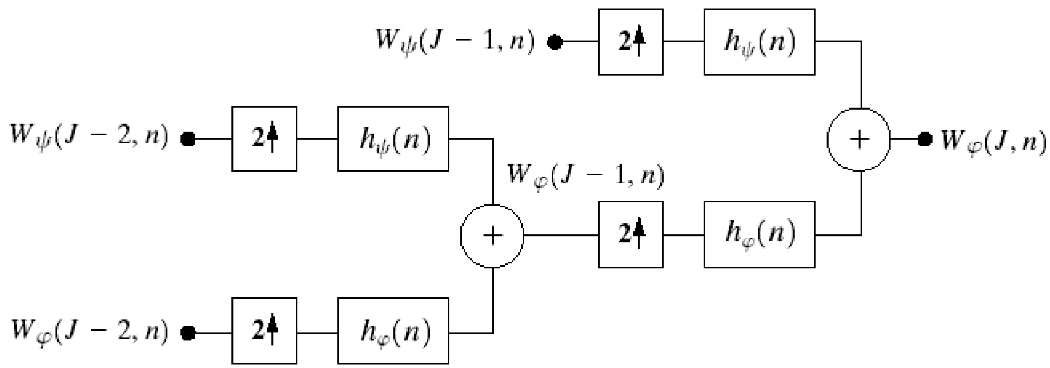


FIGURE 7.16 (a) A two-stage or two-scale FWT analysis bank and (b) its frequency splitting characteristics.

A Two-stage (Two-scale) FWT: Synthesis

FIGURE 7.19 A two-stage or two-scale FWT^{-1} synthesis bank.



Example: Two-scale FWT by Haar

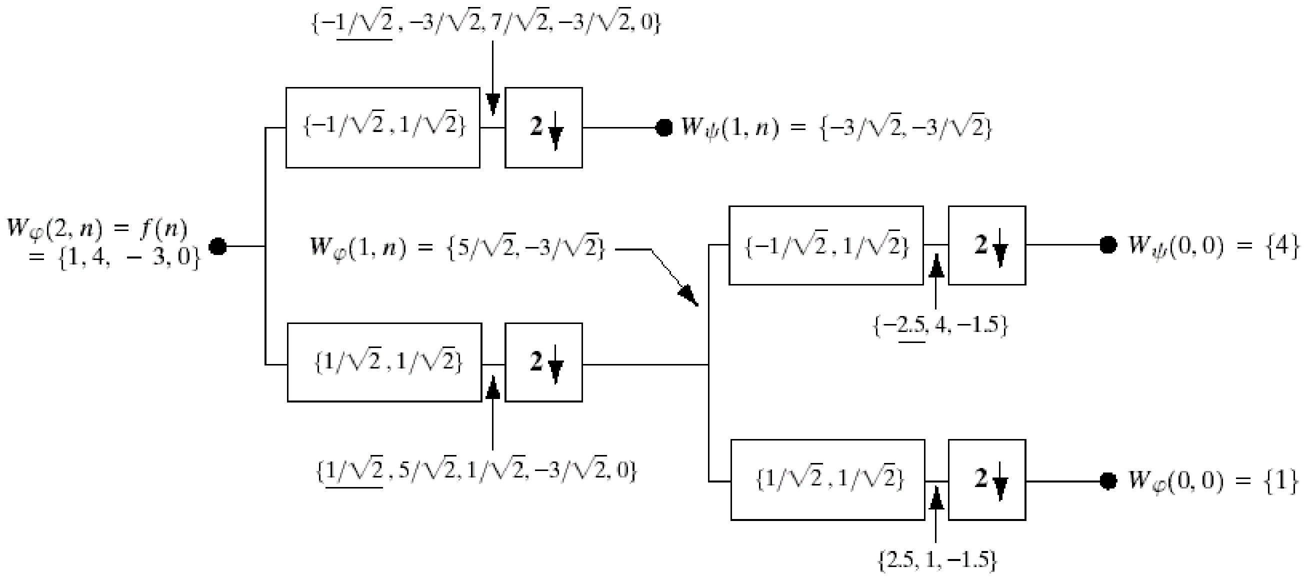


FIGURE 7.17 Computing a two-scale fast wavelet transform of sequence $\{1, 4, -3, 0\}$ using Haar scaling and wavelet vectors.

