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Analysis of Two-Dimensional Center Weighted Median Filters

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Abstract. Center weighted median (CWM) filters, which have been recognized as detail preserving filters, are an important and the simplest subclass of weighted median (WM) filters. In this paper, we analyze the root signals of two-dimensional (2-D) CWM filters. In particular, we derive the required form for a signal to be a root of a 2-D CWM filter. The required form of signals to be roots is then used to evaluate the detail preserving properties of 2-D CWM filters. As examples, the detail preserving properties of some 2-D CWM filters are compared with other detail preserving filters, i.e. multilevel median filters. The generation of binary root signals of some 2-D CWM filters is treated in the term of the smallest surviving object (SSO). It is illustrated by some examples that CWM filters with different orientation of windows can be useful in image segmentation.

Key Words: root properties, weighted median filters, two-dimensional

1. Introduction

Median based filters have been widely used in image processing because some important image details, e.g. edges, can be retained while noise can be effectively removed by the filters. Weighted median (WM) filters are a natural extension of median filters, which exploit not only rank-order information but also spatial information of input signal. Among WM filters, center weighted median (CWM) filters have received some special attentions lately [1], [2], [3], and [4]. This is because, first, CWM filters are the simplest WM filters and the easiest to design and implement. Second, CWM filters are better understood theoretically. Some theoretical analyses of CWM filters are reported in [1], [2], and [3]. Many applications of CWM filters in signal processing have been reported in the literature due to their useful properties known as detail preserving and noise suppressing, particularly heavy-tailed noise. These applications can be found in both one-dimensional (1-D) [10] and multi-dimensional cases [1], [5], [3].

Because of the nonlinear nature of median based filters, the analysis of CWM filters is mainly based on statistical and deterministic properties of the filters. The statistical properties of CWM filters have been studied to evaluate the noise suppression, edge and detail, e.g. fine lines, preservation characteristics, while the study of the deterministic properties includes root sets and convergence behavior of the filters in time domain. The statistical properties of CWM filters and adaptive CWM filters based on local statistics have been studied in [1], while the deterministic properties of 1-D CWM filters and separable CWM filters as well as their adaptive version were reported in [2] and [3].

The concept of root signals is important in the deterministic properties because they define a set of signals which is not altered by the filtering. Convergence behavior of a filter is also crucial to the usefulness of the filter because otherwise the filter could have quite a chaotic filtering behavior [6]. Even though the root structures and the convergence properties of 1-D median filters are well understood, those of WM filters are not completely well understood yet. The root properties of 1-D CWM filters, which are a special subclass of WM filters, have been successfully studied and their convergence behavior has been proven in [2], [3].

The root structures of 2-D CWM filters are also hard to analyze and characterize. In fact, even the root structures of 2-D median filters are still not clear. In [7], the generation of binary root signals of 2-D median filters with arbitrarily specified symmetric window shape is discussed. It was shown that when the filter window size is increased, small features (objects), which are roots of a filter with smaller window size, might be smoothed out after successive filtering.

In this paper, some results on deterministic properties of 2-D CWM filters are presented by considering signal structures within a filter window. We show that by changing the center weight, the deterministic properties of 2-D CWM filters can be controlled. The required forms for a signal to be a root of a given 2-D CWM filter are derived. By using these results, the detail preserving behaviors of a 2-D CWM filter are evaluated in the sense of the smallest signal structure. As examples, the detail preserving properties of some CWM filters are compared with those of other detail preserving filters, e.g. multilevel median filters [8].

The generation of binary root signals for some 2-D CWM filters is also studied. This can be considered as an extension of the results obtained for 2-D median filters [9]. It is easy to understand that with 2-D CWM filters, small objects can be retained even with a larger window mask. With this in mind, roots of 2-D CWM filters can be used in image segmentation applications. Better performance of 2-D CWM filters over 2-D median filters in signal segmentation application is demonstrated by some simulations.

2. Center weighted median filters

In the sequel, a CWM filter refers to a 2-D CWM filter, unless specified otherwise.

2.1. Definition of CWM filters

WM filters are generalization of median filters, in which each window position is assigned a weight. Each sample inside the filter window is duplicated to the number of the corresponding weight. The median value from the increased list of samples is the WM output.

