The PBF of One Weight Weighted Median Filters

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Abstract - The general form of PBFs corresponding to one weight WM filters, which are a subclass of WM filters, is derived in this paper. Based on the results, it is straightforward to derive the corresponding PBF from a given one weight WM filter or vice versa. The results can also be used to check whether an PBF corresponds to an one weight WM filters. The checking procedure is very simple and tractable due to the fact that linear programming is not needed.

1. Introduction

Weighted median (WM) filters, which can be represented by a self-dual and linear separable positive Boolean functions (PBFs), belong to stack filters [6]. A method of finding the corresponding PBF for a given WM filter (the weights are known) has been introduced in [7]. However, the reverse problem is not that easy. It was shown that complexed computation is needed to check whether a PBF corresponds to a WM filter due to the unavoidable linear programming [7] [8]. In this paper, the general form of PBFs corresponding to a subclass of WM filters: namely one weight WM (OWWM) filters, is derived. Based on the derived form, the PBF corresponding to a OWWM filter is expressed in a compact form. The result can be used to check the equivalence between a PBF and a OWWM filter. The checking procedure is very simple and tractable due to the fact that linear programming is not needed. In other words, it is straightforward to write the weights of a OWWM filter if a PBF which corresponds to the OWWM filter is given.

2. WM Filters

WM filters [1] are generalizations of median filters, where each window position is assigned a weight W_i, $i = 1, 2, \ldots, 2n+1$. Each sample inside the filter window is duplicated to the number of the corresponding weight. The median value from the increased list of samples is the WM. If the sum of weights is odd, the WM output is unique. This filtering procedure can be represented by the following equation:

$$y_k = med(W_1 \diamond x_{k+1}, \cdots, W_{n+1} \diamond x_{k+n+1}, \cdots, W_{2n+1} \diamond x_{k+2n+1})$$

where the symbol $\diamond$ is used to denote duplication, i.e.

$$n \diamond x = \overbrace{x, \cdots, x}^{n \text{ times}}$$

OWWM filters are the simplest WM filters, in which only one sample is assigned a larger weight and all other weights are equal to one. Among them, center WM (CWM) filters, which are assigned a weight only to the center sample of the window, have been found to be useful in the field of signal and image processing [2,9] In OWWM filters, three parameters are needed to specify the filters. They are the window size, the weight value and the position of the weight.

A OWWM filter can be denoted by OWWM(M;i;2K+1), where M is the window size, $2K+1$ is the weight and i is the position of the weight. M is defined as an odd number $M = 2n+1$ and $K \geq 0$ and $1 \leq i \leq 2n+1$. When $i = n+1$, the one weight WM filter is CWM filter.

CWM filters are a subclass of OWWM filters. In a CWM filter with window size $2n+1$, the center weight (assigned to the center sample) is $W_0 = 2K+1$, and all other weights $W_i = 1$ for each $i \neq n+1$, where K and n are non-negative integers. Different values of K produce different CWM filters. They range from the standard median, when $K = 0$, to the identity filter, when $K \geq n$. When $K = n-1$, the CWM filter is an idempotent filter, i.e., it produces a root signal after a single filtering pass [5].

3. The PBF of OWWM Filters

The following Theorem gives the general PBF form of OWWM filters.

Theorem 2.1: For a OWWM filter OWWM(M;i;2K+1), where $M = 2n+1$, the weighted sample is denoted as x_i and the set of samples in the window without the weighted sample x_i is denoted by X_{M-1}. Then the PBF corresponding to the OWWM filter has a form of

$$f(x_1, \cdots, x_{2n+1}) = x_i(\Upsilon^{n-K}(X_{M-1}))$$

$$+\Upsilon^{K+n+1}(X_{M-1})U(n-K-1) \qquad (1)$$

where $U(x)$ is a unit step function

$$U(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

and $\Upsilon^B(X_A)$ is defined as follows.

i) When $0 < B \leq A$, $\Upsilon^B(X_A)$ is the sum of product of all different combinations of B samples from the set which contains A samples. X_A is a set containing A samples. That is:

$$\Upsilon^B(X_A) = \sum_{\substack{l_1, l_2, \ldots, l_B = 1 \\ l_1 < l_2 \ldots < l_B}}^{A} x_{l_1} x_{l_2} \ldots x_{l_B}$$

ii) When $B \leq 0$, then $\Upsilon^B(X_A) = 1$.

○

Outline of the Proof: Without loss of generality, assume that the weight $W_i = 2k+1$ is weighted at sample x_i, then the sum of the weights is $2k+2n+1$.

Case 1: If $x_i = 1$, then 1 can be the output if and only if there are at least $n-k$ other samples having the same value as x_i. Therefore, in the MTP form of the corresponding PBF, the term $x_i(\Upsilon^{n-K}(X_{M-1}))$ appears.