Example: Two-scale FWT by Haar

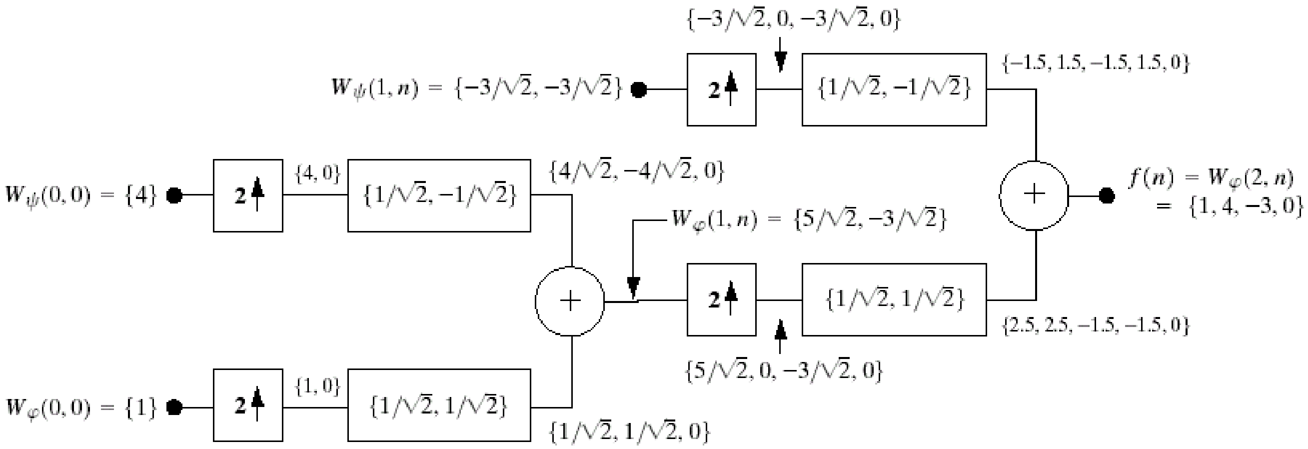


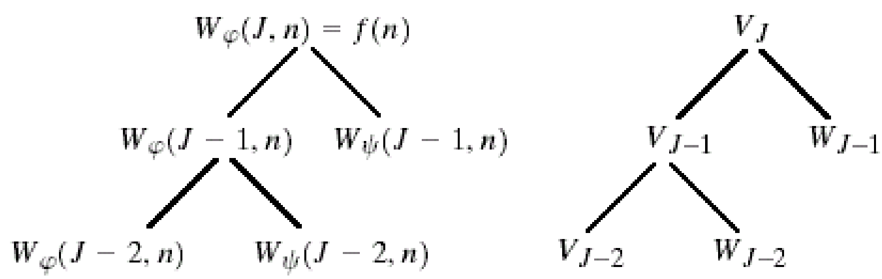
FIGURE 7.20 Computing a two-scale inverse fast wavelet transform of sequence $\{1, 4, -1.5\sqrt{2}, -1.5\sqrt{2}\}$ with Haar scaling and wavelet vectors.

Wavelet Packets

The Wavelet Transform decomposes a function into a series of logarithmically related frequency bands. Low frequencies are grouped into narrow bands, while the high frequencies are grouped into wider bands.

Wavelet Packets are generalization that allows greater control over the time-frequency plane partitioning.

