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On Root Structures and Convergence Properties of Weighted Median Filters*

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Abstract. A weighted median filter is a nonlinear digital filter consisting of a window of length $2N+1$ and a weight vector $\mathbf{W} = (W_{-N}, \ldots, W_0, \ldots, W_N)$. A root signal of a median type filter is a signal that is invariant to the filter. However, not all weighted median filters possess the convergence property. In this paper, we shall study the root structures and the convergence behavior of a subclass of weighted median filters, called *class*-1 filters, which is symmetric in its weight vector. We shall introduce an important parameter, called *feature value*, and show that any one-dimensional unappended signal of length L will converge to a root signal in at most

$$3\left\lceil \frac{L-2}{2(2N+2-p)} \right\rceil$$

passes of a *class*-1 filter with window width $2N+1$ and the *feature value* p.

1. Introduction

The median filtering for signal processing was first introduced by Tukey [24], [25], who applied it to the smoothing of statistical data. Many authors have since analyzed the statistical properties and the root structures of the standard median (SM) filters [2], [6], [8], [9], [16]–[19]. Some one-dimensional applications can be found in [4], [11], [20], [29], and some image processing applications in [5], [10], [12], [15], [21], [22], [31]. The median filter has been shown to remove both impulses, represented by up to N samples and oscillations, where the window includes $2N+1$ samples. The root of an arbitrary signal is the signal obtained by

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filtering it repeatedly until there is no further change [3], [9]. Weighted median (WM) filters, including center weighted median (CWM) filters, are generations of the SM filter and were introduced by Justusson [13], and further discussed by Brownrigg [4], Wendt *et al.* [28], and Yli-Harja *et al.* [32].

WM filters have the same advantages as the median filters: edge preservation and efficient suppression of impulsive noise. Having a set of weights, however, the weighted median filter is much more flexible in preserving desired signal structures than a median filter. It is clear that a window width $2N + 1$ median filter can only preserve the details lasting more than $N + 1$ samples. To preserve smaller details in signals, a smaller window width median filter has to be used. The smaller the window width of the median filter is, obviously, the poorer its noise reduction capability becomes [30], [33]. The weighted median filter has a set of weights that allows the control of the filter's behavior and has much greater flexibility in specification than the median filter. The choice of weights allows a certain degree of flexibility in preserving signal structures. That is, one can select to use a weighted median filter with a long enough window, say $2N+1$, to suppress noise effectively and at the same time preserve details lasting less than $N + 1$ samples. As may be inferred from this, the structures of root signals and the weights of a weighted median filter are dependent on each other. The spectral analysis of the weighted median filters gives little insight into the filtering process because the filters are nonlinear. Consequently, the deterministic and statistical properties of weighted median filters are used to describe their effect on signal. Roots or root signals play an important role in analyzing the deterministic properties of weighted median filters. Zeng [34] established the relationship between weights and the minimum length of pulses, which can be preserved by WM filters with the symmetric and, from the window center, noninreasing weights.

In this paper we shall study some root properties due to different weight structures of the weighted median filters. The necessary and sufficient conditions for any one-dimensional signal to be a root signal of these filters will be presented. Then we shall study the convergence behavior of the symmetric weighted median filters.

In Section 2, we shall first review some basic concepts and definitions about weighted median filters. The root structures and the necessary and sufficient conditions for a signal to be a root signal of the weighted median filters are shown in Section 3, with some examples given to show these conditions explicitly. According to these, the *bell*-WM filter and *class*-filter will be defined. In Section 4, we prove our convergence result for the *class*-1 filter. Finally, in Section 5, conclusions of the work are drawn.

2. Some basic concepts and definitions

Before analyzing the root properties of weighted median-type filters, it is worthwhile to review some related concepts and introduce some common terminology.

