An Efficient Design Method for Optimal Weighted Median Filtering

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ABSTRACT

Earlier research has shown that the problem of optimal weighted median filtering with structural constraints can be formulated as a nonconvex nonlinear programming problem in general. However, its high computational complexity and poor performance due to its nonconvex nature prohibit it from practical applications. In this paper, we shall show that the design problem can be formulated as a convex quadratic programming problem. The new algorithm is very efficient in the sense of computational complexity. The algorithm is also efficient in the sense of its capability to approach the global minimum. Using the algorithm optimal 1-D weighted median filters preserving pulses of length 3, 4 and 5 are tabulated.

INTRODUCTION

Recently, a theory which combines the estimation approach and the structural approach was developed for stack filtering and weighted median (WM) filtering. The problem is called the *optimal filtering with structural constraints* [1]-[4]. Based on statistical properties of WM filters, Yang *et al.* showed that optimal WM filters can be found by solving a number of linear inequalities in some simple cases [4, 5]. However, for general cases, the optimal problem has to be formulated as a nonconvex programming problem. This algorithm is computationally complex since the number of the evaluation times of a nonlinear sigmoidal function is 2^N, where N is the window size. So it is quite time consuming. For example, it took tens of hours to design a 5×5 WM filter. This algorithm also suffers from poor quality of the solution since when N becomes large, the algorithm often converges to a local minimum which may be quite far from the global minimum.

In this paper, we show that, by using the asymptotic result of WM filters [7, 6], the problem of optimal WM filters with structural constraints can be formulated as a quadratic programming problem which has a unique global minimum. This algorithm is quite efficient both in the terms of computational complexity and goodness of the solutions. Applying it to 1-D signal processing, we shall design optimal WM filters preserving pulses of length 3, 4 and 5 for several window sizes and tabulated them.

In the next section, we first introduce the definition of WM filters, then give a brief description about the previous setting of the optimal problem. Using an asymptotic result of WM filters we show that the optimal problem can be solved by quadratic programming. This algorithm is then applied for one-dimensional signal processing. Conclusions are contained in the last section.

THE ALGORITHM

Weighted median filters can be defined either in the integer or real domain. It has be shown that any WM filter with real number weights can be converted to a WM filter with integer weights, see [8].

It has been shown that the design problem of optimal WM filters with structural approach can be formulated as follows [4], can be formulated as follows[4, 10],

$$\text{Minimize} \quad \sum_{i=1}^{K} L_i M_i = \sum_{i=1}^{K} \sum_{b \in S_i} L_i U(\underline{W}^t b - T) \quad \text{subject to} \quad \mathbf{A}\,\underline{W} \geq 0, \ \text{(structural constraints)} \tag{1}$$

where $\mathbf{A}$ is an $I \times N$ matrix, $\mathbf{A}\underline{W} \geq \underline{0}$ denotes I-inequalities which represent the structural constraints. L_i's are functions of input distribution and independent of weights, M_i's are functions of the weights. $U(\cdot)$ is the unit step function and S_i is defined as

$$S_i = \{\underline{X} | \underline{X} \in \{0,1\}^N; H(\underline{X}) = i\}. \tag{2}$$

where $H(\underline{X})$ denotes the Hamming weight of $\underline{X}$.

If we approximate the unit step function by a sigmoidal function, the above problem becomes a nonlinear programming problem in which the objective function has first and second derivatives. The problem can be solved by successive quadratic programming[9]. However, the algorithm bears two severe drawbacks. First, its computational complexity, we have to calculate the objective function and its first derivative in each iteraion where the number of the times to evaluate a sigmoidal function is of 2^N order, where N is the window size. For example, it takes tens of hours to get the final result for a 5×5 window WM filter. Sometime its heavy computation load and huge swap memory requirement make solving the problem impossible. The second drawback is its poor quality of the solutions. Since its objective function is non-convex, the algorithm often converges to a local minimum which may be far away from the global minimum. The poor performance becomes even more severe with increasing of window size.

Let us return to the original problem. It is true that finding optimal WM filters with structural constraints is equivalent to finding a WM filter whose output variance is minimized subject to the structural constraints. Using the result in [7, 6], which is stated in the following, we can formulate the optimal problem into convex programming.

For i. i. d. inputs with common distribution function $F(t)$ and density $f(t)$, the output of a WM filter with large window is asymptotically normal and the output variance is

$$\sigma_{wm}{}^2 = \frac{\sum_{i=1}^N W_i^2}{4f(t_0)(\sum_{i=1}^N W_i)^2} \tag{3}$$

where $t_0 = F^{-1}(\frac{1}{2})$. Note that $f(t_0)$ is independent of weights.