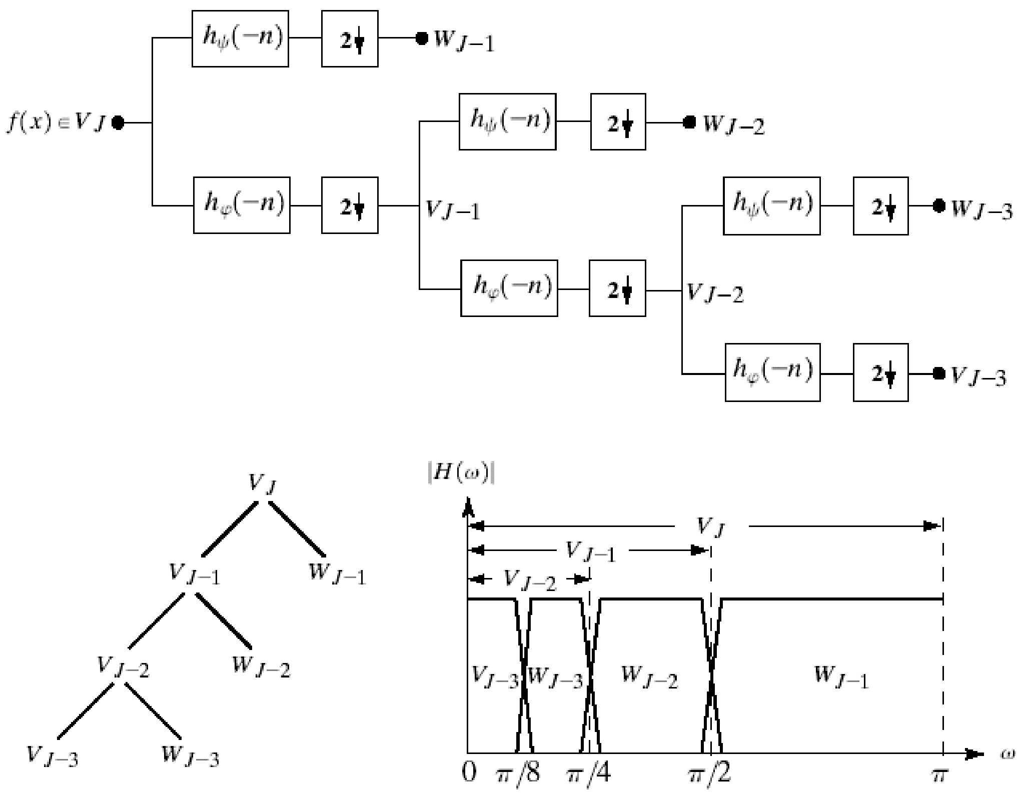
Consider the two-scale filter bank as a *binary tree.* The wavelet coefficients are at the nodes of the tree. The *root node* contains the highest-scale approximation coefficients (i.e. the sampled signal itself). Each node contains coefficients representing different subspaces – *subspace analysis tree.*



a b

FIGURE 7.27 A coefficient (a) and analysis (b) tree for the two-scale FWT analysis bank of Fig. 7.16.

WT: Filter Bank, Analysis Tree, and Spectrum Splitting



a
b c

FIGURE 7.28 A three-scale FWT filter bank: (a) block diagram; (b) decomposition space tree; and (c) spectrum splitting characteristics.

Wavelet Packets

Wavelet packets are conventional wavelet transforms in which the *details* are also iteratively filtered.

Subscripts in the figure show the scale and a string of *A*'s and *D*'s encoding the path from the parent to the node.

The packet tree almost triples the number of decompositions.