2.1. Threshold decomposition and stacking property.

Consider a signal $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_L)$, where $\mathbf{x}_k \in \{0, 1, 2, \ldots, M\}$, L is the length of the signal. $\mathbf{X}$ can be decomposed to M binary signals $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^m$ fefined by

$$\mathbf{x}_n^m = \begin{cases} 1 & \text{if } \mathbf{x}_n \geq m \\ 0 & \text{otherwise} . \end{cases} \tag{1}$$

Then the following holds:

$$\mathbf{X} = \sum_{m=1}^{M} \mathbf{x}^m . \tag{2}$$

Two binary signals, $\mathbf{u}$ and $\mathbf{v}$ "stack" if $\mathbf{u}_n \geq \mathbf{v}_n$ for each $n \in \{1, \ldots, L\}$. Consider $\mathbf{u}$ and $\mathbf{v}$ that are filtered with a binary filter of window width $2N + 1$; i.e., Boolean function f: $\{0, 1\}^{2N+1} \rightarrow \{0, 1\}$, the filtering operation being denoted as $\mathbf{x} = f(\mathbf{u})$, $\mathbf{y} = f(\mathbf{v})$, where $\mathbf{x}$ and $\mathbf{y}$ are binary output signals. Now the binary filter f is said to possess the stacking property if and only if

$$f(\mathbf{u}) \geq f(\mathbf{v}) \quad \text{whenever } \mathbf{u} \geq \mathbf{v} . \tag{3}$$

The whole procedure of threshold decomposition and stacking can be applied to real-valued signals if the number of threshold levels is thought to be infinite. However, in this paper we mainly consider integer-valued signals and filters.

2.2 Definitions.

In the integer domain, WM filters can be defined in two distinct ways; however, both definitions give exactly the same output if the weights $\mathbf{w}_i$ are restricted to be positive integers. Therefore, we only give a definition of a WM filter as follows.

For the discrete-time continuous-valued input vector with central location in i,

$$\mathbf{X}_i = (x_{i-N}, \ldots, \mathbf{x}_i, \ldots, \mathbf{x}_{i+N}) . \tag{4}$$

The output y_i of the WM filter of span $2N + 1$ associated with the integer weights

$$\mathbf{W} = (\mathbf{w}_{-N}, \ldots, \mathbf{w}_0, \ldots, \mathbf{w}_N) \tag{5}$$

is obtained by

$$\mathbf{y}_i = \mathrm{MED}\{\mathbf{w}_{-N} \diamond \mathbf{x}_{i-N}, \ldots, \mathbf{w}_0 \diamond \mathbf{x}_i, \ldots, \mathbf{w}_N \diamond \mathbf{x}_{i+N}\} \tag{6}$$

where MED$\{\cdot\}$ denotes the median operation, and $\diamond$ denotes the duplication

$$n \diamond \mathbf{x} = \underbrace{\mathbf{x}, \mathbf{x}, \ldots, \mathbf{x}}_{n \text{ times}} . \tag{7}$$

The threshold of a WM filter, denoted by T, is defined by

$$T = \frac{1}{2}\left(1 + \sum_{i=-N}^{N} W_i\right) . \tag{8}$$

The WM filter usually can be expressed by

$$\{\mathbf{w}_{-N}, \ldots, \mathbf{w}_0, \ldots, \mathbf{w}_N\} \tag{9}$$

and the procedure to choose its output can be stated as follows: duplicate each sample $\mathbf{x}_{i-k}$ of input signal $\mathbf{X}$ to the number of the corresponding weight $\mathbf{w}_k$, sort the expanded vector of $\sum_{k=-N}^{N} \mathbf{w}_k$ points, and choose the median from the sorted vector.

3. Root structures

It was found from the threshold decomposition and stacking property that an arbitrary input signal can be decomposed into many binary signals, and each binary signal will be filtered by a corresponding stack filter [7]. After filtering with this stack filter, all binary output signals will be added together to form the output signal. This operation is equivalent to that in which the input signal is filtered by the filter directly. For a finite signal with length L, usually we must append N samples to the beginning of the signal, assigning them a value that is equal to the value of the first sample, and N samples to the end that are equal to the last sample value. We call the new signal an appended signal; the original signal is the unappended signal. Therefore, we will concentrate our analysis on the binary and appended input signal in this paper.