Then the optimal WM filtering with structural constraints is formulated as follows

$$\begin{gathered} \text{Minimize} \quad \sum_{i=1}^N W_i^2 \\ \text{subject to} \\ \sum_{i=1}^N W_i = 1 \\ \mathbf{A}\, \underline{W} \geq 0, \ \text{(structural constraints)} \end{gathered} \tag{4}$$

This is a quadratic programming problem which has an unique global minimum. It is, actually, a very efficient algorithm to design optimal WM filters. The design example shown earlier, which took tens of hours to get the final result by the old algorithm, now takes only a couple of seconds! The new algorithm can be easily used to design optimal WM filters with large window sizes, e.g. window size 41, which will be shown in the next section and is impossible by the old algorithm. The second improvement is its good quality of the solutions. Due to its convexity, the algorithm always converges to a unique global minimum. For large window sizes, the solutions of algorithm (4) are well approaching the solution of the original problem because of its asymptotic nature. While for small window sizes, we tested some design examples both by the new algorithm and some deterministic method and found that the solutions by the new algorithm reach or are near the global minimum. For example, it has been shown that the weight vector of symmetric WM filters preserving pulses of length 2 have the following form [4]

$$\underline{W} = (1, 1, \cdots, 1, K, \underline{2K-1}, K, 1, \cdots, 1, 1,)$$

where the window size $N = 2K + 1$. Using the algorithm (4), we can obtain the same result. It gives us some insight about the new algorithm's robustness to reach or near the global minimum of the original problem.

We conclude that the new algorithm is very efficient because of its low computational complexity and its robustness to obtain global or near global minimum. It lifts the restriction by the earlier algorithm that WM filters with only relative small window sizes can be designed due to its huge complexity, and makes designing optimal WM filters with large window sizes realistic for practical applications.

APPLICATIONS

Symmetric WM filters are widely used in practice to avoid introducing undesired shift in the filtering process. For one dimensional signal processing, pulses in the binary domain represent many useful signal structures such as ramps, pulses, monotonic regions in multilevel domain, due to the threshold decomposition of WM filtering [11]. In the following we investigate optimal WM filters for preserving pulses with some lengths.

Example 1: Find the optimal WM filter preserving pulses of length 3 with window width 9. Suppose the weight vector has the following form

$$\underline{W} = (W_4, W_3, W_2, W_1, \underline{W_0}, W_1, W_2, W_3, W_4)$$

In order to preserve pulses of length 3, weights must satisfy the following

$$W_0 \geq 2(W_3 + W_4). \tag{5}$$

Together with

$$W_0 + 2\sum_{i=1}^4 W_i = 1 \tag{6}$$

we obtain the solution using the algorithm (4).

$$\underline{W} = \begin{matrix} (0.0488, 0.0488, 0.1463, 0.1463, \underline{0.2195}, 0.1463, \\ 0.1463, 0.0488, 0.0488) \end{matrix} \tag{7}$$

Its corresponding $\underline{M}$ vector is $\underline{M} = (0,0,6,29)$.

Remember that the $\underline{M}$ vector characterizes the output variance of a WM filter and the original optimal problem is to minimize the linear combination of its components [4, 10]. In other word, $\underline{M}$ vector can be taken as a measure to evaluate the goodness of the solutions produced by the new algorithm.

For small window width, the optimal problems can be solved using the deterministic method, i.e. check inequalities manually, certainly, it is a tedious procedure [10]. However, the solutions obtained are guaranteed to be global minimum solutions of the original problem. Using the deterministic method, we found the global minimum for *Example 1*

$$\underline{W} = \begin{matrix}(0.0513, 0.0513, 0.1282, 0.1538, \underline{0.2308}, 0.1538, \\ 0.1282, 0.0513, 0.0513)\end{matrix} \quad (8)$$

with its $\underline{M} = (0,0,5,29)$.

In this case, the solution produced by the algorithm (4) does not reach the global minimum but is quite near to it. If we add one more constraint

$$W_1 > W_2, \quad (9)$$

together with constraints (5) and (6) in *Example 1*, the algorithm (4) produces the exact solution in (8). Using this strategy, that is, adding one more constraint to the set of constraints, it seems that we can always obtain the global minima by (4). Though it has not been proven in the mathematical sense, it has been verified by many examples where the deterministic procedure can be used to derive the global minima.

