Idempotent Nonlinear Filters

R.K. Pearson and M. Gabbouj
Tampere International Center for Signal Processing
Tampere University of Technology
P.O. Box 553, FIN-33101 Tampere, Finland
E-mail: {pearson,gabbouj}@cs.tut.fi

Abstract

Idempotent nonlinear filters may be viewed as extensions of the class of (non-realizable) ideal linear filters, and one of their characteristic features is that they reduce any input sequence $\{x_k\}$ to a root after one pass. This paper explores some new constructions of idempotent nonlinear filters, using ideas from algebraic category theory, which provides a natural basis for considering cascade filter structures. In addition, this paper also examines some related filter classes, including some extensions of the class of *gentle filters* considered previously by Haavisto, Gabbouj and Neuvo, and some extensions of the idea of *involutory matrices* to nonlinear filters.

1 Introduction

Recently, nonlinear filters have been dominating the signal processing literature, especially in the area of image processing. The success of these filters is partly due to the many useful classes of nonlinear filters that have been proposed and analyzed for specific applications, including order statistic filters, morphological filters, and nonlinear mean filters, among others. A major challenge has been the lack of a unifying theory covering different nonlinear filter classes that would allow them to be analyzed, optimally designed, and implemented for a specific application. Many of the important nonlinear filter classes discussed to date have been defined on the basis of structural characteristics and, although specific analysis and design results have been successfully developed in many cases, these isolated results do not solve all problems of interest. As a specific example, suppose that idempotence, a behavioral characterization defined in Sec. 2, were desired for a particular nonlinear filtering application: what filters or filter classes represent useful design candidates? In this paper, we attempt to address this problem using *category theory*, which provides a basis for extending certain important ideas from linear algebra to the more general setting of cascade interconnected structures (Pearson, 1999, ch. 7). Essentially, category theory is concerned with the behavior of classes of objects and transformations between those objects that preserve certain important properties. Some nonlinear filter classes (for example, the class of nonlinear FIR filters) define categories and category-theoretic results frequently generalize useful linear algebra results, as in the case of full-rank factorizations of matrices. Further, since categories of systems can be defined both structurally and behaviorally, category theory represents a possible basis for a general framework to characterize multiple classes of nonlinear filters. As a specific example, although the behaviorally-defined class of idempotent filters considered here does not by itself define a category, these filters may be imbedded in larger filter categories, permitting full-rank factorization results for idempotent matrices to be extended to factorization results for idempotent nonlinear filters.

2 Idempotent filters

A filter $\mathbf{F}$ is *idempotent* if $\mathbf{F} \circ \mathbf{F} = \mathbf{F}$, where $\circ$ denotes the cascade connection of two filters, corresponding in this case to a two-pass iterative application of the filter $\mathbf{F}$. An important characteristic of idempotent filters is that they immediately reduce any input sequence $\{x_k\}$ to a *root sequence* $\{r_k\}$ that is invariant to further filtering by $\mathbf{F}$:

$$\mathbf{F}\{r_k\} = \mathbf{F} \circ \mathbf{F}\{x_k\} = \mathbf{F}\{x_k\} = \{r_k\}.$$

Because these filters are *behaviorally defined*, it is not obvious which *structures* exhibit idempotence, although some results are known. For linear filters with transfer function $H(f)$, idempotence implies

$$H^2(f) = H(f) \ \Rightarrow \ H(f) \in \{0, 1\},$$

for all f. This class includes all ideal linear filters, none of which are finitely realizable except for two cases of no practical interest: the identity filter $H(f) \equiv 1$ and the null filter $H(f) \equiv 0$. Conversely, practical nonlinear idempotent filters include the recursive median filter (Nodes and Gallagher, 1982), the recursive Class 1 and nonrecursive Class 2 filters described by Haavisto et al. (1991), and morphological opening and closing filters (Serra, 1988).