As for the SM filter, WM filters may map a class of input signals into an associated set of root sequences. If so, each of these root signals is, by definition, invariant to additional filtering passes. In WM filtering the equivalence between two filters cannot be checked simply by comparing their weights: WM filters with different weights may have identical outputs. For example, all WM filters of odd length and the weights $\mathbf{w}_{-N} = \cdots = \mathbf{w}_0 = \cdots = \mathbf{w}_N = c$, where c is some constant, are SM filters. Another example, which is more striking, is that a WM filter with weights $\mathbf{w}_{-1} = 50$, $\mathbf{w}_0 = 50$, $\mathbf{w}_1 = 99$ is a SM filter of span 3. Clearly, for a WM filter the weights do not play the role of the impulse response coefficients as they do in the case of the linear filter. However, root structures depend on weights of WM filters and their discussion is thus required.

Theorem 3.1. *The right displacement of an edge in an input signal will occur if the input is filtered by a WM filter with*

$$\sum_{n=-N}^{-1} \mathbf{w}_j \geq \frac{1}{2}\left(\sum_{k=-N}^{N} \mathbf{w}_k + 1\right) = T \tag{10}$$

where T is the threshold, which makes "edge" move to the right side bit by bit with each filtering pass, until there is no further change in the signal. The final output is a root signal, called the constant signal to leftmost sample, of this WM filter with the weights satisfying the condition in (9).

Proof. Suppose that, without loss of generality, the filter's window is centered at a point, say $\mathbf{x}_i$, that belongs to an edge of the leftmost point of the monotonic change. Because

$$\sum_{n=-N}^{-1} \mathbf{w}_j \geq T$$

obviously, we have

$$\sum_{k=0}^{N} \mathbf{w}_k < T$$

and

$$\sum_{j=0}^{N} \mathbf{w}_j < \sum_{k=-N}^{-1} \mathbf{w}_k .$$

Therefore, all $\sum_{j=-N}^{-1} \mathbf{w}_j$ points to the left of $\mathbf{x}_i$, in the enlarged window, have the same value but different from $\mathbf{x}_i$. This means that all $\sum_{j=-N}^{-1} \mathbf{w}_j$ points belong to a constant neighborhood. Then the output is the value that is equal to the constant neighborhood's value. Consequently, the leftmost point (or first point of the *edge*) of the monotonic change (*edge*) moves to the right side by one bit. It is easy to see that each time the signal is filtered the *edge* moves to the right side by one bit. Obviously, if the length between the leftmost point and the end of the unappended input signal is $L - I$, where L denotes the length of the input signal and I is the distance from the beginning point of the input signal to the leftmost point of an edge, no change in the signal will occur after $L - I$ filtering passes. □

Examples 3.1. Let us consider a WM filter $f = \{3, 3, 1, 1, 1\}$; obviously; $N = 2$,

$$\sum_{j=-2}^{2} \mathbf{w}_j + 1 = 2T = 10$$

and

$$\sum_{j=-2}^{-1} \mathbf{w}_j = 6 > T = 5 .$$

Consider a binary signal of length 15

$$* * 000001111100000 * *$$

Filtering it one time, we have

$$* * 000000111110000 * *$$

After two times, it becomes

$$* * 000000011111000 * *$$

After 10 filtering passes, the output is as follows:

$$* * 000000000000000 * *$$

Here *star* $*$ denotes the appended bits to the signal. Because $I = 5$ and $L = 15$, there is no further change in the signal after filtering it 10 times.