In a CWM filter, the center sample is assigned a larger weight, i.e. $w(0,0) = 2K + 1$ where $K \geq 0$, and all other non-zero weights are equal to one, i.e. $w(i,j) = 1$ for $i \neq 0$

and $j \neq 0$. K is a nonnegative integer. As an example, the weights of a CWM filter with connected square shape window has the form as follows:

$$\overline{W} = \begin{pmatrix} 1 & \dots & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & 2K+1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & 1 & \dots & 1 \end{pmatrix}.$$

A CWM filter is completely specified by two parameters, the window size and the center weight. The filtering behavior of a CWM filter will thus be controlled by these two parameters. Denote a CWM filter with center weight $2K+1$ by CWM$(M; 2K+1)$, where M is the number of the samples in the window, e.g., $M = (2N+1) \times (2N+1)$ for a connected square window.

2.2. *Basic properties of CWM filters*

CWM filters are recognized as detail preserving filters comparing with 2-D median filters because the latter tend to remove image details such as thin lines when reducing noise. The behavior of CWM filters can be easily adjusted by changing the center weight (parameter K), when a filter has a fixed window size. CWM filters can vary between the 2-D median filter ($K = 0$) and the identity filter ($K \geq (M-1)/2$), in which case, the output is the same as the input. When $K = ((M-1)/2) - 1$, the CWM filter becomes idempotent, i.e. it produces root signals after a single filtering pass [10].

CWM filters have a simpler implementation than other WM filters. Once the inputs are ordered, the output of a CWM filter can be found by the following property [1].

Property 2.1. The output $y(i, j)$ of a CWM filter with window size M and center weight $2K+1$ can be represented by

$$y(i,j) = med\{x(((M+1)/2) - K; M), x(i,j), x(((M+1)/2) + K; M)\},$$

where $x(r; M)$ denotes the *r*th smallest sample among the M samples within the window centered at (i, j), and $x(i, j)$ is the input value at the center of the window.

This property shows that the output of a CWM filter is always the result of a three-point median taken over three samples inside the filter window. This property is also useful to prove some other properties of CWM filters. Examples can be found in [2].

Although CWM filters are the simplest subset of WM filters, their root structures are still hard to analyze and are still not completely characterized. It is already difficult to find the root structures for a CWM filter with small window size. In the next section, the required forms for signals to be roots of CWM filters are studied. The results are then used to evaluate the detail preserving properties of the filters. In this paper, all of the analysis of the root properties of 2-D CWM filters are in binary domain. The results can be extended to multi-level domain using the threshold decomposition and the stacking properties [11].

3. Deterministic properties of CWM filters

The required form for a signal to be a root of a CWM filter is derived first as an extension of 1-D case [2]. The necessary and sufficient condition for an arbitrary 2-D binary signal to be a root of CWM filter is given as follows.

THEOREM 3.1 *A binary signal is a root signal of CWM filter CWM(M; 2K+1) if and only if, at any window position, the window contains no less than* $((M-1)/2)-K$ *samples with the same value as the center sample.*

Proof: According to Property 2.1, the output of this CWM filter is the median of the following three samples: the center sample, the r_1th smallest and the r_2th smallest sample in the window, where $r_1 = ((M+1)/2)-K$ and $r_2 = ((M+1)/2)+K$. Suppose that the window is centered at sample $x(i,j)$. This sample is unchanged by the CWM filter iff the following inequality holds: $x(r_1; M) \leq x(i,j) \leq x(r_2; M)$. That is, $x(i,j)$ is unchanged by the CWM filter when there are at least $r_1 - 1$ or $M - r_2$ samples with the same value as the sample at the center, i.e. $((M-1)/2)-K$ samples or more must have the same value as the center sample. ■

With this theorem, one can test if a 2-D signal is a root of a particular CWM filter. For example, the CWM filter CWM(9; 3) with a square window will preserve a 2-D signal if there are always three or more other samples in the window having the same value as the center one, when the filter window is positioned at any sample. It can also be seen that the root structures of CWM filters are directly related to the center weight and the window size; therefore, it is easy to control the root structures by changing the center weight for a fixed window size.

