Teager Energy and the Ambiguity Function

R. Hamila, J. Astola, F. Alaya Cheikh, M. Gabbouj, and M. Renfors

***Abstract*—The connection between the Teager energy operator and the ambiguity function is established in this correspondence after defining the Teager operator over complex-valued signals. This relation allows the use of the Teager energy in estimating the second moment angular bandwidth and the moments of a signal duration (spread) and that of its spectrum.**

I. Introduction

The Teager energy (TE) operator has been used in a number of applications since its introduction in [1] and [2], including one-dimensional (1-D) signal processing (mainly speech signal processing [3]), image processing [4], and color image processing [5]–[7]. Maragos *et al.* [3] referred to this as the nonlinear energy-tracking signal operator and used it to estimate the amplitude envelope of an AM signal and the instantaneous frequency of an FM signal.

In this correspondence, we first extend the definition of the Teager operator over complex-valued signals [8] and then establish the connection between this operator and the ambiguity function (AF) [9]. The ambiguity function is a time–frequency correlation function that is useful in many signal communication systems, especially radar signals. Physically, the ambiguity function represents the energy in a received signal as a function of time delay and Doppler frequency [10]. This function describes the local ambiguity in locating targets in range (time delay) and in velocity (Doppler frequency).

Using the above relation, the TE operator can then be used to estimate the second moment angular bandwidth of a signal and the moments of a signal duration and that of its spectrum.

This correspondence is organized as follows. Section II defines the Teager energy operator for complex signals and the ambiguity function. The connection between the Teager energy operator and the ambiguity function is established in Section III. The Teager energy operator is then used to estimate the second moment angular bandwidth in Section IV and the moments of a signal duration and that of its spectrum in Section V, respectively. Conclusions are drawn in Section VI.

II. Teager Energy and Ambiguity Function: Definitions

A. TE Operator

As mentioned in the Introduction, the TE operator was first defined over real-valued signals. Maragos and Bovik [8] later defined a TE operator for multidimensional continuous-domain signals and used it for image demodulation. In the following, a similar approach defining this operator over complex-valued signals and deriving some of its important properties is presented.

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Definition: The TE operator of a complex-valued signal $x(t)$ is defined

$$\Psi_C[x(t)] = \dot{x}(t)\dot{x}^*(t) - \frac{1}{2}[\ddot{x}(t)x^*(t) + x(t)\ddot{x}^*(t)]. \quad (1)$$

When $x(t)$ is real, (1) reduces to the TE of a real-valued signal, which is defined in [1] as

$$\Psi_R[x(t)] = \dot{x}^2(t) - x(t)\ddot{x}(t). \quad (2)$$

Furthermore, writing the complex signal $x(t)$ as a function of its real and imaginary parts $x(t) = x_r(t) + jx_i(t)$, then applying the complex TE operator of (1), we obtain

$$\begin{aligned}\Psi_C[x(t)] &= \Psi_C[x_r(t) + jx_i(t)] \\ &= \dot{x}_r^2(t) + \dot{x}_i^2(t) - x_r(t)\ddot{x}_r(t) - x_i(t)\ddot{x}_i(t). \quad (3)\end{aligned}$$

Hence, the TE of a complex signal is equal to the sum of the Teager energies of its real and imaginary parts

$$\Psi_C[x(t)] = \Psi_R[x_r(t)] + \Psi_R[x_i(t)]. \quad (4)$$

Maragos and Bovik [8] proposed the following TE operator for complex-valued signals:

$$\Psi_C[x(t)] = \|\dot{x}(t)\|^2 - \mathrm{Re}[x^*(t)\ddot{x}(t)]. \quad (5)$$

Substituting the complex-valued signal $x(t)$ by its real and imaginary parts in (5), it can easily be show that (4) holds also for Maragos and Bovik's definition, although ours exhibits the symmetry of the operator more clearly. Note that both definitions yield a real quantity, as expected for an energy operator.

B. Ambiguity Function

The ambiguity function of a signal $x(t)$ is the integral [9, ch. 8]

$$\Xi(u,\tau) = \frac{1}{E}\int_{-\infty}^{+\infty} x\Big(t+\frac{\tau}{2}\Big)x^*\Big(t-\frac{\tau}{2}\Big)e^{-jut}\,dt \quad (6)$$

where E is the energy of the signal $x(t)$ (assumed to be finite)

$$E = \int_{-\infty}^{+\infty} |x(t)|^2\,dt < \infty. \quad (7)$$

A variant of the above definition, which can be found in the literature, is to normalize the ambiguity function by the signal energy. The results obtained in the sequel hold for this case as well by setting $E = 1$.

