

Stack Filters and Free Distributive Lattices

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ABSTRACT — Stack filters are a class of discrete-time, nonlinear filters which are defined in terms of positive Boolean functions. Positive Boolean functions of n variables form a partially ordered set which is called a Free Distributive Lattice on n generators, and is denoted by $FDL(n)$. In this paper, we investigate several properties of the class of stack filters by examining $FDL(n)$. The results obtained include: the exact number of stack filters of n variables for $n = 2$ through 7; the number of stack filters with exactly k minterms for all possible values of k , for stack filters of 7 variables; and, an accurate estimate of the number of stack filters of n variables for arbitrary even n . This last result is of particular interest since the estimate compares very well with others currently available, but is much more easily derived.

1 Introduction and Definitions

Let $\tilde{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\tilde{\beta} = (\beta_1, \dots, \beta_n)$ be n -element binary vectors (primes). We say that $\tilde{\alpha}$ precedes $\tilde{\beta}$, denoted $\tilde{\alpha} \preceq \tilde{\beta}$, if $\alpha_i \leq \beta_i$ for $1 \leq i \leq n$. If $\tilde{\alpha} \neq \tilde{\beta}$, we write $\tilde{\alpha} \prec \tilde{\beta}$. The set of such vectors is a partially ordered set with the above-mentioned ordering relation. A boolean function $f(x_1, \dots, x_n)$ is called *monotone* (sometimes called *positive*) if for any $\tilde{\alpha}$ and $\tilde{\beta}$ such that $\tilde{\alpha} \prec \tilde{\beta}$, we have $f(\tilde{\alpha}) \leq f(\tilde{\beta})$.

The set of all monotone, i.e. positive, boolean functions of n variables is also ordered by the same relation and is called the Free Distributive Lattice on n generators, denoted by $FDL(n)$. It consists of all closed from below subsets of the Boolean algebra on n elements, E^n , where E^n is the set of all subsets of an n -element set. A familiar definition of E^n is the n -cube and the primes are its vertices. A closed from below subset X is one which has the following property: if $A \in X$ and B is a subset of A , then $B \in X$. This property is nothing more than the monotonicity of the boolean functions, and since elements of $FDL(n)$ are subsets, they are naturally ordered. The set

E^n splits up into $n+1$ levels: $0^{th}, 1^{st}, \dots, n^{th}$. The k^{th} level ($0 \leq k \leq n$) contains only those primes which consist of exactly k components equal to 1. The set of all primes on the k^{th} level will be denoted by $E^{n,k}$.

It is well known that a monotone boolean function is uniquely defined by its set of *minimal primes*, where a prime $\tilde{\alpha} \in E^n$ is *minimal* if $f(\tilde{\alpha}) = 1$ and $f(\tilde{\beta}) = 0$ for all $\tilde{\beta} \in E^n$ such that $\tilde{\beta} \prec \tilde{\alpha}$ [1]. Let $\phi_n(k)$ be the number of n -variable monotone boolean functions with k minimal primes. Also, a prime $\tilde{\alpha}$ will be called *free* if for all primes $\tilde{\beta} \prec \tilde{\alpha}$, $f(\tilde{\beta}) = 0$ and for all primes $\tilde{\gamma}$ such that $\tilde{\alpha} \prec \tilde{\gamma}$, $f(\tilde{\gamma}) = 1$. The reason it is called free is because $f(\tilde{\alpha})$ can either be 0 or 1. Naturally, for a prime to be minimal, it must be free.

Let $\tilde{\alpha} \in E^{n,k}$ be an arbitrary prime. Then the set of all primes $\tilde{\beta} \in E^{n,s}$, $s > k$, such that $\tilde{\alpha} \prec \tilde{\beta}$, will be called the *shadow* of $\tilde{\alpha}$, and denoted by $T(\tilde{\alpha})$. If $\tilde{\alpha}$ happens to be a minimal prime, then a monotone boolean function f takes on the value 1 on all $\tilde{\beta} \in T(\tilde{\alpha})$.

