GENERATING IDEMPOTENT FILTERS

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ABSTRACT

Idempotent filters produce a root signal in a single filter pass, i.e., the filter output is invariant to further filterings with the same filter. In this paper median based idempotent filter structures are introduced. Two approaches to generate these filters are studied: weighted median filters and median filter cascades.

Two subclasses of n-dimensional idempotent weighted median filters, called Class 1 and Class 2 filters in the paper, are introduced. It is shown that both Class 1 and Class 2 filters suppress impulsive noise from n-dimensional input signals and yet have almost no effect on the non-corrupted parts of the signal. These filters are therefore well suited for example for pre-processing purposes. Likely applications of these filters are in speech processing, image processing and, also, in image sequence processing, where the filter mask is typically 3-dimensional.

Sufficient conditions for a filter cascade to be idempotent are given. Two idempotent median filter cascades and their advantages are discussed.

1. INTRODUCTION

Idempotent filters produce a root signal after a single filter pass over the input signal. This might be of great importance in certain situations where repeated filtering is undesirable when, for instance, processing time is crucial. The standard median filter itself, when used recursively [7], and the basic morphological filters, e.g., opening and closing all fall into this category [9].

In this paper, we shall present two approaches to generate idempotent filters based on the median operation. The first approach involves weighted median filters [1] [5] [6] [8] [11] [12], which are stack filters [10]. We shall define two subclasses of weighted median filters: Class 1 and Class 2, based on the shape of the filter window. We shall then study the noise attenuation capability of these filters. In particular, we will show that all filters in Class 1 filter out impulses from the input signals, when used in the recursive mode. We will also show that all filters in Class 2 filter out impulses from the input signals, when used in the recursive and the non-recursive modes.

As will be shown in the paper, filters in Class 1 and Class 2 belong to the class of "gentle" filters. Their noise attenuation consists exclusively on suppressing those "isolated" impulses in the signal. On the other hand, recursive filters, e.g., recursive median filters, belong to the class of "strong" filters. They tend to attenuate fine details in the signal by reacting relatively slowly to changes in the signal. This leads, among other things, to significant edge jitter.

Filters falling between "gentle" filters and "strong" filters would be of great interest in many signal processing applications. They should have better noise attenuation than "gentle" filters and yet preserve fine details better than "strong" filters. They should also have more interesting roots than those of the other two types.

Since this paper deals with idempotent filters, we are interested in filters which fall between "gentle" filters and "strong" filters and which would be idempotent. This leads to the second approach to generate idempotent filters. This approach involves cascades of filters. A necessary and sufficient condition for a filter cascade to be idempotent will be derived. Two simple idempotent cascade structures will be investigated. The first structure is composed of a cascade of identical non-recursive median filters of the same window width terminated by a recursive median of the same window width. The second structure is made of a cascade of non-recursive median filters of increasing window widths and again terminated by a recursive median of window width equal to that of the preceding filter.

This paper is organized as follows. In the next section, we define two subclasses of weighted median filters, which we call Class 1 and Class 2 filters. We shall study their root structures and their convergence behavior and show that these filters are indeed idempotent. Section 3 presents the second approach to generate idempotent filters using filter cascades. Section 4 contains some concluding remarks and future research.

2. IDEMPOTENT WEIGHTED MEDIAN FILTERS

In this section we define two weighted median filter classes [13]. We shall use the same notation as in [13]. These classes will be defined in terms of the filter window, i.e., the shape and the weights of the window. The filters in both classes can be operated either recursively or non-recursively. These filters have the property that they can remove impulses from the input signal with minimal effect on the non-distorted parts of the signal. This is a

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very desirable filter behavior in certain applications. Because of this property, they are also suitable as prefilters to other signal processing algorithms.

First, we shall define two types of window symmetries: shape symmetry and weight symmetry. Then, based on these definitions, we will define two subclasses of weighted median filters which will be the subject of this section.

Definition 1:

Shape symmetry. A filter window is said to be symmetric w.r.t. the center point of the window if

$$w(i_1, i_2, \ldots, i_d) \neq 0 \Rightarrow w(-i_1, -i_2, \ldots, -i_d) \neq 0, \tag{1}$$

where $-l_j \leq i_j \leq r_j$, $j = 1, \ldots, d$.

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Definition 2:

Weight symmetry. A filter window is said to be symmetrically weighted if

$$w(i_1, i_2, \ldots, i_d) = w(-i_1, -i_2, \ldots, -i_d), \tag{2}$$

where $-l_j \leq i_j \leq r_j$, $j = 1, \ldots, d$.

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Note that a symmetrically weighted window always produces a symmetric window but the opposite is not generally true. The following is a useful property of symmetric windows and could alternatively be used to define shape symmetry.

Property 1:

If the filter window is symmetric then for any two inputs x_i and x_j,

$$i \in \Omega_j \Rightarrow j \in \Omega_i, \tag{3}$$

where Ω_k is the set of indices of the samples in the filter window when it is positioned at sample x_k.

