Optimal Weighted Order Statistic Filters under Structural Constraints

Jacek Niewęgłowski, Lin Yin, Moncef Gabbouj, and Yrjö Neuvo

Tampere University of Technology
Signal Processing Laboratory
P. O. Box 553, SF–33101 Tampere, Finland

Abstract – Both adaptive and nonadaptive methods have been developed in the past for determining optimal Weighted Order Statistic (WOS) filters. In practical applications there is very often a need to preserve or to remove certain structures in filtered image, which in case of WOS filters can easily be expressed in terms of linear inequalities relating the weights of the filter. In this paper we present a procedure based on Quadratic Programming (QP) for finding the optimal WOS filter under Mean Absolute Error (MAE) criterion satisfying imposed structural constraints. In this way we obtain filters, which simultanously meet e.g. the desired detail preservation constraints and give a good noise attenuation. The algorithm is characterized by low computational complexity and simplicity of constraints formulation.

1.1 Introduction

In this paper we derive an algorithm for finding a WOS filter minimizing the MAE between the output of the filter and some desired signal. Let $R(n)$ be the discrete-time received process at the input of WOS filter. The filtering process can be visualized as sliding the filter window, of width N, along $R(n)$. Let $\underline{R}(n)$ be the vector of N samples in the filter window at given moment of time n.

$$\underline{R}(n) = [R(n-N_1), R(n-N_1+1), ..., R(n), ..., R(n+N_2)]^T, \quad (1)$$

where N_1, N_2 are integers such that $N = N_1 + N_2 + 1$. The output of the WOS filter $F_{WOS}(\cdot)$ with positive integer weights $\underline{W} = [W_1, W_2, ..., W_N, T]$ is given by

$$\hat{S}(n) = F_{WOS}(\underline{R}(n)) = T \text{ :th largest sample from the set}$$
$$\Big[\overbrace{R(n-N_1), ..., R(n-N_1)}^{w_1 \text{ times}}, \overbrace{R(n-N_1+1), ..., R(n-N_1+1)}^{w_2 \text{ times}}, ..., \overbrace{R(n+N_2), ..., R(n+N_2)}^{w_N \text{ times}}\Big]. \quad (2)$$

If $R(n) \in (0, 1, ..., K-1)$ for all n, then such signal may be decomposed into K binary signals using the threshold decomposition operation:

$$r^m(n) = T^m(R(n)) = \begin{cases} 1, & \text{if } R(n) \geq m \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

The original signal may be recovered from the K-1 binary threshold signals by

$$R(n) = \sum_{m=1}^{K-1} r^m(n) \quad (4)$$

Binary WOS filters may be applied to these binary signal, which yields

$$\hat{s}^m(n) = F_{WOS}(r^m(n)), \text{ where m=1,2,...,K-1} \quad (5)$$

The output of the WOS filter in the multilevel domain may then be computed by

$$\hat{S}(n) = \sum_{m=1}^{K-1} \hat{s}^m(n). \quad (6)$$

In the binary domain, a WOS filter is specified by a Positive Boolean function (PBF)

$$\hat{s}^m(n) = \begin{cases} 1, & \text{if } \sum_{i=-N_1}^{N_2} r^m(n+i)w_i \geq T \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

Notice that this definition is also valid for any positive real-valued weights. In such a case, the output of the WOS filter is obtained as follows. First, samples inside the filter window are sorted. Starting from the lower end of this sorted set, the weights corrresponding to each sample are added until the cumulative sum $S_W \geq T$. The sample corresponding to last added weight is the output of WOS filter. Using vector notation

$$\tilde{r}^m(n) = [r^m(n-N_1), ..., r^m(n), ...r^m(n+N_2), -1], \quad (8)$$

the PBF of the filter may be expressed compactly

$$\hat{s}^m(n) = U(\underline{W}^T \underline{\tilde{r}}^m(n)), \quad (9)$$

where $U(\cdot)$ is the unit-step function.