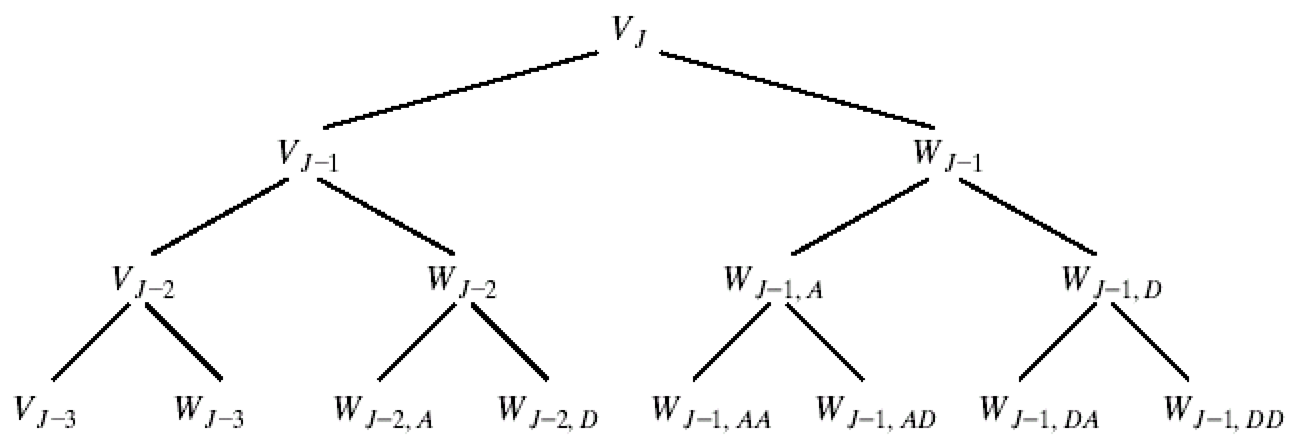
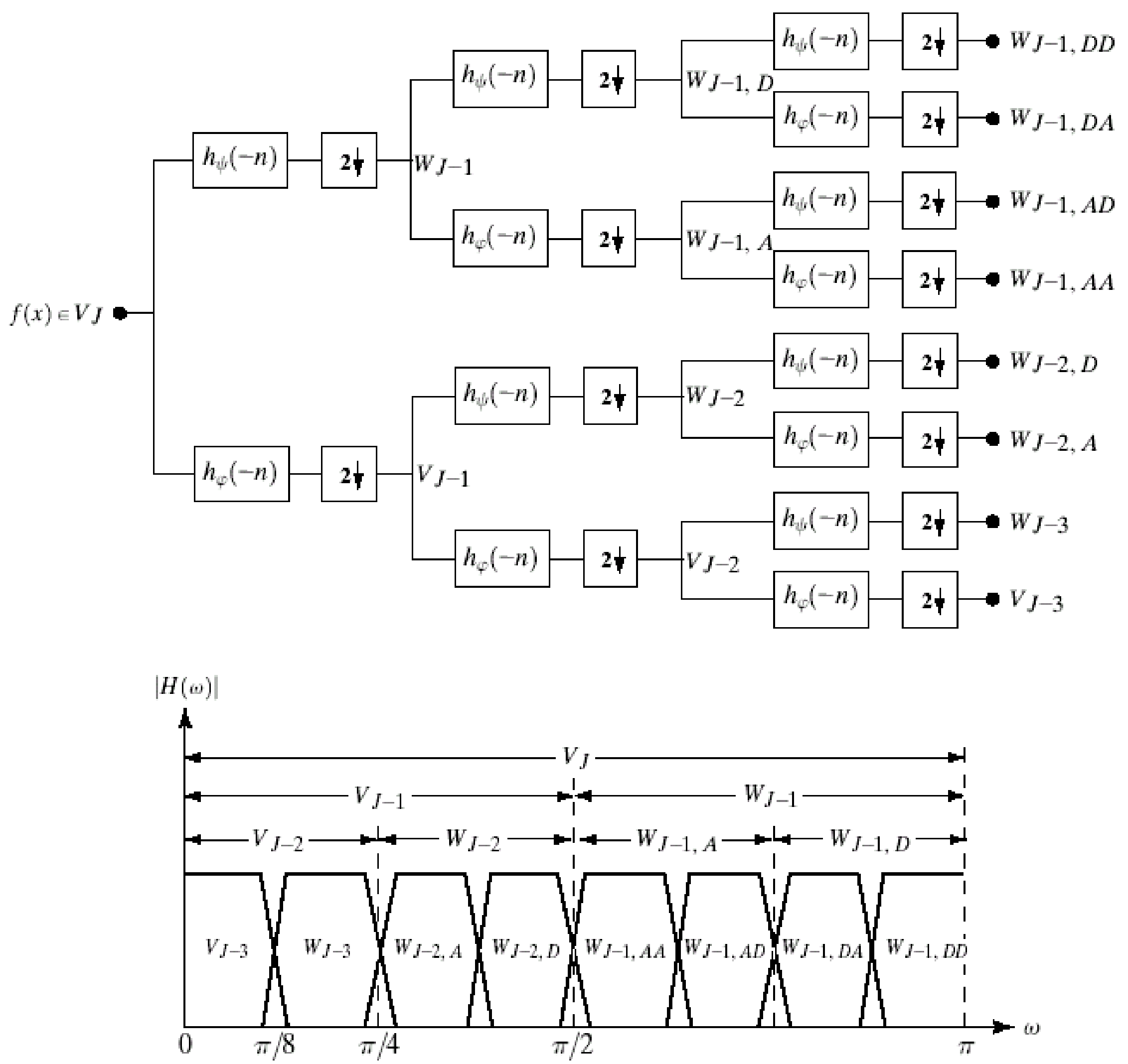


FIGURE 7.29 A three-scale wavelet packet analysis tree.

