A Simple Adaptive Filter for the Restoration of Nonstationary Signals

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Abstract

A simple adaptive filter is proposed for the restoration of smoothly varying signals in a nonstationary noisy background. The filter switches between the median filter and the running mean filter according to local estimates of the tail behavior (impulsivity) of the noise distribution. A theoretical threshold is derived against which the local estimate is compared. The local median is selected if the threshold is exceeded; otherwise, the local mean is used. Simulations using a family of generalized exponential distribution were performed to compare the median, the mean filter, the γ -filter, the adaptive γ-filter and the proposed filter.

1 Introduction

The median filter provides a very good performance over signal regions where the noise is impulsive, but its efficiency in the suppression of Gaussian noise is inferior to that of the moving average. In practice, noise can be nonstationary due to the presence of more than one noise source with quite different distributions. An adaptive filtering algorithm is proposed for the restoration of smoothly varying signals in a nonstationary noisy background.The algorithm is based on the switching between the median filter and the running mean filter according to local estimates of the tail behavior (impulsivity) of the noise distribution. Simulation results compare this filter with the γ-filter proposed by Astola and Neuvo [2] and the adaptive γ-filter proposed by Rahim and Dobrowiecki [4].

2 Assumptions

Assumptions underlying the definitions of the presented filter:

1. The useful signal is constant throughout the window.We consider smoothly varying signals.
2. The signal is corrupted by exponential noise .

We consider the family of generalized exponential densities:

$$f(x) = \alpha e^{-\beta|x|^{\gamma}} \quad (1)$$

$$\alpha = \frac{\gamma}{2A(\gamma)\Gamma\left(\frac{1}{\gamma}\right)}$$

$$\beta = \left(\frac{1}{A(\gamma)}\right)^{\gamma}$$

$$A(\gamma) = \left[\sigma_x^2 \frac{\Gamma(1/\gamma)}{\Gamma(3/\gamma)}\right]^{1/2}$$

σ_x^2 – variance

This family of densities represents a broad range of noise behaviors ranging from very impulsive ($\gamma < 1$), to shallow tailed ($\gamma > 2$).

3 Test statistic

The measure of tail behavior is the test statistic suggested by Hogg [1]:

$$Q(\alpha) = [U(\alpha) - L(\alpha)]/[U(0.5) - L(0.5)] \quad (2)$$

$M = 2N + 1$-is the length of the window ($N > 1$)
$U(\alpha)$ and $L(\alpha)$ are L-filters:

$$U(\alpha) = \sum_{i=1}^{M} a_i x_{(i)}$$

$$L(\alpha) = \sum_{i=1}^{M} b_i x_{(i)}$$

$$a_i = \begin{cases} \frac{1}{M\alpha} & 2N+2-\lfloor M\alpha \rfloor \le i \le 2N+1 \\ 1 - \frac{\lfloor M\alpha \rfloor}{M\alpha} & i = 2N+1-\lfloor M\alpha \rfloor \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

$$b_i = \begin{cases} \frac{1}{M\alpha} & 1 \le i \le \lfloor M\alpha \rfloor \\ 1 - \frac{\lfloor M\alpha \rfloor}{M\alpha} & i = \lfloor M\alpha \rfloor + 1 \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

For $\alpha < \frac{1}{M}$ we can find simpler form for this statistic:

$$Q(\alpha) = \frac{M}{2} \frac{x_{(2N+1)} - x_{(1)}}{\sum_{i=N+2}^{2N+1} x_{(i)} - \sum_{i=1}^{N} x_{(i)}} \quad (5)$$

For $\alpha = 0.05$ the statistic $Q(\alpha)$ performs well as the measure of tail behavior if the underlying population ranges anywhere from uniform to very impulsive noise.The statistic $Q(\alpha)$ is both location and scale independent.

Proof of Eq. (5)

1)$\alpha < \frac{1}{M}$

$$\lfloor M\alpha \rfloor = 0$$

After substituting this into Eqs. (3) and (4) the coefficients of $U(\alpha)$ are:

$$a_i = \begin{cases} 1 & i = 2N+1 \\ 0 & \text{otherwise} \end{cases}$$

and those of $L(\alpha)$ are

$$b_i = \begin{cases} 1 & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Now $U(\alpha)$ and $L(\alpha)$ are:

$$U(\alpha) = x_{(2N+1)}$$

$$L(\alpha) = x_{(1)}$$

2)$\alpha = 0.5$

$$\lfloor M\alpha \rfloor = \left\lfloor \frac{M}{2} \right\rfloor = \left\lfloor \frac{2N+1}{2} \right\rfloor = \lfloor N + 0.5 \rfloor = N$$