For optimal WM filters preserving pulses of length 4, the additional constraints are as follows

$$W_1 > W_2 > W_3. \quad (10)$$

In general, for preserving pulses of length L, the additional constraints are the following

$$W_1 > W_2 > \cdots > W_{L-1} \quad (11)$$

Using this strategy, we obtained optimal WM filters preserving pulses of length 3, 4 and 5, respectively, for various window sizes. They are listed in Table 1 through Table 2 (for length 3, it is found the optimal WM filters are identical to those solved by some deterministic method [10] and thus they are not listed here). Filters in each table preserve pulses of same length and have different noise attenuation capability. The wider the window, the better the noise attenuation capability.

Due to its low computational complexity, the algorithm can used for large window sizes, which are impossible by previous algorithms.

Example 2: Design a optimal WM filter with window size $N = 41$ preserving pulses of length 10.

The constraints are the following

$$\begin{aligned} & W_0 + 2\sum_{i=1}^{20} W_i = 1 \\ & W_0 > 2\sum_{i=10}^{20} W_i \\ & W_1 > W_2 > \cdots > W_9 \end{aligned} \quad (12)$$

The solution is obtained as follows.

$$\begin{aligned} \underline{W} = 10^{-2}(&0.398 \diamond 11, 4.383, 4.433, 4.483, 4.583, 4.633, \\ &4.683, 4.733, 4.783, \underline{8.767}, 4.783, 4.733, 4.683, \\ &4.633, 4.583, 4.533, 4.483, 4.433, 4.383, 0.398 \diamond 11) \end{aligned} \quad (13)$$

The corresponding $\underline{M}$ vector is as follows

$$\begin{aligned} \underline{M} = (&0,0,0,0,0,0,0,0,0,25994, 107184, 12934314, \\ &93469910, 484299740, 1.9174e+09, 6.0312e+09, \\ &1.5475e+10, 3.3003e+1, 6.0245e+10, 1.0675e+11) \end{aligned} \quad (14)$$

CONCLUSIONS

It has been shown that the optimal WM filtering problem can be formulated as a convex programming problem. Its simplicity, low computational complexity and its global or near global solutions make the new approach very attractive. Examples are presented to demonstrate its efficiency and some useful strategy is indicated. Some optimal WM filters preserving pulses of length 3, 4 and 5 are listed in the tables. The new algorithm is expected to have good applications in image processing, where large window masks can be designed for better noise reduction while small image details can still be preserved.

1 REFERENCES

[1] E. Coyle, J.-H. Lin, and M. Gabbouj, "Optimal Stack Filtering and the Estimation and Structural Approachs to Image Processing", *IEEE Trans. On Acoust., Speech and Signal Processing*, vol.37, no.12, Dec. 1989.

[2] M. Gabbouj and E.J. Coyle, "Minimum mean absolute error stack filtering with structural constraints and goals," *IEEE Trans. on Acoust., Speech and Signal Processing*, vol.38, no.6, June 1990.

[3] J. Niewgelowski, L. Yin, M. Gabbouj, and Y. Neuvo, "Optimal weighted order statistic filters under structural constraints," *IEEE Int. Symp. on CAS*, May 10-13, 1992, San Diego, USA, pp.2621-2624.

[4] R. Yang, L. Yin, M. Gabbouj, J. Astola, and Y. Neuvo, "Optimal weighted median filters under structural constraints", *Proc. of IEEE Int. Symp. on CAS*,

Table 1: *Optimal WM filters preserving pulses of length 4*

N	Weight Vector	*M*
9	(0.0769, 0.1149, 0.1154, 0.1159 0.1539, 0.1159, 0.1154, 0.1149, 0.0769)	(0, 0, 0, 14)
11	(2⋄0.0434, 0.1082, 0.1087, 0.1092 0.1740, 0.1092, 0.1087, 0.1082, 2⋄0.0434)	(0, 0, 0, 14, 101)
13	(3⋄0.0303, 0.1056, 0.1061, 0.1066 0.1819, 0.1066, 0.1061, 0.1056, 3⋄0.0303)	(0, 0, 0, 14, 141, 433)
15	(4⋄0.0232, 0.1042, 0.1047, 0.1052, 0.1861, 0.1052, 0.1047, 0.1042, 4⋄0.0232)	(0, 0, 0, 14, 181, 735, 1765)
17	(5⋄0.0189, 0.1032, 0.1037, 0.1042, 0.1886, 0.1042, 0.1037, 0.1032, 5⋄0.0189)	(0, 0, 0, 14, 221, 1117, 3416, 7045)
19	(6⋄0.0159, 0.1027, 0.1032, 0.1037, 0.1905, 0.1037, 0.1032, 0.1027, 6⋄0.0159)	(0, 0, 0, 14, 261, 1579, 5871, 14994, 27841)
21	(7⋄0.0137, 0.1023, 0.1028, 0.1033, 0.1918, 0.1033, 0.1028, 0.1023, 7⋄0.0137)	(0, 0, 0, 14, 301, 2121, 9290, 28315, 63700, 109473)
23	(8⋄0.0120, 0.1020, 0.1025, 0.1030, 0.1929, 0.1030, 0.1025, 0.1020, 8⋄0.0120)	(0, 0, 0, 14, 341, 2743, 13833, 49016, 129620, 265188, 429364)