Since CWM filters have been recognized as detail preserving filters, some methods are needed to evaluate the detail preserving ability of the filters. Theorem 3.1 can be used to evaluate the detail preserving property of CWM filters in the sense of the smallest structure that can be preserved. In order to evaluate the detail preserving property, we first define the terminology "same value neighborhood"(SVN) in the binary domain; a notion that is similar to but not the same as the constant neighborhoods (CN) for 1-D median filtering.

Definition. If there are $A-1$ samples having the same value as the center sample in the window, the window is said to contain an SVN of value A.

Note that an SVN in a window is defined only by the number of samples but not their position in the window. An SVN can be of any shape as long as the number is correct. Therefore, it is suitable to use SVNs to evaluate the detail preserving properties of CWM filters. This stems from an interesting property of CWM filters; CWM filters can preserve signals with any structural shape that satisfy Theorem 3.1. There is a fundamental difference between an SVN and an CN. The former (SVN) is only defined by the samples inside the filter window but not by other signal samples; while, the latter (CN) may extend beyond the samples inside a window. The following property can be described by SVNs.

Table 1. Minimal SVN for 3 by 3 window CWM filters with different center weights

minimal SVN	center weight $(2K + 1)$
5	1
4	3
3	5
2	7

Property 3.1. If the center sample is preserved by a CWM filter when the filter window contains an SVN of value A, then the center sample is also preserved by the CWM filter when the filter window contains any SVN of value B, provided that $B \geq A$.

The proof of this property is easy based on the definition of an SVN, and it is therefore omitted. This property gives nested relations among different SVNs for a fixed CWM filter in order to preserve the center sample. Theorem 3.1 can alternatively be expressed in term of SVNs: *A signal is a root of a CWM filter if and only if, at any window position, the filter window contains a SVN B with $B \geq A$, where A is the minimal SVN which the CWM filter can preserve.*

In order to evaluate the detail preserving properties of a CWM filter, the most important condition is the *minimal SVN* value in a filter window which guarantees the preservation of the center sample by the given CWM filter. The minimal SVN for a given CWM filter can be calculated by the window size and the center weight of the filter. As examples, the minimal SVN values of 3 by 3 CWM filter with different center weights are tabulated in Table 1.

Before further discussion on detail preservation, an important property of root signal sets of CWM filters is considered. The relationship between root signal sets of CWM filters with the same window size and different center weights is shown in Property 3.2.

Property 3.2. Roots of 2-D CWM filter CWM(M; $2K_1 + 1$) are also roots of 2-D CWM filter CWM(M; $2K_2 + 1$) whenever $K_2 \geq K_1$. That is, if R_i^M denotes the root set of CWM(M; 2i+1), then

$$\cdots \subseteq R_{i-1}^M \subseteq R_i^M \subseteq \cdots \subseteq S$$

where S is the set of all signals.

Proof: Assume that the center point of the window is positioned at sample $X(i, j)$. If one signal is a root of the CWM filter with center weight $2K_1 + 1$, the following inequality must be hold, $x(r_1; M) \leq X(i, j) \leq x(r_2; M)$, where $r_1 = (M + 1)/2 - K_1$ and $r_2 = (M + 1)/2 + K_1$. Now, for the following CWM filter with center weight $2K_2 + 1$, where $K_2 \geq K_1$, the inequality $x(r_1; M) \leq X(i, j) \leq x(r_2; M)$ must also hold for $r_1 = (M + 1)/2 - K_2$ and $r_2 = (M + 1)/2 + K_2$. Therefore, this signal is also a root of the latter CWM filter. ■

This property explains that a CWM filter with larger center weight can preserve more details than a CWM filter with a smaller center weight of the same window size. On the other hand, noise suppression ability of the filter decreases as a result. Property 3.2 implies that a CWM filter with larger center weight has a smaller minimal SVN than a CWM filter with smaller center weight. Table 1 shows some examples.

The detail preserving evaluation of a CWM filter is based on the fact that if a CWM filter has a smaller minimal SVN, the filter has a better detail preserving behavior. The concept of SVN can be used to evaluate the detail preserving property of not only CWM filters but also some other detail preserving filters. As examples, the detail preserving properties of 3×3 unidirectional multilevel median filters [8] are analyzed based on its equivalence with the CWM filter CWM(9; 7); and the detail preserving properties of CWM filter CWM(9; 5) are compared with another detail preserving filter which is 3×3 bidirectional multilevel median filters in the following.