A monotone boolean function f is on the m^{th} level of $FDL(n)$ if it selects m primes, that is, there exist exactly m primes on which $f = 1$. Let the number of monotone boolean functions on the m^{th} level of $FDL(n)$ be denoted by $\varphi_n(m)$. There are 2^{n-1} levels of $FDL(n)$ and the maximum number of minimal primes of a function of n variables is $\binom{n}{n/2}$ [3].

Monotone boolean functions have been extensively studied in the area of non-linear digital filtering, specifically stack and morphological filtering. In fact, any Stack Filter of window-width n is uniquely specified by a monotone boolean function of n variables. Similarly, the *Stacking Property* obeyed by all stack filters is the monotonicity of the monotone boolean functions [4].

We can conclude, therefore, that the enumeration of the number of monotone boolean functions

is equivalent to the enumeration of the elements of the free distributive lattice, which in turn is equivalent to enumeration of the set of stack filters. Let $\Psi(n)$ denote the number of elements of $FDL(n)$. Clearly,

$$\Psi(n) = \sum_{m=1}^{2^n-1} \varphi_n(m) = \sum_{k=1}^{\binom{n}{n/2}} \phi_n(k)$$

2 Enumeration of $FDL(n)$

The problem of determining $\Psi(n)$ was proposed by Dedekind in 1897, who determined it for $n \leq 4$. $\Psi(5) = 7,581$ was obtained by R. Church in 1940 and $\Psi(6) = 7,828,352$ by M. Ward in 1946. A result for $\Psi(7) = 2,414,682,040,998$ was reported by Church in 1965 [5], after which a different result by F. Lunnon (1971) [2] was published. It was not until 1976 that Berman & Köhler [6] confirmed that Church's result was the correct one. The reason for this difficulty is the computational complexity encountered in calculating $\Psi(n)$. In fact, only in 1991, was

$$\Psi(8) = 56,130,437,228,687,557,907,788$$

computed by D. Wiedemann [7], using about 200 hours of CPU time on a Cray-2. $\Psi(n)$ for $n \geq 9$ is unknown, and will probably remain so for the near future because of the extreme growth rate of $\Psi(n)$. For this reason, much work has gone into asymptotic enumeration of $FDL(n)$.

2.1 Asymptotic Enumeration of $FDL(n)$

Ward, Gilbert, Kleitman, and many others have derived bounds and asymptotics for $\Psi(n)$, with the most accurate asymptotic given by A. D. Korshunov (1981). In his paper, he states that for even $n \rightarrow \infty$:

$$\Psi(n) \sim 2^{\binom{n}{n/2}} \exp\left(\left(\frac{n}{2}-1\right) \cdot \left(\frac{1}{2^{n/2}} + \frac{n^2}{2^{n+3}} - \frac{n}{2^{n+4}}\right)\right) \quad (1)$$

and for odd $n \rightarrow \infty$:

$$\Psi(n) \sim 2 \cdot 2^{\binom{n}{(n-1)/2}} \exp\left(\left(\frac{n}{2}\right) \cdot \left(\frac{1}{2^{(n+3)/2}} - \frac{n^2}{2^{n+6}} - \frac{n}{2^{n+3}}\right) + \binom{n}{(n-1)/2} \left(\frac{1}{2^{(n+1)/2}} + \frac{n^2}{2^{n+4}}\right)\right) \quad (2)$$

These results are obtained using combinatorial analysis and the theory of *shadow sets* [1]. We

have derived a similar asymptotic based on Korshunov's results about shadow sets and have developed a statistical description of the elements of the ordered set E^n . We shall hereafter focus on the development of the asymptotic for even n . Korshunov shows in his article that almost all monotone boolean functions of n variables have minimal primes located in $E^{n,n/2-1}$, $E^{n,n/2}$, and $E^{n,n/2+1}$ for even n [1]. Therefore, as n becomes large, the number of functions with minimal primes on those three levels of E^n approaches the total number of monotone boolean functions. In view of this fact, we may assume from now on that all functions f have this property. If we restrict our attention to only those functions which contain their minimal primes on those three levels, then by monotonicity, they must take on the value 1 on all primes in $E^{n,k}$ for $k > n/2 + 1$ and the value 0 on all primes in $E^{n,k}$ for $k < n/2 - 1$.