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Two subclasses of weighted median filters are defined below based on the shape and the weights of the filter window. The root sets of these filters will be investigated soon after.

Definition 3:

Class 1: Any weighted median filter whose window is symmetric and the sum of the weights $S_w = 2w_c + 1$, where the center weight $w_c \geq 1$, belongs to Class 1.

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Definition 4:

Class 2: Any weighted median filter whose window is symmetric, $S_w = 2w_c + 1$, where $w_c \geq 1$, and where for each input sample x_i there exist two input points x_j and x_k, such that

$$j \in \Omega_i, \quad k \in \Omega_i, \quad j \in \Omega_k, \tag{4}$$

where $j \neq k, j \neq i, k \neq i$, belongs to Class 2.

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It is clear that Class 2 is a proper subset of Class 1 since the conditions for Class 2 filters are stricter than those in Class 1. Some examples of filters belonging to Class 2, and hence to Class 1, are given below,

$$W_1 = \langle 1 \quad 1 \quad \underline{3} \quad 1 \quad 1 \rangle, \quad W_2 = \left\langle \begin{matrix} 1 & 1 & 1 \\ 1 & \underline{7} & 1 \\ 1 & 1 & 1 \end{matrix} \right\rangle,$$

$$W_3 = \left\langle \begin{matrix} 0 & 1 & 1 \\ 1 & \underline{5} & 1 \\ 1 & 1 & 0 \end{matrix} \right\rangle. \tag{5}$$

However, there are filters which belong to Class 1 but not to Class 2. The following are some examples,

$$W_1 = \langle 1 \quad \underline{1} \quad 1 \rangle, \quad W_2 = \left\langle \begin{matrix} 0 & 1 & 0 \\ 1 & \underline{3} & 1 \\ 0 & 1 & 0 \end{matrix} \right\rangle,$$

$$W_3 = \left\langle \begin{matrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & \underline{5} & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{matrix} \right\rangle. \tag{6}$$

2.1. Root structures of Class 1 and Class 2 filters

Before discussing the root structures of our Class 1 and Class 2 filters, we shall consider the root sets of general weighted median filters when used in both recursive and non-recursive modes. It was shown in [7] that an arbitrary signal is a root of a non-recursive median filter if and only if it is a root of the recursive median filter of the same window width. As the next theorem shows, this result can be extended to relate the root sets of recursive and non-recursive weighted median filters.

Theorem 1:

X is a root signal of a recursive weighted median filter iff it is a root of the same non-recursive filter.

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We will now give several properties of Class 1 and Class 2 filters. The following property determines the root structure for Class 1 filters.

Property 2:

Any sample to change value when filtering with a Class 1 filter must be unique inside the filter window. Also, any sample which is unique inside the filter window will change value when filtered with a Class 1 filter.

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This property holds for both recursive and non-recursive filtering. To show this, assume that the center point of the window is positioned at sample x_i whose value is 1. If there is another sample of value 1 in the window then the amount of weight on that value will be at least $w_c + 1$ which is more than half of the total weight $S_w = 2w_c + 1$. Median filtering in the binary case is equivalent to majority decision and the filter output value will be unchanged, i.e., $y_i = x_i = 1$.

A filter window is said to contain an impulse if it is centered at an input sample whose value is unique in the window. Property 2, then, explains the impulse-suppressing property of Class 1 filters. The filter output y_i will be equal to the input x_i unless x_i is unique, in value, inside the window, i.e., positive and negative going impulses are removed. If there are no values significantly different from all others in the filter window, the output will be practically the same as the input. Class 1 filters only cause a minimal amount of blurring in the input signal.

Property 2 is reformulated in the following theorem to describe the root structure of Class 1 filters.

Theorem 2:

X is a root signal of a Class 1 filter iff for all x_i there exists x_j such that $j \neq i$, $j \in \Omega_i$, and $x_j = x_i$.

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2.2. Convergence behavior; Idempotence

The convergence behavior of Class 1 and Class 2 filters is studied in this section. Two properties concerning the filtering behavior of Class 1 and Class 2 filters are stated next.

Property 3:

Of any two input samples x_i and x_j such that $i \neq j$ and $j \in \Omega_i$, only one can change in value when filtered with a recursive Class 1 filter.

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The same property can be shown to apply for non-recursive Class 2 filters. Since Class 2 is a subclass of Class 1, Property 3 clearly applies for recursive Class 2 filters.

Property 4:

Of any two input samples x_i and x_j such that $i \neq j$ and $j \in \Omega_i$, only one can change in value when filtered with a non-recursive Class 2 filter.