Wavelet Packets: Filter Structure and Spectrum Splitting



a
b

FIGURE 7.30 The (a) filter bank and (b) spectrum splitting characteristics of a three-scale full wavelet packet analysis tree.

Wavelet Packets

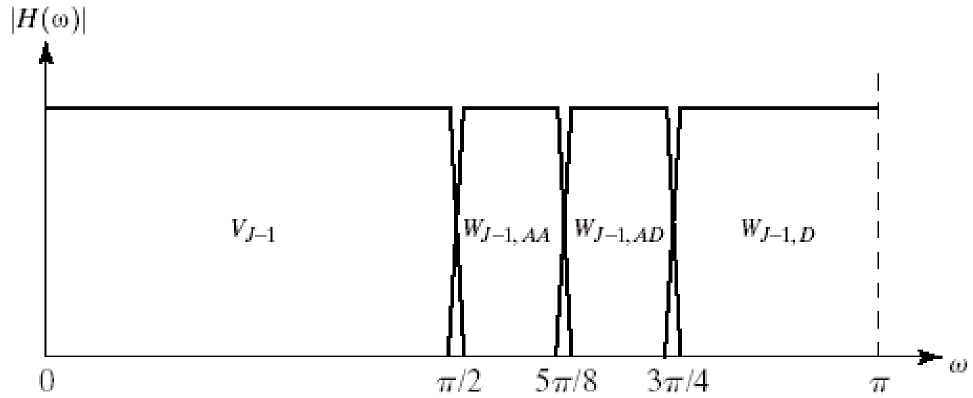


FIGURE 7.31 The spectrum of the decomposition in Eq. (7.6-5).

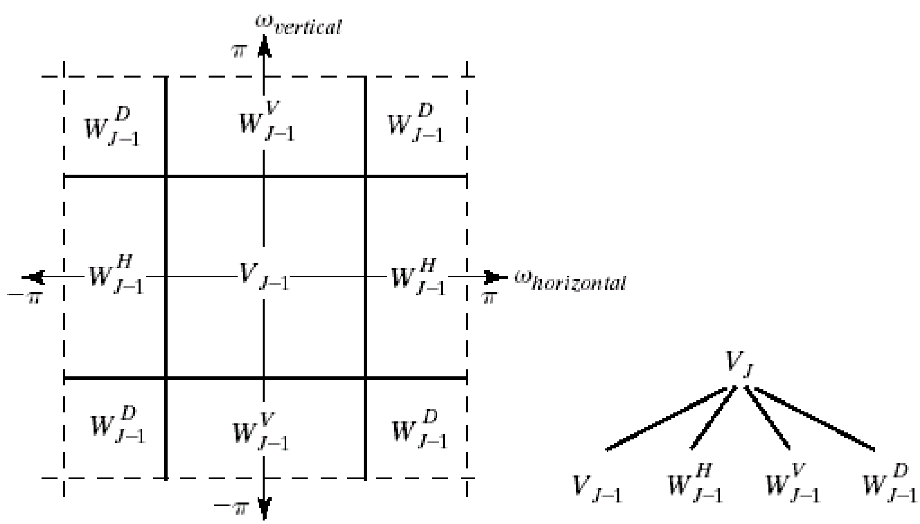
The frequency selectivity can be increased toward high frequencies

2-D Wavelet Packets

In 2-D, separable implementation of wavelet packets leads to a quad-tree structure.
The frequency spectrum is divided into four areas. The low-frequency area is in the centre.

a b

FIGURE 7.32 The first decomposition of a two-dimensional FWT: (a) the spectrum and (b) the subspace analysis tree.