From the definition, it is clear that $\Xi(u,\tau)$ is the Fourier transform with respect to the variable t of the function

$$\gamma(t,\tau) = \frac{1}{E}x\Big(t+\frac{\tau}{2}\Big)x^*\Big(t-\frac{\tau}{2}\Big). \quad (8)$$

Denote by $\Gamma(u,v)$ the Fourier transform of $\Xi(u,\tau)$ with respect to τ

$$\gamma(t,\tau) \overset{t}{\longleftrightarrow} \Xi(u,\tau) \overset{\tau}{\longleftrightarrow} \mathbf{\Gamma}(u,v). \quad (9)$$

Thus, $\Gamma(u,v)$ represents the 2-D Fourier transform of $\gamma(t,\tau)$.

$$\gamma(t,\tau) \Longleftrightarrow \Gamma(u,v) \quad (10)$$

III. Connection Between TE and the Ambiguity Function

The Fourier transform $\mathbf{\Gamma}(u,v)$ of the ambiguity function can be related to the signal spectrum as

$$\Gamma(u,v) = \frac{1}{E}\mathcal{X}\Big(v+\frac{u}{2}\Big)\mathcal{X}^*\Big(v-\frac{u}{2}\Big) \tag{11}$$

where $\mathcal{X}(v)$ is the Fourier transform of $x(t)$.

Similarly, the TE operator $\Psi_C[x(t)]$ can be related to the signal spectrum as follows. Using (10), we can easily derive

$$\frac{\partial^2\gamma(t,\tau)}{\partial\tau^2} \Longleftrightarrow (jv)^2\Gamma(u,v) \tag{12}$$

or it could be simply expressed by

$$\frac{\partial^2\gamma(t,\tau)}{\partial\tau^2} = \left(\frac{1}{4\pi^2}\right)\iint_{-\infty}^{+\infty}(jv)^2\Gamma(u,v)e^{j(ut+v\tau)}\,du\,dv. \tag{13}$$

It is easy to relate $\Psi_C[x(t)]$ to $\gamma(t,\tau)$ in (8) as follows.

Differentiating both sides of (8) twice with respect to τ yields

$$\frac{\partial^2\gamma(t,\tau)}{\partial\tau^2} = \frac{1}{4E}\Big[\ddot{x}\Big(t+\frac{\tau}{2}\Big)x^*\Big(t-\frac{\tau}{2}\Big)+x\Big(t+\frac{\tau}{2}\Big) \times \ddot{x}^*\Big(t-\frac{\tau}{2}\Big)-2\dot{x}\Big(t+\frac{\tau}{2}\Big)\dot{x}^*\Big(t-\frac{\tau}{2}\Big)\Big].$$

Therefore

$$\Psi_C[x(t)] = -2E\frac{\partial^2\gamma(t,\tau)}{\partial\tau^2}\bigg|_{\tau=0} \tag{14}$$

$$= -2E\left(\frac{1}{4\pi^2}\right)\iint_{-\infty}^{+\infty}(jv)^2\mathbf{\Gamma}(u,v)e^{jut}\,du\,dv. \tag{15}$$

Substituting (11) for $\Gamma(u,v)$ in (15), we obtain

$$\Psi_C[x(t)] = \frac{1}{2\pi^2}\iint_{-\infty}^{+\infty}v^2\mathcal{X}\Big(v+\frac{u}{2}\Big) \times \mathcal{X}^*\Big(v-\frac{u}{2}\Big)e^{jut}\,du\,dv. \tag{16}$$

Important Remark: Examining (16) reveals an important link between the TE and the Wigner distribution of a signal (this observation is credited to one of the anonymous reviewers). In fact, it shows that the product of the second-order conditional moment in frequency of the Wigner distribution with the time marginal is proportional to the TE of a signal (the proportionality factor is π). Furthermore, this link helps explain the curious results obtained via TE (negative "energy") and the negative conditional second moment via the Wigner distribution. Besides, this relation suggests possible generalization of the TE operator by utilizing different characteristic functions corresponding to different time–frequency distributions.

Observe from (16) that $\Psi_C[x(t)]$ is the inverse Fourier transform of

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}v^2\mathcal{X}\Big(v+\frac{u}{2}\Big)\mathcal{X}^*\Big(v-\frac{u}{2}\Big)\,dv$$

and thus

$$\mathcal{F}\{\Psi_C[x(t)]\}(u) = \frac{1}{\pi}\int_{-\infty}^{+\infty}v^2\mathcal{X}\Big(v+\frac{u}{2}\Big)\mathcal{X}^*\Big(v-\frac{u}{2}\Big)\,dv. \tag{17}$$

Now, the ambiguity function results from taking the inverse Fourier transform of $\Gamma(u,v)$ with respect to τ [see (9) and (11)]

$$\Xi(u,\tau) = \frac{1}{2\pi E}\int_{-\infty}^{+\infty}\mathcal{X}\Big(v+\frac{u}{2}\Big)\mathcal{X}^*\Big(v-\frac{u}{2}\Big)e^{jv\tau}\,dv. \tag{18}$$

Equation (18) above relates the ambiguity function to the signal spectrum. In fact, $\Xi(u,\tau)$ represents the ambiguity function of the signal spectrum if the variables u and τ are interchanged.