3 Categories

A *category* is a collection of mathematical *objects*, together with sets of *morphisms* relating pairs of objects (Blyth, 1986). Perhaps the simplest example is the category of linear vector spaces, where the objects are the spaces R^n and the morphisms are matrices mapping one space into another. To qualify as a category, there must also be a *composition law* under which the morphism sets are closed. In the category of linear vector spaces, this composition law corresponds to matrix multiplication: if $\mathbf{M} : R^n \to R^m$ and $\mathbf{N} : R^m \to R^p$, the matrix product $\mathbf{NM}$ is well-defined, mapping R^n into R^p. In addition, there must be an *identity morphism* associated with each object; in the category of linear vector spaces, the identity morphism associated with R^n is the $n \times n$ identity matrix, $\mathbf{I}_n$. Finally, composition of morphisms must satisfy the following associativity condition:

$$X \circ (Y \circ Z) = (X \circ Y) \circ Z. \tag{1}$$

In the categories of primary interest here, the object class $\mathcal{O}$ is some collection of real-valued sequences $\{x_k\}$, and the morphisms consist of filters $\mathbf{F}$, viewed as maps from an input sequence $\{x_k\}$ to an output sequence $\mathbf{F}\{x_k\}$. Composition of morphisms is denoted by $\circ$ and corresponds to cascade (i.e., series) interconnection of these filters or, equivalently, to the functional composition of sequence maps:

$$[\mathbf{F} \circ \mathbf{G}]\{x_k\} = \mathbf{F}[\mathbf{G}\{x_k\}].$$

Identity morphisms correspond to the identity filter $\mathbf{F}\{x_k\} = \{x_k\}$ for all $\{x_k\}$, and the associativity condition (1) follows from the associativity of functional compositions. To qualify as a category, then, a family $\mathcal{F}$ of filters must satisfy the conditions:

1. $\mathbf{F}\mathcal{O} \subset \mathcal{O}$ for all filters $\mathbf{F} \in \mathcal{F}$,
2. $\mathcal{F}$ must include the identity filter,
3. $\mathcal{F}$ must be closed under cascade interconnection.

Because it is not closed under cascade interconnection, the behaviorally-defined class of idempotent filters does *not* define a category, although one of the main points of this paper is that the category-theoretic formulation may be used to obtain some general results for this nonlinear filter class, along with some other closely related classes. To see that idempotence is not preserved under cascade interconnection, suppose the object class $\mathcal{O}$ consists of real-valued sequences defined for $k > 0$, define $\mathcal{D}$ as the set of all strictly decreasing sequences in $\mathcal{O}$, define $\mathcal{I}$ as the set of all strictly increasing sequences in $\mathcal{O}$, and consider the following two filters, both of which are easily shown to be idempotent:

$$\begin{aligned} K\{x_k\} &= \begin{cases} \{-2^{1/k}\} & \{x_k\} \in \mathcal{I}, \\ \{2^{1/k}\} & \{x_k\} \in \mathcal{D}, \\ \{1\} & \text{otherwise}, \end{cases} \\ L\{x_k\} &= \begin{cases} \{1\} & x_k > 0 \text{ for all } k > 0 \\ \{-1\} & x_k < 0 \text{ for all } k > 0 \\ \{0\} & \text{otherwise.} \end{cases} \end{aligned}$$

The composition $L \circ K$ is given explicitly as:

$$\begin{aligned} L \circ K\{x_k\} &= \begin{cases} \{1\} & K\{x_k\} > 0 \text{ for all } k > 0 \\ \{-1\} & K\{x_k\} < 0 \text{ for all } k > 0 \\ \{0\} & \text{otherwise} \end{cases} \\ &= \begin{cases} \{-1\} & \{x_k\} \in \mathcal{I}, \\ \{1\} & \text{otherwise.} \end{cases} \end{aligned}$$

To see that this composition is not idempotent, note that $(L \circ K)^2$ is given by

$$\begin{aligned} (L \circ K)^2\{x_k\} &= \begin{cases} \{-1\} & (L \circ K)\{x_k\} \in \mathcal{I}, \\ \{1\} & \text{otherwise.} \end{cases} \\ &= \{1\} \text{ for all } \{x_k\}. \end{aligned}$$

Conversely, if K and L are idempotent filters that *commute,* their composition is idempotent:

$$\begin{aligned} (K \circ L)^2 = K \circ L \circ K \circ L &= K \circ L \circ L \circ K \\ &= K \circ L \circ K \\ &= K \circ K \circ L = K \circ L. \end{aligned}$$