4. Comparison of CWMs and MMFs

Multilevel median filters (MMFs) are very promising detail preserving filtering structures [8] since it was shown that every subfilter will preserve signal details within their subwindows. MMFs are divided into two classes in general: unidirectional MMFs and bidirectional MMFs. The structures for a 3×3 window unidirectional MMF and a bidirectional MMF are shown in Figure 1 and Figure 2, respectively.

Unidirectional MMFs are designed to preserve image details along the vertical, horizontal and the two diagonal directions. Therefore, the samples of SVN must be located along those directions in order to preserve the center sample by unidirectional MMFs. It was shown that

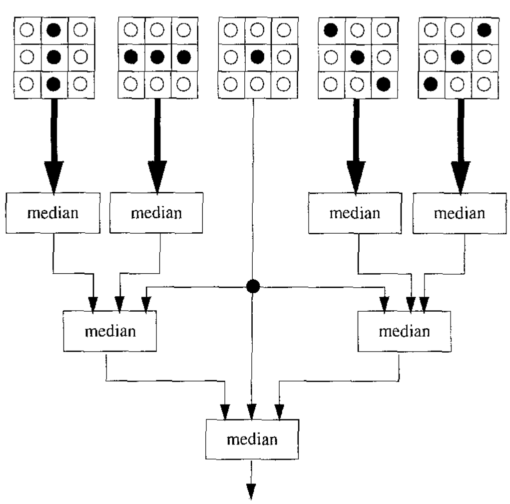

Figure 1. Structure of 3 by 3 window unidirectional MMF.

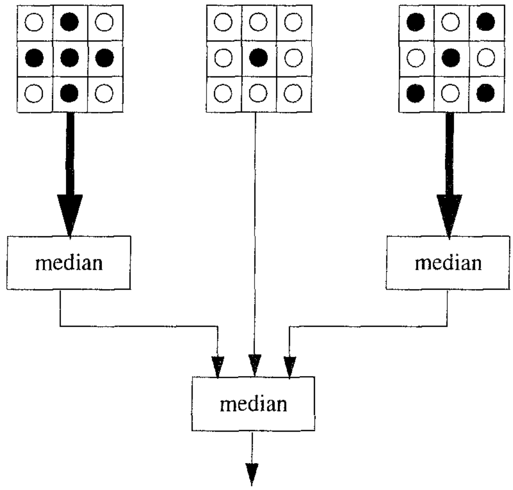

Figure 2. Structure of 3 by 3 window bidirectional MMF.

a 3×3 square window unidirectional MMF is equivalent to CWM filter CWM(9; 7) [1]. Based on this equivalent relation, the root properties of 3×3 square window unidirectional MMF can be better understood. Since the minimal SVN of CWM(9; 7) is SVN 2, the minimal SVN for 3×3 unidirectional MMF is also SVN 2. Therefore, two connected identical samples are root of 3×3 unidirectional MMF. An interesting consequence is that the 3×3 square window unidirectional MMF is not the only multilevel decomposition of CWM(9; 7). Other decompositions do exist which one may choose for analysis or implementation depending on the particular task and application at hand.

Observation 3.1. For a 3×3 square window mask, a unidirectional MMF is equivalent to a structure containing four subwindows; each containing the center sample and two other samples. The samples other than the center sample can only appear once in the four subwindows. The multilevel median filtering procedure is the same as the unidirectional MMF.

Proof: It was shown that 3×3 square window unidirectional MMF is equivalent to 2-D CWM filter CWM(9; 7) [1]. It is also easy to show the equivalence between the filter structure described in Observation 3.1 and the 2-D CWM filter CWM(9; 7) using the same method. Boolean function representation of the filters can be used to easily show the equivalence. ■

CWM(9; 7) as well as its corresponding 3×3 MMFs can all preserve minimal SVN 2. Based on this equivalence, we derive the following theorem.