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The most important property of these impulse-suppressing filters is, perhaps, the fact that their output is invariant to further filterings with the same filter. For standard median filters, it is known that, when there are long periods of oscillation in the input signal, it takes several passes for the signal to converge to a root. When the input signal has more than one dimension, it is even possible that the output will oscillate indefinitely. The property to filter any signal to a root in a single filter pass is referred to as idempotence and is defined below.

Definition 5:

A filter is said to be idempotent if it produces roots in one filter pass.

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Therefore, idempotent filters can make changes to the input signal only during the first pass of the filter. Subsequent filter passes do not have any effect on the signal. A trivial idempotent filter is the identity filter whose output is always equal to the input. Another well-known idempotent filter is the one-dimensional recursive median filter.

The following theorem states that all Class 1 filters possess this property.

Theorem 3:

Any Class 1 recursive filter is idempotent.

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The same result also holds for Class 2 filters, as stated by the following theorem.

Theorem 4:

Any Class 2 non-recursive filter is idempotent.

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3. IDEMPOTENT FILTER CASCADES

Recursive median filters, the basic morphological filters opening and closing all belong to the class of "strong" filters. They attenuate noise so harshly that they loose fine details in signals and images.

Class 1 and Class 2 filters, on the other hand, are "gentle" filters. They have less noise attenuation capability, but do preserve more image details.

In this section, we shall seek a class of filters which are idempotent and fall between the previous two types of filters. These filters should have more "interesting" root structures and their noise attenuation capability should be somewhere between those of the "gentle" and the "strong" filters.

The second approach to generate idempotent filters is an attempt in this direction. The approach is based on cascading n filters, $f_1, \ldots, f_n$, where the last filter f_n is idempotent. The idea here is to attenuate noise gradually. The fact that the last filter in the cascade is idempotent, however, does not guarantee that the cascade is idempotent.

Let $s_1, s_2, \ldots, s_R$, be the root signals of f_n. Here, R is thus the number of different roots and is always finite. Let also S_i be the set of those signals that are reduced to root s_i by filter f_n. In other words, if the input to the idempotent filter f_n is $X \in S_4$, then the output of the filter is s_4.

Theorem 5:

A filter cascade $f_1, \ldots, f_n$ is idempotent if the filter f_n is idempotent, and if the cascade of the first $n-1$ filters is such, that when the input to it is s_i the output of the first cascade $y_i \in S_i$.

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A simple way to guarantee idempotence of the cascade is to design the filters so that if a signal is a root of f_i it is also a root of f_{i-1}, $i \geq 2$.

In median filtering, a useful filter cascade is obtained when identical non-recursive filters are cascaded and the last filter in the cascade is a recursive median filter of the same length:

Definition 6:

Cascade 1.

$f_i =$ Median filter of length $2k+1$, $\quad 1 \leq i < n$;

$f_n =$ Recursive median filter of length $2k+1$.

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The one-dimensional recursive median filter has the drawback that it tends to be reluctant to react to changes in the input signal. In noisy signals this often leads to significant edge jitter. Typically, the root signal obtained by repeated median filtering is preferred over that obtained by a single pass of the recursive median filter.

Usually the number of filter passes required to guarantee that repeated median filtering produces a root is impractically large. Cascade 1 can be used as a compromise solution. It always produces a root signal which is closer to the one obtained by repeated median filtering than to the recursive median filter root.

Another idempotent cascade that could have desirable properties is presented next.

Definition 7:

Cascade 2.

f_i = Median filter of length $2i+1, \quad 1 \leq i < n$;

f_n = Recursive median filter of length $2n-1$.

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It is not very difficult to show that the filters in Cascade 2 are indeed idempotent. The proof is based on the following two facts. First, the roots of median filters of large window widths are also roots of median filters of smaller window widths, [3]. Second, a signal which is a root of a median filter is also a root of the recursive median filter of the same window width [7].

The idea is to start the cascade with short median filters and increase the filter length towards the end of the cascade. The noise attenuation of this cascade is approximately the same as that of the longest filter in the cascade. However, long median filters cause some blurring near noisy signal edges. In Cascade 2, short filters perform initial smoothing near the edge after which longer filters can filter edges with less blurring.

4. CONCLUSIONS

Two weighted median filter subclasses were defined in this paper. The root structure of these filters was determined. It was shown that the main property of these filters is their ability to suppress impulsive noise without much effect on the input signal. We then analyzed the convergence behavior of these filters and showed that the second most important property of these filters is that they remove all impulses from input signals in a single filter pass, that is, these filters are idempotent.

Another class of idempotent filters was investigated. Filters in this class are composed of cascaded median filters with different window widths. Sufficient conditions for a filter cascade to be idempotent were derived. An easy and useful filter cascade is obtained when identical non-recursive median filters are cascaded and the last filter in the cascade is a recursive median filter of the same window width. A different idempotent structure is obtained by cascading median filters of increasing window widths and terminating the cascade by a recursive median filter of window width equal to that of the filter in the previous stage.

These are but a few types of idempotent filters, other structures are currently being investigated [14].

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