Portion of 2-D Wavelet Packet Tree

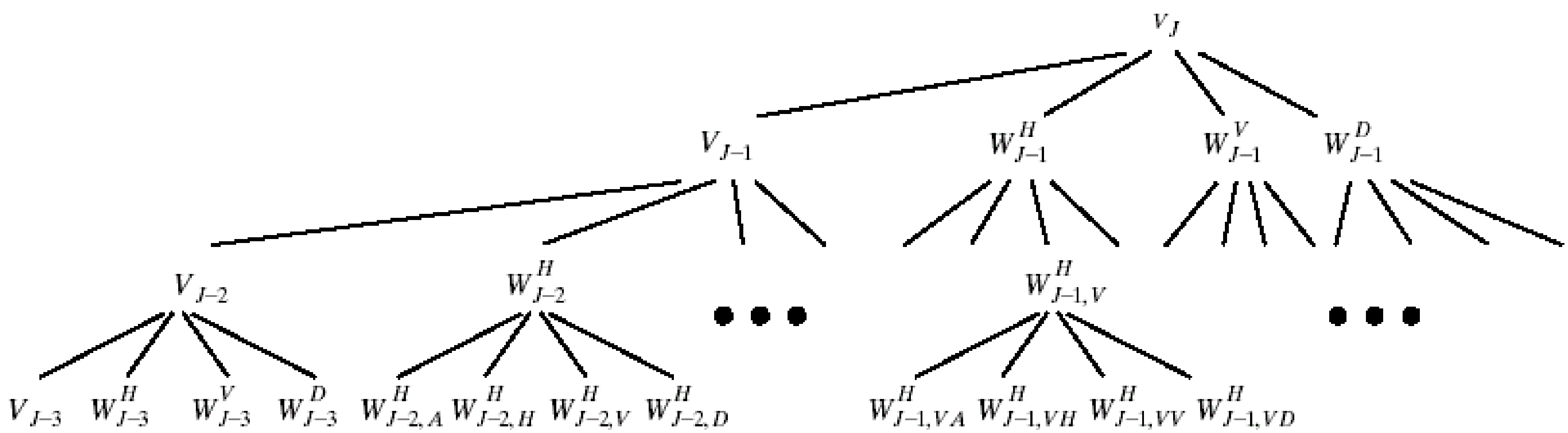
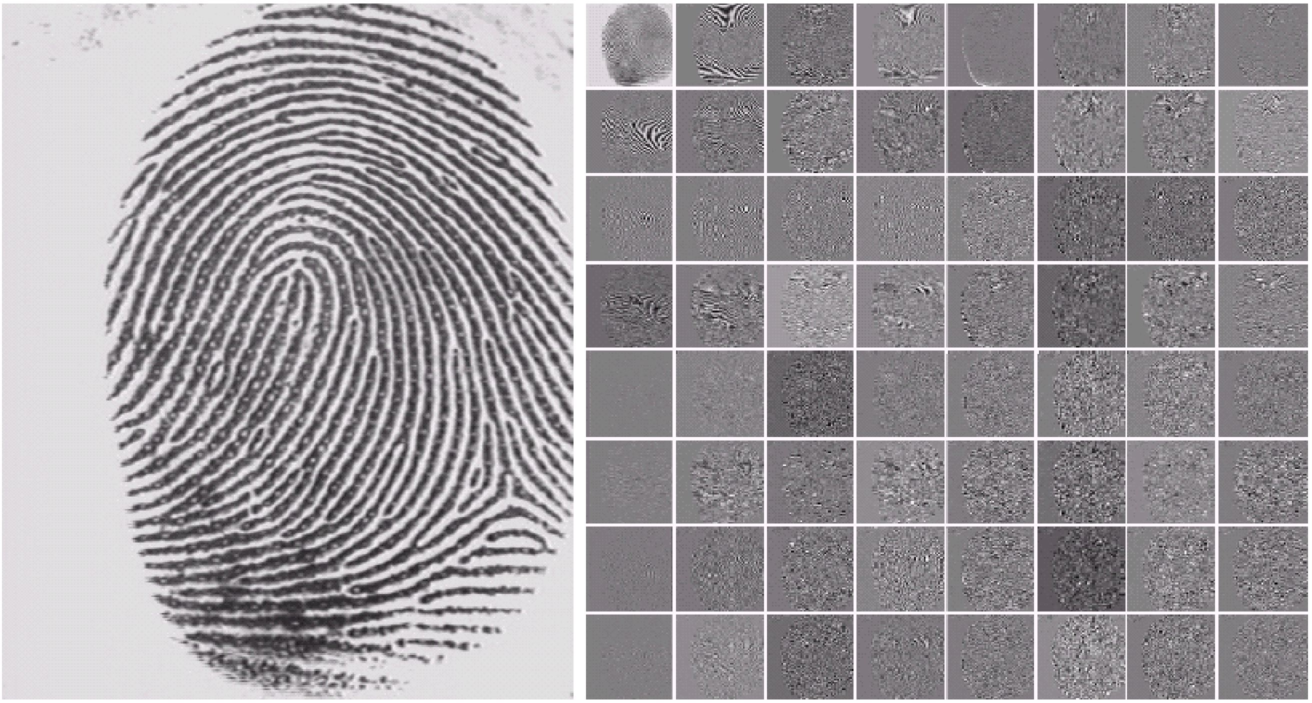