1.2 Optimal WOS filters under the MAE criterion

The goal of optimal WOS filtering is find a WOS filter which minimizes the average mean absolute error between the filter's output and some desired signal. Let $S(n)$ be a desired signal and suppose, that $S(n)$ and $R(n)$ are jointly stationary. Then the cost function to be minimized is

$$J(\underline{W}) = E[S(n) - \hat{S}(n)]. \quad (10)$$

0-7803-0593-0/92 $3.00 1992 IEEE

Applying the threshold decomposition property

$$J(\underline{W}) = \sum_{m=0}^{K-1} E[s^m(n) - U(\underline{W}^T\tilde{\underline{r}}^m(n))]$$

$$= \sum_{m=0}^{K-1} E[(s^m(n) - U(\underline{W}^T\tilde{\underline{r}}^m(n)))^2]. \quad (11)$$

Notice that for binary valued data the MAE is equal to the MSE. The above cost function, however is not a continuous function since it involves the unit-step function. Such optimization problems can be quite difficult to solve. The problem, however, can be restated using some approximation of $U(\cdot)$. In our approach we choose a linear function to approximate $U(\cdot)$. The new cost function takes the following form:

$$J(\underline{W}) = \sum_{m=0}^{K-1} E[(s^m(n) - \underline{W}^T\tilde{\underline{r}}^m(n))^2]. \quad (12)$$

The basic assumption is that weights are nonnegative real numbers. Thus, optimization has to be done subject to positivity constraints

$$W_i \geq 0, \text{ where } i = 1, 2, ..., N \text{ and } T \geq 0 \quad (13)$$

The solution to this problem is:

$$\min J(\underline{W}) = \left\{\frac{1}{2}\underline{W}^T\mathbf{R}\underline{W} - \underline{W}^T\underline{R}^s + \frac{1}{2}E[s^2(n)]\right\} \quad (14)$$

where

$$\mathbf{R} = \sum_{m=1}^{K-1} E[\tilde{\underline{r}}^m(n)\tilde{\underline{r}}^m(n)^T], \ \underline{R}^s = \sum_{m=1}^{K-1} E[s^m(n)\tilde{\underline{r}}^m(n)] \quad (15)$$

Using the linear function as an approximation of $U(\cdot)$ requires modification of threshold the T to obtain symmetry between 0 and 1. If $W^* = [W_1^*, W_2^*, ..., W_N^*, T^*]$ is the solution to the above optimization problem, then desired filter will be specified by vector:

$$W^{*'} = \left[W_1^*, W_2^*, ..., W_N^*, T^* + \frac{1}{2}\right] \quad (16)$$

1.3 Method of Solution

The optimization problem formulated in Eq.(14) is very well suited for Quadratic Programming (QP). The algorithm solves convex QP problems subject to general linear equality and inequality constraints. Such problems can be expressed in this form

$$\min_{x \in \Re^N} g^T x + \frac{1}{2}x^T H x$$

subject to:

$$\begin{aligned} A_1 x &= b_1 \\ A_2 x &\geq b_2 \end{aligned} \quad (17)$$

where x is a vector of variables, A_1, A_2 and H are matrices; and b_1 and b_2 are vectors of constants. Matrix H is required to be positive definite. Our optimization problem in Eq(14) can be rewritten as Eq. (17). The algorithm gives a unique solution as long as the linear constraints are consistent.

The important advantage of this method, is that the search for an optimal solution is done only within the feasible region specified by linear constraints. In the previous method [10], if any of the weights of optimal WOS filter becomes negative after some iteration, then it was forced to be zero.

In the next sections, we show how to estimate $\mathbf{R}$ and $\underline{R}^s$ and how to exploit the linear contraints, which may be imposed on the weights.

1.4 Parameters Estimation. The parameters to be estimated are the entries in the autocorrelation matrix $\mathbf{R}$ of the samples inside the filter window and those in the crosscorrelation vector $\underline{R}^s$ of the received process and the desired process. Suppose we have N_s long observations of $\underline{R}(j)$ and $S(j)$, $j = 0, 1, 2, ..., N_s - 1$. Let $R_{min}(j)$ denotes the smallest element in $R(j)$. Then all binary vectors $r^m(j)$, $m = 1, 2, ..., R_{min}$ are equal to 1 for each observation $R(j)$. This vectors would be over-emphasized by our cost function due to linearization of the unit step function . This would yield a solution which is far from the optimal one . In order to lessen the effect of these vectors we will use the following estimates of $\mathbf{R}$ and $\underline{R}^s$.

$$\widehat{\mathbf{R}} = \frac{1}{N_s}\sum_{j=0}^{N_s-1}\sum_{m=R_{min}(j)+1}^{K-1} \tilde{\underline{r}}^m(j)\tilde{\underline{r}}^m(j)^T \quad (18)$$

$$\widehat{\underline{R}}^s = \frac{1}{N_s}\sum_{j=0}^{N_s-1}\sum_{m=R_{min}(j)+1}^{K-1} s^m(j)\tilde{\underline{r}}^m(j) \quad (19)$$