Theorem 3.2. *The left displacement of an edge in an input signal will occur if the input is filtered by a WM filter and*

$$\sum_{j=1}^{N} \mathbf{w}_j \geq \frac{1}{2}\left(\sum_{k=-N}^{N} \mathbf{w}_k + 1\right) = T \tag{11}$$

where T is the threshold, which makes "edge" move to the leftside bit by bit with each filtering pass, until there is no further change in the signal. The resulting signal is a root signal, called constant signal to the right-most sample, of this WM filter with weights satisfying the condition in (10).

Theorem 3.2 is proved similarly to Theorem 3.1.

Example 3.2. Let us consider a WM filter $f = \{1, 1, 1, 3, 3\}$; obviously, $N = 2$,

$$\sum_{j=-2}^{2} \mathbf{w}_j + 1 = 2T = 10$$

and

$$\sum_{j=2}^{1} \mathbf{w}_j = 6 > T = 5\ .$$

Consider a binary signal of length 15

$$**\,000001111100000\,**$$

Filtering it one time, we have

$$**\,000011111000000\,**$$

After two times, it becomes

$$**\,000111110000000\,**$$

After 10 filtering passes, the output is

$$**\,000000000000000\,**$$

Remark 3.1. In the special cases of Theorem 3.1 and Theorem 3.2, the WM filter is called *the fixed left-shifting identity filter, the fixed right-shifting identity filter, or the center identity filter*, respectively, when the following conditions are satisfied:

(1) the fixed left-shining identity filter, if there is only one weight satisfying $\mathbf{w}_j \geq T$, $j \in \{1, 2, \ldots, N\}$;
(2) the fixed right-shifting identity filter, if there is only one weight such that $\mathbf{w}_j \geq T$, $j \in \{-N, -N+1, \ldots, -1\}$;
(3) the center identity filter, if $\mathbf{w}_0 \geq T$.

The number of shifting bits equals to $|j|$ in each filtering pass.

Example 3.3. Let us consider three WM filters, $f_1 = \{1, 2, 3, 2, 11\}$ *(fixed left-shifting identity filter)*, $f_2 = \{11, 2, 3, 2, 1\}$ *(fixed right-shifting identity filter)*, and $f_3 = \{1, 2, 7, 2, 1\}$ *(center identity filter)*, and an input signal (unappended signal) 0000011111. Obviously, the length L of the signal is 10; $N = 2$; $T_1 = 10$ and $\mathbf{w}_2 = 11 > T_1$ for f_1, $T_2 = 10$ and $\mathbf{w}_{-2} = 11 > T_2$ for f_2, and $T_3 = 7$ and $\mathbf{w}_0 = 7 \geq T_3$ for f_3. The output y_i of filtering one time by WM filter f_i is, respectively,

$$\begin{aligned} y_1 &= 0001111111 \\ y_2 &= 0000000111 \ . \\ y_3 &= 0000011111 \end{aligned}$$

It is clear that two bits have shifted to the left by f_1, two bits to the right by f_2, and no bits have shifted by f_3.

In most practical situations, the weights of a WM filter are distributed symmetrically about the center to avoid introducing unwanted bias in the filtered signal. Because the center-located point in the window plays an important role in the filtering procedure, the weights of a WM filter should be nonincreasing from the center.

Definition 3.1. A *bell*-WM (b-WM) filter is any symmetric WM filter whose weights are nonincreasing from the center, which usually is denoted by

$$\mathbf{W} = \{\mathbf{w}_N, \mathbf{w}_{N-1}, \ldots, \mathbf{w}_1, \mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_{N-1}, \mathbf{w}_N\} \tag{12}$$

where $\mathbf{w}_k \geq \mathbf{w}_{k+1}$, for $k = 1, 2, \ldots, N - 1$.

These restrictions on the weights of WM filters are suitable for applications where prior information is absent; actually they are commonly used in practice. In addition, they reduce the design complexity by reducing the number of parameters.

