Image Restoration using Boolean and Stack Filters

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ABSTRACT — **This paper presents an overview of our investigations in the field of optimal Boolean and stack filter design aimed for image restoration applications. We develop three tractable and distinct approaches to optimal stack filter design: a training framework, a model based approach and a mixed training- and model based- approach. The links between the three approaches are investigated in this paper and an experimental study is undertaken in order to illustrate the effectiveness of these approaches in image restoration applications.**

1 Introduction

One of the most attractive applications of median type nonlinear filters is image restoration for the case of impulsive noise. There are many nonlinear filter classes encompassing the median filter class. Stack filter class encompasses besides the median filter class other widespread nonlinear filters such as Weighted Median, Weighted Order Statistics and flat morphological filters and can therefore provide performances at least as good as any filter subclass.

As a drawback, since stack filters generalize as many filter classes, the number of stack filters, even for moderate window sizes is extremely large. The existence of optimal design procedures for selecting the most appropriate stack filter for a given application is therefore one of the most important theoretical and practical problems.

Our recent work was devoted to the investigation of various setting for the optimal design problem which can receive effective solutions and for which analytical results can be derived. We can identify three distinct approaches, which will be denoted in the sequel corresponding to the initial specification of the problem.

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2 Three approaches to Optimal Boolean and Stack Filter Design

2.1 $\mathcal{T}$ – The training approach

In the training approach[3][7][8], a representative training set, $\mathcal{T} = \{D(t), X(t)\}_{t=1:T}$, is given, containing the desired signal $D(t)$ and the corrupted signal $X(t)$. There are no constraints on the type of signals which define the training set, any raw data (e.g. images) being therefore allowed to define the initial specification.

The optimal design problem requires finding the parameters ψ of the Boolean or stack filter f_ψ which minimize the criterion

$$J_{\mathcal{T}}(\psi) = \frac{1}{T}\sum_{t=1}^{T} |D(t) - Y(f_\psi, \underline{X}(t))|, \quad (1)$$

where $\underline{X}(t)$ is the N-length processing window and $Y(f_\psi, \underline{X}(t))$ is the output of the filter. For stack filters, this criterion was shown to be equivalent to

$$J_{\mathcal{T}}(\psi) = \sum_{i=1}^{2^N} c(\underline{x}_i) f_\psi(\underline{x}_i) + C_0, \quad (2)$$

with

$$c(\underline{x}_i) = \frac{1}{T}(Card\{(t,m)|\underline{X}^m(t)=\underline{x}_i, D^m(t)=0\} - Card\{(t,m)|\underline{X}^m(t)=\underline{x}_i, D^m(t)=1\}), \quad (3)$$

where the superscript denotes the thresholding at level m, and $\underline{x}_i \in \{0,1\}^N$. The minimization of criterion (2) can be accomplished using very simple procedures, see Section 3.

2.2 $\mathcal{M}$ – The model based approach

In the model based approach[6] some models are assumed to describe the desired and the corrupted signals, in the threshold domain. Since the leading papers on optimal stack filter design were introduced, Markov chain models were considered as initial specification, due to the mathematical tractability of the optimal problem in this setting. However, the connection with the image restoration problem becomes explicit only in our recent work,[6].

Now we consider the data generating system to be given in the form of the MC model $\mathcal{M} = \{P, f^X, f^D\}$. A training set can be generated simulating the dynamic evolution of the MC model, resulting in the sequence of states $\underline{q}(1:T) = \{q(1), q(2), \ldots, q(T+N)\}$, the sequence of target values $\underline{D}(1:T)$ and the input sequence $\underline{X}(1:T)$. The input window to be processed is denoted by $\underline{X}(t) = [f^X(q(t+1)), \ldots, f^X(q(t+N))]$ and the target signal (which ideally must be attained by the filter output $f_\psi(\underline{X}(t))$ is $D(t) = f^D(q(t+N_c))$. Therefore, a training set generated by $\mathcal{M}$ is $\mathcal{T}_{\underline{q}(1:T)} = \{(D(t), \underline{X}(t)\}_{t=1}^T$.