It is also possible to compute this estimates in the multilevel domain

$$\widehat{R}_{i,j} = \frac{1}{N_s}\sum_{n=0}^{N_s-1}(min\{R(n-N_1+i-1), R(n-N_1+j-1)\} - $$
$$-R_{min}(n)) \quad (20a)$$

$$\widehat{R}_{N+1,j} = \widehat{R}_{j,N+1} = -\frac{1}{N_s}\sum_{n=0}^{N_s-1}(R(n-N_1+j-1) - R_{min}(n)) \quad (20b)$$

where $i, j = 1, 2, ..., N$

$$\widehat{R}_{N+1,N+1} = \frac{1}{N_s}\sum_{n=0}^{N_s-1}(R_{max}(n) - R_{min}(n)) \quad (20c)$$

$$\widehat{R}_j^s = \frac{1}{N_s}\sum_{n=0}^{N_s-1}(\min\{S(n), R(n-N_1+j-1)\} - R_{min}(n)) \quad (21a)$$

where $j = 1, 2, ..., N$

$$\widehat{R}_{N+1}^s = -\frac{1}{N_s}\sum_{n=0}^{N_s-1}(S(n) - R_{min}(n)). \quad (21b)$$

In fact $R_{i,j}$ is the morphological correlation defined in [5].

2. Structural Approach to Filtering.

In this paper we focus on filters which can be constrained to posses certain deterministic properties. Previous work in this area consisted in developing algorithms to design optimal stack filters with structural constraints and goals, see [2].

W_1	W_2	W_3
W_4	W_5	W_6
W_7	W_8	W_9

Fig.1 3×3 *WOS filter mask*

As mentioned earlier it is possible to impose certain constraints on filter weights so that desired structures will always be preserved by this filter. By structures we understand here shapes. Notice that shapes analysis can be done in the binary domain. If some shape is to be preserved in the picture, it has to be preserved in the binary signals. Therefore it is sufficient to consider binary patterns in the picture.

3. Experimental Results.

In our simulations we tried to find an optimal Weighted Median (WM) filter with a 3×3 window (Fig. 1). WM filters is a special case of WOS filters, such that the threshold T is specified by

$$T = \frac{\sum_{i=1}^{N} W_i}{2}.$$

The 3D graph of the original image is depicted in Fig 2.

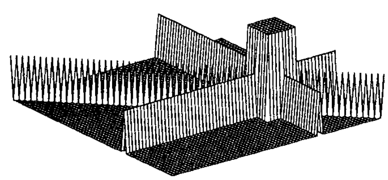

Fig. 2 *Original Picture*

The picture was corrupted with additive impulsive noise with probability 10 % (Fig. 3).

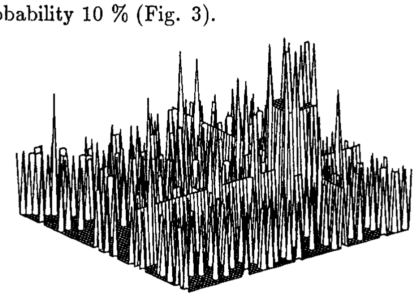

Fig. 3 *Picture corrupted with 10 % impulsive noise*

We used the upper left part of the picture to train the WOS filter. Notice, that this part of the picture contains only diagonal line and noise. We obtain following solution:

$$\mathbf{W} = \begin{pmatrix} 0.1504834 & 0.0014417 & 0.0298288 \\ 0.0412991 & 0.2142351 & 0.0661733 \\ 0.0958055 & 0.0484007 & 0.2466231 \end{pmatrix}$$

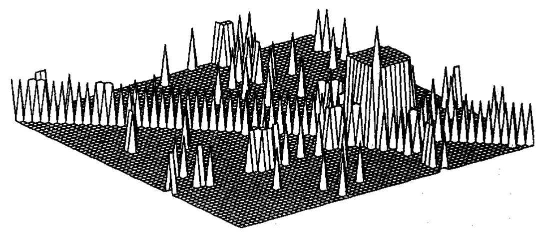

Fig. 4 *Unconstrained filter; MAE=0.2576, MSE=0.06139*

The result of filtering is shown in Fig. 4. MSE stands for the value of Mean Square Error and MAE for Mean Absolute Error. Now suppose we are interested in removing the noise but preserving one pixel-wide vertical and horizontal lines. This means, that following inequalities have to be satisfied:

$$\begin{aligned} W_2 + W_5 + W_8 &\geq T \text{ (vertical lines)} \\ W_4 + W_5 + W_6 &\geq T \text{ (horizontal lines)} \end{aligned} \tag{20}$$

The procedure yields the following filter:

$$\mathbf{W} = \begin{pmatrix} 0.0934983 & 0.03018536 & 0.0642875e-7 \\ 0.0451461 & 0.32653769 & 0.0689583 \\ 0.0447314 & 0.08391915 & 0.1883078 \end{pmatrix}$$

The above coefficients are truncated for easier represantation. The result of filtering under imposed constraints is shown in Fig 5.