THEOREM 3.2 *The* 3×3 *square window unidirectional MMF filter is idempotent.*

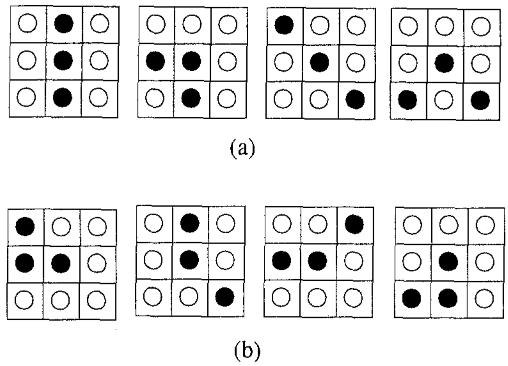

Figure 3. (a) roots of 3 × 3 bidirectional MMF and CWM filter CWM(9; 5), (b) roots of CWM filter CWM(9; 5) but not 3 × 3 bidirectional MMF.

The proof of Theorem 4.2 is omitted. Interested readers can find a similar proof in [10].

It is not difficult to show that the 3 × 3 bidirectional MMF is not equivalent to a CWM filter. However, it is easy to see that the 3 × 3 bidirectional MMF can preserve the center sample when the filter window contains SVN 3 and all these three samples, one of which is the center sample, are located in one of the subwindows. This can be considered as the minimal SVN for the 3 × 3 bidirectional MMF. From this point of view, its performance is similar to CWM(9; 5) which can also preserve the center sample because SVN 3 is the minimal SVN. As seen earlier, CWM filters do not care about the positions of the sample in the SVN; whereas, bidirectional MMFs do.

Figure 3 shows that CWM(9; 5) preserves not only all root structures of the 3 × 3 bidirectional MMF, but also some additional ones. Therefore, CWM(9; 5) can preserve more details than 3 × 3 bidirectional MMF. If one wants to implement CWM(9; 5) using multilevel structures, more subwindows are needed, e.g. the structure shown in Figure 4. Bidirectional MMFs are sensitive to the shape of minimal SVN, while CWM filters are not. Therefore, one can conclude that CWM filter CWM(9; 5) has better detail preserving properties than the 3 × 3 bidirectional MMF.

5. Root signals of CWM filters

Döhler [7] derived the rules for the generation of binary roots of a special case of CWM filters: 2-D median filters with arbitrarily symmetrically shaped windows. In this section, the generation of binary root signals of some CWM filters is studied. Since the whole root structures for 2-D median filters are not clear yet, the root structures for CWM filters are even more hard to describe. Therefore, it is not easy to compute all those invariant patterns in limited time, already for small windows. It is necessary to restrict our study to some particular but useful root signal structures. The concept of the smallest surviving object (SSO) is used in [7] for 2-D median filters and it was shown that the SSO is proportional to the window size, i.e. the larger the median window size, the larger the SSO. In this paper,

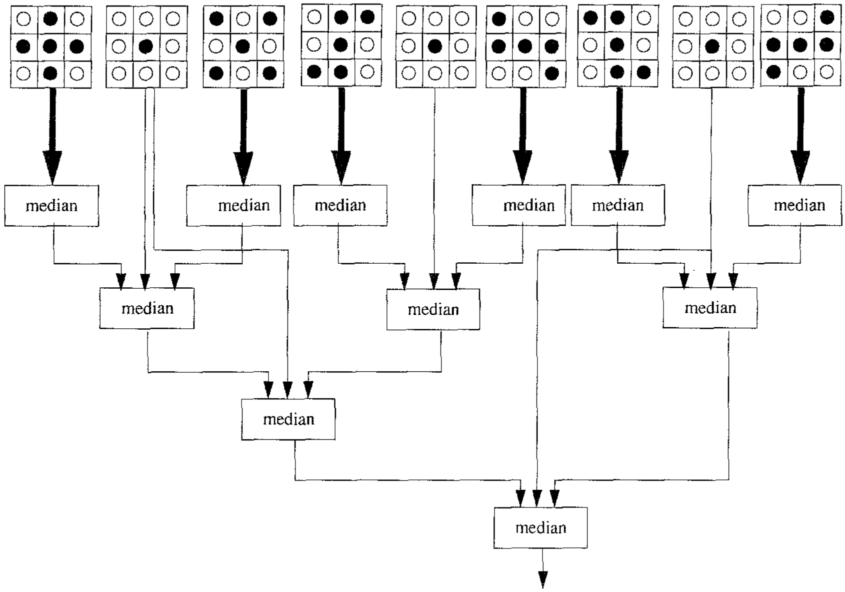

Figure 4. Multilevel structure which is equivalent to CWM(9,5) with 3 by 3 square window.

the generation of binary root signals of some CWM filters is also treated in the term of SSO and it is shown that SSO can be useful in signal segmentation applications.