Property of b-WM filter. The weights of any b-WM filter with window width $2N + 1$ satisfy the following relations [34]:

$$2\sum_{i=p}^{N} \mathbf{w}_i < \mathbf{w}_0 < 2\sum_{i=p-1}^{N} \mathbf{w}_i \tag{13}$$

where $1 \leq p \leq N + 1$. p is an integer called the *feature value* of the b-WM filter.

This parameter p plays a crucial role in the convergence behavior analysis of b-WM filters in the sequel. A careful examination of the above relation reveals that this parameter, the *feature value* of a b-WM filter, corresponds to the minimum length of a pulse that can be preserved by the b-WM filter. Note that any b-WM filter with window width $2N + 1$ has its own p parameter valued from 1 to $N + 1$.

Theorem 3.3. *The b-WM filter with window width 3 is the same as an SM filter with the same window size.*

Proof. For a b-WM filter with window width 3, say that $\{\mathbf{w}_1, \mathbf{w}_0, \mathbf{w}_1\}$, $\mathbf{w}_0 > \mathbf{w}_1$, and $2\mathbf{w}_1 + \mathbf{w}_0$ is odd. It is clear that $\mathbf{w}_0$ must be an odd integer. The corresponding Boolean expression can be written as $f(\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}) = \mathbf{x}_{i-1}\mathbf{x}_i + \mathbf{x}_{i-1}\mathbf{x}_{i+1} + \mathbf{x}_i\mathbf{x}_{i+1}$. The expression is the same as that of an SM filter with window width 3.□

Note that we do not consider the identity filter in this proof.

Assume that a signal of length L is filtered with a window of width $2N + 1$ and weight $\{\mathbf{w}_N, \ldots, \mathbf{w}_0, \ldots, \mathbf{w}_N\}$. As noted previously, we always append to the beginning of the signal an additional N samples that equal to the value of the first sample of the signal. Similarly, N constant points are appended to the end of the L-length signal. By doing this, we ensure that, when the initial signal's first or last sample is in the center of the enlarged window, the filter output equals this sample value. If a signal passes through a WM filter unchanged, it means that the central sample value for each enlarged window position is itself the median of the samples within the enlarged window.

In general, if P and Q are subsets of $\mathbf{R}^n$, $f : P \to Q$ is an arbitrary function, and $\mathbf{x}_n \in P$ obeys the condition in the following equation:

$$\mathbf{x}_n = f(\mathbf{x}_n) \tag{14}$$

i.e., a root of a filter f is an appended signal $\mathbf{x}_n$ that is invariant to f. We call $\mathbf{x}_n$ a *root* of f.

In order to study root properties of WM filters, we define the following signal characteristics. Note that there are differences between these definitions and those for SM filters [9].

(1) *Constant Neighborhood (CN):* This consists of at least $p + 1$ consecutive identically valued points, where p, *the feature value* of the b-WM filter, is defined by the Property of b-WM filter.

(2) *Edge:* A monotonic region between two constant neighborhoods of different value. The connecting monotonic region cannot contain any constant neighborhood.

(3) *Impulse:* A constant neighborhood followed by at least one but no more than p points that are then followed by another constant neighborhood having the same value as the first constant neighborhood.

(4) *Oscillation:* A sequence of points that is not part of a constant neighborhood, an edge, or an impulse.

Theorem 3.4. *The sufficient condition for a 1-D signal having length L and an M-valued sequence to be a root of the b-WM filter is that the signal consists only of constant neighborhoods and edges.*

Proof. Now assume that the extended binary signal contains only constant neighborhoods and edges. If the center of the window of width $2N + 1$ is at any signal sample, then the points in the enlarged window are either monotone or nonmonotone. If the samples are monotonic, then they will not be changed by the b-WM filter. If they are nonmonotonic, then the window must be centered on a point in the constant neighborhood shared by two edges. Of the $2\sum_{i=i}^{N} \mathbf{w}_i + \mathbf{w}_0$ points in the enlarged window, at least $p+1$ of them are equal to the center point, and thus the center point is unchanged by the b-WM filter. The sufficient condition is quite obvious. □

Because the weights of b-WM filters can be selected in various ways, the necessary condition of Theorem 3.4 has not been proven until now. However, the necessary condition of the *class*-1 filter, a special modification of the b-WM filter, can be proven as follows.