This result is closely related to root sequence characterizations. Specifically, let $\mathcal{R}_{\mathbf{F}}$ denotes the root set for filter $\mathbf{F}$ and note that, since $K\{x_k\} \in \mathcal{R}_K$ if K is idempotent, $(K \circ L)\{x_k\} \in \mathcal{R}_K$. Similarly, if L is idempotent, $(L \circ K)\{x_k\} \in \mathcal{R}_L$, so if K and L commute, $(K \circ L)\{x_k\}$ must belong to the intersection $\mathcal{R}_K \cap \mathcal{R}_L$. Since the composition is idempotent if K and L commute, this intersection cannot be empty. Hence, two idempotent filters can commute only if $R_K \cap R_L \neq \emptyset$. More generally, if $\mathcal{R}_L \subset \mathcal{R}_K$, then for any $\{x_k\}$ we have $L\{x_k\} \in \mathcal{R}_L$ by the idempotence of L, implying $L\{x_k\} \in \mathcal{R}_K$. It then follows that $K \circ L$ is idempotent since, for all input sequences $\{x_k\}$,

$$\mathcal{R}_L \subset \mathcal{R}_K \;\Rightarrow\; K \circ L\{x_k\} = L\{x_k\} \;\Rightarrow\; K \circ L = L.$$

Conversely, consider the case where $\mathcal{R}_K \subset \mathcal{R}_L$. Suppose $\{x_k\}$ is an arbitrary input sequence and define $\{z_k\} = K \circ L\{x_k\}$, noting that since $\{z_k\}$ is the response of the idempotent filter K, then $\{z_k\} \in \mathcal{R}_K \subset \mathcal{R}_L$. Hence,

$$\begin{aligned} (K \circ L)^2\{x_k\} &= K \circ L\{z_k\} \\ &= K\{z_k\} = \{z_k\} = K \circ L\{x_k\}, \end{aligned}$$

again establishing the idempotence of the composition $K \circ L$. Overall, these results suggest that cases where this composition is *not* idempotent are those where the root sets $\mathcal{R}_K$ and $\mathcal{R}_L$ are "too disparate."

4 Factorizations

An important result from linear algebra is the existence of a *full-rank factorization* for any $n \times m$ matrix $\mathbf{M}$ (Ben-Israel and Greville, 1974, p. 22). If $\mathbf{M}$ has rank r, there exists an $n \times r$ matrix $\mathbf{P}$ with linearly independent columns (i.e., full column rank) and an $r \times m$ matrix $\mathbf{Q}$ with linearly independent rows (i.e., full row rank) such that $\mathbf{M} = \mathbf{PQ}$. A square matrix $\mathbf{M}$ is idempotent if $\mathbf{MM} = \mathbf{M}$, and a rank r, square matrix $\mathbf{M}$ with full-rank factorization $\mathbf{M} = \mathbf{PQ}$ is idempotent if and only if $\mathbf{QP} = \mathbf{I}_r$ (Ben-Israel and Greville, 1974, p. 49). Generalizing the notion of full column rank, a morphism m in a category $\mathcal{C}$ is *monic* if it is *left-cancellable*, meaning that

$$m \circ f = m \circ g \;\Rightarrow\; f = g.$$

The generalization of a full row rank matrix is an *epic* morphism e, which is *right-cancellable:*

$$f \circ e = g \circ e \;\Rightarrow\; f = g.$$

Generalizing the notion of full-rank decomposition of matrices, a morphism f in a category $\mathcal{C}$ is said to be *factorizable* if it can be written as $f = m \circ e$ where m is monic and e is epic. For an idempotent morphism f, we have the following result.

Lemma:

A factorizable morphism $f = m \circ e$ is idempotent if and only $e \circ m = \mathrm{id}$.

Proof:

If $f = m \circ e$ and $e \circ m = \mathrm{id}$, it follows that

$$f^2 = m \circ e \circ m \circ e = m \circ \mathrm{id} \circ e = m \circ e = f.$$

Conversely, suppose $f = m \circ e$ is idempotent where m is monic and e is epic. Then

$$f^2 = f \;\Rightarrow\; m \circ e \circ m \circ e = m \circ e.$$

Since m is left-cancellable, this result implies

$$e \circ m \circ e = e = \mathrm{id} \circ e.$$

Similarly, since e is right-cancellable, it follows from this result that $e \circ m = \mathrm{id}$.

||

This result generalizes easily to the extensions of idempotent filters introduced in the next section.