The number of primes on $E^{n,n/2}$ is $\binom{n}{n/2}$. If a function f is allowed to take on any value on any of these primes, there are $2^{\binom{n}{n/2}}$ possibilities or *configurations*. The probability that any prime on $E^{n,n/2+1}$ is free is equal to the probability that the function takes on the value 0 on all of the primes preceding it, by definition. Any prime on $E^{n,n/2+1}$ has exactly $n/2 + 1$ primes preceding it. Also, suppose a function has equal probability of taking on 1 or 0 on any prime on $E^{n,n/2}$. Then, the probability that a prime on $E^{n,n/2+1}$ is free is just $(\frac{1}{2})^{n/2+1}$ and is distributed as a Bernoulli random variable. This, of course, is equal to the probability that a prime on $E^{n,n/2-1}$ is free, since it precedes exactly $n/2 + 1$ primes.

The total number of primes on $E^{n,n/2-1}$ and $E^{n,n/2+1}$ is

$$\binom{n}{n/2-1} + \binom{n}{n/2+1} = 2 \binom{n}{n/2+1} \quad (3)$$

Therefore, given a particular configuration on $E^{n,n/2}$, the number of free primes on $E^{n,n/2-1}$ and $E^{n,n/2+1}$ is distributed as

$$\text{Binomial} \left(2 \binom{n}{n/2+1}, \left(\frac{1}{2}\right)^{n/2+1} \right) \quad (4)$$

When p is small, a *Binomial*(n, p) random variable is well approximated by a *Poisson* random variable with parameter $\lambda = np$. For large n , $(\frac{1}{2})^{n/2+1}$ is small and the above *Binomial* random variable is approximately

$$\text{Poisson} \left(\lambda = 2 \binom{n}{n/2+1} \left(\frac{1}{2}\right)^{n/2+1} \right) \quad (5)$$

Therefore, if K is the number of free primes on $E^{n,n/2-1}$ and $E^{n,n/2+1}$, then

$$K \sim \text{Poisson} \left(\lambda = \binom{n}{n/2+1} \left(\frac{1}{2} \right)^{n/2} \right) \quad (6)$$

The number of monotone boolean functions with a particular configuration on $E^{n,n/2}$ is clearly 2^K , where K is distributed as above. So, if we could sum over all possible configurations on $E^{n,n/2}$, we would have an estimate for $\Psi(n)$. Unfortunately, doing that would be practically equivalent to calculating the exact number of monotone boolean functions. The next best thing is to compute the average number of functions with a particular configuration and multiply by the total number of configurations on $E^{n,n/2}$. Using a moment generating function, we can find

$$E[2^K] = \exp \left(\binom{n}{\frac{n}{2}+1} \left(\frac{1}{2} \right)^{n/2} \right) \quad (7)$$

Using the fact that there are $2^{\binom{n}{n/2}}$ configurations on $E^{n,n/2}$,

$$\Psi(n) \sim 2^{\binom{n}{n/2}} \exp \left(\binom{n}{\frac{n}{2}-1} \left(\frac{1}{2^{n/2}} \right) \right) \quad (8)$$

which is just equation 1 save the last two terms. One interesting aspect of this new asymptotic expression is the technique used to obtain it; namely, the statistical, as opposed to combinatorial, approach.

2.2 Exact Enumeration and Generation of $FDL(n)$

The algorithm used for exact enumeration of $FDL(n)$ is well documented in [2] despite his erroneous results. Essentially, it is necessary to have $FDL(n-2)$ generated in order to enumerate $FDL(n)$. It should be noted that it is possible to use $FDL(k)$, $1 < k < n$, to enumerate $FDL(n)$, but $k = n-2$ turns out to be most useful (efficient) [8]. The distinction here between generating and enumerating is important. When we generate a lattice, we produce each element of it, thus storing every monotone boolean function in the case of the free distributive lattice. When we enumerate a lattice, we simply count the number of elements it contains without actually producing the elements themselves. So, in order to determine $\Psi(9)$, it is actually necessary to have $FDL(7)$ stored. The generation of

$FDL(7)$ proved to be a difficult task, demanding great amounts of computational power as well as storage space. However, the fact that the problem is naturally parallel came to the rescue.