In order to obtain close form expressions for the cost coefficients (3), directly related to the MC model, without generating the training set, two possible modifications of criterion (1) will be considered. First, the size of the training set will be kept fixed, but the optimization will be performed with respect to all possible realizations:

$$J_{ET}(\psi) = E_{\underline{q}(1:T)} \frac{1}{T} \sum_{t=1}^{T} |D(t) - f_\psi(\underline{X}(t))|. \quad (4)$$

Second, the case of a single realization $\underline{q}(1:T)$ will be considered, but in the asymptotic case when $T \to \infty$:

$$J_{T\infty}(\psi) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} |D(t) - f_\psi(\underline{X}(t))|. \quad (5)$$

Since J_T given in (1) reduces to (2), the modified performance criterion J_{ET} will reduce to

$$J_{ET}(\psi) = \sum_{i=1}^{2^N} c_{ET}(\underline{x}_i) f_\psi(\underline{x}_i) + C_{0ET}, \quad (6)$$

where

$$c_{ET}(\underline{x}_i) = \frac{1}{T} E_{\underline{q}(1:T)}(Card\{t|\underline{X}(t)=\underline{x}_i, D(t)=0\} - Card\{t|\underline{X}(t)=\underline{x}_i, D(t)=1\}t). \quad (7)$$

Similarly the criterion $J_{T\infty}$ will reduce to

$$J_{T\infty}(\psi) = \sum_{i=1}^{2^N} c_{T\infty}(\underline{x}_i) f_\psi(\underline{x}_i) + C_{0T\infty}, \quad (8)$$

where

$$c_{T\infty}(\underline{x}_i) = \lim_{T\to\infty} \frac{1}{T}(Card\{t|\underline{X}(t)=\underline{x}_i, D(t)=0\} - Card\{t|\underline{X}(t)=\underline{x}_i, D(t)=1\}). \quad (9)$$

Theorem (Connection between training approach and model based approach)[4]

1. $c_{ET}(\underline{x}_i) = c_{T\infty}(\underline{x}_i)$.

2. The cost coefficients from (7) can be computed using the limiting probabilities of an extended Markov chain states as

$$c_{ET}(\underline{x}_i) = \pi^{X,0}(\underline{x}_i) - \pi^{X,1}(\underline{x}_i). \quad (10)$$

Since the cost coefficients (10) are those derived in classical model based optimal stack filter design[1], the result of the above theorem connects the model based approach with the training approach.

The evaluation of c_{ET} for a given Markov chain can be accomplished using fast algorithms[5].

2.3 $\mathcal{D}, \Phi$ – The mixed training and model based approach

In the mixed training and model based approach[4], a representative target signal $\mathcal{D}$ and the model of the corrupting process are given. This approach generalizes the elaborated analytical studies carried under the assumption of *independent and identically distributed* signal to a case which fits better the image restoration applications: the case of *independent* errors. This approach allows tractable solutions for any given discrete probability distribution function and any given raw data as a target signal.

The optimal design problem can be formulated as follows. *Given* the target deterministic signal $\mathcal{D} = \{D(t)\}_{t=1}^T$ (with integer values $0 \leq D(t) \leq M$) and the distribution functions (which may depend on t) of the discrete random variables $e(t)$, $\Phi = \{Prob(e(t) \leq u)\}_{t=1}^T = \{\phi_{e(t)}(u)\}_{t=1}^T$ ($e(t)$ is the additively corrupting noise) *find* the positive Boolean function f, associated with the stack filter, which minimizes the criterion

$$J_{\mathcal{D},\phi} = \frac{1}{T} \sum_{t=1}^{T} E|D(t) - Y(f, \underline{X}(t))|, \quad (11)$$

where $\underline{X}(t) = \underline{D}(t) + \underline{e}(t)$ denotes the input of the stack filter at "moment" t and $Y(f, \underline{X}(t))$ denotes the output of the stack filter .