5 Variations and extensions

One relaxation of idempotence is based on the following observation: if $f^{n+1} = f^n$ for some $n > 1$, we obtain a filter that reduces any input sequence $\{x_k\}$ to a root in n steps. Specifically, if $y_k = f^n x_k$, then

$$f y_k = f^{n+1}\{x_k\} = f^n\{x_k\} = y_k.$$

In fact, this idea leads to an interesting ordering on the collection of all filters: define Φ_0 as the class containing the identity filter, Φ_∞ as the class of all filters, and for each $n \geq 1$, define Φ_n as the set of all filters satisfying $f^{n+1} = f^n$. Hence, Φ_1 is the class of idempotent filters and these classes are nested as

$$\Phi_0 \subset \Phi_1 \subset \Phi_2 \subset \cdots \subset \Phi_\infty.$$

If we consider the restriction of this chain of classes to any category **C**, it induces an ordering on that category (specifically, define Φ_n^C as the class of morphisms f in **C** that satisfy $f^{n+1} = f^n$). Further, note that if $f \in \Phi_n$ and f^{-1} exists, then

$$f^{n+1} = f^n \;\Rightarrow\; f^n = f^{n-1} \;\Rightarrow\; f \in \Phi_{n-1}.$$

Repeating this argument ultimately leads to $f \in \Phi_0$, implying that the only invertible filter contained in the set Φ_n for any finite n is the identity filter.

A second extension of the class of idempotent filters arises from a consideration of the class of *involutory matrices* $\mathbf{A}$, satisfying the condition $\mathbf{A}^2 = \mathbf{I}$. These matrices are closely related to idempotent matrices, since every involutory matrix is *tripotent:* $\mathbf{A}^3 = \mathbf{A}$, although the converse implication does not hold (Tian and Styan, 2001). More generally, in any category **C**, define an n-potent morphism K as one for which $K^n = K$, implying:

$$K^{n-2} \circ K^n = K^{n-2} \circ K \;\Rightarrow\; K^{2n-2} = K^{n-1},$$

from which it follows that $L = K^{n-1}$ is idempotent. In other words, every n-potent morphism K may be viewed as the $n-1$ root of some idempotent morphism L. Taking $n = 3$ and $L = \text{id}$ leads to the characterization of idempotent morphisms given by the Lemma in Sec. 4. More generally, it is an easy extension of this result that if f is factorizable as $f = m \circ e$, then $f \in \Phi_n$ if and only if $e \circ m \in \Phi_{n-1}$. Finally, note that if A is an n-th order involutory filter for some $n > 1$ (i.e., $A^n = \text{id}$) and K is any idempotent filter, the following construction leads to another idempotent filter:

$$B = A^r K A^{n-r}, \tag{2}$$

where $1 \leq r < n$. Unless K communtes with A, the idempotent filter B is distinct from K.

6 Functors

A *covariant functor* (Blyth, 1986, p. 73) $\mathcal{F}$ from a category **C** to a category **D** may be viewed as two mappings, one taking every object A in **C** to an object $\mathcal{F}A$ in **D**, and the other taking every morphism f in **C** into a morphism $\mathcal{F}f$ in **D**. In addition, these mappings must preserve both identities and compositions: if $f \circ g$ is defined in **C**, $\mathcal{F}(f \circ g)$ must represent a morphism in **D** and $\mathcal{F}(f \circ g) = \mathcal{F}f \circ \mathcal{F}g$. An *isomorphism* is a morphism g that is invertible, meaning that there exists a unique morphism g^{-1} such that $g \circ g^{-1} = \text{id}$ and $g^{-1} \circ g = \text{id}$. If g is an isomorphism, the following construction defines a functor $\mathcal{F}_g$ between two categories **C** and **D**, provided all of the required objects and morphsism are well-defined:

- every object A in **C** is mapped into the object $g^{-1}A$ in **D**,
- every morphism f in **C** is mapped into the morphism $g^{-1} \circ f \circ g$ in **D**.

If g is any invertible scalar function $g(x)$, and $\mathcal{L}$ is the category of linear, time-invariant (LTI) systems, the category $\mathcal{F}_g\mathcal{L}$ corresponds to the class of homomorphic systems (Oppenheim et al., 1969; Pitas and Venetsanopoulos, 1990).

To see the utility of this idea, note that if K is any idempotent morphism in **C**, it follows that

$$K^2 = K \;\Rightarrow\; \mathcal{F}K = \mathcal{F}K^2 = \mathcal{F}K \circ \mathcal{F}K,$$

establishing that $\mathcal{F}K$ is also idempotent. As a specific example, suppose K is either a recursive median filter or any of the nonlinear idempotent filters considered by Haavisto et al. (1991), and take g^{-1} as a linear autoregressive lowpass filter, implying that g is an FIR highpass filter. Applying the functor defined by g then yields the novel filter structure $g^{-1} \circ K \circ g$, consisting of a first-stage high-pass (sharpening) filter, followed by a nonlinear impulsive noise reduction filter, followed by a lowpass smoothing filter.