5.1. Connected smallest surviving object

Before going further, some helpful definitions and notations are given. Our study here is restricted to binary images case without imposing restrictions on the multi-value or universal validity.

Digital images are composed of *pixels*, which are associated with their integer coordinates i, j of their corresponding lattice point and with their values (black (1) and white (0)). A finite set of connected pels with the same value is called a *digital region R*. Its complement $\overline{R}$ containing all points which are not in *R* is called the *background* of *R*. The number of pixels belonging to *R* is denoted by $N(R)$. A digital region is called *convex*, if the straight line between every pair of pels must not pass through lattice points belonging to the background. A connected SSO is a convex digital region that cannot be removed by repeated filtering.

Let W be a median window with area [7]

$$N(W) = 2n + 1 \text{with } n \in \{1, 2, \cdots\}$$

If the window has a central symmetric shape, the conditions of R to be a root signal of the median filter with window W, for every pixel P of R, are [7]

$$N(W \cap R)_P > n, \quad \text{if } P = 1,$$
$$N(W \cap \overline{R})_P > n, \quad \text{if } P = 0.$$

Property 5.1. The conditions of R to be a root signal of a CWM filter with center weight $2K + 1$ and a central symmetric window W_c that contains $M = 2n + 1$ pixels, for every pixel P of R, are

$$N(W_c \cap R)_P > n - K, \quad \text{if } P = 1,$$
$$N(W_c \cap \overline{R})_P > n - K, \quad \text{if } P = 0.$$

It is not difficult to prove this property which is just a consequence of Theorem 3.1. The proof is omitted.

It can be seen that the size of SSO of CWM filters depends not only on the window size, but also on the center weight of the filter. Therefore, the SSO can be small when the center weight is larger. The following two properties illustrate this fact.

Property 5.2. The SSO of idempotent CWM filters is any object with two connected pixels.

Property 5.3. A single pixel is an SSO for CWM filters if the CWM filters are identity filters, i.e. $K \geq (M - 1)/2$.

In the following, we focus our attention to some CWM filters with a square window shape. Recall that the window size of a square shape window is defined as $M = (2N + 1) \times (2N + 1)$, where N is a non-negative integer. The following property is derived.

Property 5.4. A CWM filter with a square window $M = (2N + 1) \times (2N + 1)$ preserves an SSO which is also an SSO of a median filter with a smaller square window $M_i = (2(N - i) + 1) \times (2(N - i) + 1)$, where $0 \leq i \leq N$, if the center weight is equal to $K = 2i(2N - i + 1)$.

Proof: Since the square window contains the following pixels,

$$M = (2N + 1) \times (2N + 1) = 4(N^2 + N) + 1.$$

Therefore, the parameter

$$n = 2(N^2 + N).$$

Similar results can be derived for M_i.

$$M_i = (2(N - i) + 1) \times (2(N - i) + 1) = 4(N^2 + N - 2iN + i^2 - i) + 1$$

and

$$n_i = 2(N^2 + N - 2iN + i^2 - i).$$

Based on Property 5.1, it is easy to get

$$K = 2i(2N - i + 1).$$

■

Property 5.4 implies that a CWM filter with larger window size may have a smaller SSO if the center weight is properly chosen. The advantage is that better noise attenuation may be achieved by a larger window size.

Another interesting consequence of Property 5.4 is that the algorithm derived in [7] to generate binary root signals for 2-D median filters can also be directly used for these CWM filters. More details about the algorithm can be found in [7].