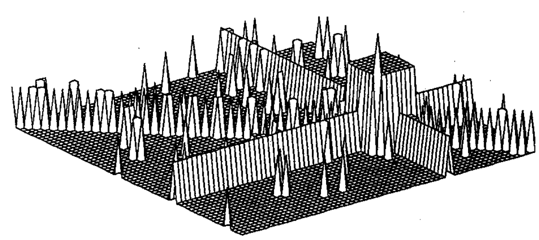

Fig. 5 *Constrained filter; MAE=0.1379, MSE=0.01708*

Notice, that filter preserves the diagonal line, which we are not interested in. It is so because filter was trained on the fragment containing this line. In next experiment we train the filter on the same part of the image. We add the constraint

$$W_1 + W_5 + W_9 < T.$$

This gives the filter the property of removing the diagonal lines. To obtain not strict inequality constraint, we use

$$W_1 + W_5 + W_9 \leq T - \epsilon,$$

where $\epsilon > 0$ is a small number. The filter is constrained as previously to preserve horizontal and vertical lines using Eq. (3.1). The solution is

$$\mathbf{W} = \begin{pmatrix} 0.0414336 & 0.0625557 & 0.0143002 \\ 0.0773116 & 0.2694962 & 0.0985873 \\ 0.0792970 & 0.1133431 & 0.1344643 \end{pmatrix}$$

The result of filtering is shown in Fig. 6.

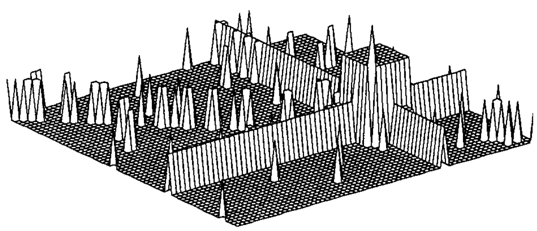

Fig. 6 *Constrained Filter; MAE=0.1354, MSE=0.01232*

The MAE is lower then previously, because preserving the diagonal edge decreases noise attenuation.

Conclusions

The described method has several advantages over the ones studied previously. It is capable of handling large window size problems. It is fast and simple conceptually. Filters obtained in this way, have good noise attenuation properties and can remove or preserve some desired structures in the signal. One should however notice that it yields a suboptimal solution due to linearization of unit step function.

References

[1] R. Benson, *Euclidean Geometry and Convexity*, McGraw-Hill, 1966.

[2] E. Coyle, J-H Lin and M. Gabbouj "Optimal Stack Filtering and the estimation and structural approaches to image processing", *IEEE Trans. on Acoustics, Speech, and Signal Processing*, vol. ASSP-37, no.12, June 1990, pp. 2037-2066.

[3] M. Gabbouj, E. Coyle, "Minimum Mean Absolute Error Stack Filtering with Structural Constraints and Goals", *IEEE Trans. on Acoustics, Speech, and Signal Processing*, vol. ASSP-38, no. 6, June 1990, pp. 955-968.

[4] S. Muroga, *Threshold Logic and Applications*, Wiley-Interscience, 1971.

[5] P.A. Maragos, "Morphological correlation and mean absolute error criteria", *Proc. Int. Conf. ASSP*, April 1989. pp. 1568-1571.

[6] K. Saarinen and Y. Neuvo, "An adaptive weighted median filter based on LMS algorithm", in *Proceedings of the 1990 Bilkent International Conference on New Trends in Communication, Control and Signal Processing*, Ankara, Turkey, pp. 1241-1248, June 1990.

[7] P.D. Wendt, E.J. Coyle and N.C. Gallagher, " Stack Filters",*IEEE Trans. on Acoustics, Speech and Signal Processing*, vol. ASSP-34, no. 4, August 1986, pp. 898-910.

[8] L.Yin, J. Astola and Y. Neuvo, "Adaptive Stack Filtering with Applications to Image Processing", submitted to *IEEE Trans. on Signal Processing* in Oct. 1990.

[9] L.Yin, J. Astola and Y. Neuvo, "Adaptive Weighted Median Filtering under Mean Absolute Error Criterion", *Proceedings IEEE Workshop on Visual Signal Processing and Communication*, Taiwan, June 5-7, 1991, pp. 184-187, Oct. 1990.