Noting that some idempotent filters, like the recursive median filter, tend to be too aggressive for certain applications, Haavisto et al. (1991) considered an interesting class of cascade structures termed *gentle filters* that, while less effective in terms of noise attenuation, are also significantly less prone to introduce unacceptable signal distortions than "strong" filters like the recursive median filter. These filters exhibit cascade structures of the following form:

$$g = f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1, \quad (3)$$

where f_n is idempotent, with the same root set as the penultimate filter f_{n-1}, and filters f_1 through f_{n-1} are chosen so that the root set for f_i is contained in that for f_{i-1}. Since f_n is idempotent, it follows that any response of this cascade filter structure is contained in the root set for f_n, which belongs by construction to the root sets for filters f_1 through f_{n-1}. Hence, the resulting filter is idempotent. Two specific examples based on recursive and nonrecursive median filters are described by Haavisto et al. (1991). In one case, f_n is taken as the $2k+1$-point recursive median filter and f_1 through f_{n-1} are all equal to the $2k+1$-point standard median filter. In the second case, f_n is the $2n-1$-point recursive median filter, and f_i is the $2i+1$-point standard median filter for $i = 1, 2, \ldots, n-1$.

Applying any functor $\mathcal{F}$ to this filter structure yields another cascade filter structure with the same properties. In particular, note that if $\{x_k\}$ is a root sequence for any filter f defined in a category $\mathbf{C}$ and $\mathcal{F}$ is any functor from $\mathbf{C}$ to another category $\mathbf{D}$, it follows from the defining requirements of a functor that $\mathcal{F}\{x_k\}$ is necessarily a root sequence for the filter $\mathcal{F}f$. Hence, applying $\mathcal{F}$ to Eq. (3) yields the new filter structure:

$$\mathcal{F}g = \mathcal{F}f_n \circ \mathcal{F}f_{n-1} \circ \cdots \circ \mathcal{F}f_2 \circ \mathcal{F}f_1,$$

where the same relationships between the root sets of the component filters hold as in Eq. (3). Hence, the resulting filter is again idempotent. Finally, note that functors must also preserve involutory filters of all orders, so functors applied to the idempotent filter construction defined in Eq. (2) also lead to new idempotent filters.

7 Summary

This paper has considered the general question of how to design idempotent nonlinear filters. The basic construction considered here has been based on cascade interconnections of nonlinear filter components, permitting us to use category theory as a vehicle for exploiting known result from linear algebra. Specific examples include the full-rank factorization of idempotent matrices and its extension to idempotent nonlinear filters discussed in Sec. 4, and the construction of new idempotent filter structures from involutory filters discussed in Sec. 5. Similarly, the notion of functors introduced in Sec. 6 led to novel constructions of idempotent filters based on the construction underlying homomorphic systems. The key point of this paper is that category theory provides a useful framework for the analysis and design of certain types of nonlinear filters, including behaviorally-defined classes like idempotent filters.

References

Ben-Israel, A. and Greville, T. (1974). *Generalized Inverses: Theory and Applications*. Wiley.

Blyth, T. (1986). *Categories*. Longman.

Haavisto, P., Gabbouj, M., and Neuvo, Y. (1991). Median based idempotent filters. *J. Circuits, Systems, Computers*, 1:125–148.

Nodes, T. and Gallagher, N. (1982). Median filters: Some modifications and their properties. *IEEE Trans. Acoustics Speech Signal Proc.*, 30:739–746.

Oppenheim, A., Schafer, R., and Stockham, T. (1969). Nonlinear filtering of multiplied and convolved signals. *Proc. IEEE*, 56:1264–1294.

Pearson, R. (1999). *Discrete-Time Dynamic Models*. Oxford.

Pitas, I. and Venetsanopoulos, A. (1990). *Nonlinear Digital Filters*. Kluwer.

Serra, J. (1988). *Image Analysis and Mathematical Morphology*, volume 2. Academic Press.

Tian, Y. and Styan, G. (2001). Rank equalities for idempotent and involutory matrices. *Linear Algebra Appl.*, 335:101–117.