After substituting this into Eqs. (3) and (4) the coefficients of $U(0.5)$ are:

$$a_i = \begin{cases} \frac{2}{M} & N+2 \le i \le 2N+1 \\ \frac{1}{M} & i = N+1 \\ 0 & \text{otherwise} \end{cases}$$

and those of $L(0.5)$ are

$$b_i = \begin{cases} \frac{2}{M} & 1 \le i \le N \\ \frac{1}{M} & i = N+1 \\ 0 & \text{otherwise} \end{cases}$$

Now $U(\alpha)$ and $L(\alpha)$ are:

$$U(0.5) = \frac{2}{M} \sum_{i=N+2}^{2N+1} x_{(i)} + \frac{1}{M} x_{(N+1)}$$

$$L(0.5) = \frac{2}{M} \sum_{i=1}^{N} x_{(i)} + \frac{1}{M} x_{(N+1)}$$

After substituting this into Eq. (2) $Q(\alpha)$ is:

$$Q(\alpha) = \frac{M}{2} \frac{x_{(2N+1)} - x_{(1)}}{\sum_{i=N+2}^{2N+1} x_{(i)} - \sum_{i=1}^{N} x_{(i)}}$$

4 A simple adaptive filter

Examine Figs. 1-4 and notice the cross-over point of the mse curves of the median and the mean filters.If this cross-over point γ^* is somehow determined, then it would be an easy task to locally estimate γ, compare it against γ^* and select either the median or the mean filter accordingly. Therefore, the filter will switch between the median filter and the running mean filter according to local estimates of the tail behavior (impulsivity) of the noise distribution. It chooses the one for which the MSE is smaller for the signal within the current window. The measure of tail behavior is the test statistic $Q(\alpha)$.

Under the given assumptions about the noise and the signal, the mean and the median are both unbiased estimators of location and the MSE is equal to the output variance. Therefore, we can compute γ (denoted as γ^*) for which the the MSE of the mean and the median filter are equal by simply comparing the variances.We can show that γ^* is independent of the input variance σ_x^2. To find the threshold τ ($Q(\alpha)$ for the noise density $f(x, \gamma^*)$) one can use Monte Carlo simulation. The local value of the statistic $Q(\alpha)$ is compared against this threshold τ. The local median is selected if the threshold is exceeded; otherwise, the local mean is used.

The algorithm. At each time point, $Q(\alpha)$ is computed using the data within the current window and the result is compared with the threshold τ.

If $Q(\alpha) \ge \tau$ median is used

If $Q(\alpha) < \tau$ mean is used

Computation of the cross-over point.

Variance of the mean:

$$\sigma_{mean}^2 = \frac{\sigma_x^2}{M}$$

Variance of the median:

We consider a constant signal s (within the window of the size M=2N+1) corrupted by noise x_i:

$$y_i = s + x_i$$

x_i are independent, identically distributed random variables having density $f_X(x)$ (given by Eq. (1)). y_i are independent, identically distributed random variables having density $f_Y(y)$.

$$f_Y(y) = f_X(y - s)$$

$$\sigma^2_{med} = M\binom{M-1}{N}\int_{-\infty}^{+\infty}(y-s)^2 \cdot F_Y^N(y)\left[1-F_Y(y)\right]^N f_Y(y)\,dy$$

$$\sigma^2_{med} = M\binom{M-1}{N}\int_{-\infty}^{+\infty}(y-s)^2 F_X^N(y-s) \cdot \left[1-F_X(y-s)\right]^N f_X(y-s)\,dy$$

$$x = y - s \qquad dy = dx$$

$$\sigma^2_{med} = M\binom{M-1}{N}\int_{-\infty}^{+\infty} x^2 F_X^N(x) \cdot \left[1-F_X(x)\right]^N f_X(x)\,dx \qquad (6)$$

$$F_X(x) = \alpha\left[xe^{-\beta x^\gamma} + \beta^{1-k}\gamma(k, \beta x^\gamma)\right] + \frac{1}{2}$$

$$k = \frac{1}{\gamma} + 1 \quad x \in (0, +\infty) \qquad (7)$$

The integrand in Eq. (6) is symmetric about 0. Therefore,

$$\sigma^2_{med} = 2M\binom{M-1}{N}\int_0^{+\infty} x^2 F_X^N(x) \cdot \left[1-F_X(x)\right]^N f_X(x)\,dx \qquad (8)$$

Using numerical methods we can find the value of σ^2_{med} from Eq. (8) for different values of γ and then compare them with σ_x^2/M. For $\sigma_x^2 = 1$ and M=11, σ^2_{med} and σ^2_{mean} are equal for $\gamma^* = 1.251 \pm 0.001$

A Monte Carlo simulation was used to find the threshold τ. 2000 samples of size 11 each from the density f(x) for γ^* =1.251 (value for which the MSE of the median filter and the mean filter are the same) were generated. The density was chosen to have zero-mean and unity variance since the statistic $Q(\alpha)$ is both location and scale independent. The threshold $\tau = 2.3312$ is the sample mean.