FIGURE 7.33 A three-scale, full wavelet packet decomposition tree. Only a portion of the tree is provided.

Wavelet Packet Decomposition



a b

FIGURE 7.34 (a) A scanned fingerprint and (b) its three-scale, full wavelet packet decomposition. (Original image courtesy of the National Institute of Standards and Technology.)

Optimal Wavelet Packet Basis Search

Entropy-based criteria for searching the optimal (the best) wavelet packet decomposition

An *additive cost function* for comparing different decompositions

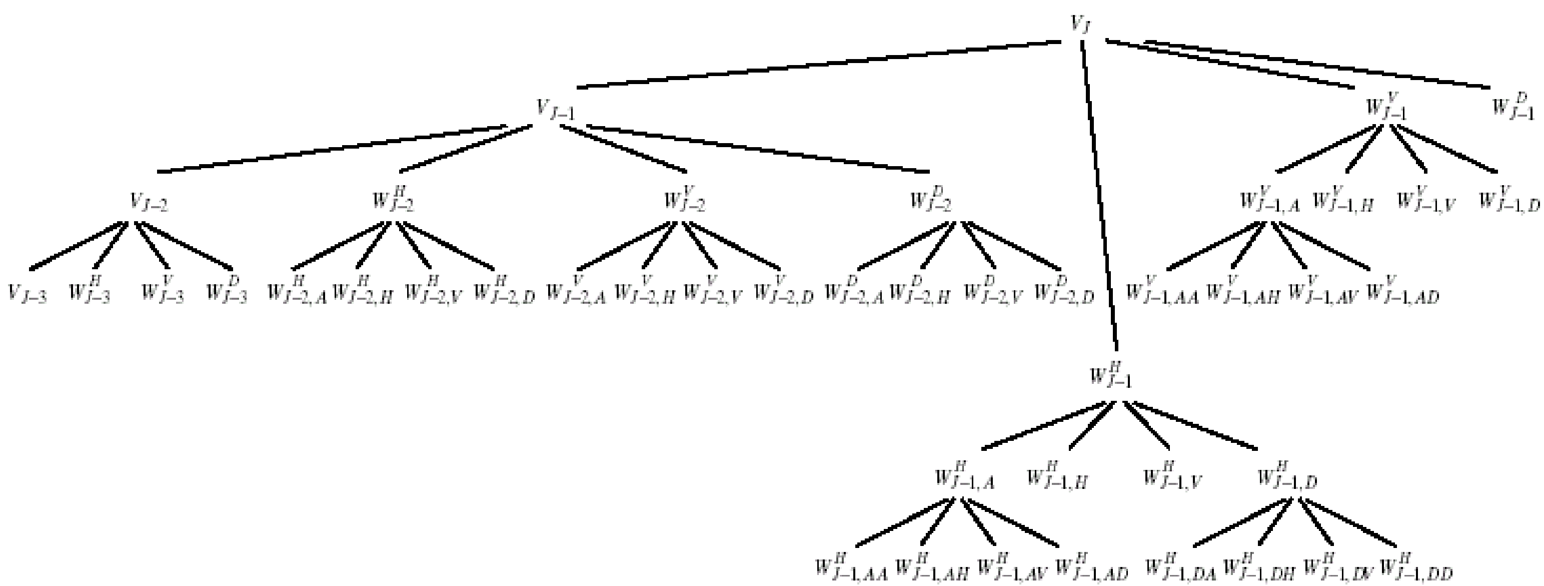


FIGURE 7.36 The optimal wavelet packet analysis tree for the decomposition in Fig. 7.35.