Case 2: If $x_i \neq 1$, then 1 can be the output if and only if there are at least $k+n+1$ other samples having the value as one. Therefore, in the MTP form of the corresponding PBF, the term $\Upsilon^{K+n+1}(X_{M-1})$ appears. However, if there is no enough such samples, there will be no such a term exit in the PBF. Therefore, the term can be writted as $\Upsilon^{K+n+1}(X_{M-1})U(n-K-1)$.

Therefore, Theorem 2.1 is correct.

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CWM filters are a special case of OWWM filters, where the weighted sample is the center sample in the window. The last theorem for CWM filters is expressed again in the following.

Theorem 2.2: For a CWM filter CWM(M;2K+1), where $M = 2n+1$, the center sample is denoted as x_{n+1} and the set of samples in the window without the center sample x_{n+1} is denoted by X_{M-1}. Then the PBF corresponding to this CWM filter has a form of

$$f(x_1, \cdots, x_{2n+1}) = x_{n+1}(\Upsilon^{n-K}(X_{M-1}))$$

$$+\Upsilon^{K+n+1}(X_{M-1})U(n-K-1) \qquad (2)$$

○

The consequence of Theorem 2.1 and Theorem 2.2 is that the equation (1) and (2) are self-dual and linear separable.

Example: For a standard median filter with window width $2n+1$, the corresponding PBF has a form of

$$f(x_1, \cdots, x_{2n+1}) = \Upsilon^{n+1}(X_{2n+1})$$

When $n = 1$, then the corresponding PBF is

$$f(x_1, x_2, x_3) = \Upsilon^2(x_1, x_2, x_3)$$

$$= x_1x_2 + x_2x_3 + x_3x_1$$

The notation $\Upsilon^B(X_A)$ is appeared quite often in PBFs of median type filters.

4. Application of the Results

The theorems proved in Section 3 can be used to check if a PBF is corresponding to a OWWM filter or not. It is also very easy to find the corresponding weights according to the theorems. The advantage is that the method does not need to check the self-dual and linear separable properties of a PBF, therefore avoiding the use of linear programming algorithm.

In the checking method, it needs only to compute the MTP of a PBF and to check if the final form fits to the form describe in theorems. If it fits to the form in the theorem then it is a OWWM filter, otherwise it is not.

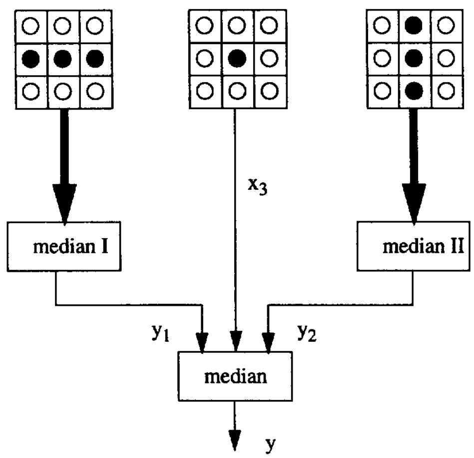

Figure 1: Multilevel 3 by 3 cross window median filter.

Example: Check if a 3 by 3 multilevel cross window median filter corresponds to a CWM filter (Figure 1).

First level median I operate on 3 samples x_1, x_3, x_5, the PBF is given by $y_1 = \Upsilon^2(x_1, x_3, x_5)$ and first level median II operate on 3 samples x_2, x_3, x_4, the PBF is given by $y_2 = \Upsilon^2(x_2, x_3, x_4)$.

The second level median operate on 3 samples y_1, x_3, y_2, the PBF is given by $y = \Upsilon^2(y_1, x_3, y_2)$.

The final output can be written as

$$y = \Upsilon^2(\Upsilon^2(x_1, x_3, x_5), x_3, \Upsilon^2(x_2, x_3, x_4)).$$

It is easy to compute that

$$y = x_1x_3 + x_3x_5 + x_2x_3 + x_3x_4 + x_1x_2x_4x_5$$

$$= x_3(x_1 + x_2 + x_4 + x_5) + x_1x_2x_4x_5$$

That is

$$y = x_3\Upsilon^1(X_4) + \Upsilon^4(X_4)$$

Therefore it is corresponding to CWM(5;3) since $n = 2$ and $K = 1$ (see equation (2)).

5. Conclusions

The important result of this paper is that for a special class of WM filters: namely one weight WM filter, the checking procedure of the equivalence between WM filters and PBFs is getting simple. Finding weight value from PBF is straightforward. More analysis of OWWM filters is possible based on the PBF representation of the filters.

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