Proof that γ^* does not depend on σ_x^2.

$$\sigma^2_{mean} = \sigma^2_{med}$$

$$\frac{\sigma_x^2}{M} = 2M\binom{M-1}{N} \cdot \int_0^{\infty} x^2 F^N(x)\left[1-F(x)\right]^N f(x)\,dx$$

We can rewrite $f(x)$ and $F(x)$ as:

$$f(x) = \frac{1}{\sigma_x} f_1\left(\frac{x}{\sigma_x}\right)$$

$$F(x) = F_1\left(\frac{x}{\sigma_x}\right)$$

$$\frac{\sigma_x^2}{M} = 2M\binom{M-1}{N}\int_0^{\infty} x^2 F_1^N\left(\frac{x}{\sigma_x}\right) \cdot \left[1 - F_1\left(\frac{x}{\sigma_x}\right)\right]^N \frac{1}{\sigma_x} f_1\left(\frac{x}{\sigma_x}\right)\,dx$$

$$x = \sigma_x t \qquad dx = \sigma_x dt$$

$$\frac{\sigma_x^2}{M} = 2M\binom{M-1}{N} \cdot \sigma_x^2\int_0^{\infty} t^2 F_1^N(t)\left[1-F_1(t)\right]^N f_1(t)\,dt$$

$$\frac{1}{M} = 2M\binom{M-1}{N} \cdot \int_0^{\infty} t^2 F_1^N(t)\left[1-F_1(t)\right]^N f_1(t)\,dt$$

Proof of Eq. (7).
From table:

$$\int_0^u x^{\nu-1}e^{-\mu x}\,dx = \mu^{-\nu}\gamma\left(\nu,\nu u\right)$$

$$re(\nu) > 0$$

$$f(x) = \alpha e^{-\beta|x|^{\gamma}}$$

$$F(u) = \int_{-\infty}^{u} f(x)\,dx = \frac{1}{2} + \int_0^u f(x)\,dx$$

$$I_1 = \int_0^u f(x)\,dx = \int_0^u \alpha e^{-\beta x^{\gamma}}\,dx$$

Using integration by part, we get:

$$I_1 = \alpha\left[x e^{-\beta x^{\gamma}}\Big|_0^u + \beta\int_0^u \gamma x^{\gamma} e^{-\beta x^{\gamma}}\,dx\right]$$

$$= \alpha\left[u e^{-\beta u^{\gamma}} + \beta I_2\right]$$

$$I_2 = \int_0^u \gamma x^{\gamma} e^{-\beta x^{\gamma}}\,dx$$

$$x = t^{\frac{1}{\gamma}} \quad dx = \frac{1}{\gamma} t^{\frac{1}{\gamma}-1} dt$$

$$I_2 = \int_0^{u^{\gamma}} t^{\frac{1}{\gamma}} e^{-\beta t}\,dt = \beta^{-k}\gamma\left(k, \beta u^{\gamma}\right)$$

$$k = \frac{1}{\gamma} + 1$$

5 Simulations

Noisy input sequences from exponential distributions with zero mean, unity variance, and parameter γ varied over the range $0.5 \leq \gamma \leq 5$ (step 0.5) or $0.2 \leq \gamma \leq 2$ (step 0.1) were filtered with the median filter, running mean filter, the γ filter (in this case we assume that γ is known), the adaptive γ -filter, and the proposed filter.The window size is 11. The size of the input sequence is 10000 for $0.5 \leq \gamma \leq 5$ and 20000 for $0.2 \leq \gamma \leq 2$ for each value of the input sequence.

The proposed filter has been used for the smoothing of the signal shown in Fig.5. For the purpose of simulations, noise with varying level of impulsivity has been added to this signal.The resulting noisy signal is shown in Fig.6. The noise is very shallow tailed ($\gamma = 5$) for $1 \leq k \leq 150$, Gaussian ($\gamma = 2$) for $151 \leq k \leq 300$, and very impulsive ($\gamma = 0.5$) for

gamma	MSE	
	mean	median
0.5	0.0905	0.0181
1.0	0.0935	0.0730
1.5	0.0899	0.1091
2.0	0.0895	0.1345
2.5	0.0910	0.1575
3.0	0.0941	0.1794
3.5	0.0940	0.1856
4.0	0.0907	0.1931
4.5	0.0953	0.2021
5.0	0.0917	0.2012