As defined before, $\phi_n(k)$ is the set of functions with k minimal primes. This corresponds to a stack filter with k minterms. Specifically, $\phi_n(1)$ is the set of stack filters with one minterm, which are just the minimum rank-order filters. Then, $\phi_n(k+1)$ can be generated by taking all $f \in \phi_n(k)$ and forming disjunctions (OR) with all $f \in \phi_n(1)$. However, it may be possible that the disjunction will not produce a new function. This is known as *subsumption*. The best way to see this is to consider E^n . Suppose we have a function with k minimal primes defined. It thus takes on the value 1 at those primes. By monotonicity, it takes on the value 1 on all the primes in the shadows of the minimal primes (the shadows need not be disjoint). Now, if we pick an arbitrary prime and decide to make it a minimal prime, we must check to see that it is not in the shadow of any minimal prime. If it is in some shadow, then it gets *subsumed* by that minimal prime. Consequently, we do not produce a unique monotone boolean function with $k+1$ minimal primes.

A recursive implementation is used to generate $\phi_n(k+1)$ from $\phi_n(k)$. We take a function $f \in \phi_n(k)$ and form a disjunction with a function $g \in \phi_n(1)$. If a subsumption does not occur, we obtain a new function belonging to $\phi_n(k+1)$ and proceed recursively using the newly generated function. If a subsumption does occur, we try the next $g \in \phi_n(1)$. Due to the independence of the branches of this recursion tree, we are able to execute the algorithm, branch by branch, in parallel. So, it is possible to have several (many) computers running the algorithm with no communication between them whatsoever.

One of the benefits of generating $FDL(7)$, besides storing it for purposes of enumerating $FDL(9)$, is the knowledge of $\phi_7(k)$, $k = 1, \dots, 35$ as well as $\varphi_7(m)$, $m = 1, \dots, 127$. In Table 1, we provide exact numbers for $\phi_7(k)$, $k = 1, \dots, 35$.

3 Conclusions

We have generated the free distributive lattice on 7 generators and as a result obtained exact numbers for the number of monotone boolean functions with k minimal primes. In addition, we provide a statistical interpretation of a monotone boolean function of n variables, for even n . This leads to an asymptotic formula for the size of the free distributive lattice on n generators, for even n . This interpretation also paves the way

$\phi_7(1) = 129$	$\phi_7(19) = 110689090540$
$\phi_7(2) = 6069$	$\phi_7(20) = 57514989406$
$\phi_7(3) = 153762$	$\phi_7(21) = 27233103758$
$\phi_7(4) = 2401910$	$\phi_7(22) = 11961687479$
$\phi_7(5) = 25400564$	$\phi_7(23) = 4908950690$
$\phi_7(6) = 192504214$	$\phi_7(24) = 1877475320$
$\phi_7(7) = 1085660748$	$\phi_7(25) = 663481350$
$\phi_7(8) = 4693213105$	$\phi_7(26) = 214126234$
$\phi_7(9) = 15946757960$	$\phi_7(27) = 62245330$
$\phi_7(10) = 43506231489$	$\phi_7(28) = 16038865$
$\phi_7(11) = 96986433726$	$\phi_7(29) = 3593660$
$\phi_7(12) = 179039339787$	$\phi_7(30) = 683543$
$\phi_7(13) = 276189432540$	$\phi_7(31) = 106932$
$\phi_7(14) = 357959527245$	$\phi_7(32) = 13160$
$\phi_7(15) = 390972758470$	$\phi_7(33) = 1190$
$\phi_7(16) = 360792368937$	$\phi_7(34) = 70$
$\phi_7(17) = 282573842684$	$\phi_7(35) = 2$
$\phi_7(18) = 189570420130$	

Table 1: $\phi_7(k)$

toward determining a statistical distribution of $\phi_n(k)$, the number of functions with k minimal primes as well as $\varphi_n(m)$, the number of elements on the m^{th} level of the free distributive lattice, which we conjecture to be Normal.

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