Table 2: *Optimal WM filters preserving pulses of length 5*

N	Weight Vector	*M*
11	(0.0624, 0.0930, 0.0935, 0.0940, 0.0945, 0.1251, 0.0945, 0.0940, 0.0935, 0.0930, 0.0624)	(0, 0, 0, 0, 44)
13	(2⋄0.0357, 0.0885, 0.0890, 0.0895, 0.0900, 0.1430, 0.0900, 0.0895, 0.0890, 0.0885, 2⋄0.0357)	(0, 0, 0, 0, 44, 364)
15	(3⋄0.0249, 0.0866, 0.0872, 0.0878, 0.0884, 0.1503, 0.0884, 0.0878, 0.0872, 0.0866, 3⋄0.0249)	(0, 0, 0, 0, 44, 504, 1590)
17	(4⋄0.0192, 0.0858, 0.0863, 0.0868, 0.0873, 0.1539, 0.0873, 0.0868, 0.0863, 0.0858, 4⋄0.0192)	(0, 0, 0, 0, 44, 644, 2668, 6569)
19	(5⋄0.0156, 0.0852, 0.0857, 0.0862, 0.0867, 0.1563, 0.0867, 0.0862, 0.0857, 0.0852, 5⋄0.0156)	(0, 0, 0, 0, 44, 784, 4026, 12549, 26491)
21	(6⋄0.0131, 0.0848, 0.0853, 0.0858, 0.0863, 0.1579, 0.0863, 0.0858, 0.0853, 0.0848, 6⋄0.0131)	(0, 0, 0, 0, 44, 924, 5664, 21305, 55615, 105546)
23	(7⋄0.0114, 0.0845, 0.0850, 0.0855, 0.0860, 0.1591, 0.0860, 0.0855, 0.0850, 0.0845, 7⋄0.0114)	(0, 0, 0, 0, 44, 1064, 7582, 33637, 104049, 238161, 417781)
25	(8⋄0.0099, 0.0840, 0.0847, 0.0854, 0.0861, 0.1601, 0.0861, 0.0854, 0.0847, 0.0840, 8⋄0.0099)	(0, 0, 0, 0, 44, 1204, 9780,49865, 178965, 479896, 998152, 1647660)

May 3-6, 1993, vol.1, pp.942-945, Chicago, Illinois, USA.

[5] R. Yang, L. Yin, M. Gabbouj, and Y. Neuvo, "2-D optimal weighted median filters," *IEEE Workshop on visual signal processing and communications*, Sept.2-3, 1992, North Carolina, USA.

[6] J. Astola, and Y. Neuvo, "An efficient tool for analyzing weighted median and stack filters", *submitted to IEEE Trans. on CAS.*

[7] B.J. Justusson, "Median filtering: statistical properties, in *Two Dimensional Digital Signal Processing II*, ed.T.S. Huang, Springer-Verlag, Berlin, 1981.

[8] O. Yli-Harja, J. Astola, and Y. Neuvo, "Analysis of the properties of median and weighted median filters using threshold logic and stack filter representation," *IEEE on Acoust. Speech and Signal Processing*, vol.39, no.2, Feb. 1991, pp.395-410.

[9] "User's Manual, Math/Library", Problem-Solving Software Systems, version 1.0, April 1987.

[10] R. Yang, L. Yin, M. Gabbouj, J. Astola, and Y. Neuvo, "Optimal weighted median filters under structural constraints," *accepted to IEEE Trans. Signal Processing.*

[11] J. Fitch, E. Coyle, and N.C. Gallagher, "Median filtering by threshold decomposition," *IEEE Trans. Acoust. Speech, Signal Processing*, vol.ASSP-32, pp.1183-1189, Dec. 1984.