If the random variables $e(t)$, $e(s)$ are independent for all $t \neq s$, then criterion (11) can be computed as follows[4]

$$J_{\mathcal{D},\phi} = C_0 + \sum_{i=1}^{2^N} c(\underline{x}_i) f(\underline{x}_i) \quad (12)$$

with

$$
\begin{aligned}
c(\underline{x}_i) = & \frac{1}{T}\sum_{t=1}^{T}\Big[\sum_{m=D(t)}^{M-1}\prod_{j=0}^{N-1}\phi_{X(t+j)}(m)^{1-x_{i_j}} \\
& \cdot[1-\phi_{X(t+j)}(m)]^{x_{i_j}} - \\
& -\sum_{m=0}^{D(t)-1}\prod_{j=0}^{N-1}\phi_{X(t+j)}(m)^{1-x_{i_j}}\,[1-\phi_{X(t+j)}(m)]^{x_{i_j}}\Big] \\
C_0 = & \frac{1}{T}\sum_{i=1}^{2^N}\sum_{t=1}^{T}\sum_{m=0}^{D(t)-1}\prod_{j=0}^{N-1}\phi_{X(t+j)}(m)^{1-x_{i_j}} \\
& \cdot[1-\phi_{X(t+j)}(m)]^{x_{i_j}}
\end{aligned} \tag{13}
$$

and the distribution $\phi_{X(t)}(u) = Prob(X(t) \leq u) = Prob(e(t) \leq u - D(t)) = \phi_{e(t)}(u - D(t))$ denotes the distribution of the input signal.

3 Optimal design starting from the cost coefficients

These three approaches to optimality require some specific tools for developing the optimal solution. The first design stages, i.e the computation of the cost coefficients, are different, reflecting the different initial specifications of the problem. The procedural aspects are common for stages starting from (2),(6),(8),(12) of the optimal design procedures. We introduced in [8] procedures for finding the optimal Boolean filter, starting from the cost coefficients, using only simple assignment and compare operations, for binary signals. If the optimal *stack* filter is required, a fast procedure checks if the Boolean filter fulfill the stacking constraints, and if not, the optimal Boolean filter is projected in stack filter class.

The *differences* expected between the results of the three approaches are very intuitive: the model based approach will provide general solutions, which prove to be good for all experimental situations according to the assumed model, while the solutions provided by the training framework will be more specific to the experimental situations and will generalize only in the limits established by their robustness.

4 Experimental Results

In order to illustrate the effectiveness of $\mathcal{T}$- and $\mathcal{M}$- approaches in image restoration applications, we present the results of $\mathcal{T}$- optimal stack filtering the image Airm0s9l01, from TUT Noisy Image Database[2]. This image is obtained by corrupting the clean image "airfield" with contaminated Gaussian noise, at a SNR=9dB. For $\mathcal{D}, \Phi$- optimal stack filtering, the probability contaminated Gaussian distribution function, with contamination $\lambda = 0.1$, and variance 599.3 was generated. The cost coefficients were computed using (13) and the optimal filter corresponding to the cost coefficients was determined. The numerical results, presented in Table 1, show very close performances obtained using the two approaches for optimal design, with only a minor difference in the favor of filter designed using the image realization. The subjective quality of both filtered images, shown in Figure 1, is very close and prove effectiveness in noise rejection.

5 Conclusions

Optimal Boolean and stack filtering can effectively be applied in image restoration applications. Three approaches, differing in the initial specification of the problem, are available and fast procedures are developed for the implementation of the design stages. The $\mathcal{T}$- approach is the least computational demanding, but its result depends on the realization in the training set. More robust solutions can be obtained using model based approach, and mixed training- and model- based approach, with the cost of a higher computational effort.