Definition 3.2. Among b-WM filters, those filters whose weights satisfy the following inequality are defined as the *class*-1 filters:

$$\mathbf{w}_0 < 2\mathbf{w}_p - 2\sum_{i=p+2}^{N} \mathbf{w}_i \,. \tag{15}$$

The b-WM filters, for example, $f_1 = \{1, 2, 3, 2, 1\}$ and $f_2 = \{1, 2, 4, 7, 4, 2, 1\}$, all fall into the set of *class*-1 filters.

Theorem 3.5. *The necessary and sufficient condition for a 1-D signal having length L and M-valued sequence to be a root of the class-1 filter is that the signal consists only of constant neighborhoods and edges.*

Proof. Suppose that we are given an L-length binary signal with first sample 0's (1's). Then N 0's (1's) are appended to the beginning of the signal. Obviously, the first sample will not change upon filtering. As the window is sliding from left to right, the first sample that will change must be either a point contained in an impulse or oscillation, because any point contained in CN of length $p+1$ or more will not change upon filtering.

In the following, we consider a special case, actually the worst case. Before point v there is a CN of length $p+1$ with 0's, the sample at the point v is 1, followed by $(p-1)$ 1's. Now the filter window is centered at v. The $2N+1$ samples inside the window are of the following form:

$$\underbrace{\overbrace{?, ?, \ldots, ?}^{N-p-1}\overbrace{0, \ldots, 0}^{p+1}\overbrace{1, \ldots, 1}^{p}\overbrace{0, ?, \ldots, ?}^{N-p+1}}_{2N+1}$$

where the ?'s denote those samples that are unknown. The window is centered at the point v, the first 1 after $p+1$ consecutive 0's. Let the weights be $\mathbf{w}_N, \ldots, \mathbf{w}_0, \ldots, \mathbf{w}_N$,. The weight sum of 0's in the window is at least

$$\mathbf{w}_p + \sum_{i=1}^{p+1} \mathbf{w}_i \,.$$

Because of (15), the output at the point v will be 0, no matter what the values of these ?'s are. Then this 0 at the point v will not change anymore upon further filtering.

This special case shows that if there are $p+1$ or more consecutive 0's (or 1's) immediately before the window center, the filter's output at this point will be 0 (or 1) unless this center is among another CN of length $p+1$. The output at this point will not change upon further filtering. The necessary condition in the theorem is proven. The sufficient condition is quite obvious from Theorem 3.4. □

4. Convergence properties

In [7], it was shown that median filtering the M-level signal k times is equivalent to median filtering each threshold signal k times, and then reconstructing the M-level signal by summing these filtered threshold signals. Therefore, the M-level signal converges in at most m passes if and only if each of its threshold signals converges in at most m passes of the filter. It follows that the maximum number of passes to a root for M-level signals of length L is bounded above by the corresponding limit for binary signals. Thus, it is enough for us to study the convergence properties of these filters with binary signals.

With the N appended bits, the left end of a binary signal looks like

$$0^{N+1}x_1x_2\cdots \tag{16}$$

or

$$1^{N+1}x_1x_2\cdots . \tag{17}$$

The right end is similar, but the bits are reversed.

Wendt [26] proved that any appended finite length signal converges to a root or to a circle of period 2 by a finite number of successive filterings of any symmetric WM filter. According to this, some, but not all, symmetric WM filters possess the convergence property. In the following theorem, we invert the lower limit above to get an upper limit on the number of passes to a root.

Theorem 4.1. *Let the length of an unappended M-valued signal be L. Then the appended signal will converge to a root in at most*

$$3\left\lceil \frac{L-2}{2(2N+2-p)} \right\rceil \tag{18}$$

passes a class-1 filter with window width $2N+1$.