5.2. Image segmentation by CWM filters

CWM filters, like median filters, are orientation sensitive filters because oriented shapes of filter windows produce oriented shapes of SSOs. Such a property can be used for image segmentation tasks, where, CWM filters with different window orientation are able to separate lines or objects located in different slopes. In this case, the objects oriented in different slopes but with the same size can be separated. It is clear that the median filter can already do the works, but with a larger window size, it is impossible to restore small objects, even they located in the right direction. According to the statistical studies, median filters with small window sizes usually have poor noise reduction capability. Therefore it might be important to use a larger window mask in some cases.

With CWM filters, even with a larger window size, the small objects are still able to be retained, if the center weight is properly chosen.

Some examples are given in Figure 5 for an artificial image. Different line objects with different orientations can be separated by appropriately adjusted filter window shape. It is shown that median filters with certain window size are able to do the segmentation job but have lost some small objects. CWM filters with the same window size can segment different larger line objects as well as the small objects. The window shapes used corresponding to different slopes of lines are shown in Figure 6. In the examples, the median and CWM filter have the same window shape for every orientation. The center weight of CWM filter is $K = 1$ for all cases. The window size is $A(W) = 9$ for all cases.

6. Conclusions

This paper contains some recent advances in the study of CWM filters. Some new results about deterministic properties of 2-D CWM filters are presented in the paper. The required form for a 2-D signal to be a root of a 2-D CWM filter is derived. A new tool called

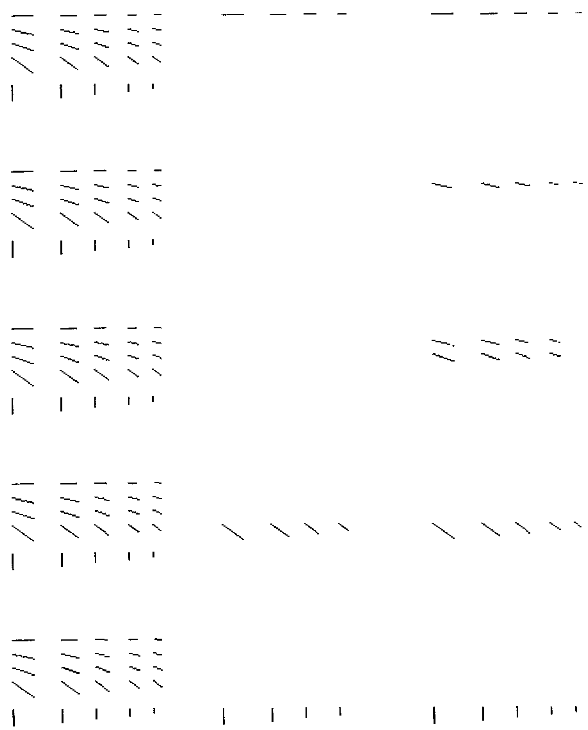

Figure 5. Segmentation by repeated median filtering with the direction sensitive filter windows. (a) input image, (b) output image of median, (c) output image of CWM filter with $K = 1$.

SVN is used to describe and evaluate the detail preserving properties of CWM filters. The detail preserving properties of 3×3 square window unidirectional MMF, which is also an idempotent filter, are analyzed based on its equivalence with CWM(9; 7). The detail preserving properties of CWM(9; 5) filter is compared with those of 3×3 window bidirectional MMFs. It is shown that the CWM filters can preserving more details than or at least as much as MMFs because they are insensitive to the shape and location of SVN in the window.

Generation of binary root signals of some CWM filters are presented by using the concept of SSO. It is shown that the algorithm for 2-D median filters can be directly used for these CWM filters. Computer simulations are given to show the potential applications of CWM filters in image segmentations.

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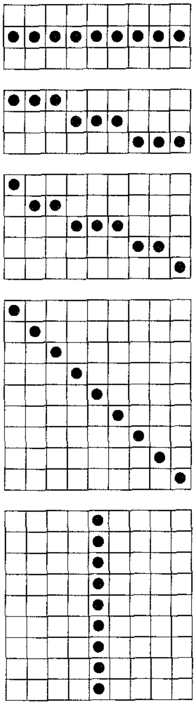

Figure 6. Filter window shape for different orientations. The position of each window is corresponding to the position of the images in Figure 5.

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