Table 1: MSE of the mean, and the median filter

gamma	MSE		
	γ -filter	adaptive γ -filter	proposed filter
0.5	0.0210	0.0307	0.0291
1.0	0.0730	0.0950	0.0751
1.5	0.0996	0.1270	0.0949
2.0	0.1080	0.1336	0.0987
2.5	0.1138	0.1449	0.1026
3.0	0.1149	0.1484	0.1067
3.5	0.1082	0.1416	0.1068
4.0	0.1083	0.1388	0.1007
4.5	0.1058	0.1389	0.1050
5.0	0.0988	0.1351	0.1028

Table 2: MSE of the γ -filter, the adaptive γ -filter and the proposed filter.

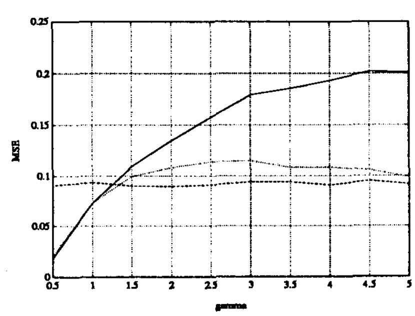

Figure 1: MSE of the γ -filter (dotted line), the mean (dashed line), and the median filter (solid line)

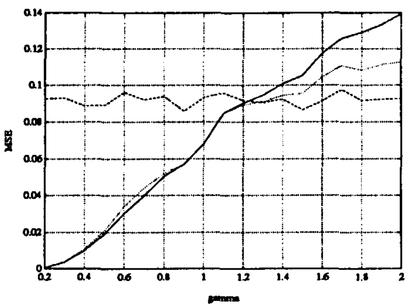

Figure 2: MSE of the γ -filter (dotted line), the mean (dashed line), and the median filter (solid line)

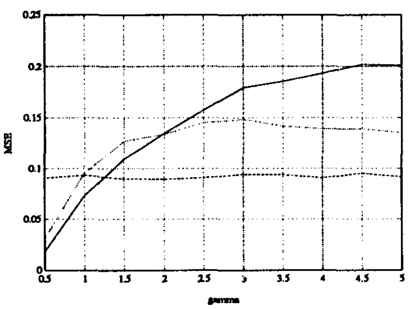

Figure 3: MSE of the adaptive γ -filter (dotted line), the mean (dashed line), and the median filter (solid line)

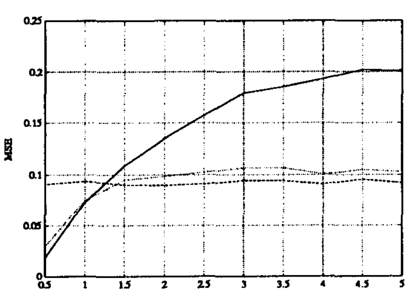

Figure 4: MSE of the proposed filter (dotted line), the mean (dashed line), and the median filter (solid line)

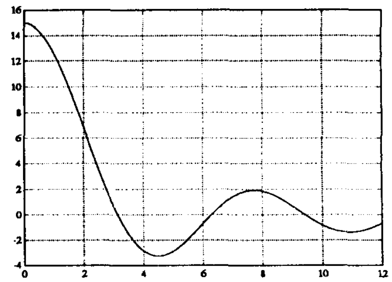

Figure 5: Noise Free Signal

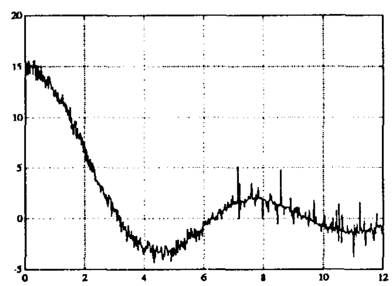

Figure 6: Noisy Input Signal

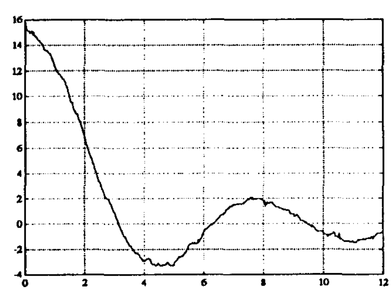

Figure 7: Output Signal of the Proposed Filter

$301 \leq k \leq 600$. The noisy signal is used as input to the switch filter with window size 11. The output signal is shown in Fig.7.

6 Conclusion

We presented in this paper a simple adaptive filter for smoothing nonstationary signals. The filter switches between the median filter and the running mean filter according to local estimates of the noise impulsivity. Simulation results show that this filter produces far better results then the mean or the median filter. It is also much simpler to compute then the adaptive γ-filter.

References

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