Proof. The first step in our proof of the limit on the number of passes is to find when a constant end region will extend inwards by exactly k bits on one pass of the *class*-1 filter. Suppose that $1^{N+1+g}0x_1x_2\cdots \to 1^{N+1+g+k}0x_1x_2\cdots$ in one pass, where $g \geq 0$. In other words, the end extends inwards by exactly k bits. Then the original sequence is called a *k-state*. This definition is determined in the same way as that of the median filter in [27]. However, here $k = k_0 + N - p$ because the CN of a b-WM filter is $p+1$, but not $N+1$, where k_0 corresponds to the median filter.

Assume that $x_1 \cdots x_N \neq 0^N$. Obviously, the signal $1^{N+1}0x_1 \cdots x_N$ is not a *k-state* for any $k \geq N$. Then it is not a *k-state* for any $k > 0$, and the initial 0 is preserved after one pass. This is impossible unless $x_1 \cdots x_N = 0^N$. Therefore, if the left end of a signal is $1^{N+1}0x_1 \cdots x_N \cdots$, such that $x_1 \cdots x_N \neq 0^N$ and it is not a *k-state* for any $k < N$, then it is a *k-state* for some $k \geq N$.

Suppose that in three passes of the *class*-1 filter, a k_1-state is followed by a k_2-state, and then by a k_3-state. Obviously, $k_1+k_2+k_3 \geq k_{01}+k_{02}+k_{03}+N-p$, if one of the k_{0i} is N or more and the other two are k_{0i} are at least 1. It was proven in [27] that $k_{01}+k_{02}+k_{03} \geq N+2$. And so $k_1+k_2+k_3 \geq 2N+2-p$.

As discussed above, the slowest possible convergence for any signal would occur when each end root region extends inwards by exactly $2N+2-p$ bits in every three consecutive passes of the filter. The signal converges to a root when the two end regions meet in the middle of the signal. After $m = 3k$ passes, the right-hand boundary of the left end root region is $1+(m/3)(2N+2-p)$, whereas the left boundary of the right end root region is $L-(m/3)(2N+2-p)$. The signal converges when the two end root regions meet, so that

$$L - \frac{m}{3}(2N+2-p) \leq 1 + \frac{m}{3}(2N+2-p) + 1 .$$

Solving the inequality, we have

$$m \geq \frac{3(L-2)}{2(2N+2-p)} .$$

In general, a signal of length L will always converge to a root in at most

$$3\left\lceil \frac{L-2}{2(2N+2-p)} \right\rceil$$

passes. □

Obviously, if $\mathbf{w}_i = 1$ for $i = 0, 1, \ldots, N$, then $p = N$, the previous equation becomes

$$3\left\lceil \frac{L-2}{2(N+2)} \right\rceil \tag{19}$$

which is a form for the median filter [27].

Example 4.1. Consider a *class*-1 filter $f = \{1, 2, 3, 7, 3, 2, 1\}$; obviously the *feature value* $p = 2$. Let $L = 13$ and input signal $\mathbf{X} = 0101110001110$. According to Theorem 4.1, the signal will converge to a root in at most two passes of f. After filtering one time, we have the root signal $\mathbf{Y} = 0001110001110$.

5. Conclusions

In this paper, we have studied the root structures and the convergence behavior of a subclass of WM filters. We have defined an important parameter, called the *feature value*, and have shown that any one-dimensional unappended signal will converge to a root signal in at most

$$3\left\lceil \frac{L-2}{2(2N+2-p)} \right\rceil$$

passes of a *class*-1 filter with window width $2N+1$.

The weights of a WM filter can be selected in various ways; thus, the root structures and convergence behavior of WM filters, even for b-WM filters, are much more complex. Some theoretical analysis of the root structures and convergence behavior of WM filters has to be done in the future.

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