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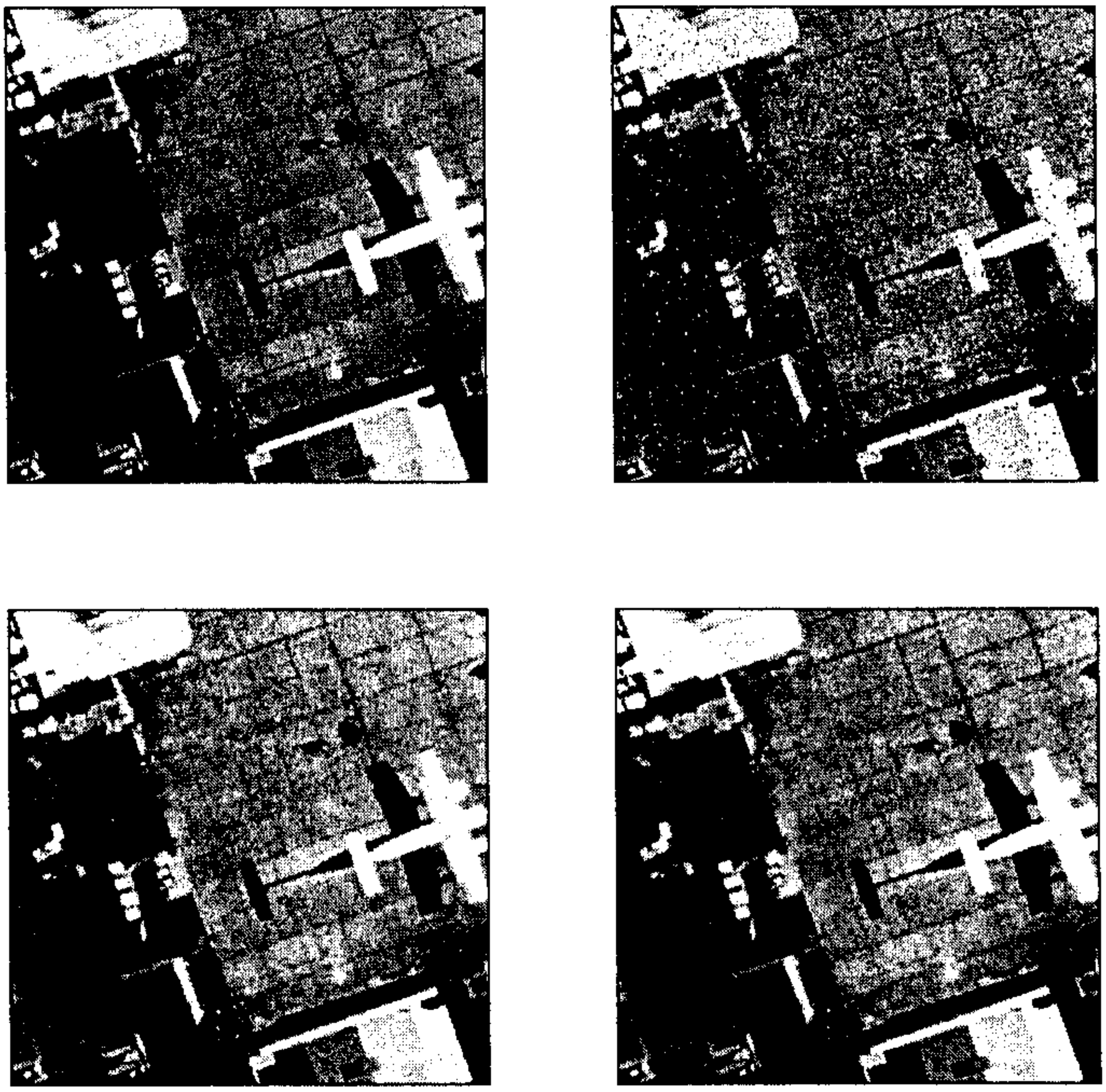
Image	$\hat{\sigma}_\nu^2$	$\widehat{SNR}$	$\widehat{PSNR}$	$\widehat{MAE}$
Airm0s9l01	451.121	9.013	21.587	10.335
$\mathcal{T}$- optimal stack filtered	85.294	16.398	28.856	6.232
$\mathcal{D}, \Phi$- optimal stack filtered	85.107	16.407	28.865	6.235

Table 1: Performance indexes for the noisy image Airm0s9l01 and for the filtered images



a	b
c	d

Figure 1: a) Original airfield image (256 × 256 detail);b) Corrupted image Airm0s9l01;c) Filtered image with the $\mathcal{T}$- optimal stack filter; d) Filtered image with the $\mathcal{D}, \Phi$- optimal stack filter