Optimal Wavelet Packet Basis

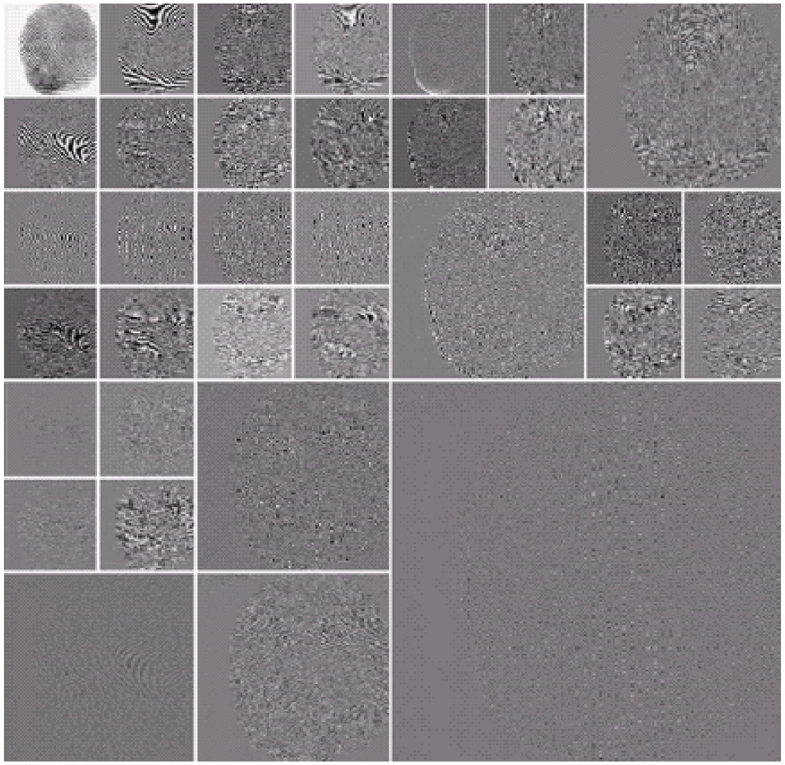


FIGURE 7.35 An optimal wavelet packet decomposition for the fingerprint of Fig. 7.34(a).

Some applications

- FBI fingerprints.
- JPEG2000.
- Image indexing and image search engines
- Image modeling
- Image denoising, deblurring and restorations.
- Texture analysis
- Direct processing tools on the wavelets domain
- Time series analysis, etc.

Some extensions

- Random Wavelets Expansion (RWE) by Mumford-Gidas **[2001],** to model the scale-invariance of general images.
- Geometric Wavelets:
  - D. Donoho (Stanford): ridgelets, wedgelets, curvelets.
  - S. Mallat (Ecole Polytechnique,France)[2001]: beamlets.
  - T. Chan & H.-M. Zhou (UCLA) [2000], A. Cohen [2002]: integrate computational PDE techniques such as the ENO scheme into wavelet transforms, to better capture shocks (discontinuities).

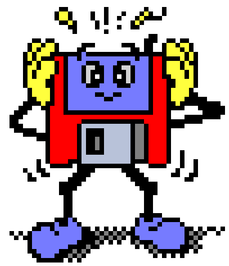
Extra material for reading is in the webpage of the course

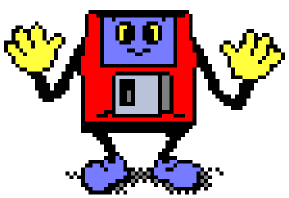
Let's repeat what we have learned.

Watch the WaveletMovie

(http://www.loni.ucla.edu/~dinov/WAIR.dir/WaveletMovie_caption.html)

QUESTIONS?





See you next week