Differentiating (18) twice with respect to τ yields

$$\frac{\partial^2\Xi(u,\tau)}{\partial\tau^2} = -\frac{1}{2\pi E}\int_{-\infty}^{+\infty}v^2\mathcal{X}\Big(v+\frac{u}{2}\Big) \times \mathcal{X}^*\Big(v-\frac{u}{2}\Big)e^{jv\tau}\,dv. \tag{19}$$

Finally, (17) and (19) give the explicit relation between the Teager energy operator and the ambiguity function as

$$\mathcal{F}\{\Psi_C[x(t)]\}(u) = -2E\frac{\partial^2\Xi(u,\tau)}{\partial\tau^2}\bigg|_{\tau=0}. \tag{20}$$

The following sections show the application of relation (20) to the estimation of the second moment angular bandwidth and the duration of a signal.

IV. Estimation of the Second Moment Angular Bandwidth

The use of the TE operator in estimating the second moment angular bandwidth of a signal is developed next. The relation of the second moment angular bandwidth to the time delay estimation is derived in [11].

In general, the second moment angular bandwidth of a signal [11] is defined in terms of the second partial derivative of the modulus of the AF as

$$\left[\frac{\partial^2|\Xi|}{\partial\tau^2}\right]_{\substack{u=0\\ \tau=0}} = -\frac{1}{2\pi}\int_{-\infty}^{+\infty}v^2|\mathcal{X}(v)|^2\,dv + \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty}v|\mathcal{X}(v)|^2\,dv\right)^2. \tag{21}$$

The above equation is interpreted as the negative bandwidth of a signal or the negative spectral variance.

Given

$$\overline{\omega} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}v|\mathcal{X}(v)|^2\,dv$$

$\overline{\omega}$ is called the angular frequency centroid of the signal [11], and using (20) and (21), the second moment angular bandwidth of a signal is related to the TE as

$$\left[\frac{\partial^2|\Xi|}{\partial\tau^2}\right]_{\substack{u=0\\ \tau=0}} = -\frac{1}{2}\mathcal{F}\{\Psi_C[x(t)]\}(u)|_{u=0}+\bar{\omega}^2. \tag{22}$$

V. Moments of a Signal Duration

Measures of the durations of a signal $x(t)$ and that of its FT $\mathcal{X}(\omega)$ are determined by the moments [3, ch. 8]

$$w = \frac{1}{E}\int_{-\infty}^{+\infty}t^2|x(t)|^2\,dt \tag{23}$$

$$W = \frac{1}{2\pi E}\int_{-\infty}^{+\infty}v^2|\mathcal{X}(v)|^2\,dv. \tag{24}$$

Evaluating (17) above at $u=0$ yields

$$W = \frac{1}{2E}\mathcal{F}\{\Psi_C[x(t)]\}(u)|_{u=0}. \tag{25}$$

As for the signal duration w, (23), can be obtained by differentiating the AF [see (6)] twice with respect to u as

$$\left[\frac{\partial^2\Xi(u,\tau)}{\partial u^2}\right]_{\substack{u=0\\ \tau=0}} = \frac{1}{E}\int_{-\infty}^{+\infty}t^2|x(t)|^2\,dt. \tag{26}$$

Using (18) and (26), the signal duration is next shown to be related to the TE operator of the signal spectrum.

Differentiating (18) twice with respect to u, we get

$$\left[\frac{\partial^2 \Xi(u,\tau)}{\partial u^2}\right]_{\substack{u=0\\ \tau=0}} = \frac{\partial^2}{\partial u^2}\left[\frac{1}{2\pi E}\int_{-\infty}^{+\infty} \mathcal{X}\Big(v+\frac{u}{2}\Big)\mathcal{X}^*\Big(v-\frac{u}{2}\Big)e^{jv\tau}\,dv\right]\Bigg|_{\substack{u=0\\ \tau=0}} \quad (27)$$

$$= \frac{1}{2\pi E}\int_{-\infty}^{+\infty} \frac{\partial^2}{\partial u^2}\Big[\mathcal{X}\Big(v+\frac{u}{2}\Big)\,\mathcal{X}^*\Big(v-\frac{u}{2}\Big)\Big]\Big|_{u=0}\;dv. \quad (28)$$

Note that the integrand is just the TE operator of the signal spectrum, resulting in

$$w = \frac{-1}{4\pi E}\int_{-\infty}^{+\infty} \Psi_C[\mathcal{X}(v)]\,dv. \quad (29)$$

VI. Conclusion

In this correspondence, we defined the TE operator for complex-valued signals and established the link between this TE operator and the ambiguity function. This relation allows the use of the TE operator to estimate the second moment angular bandwidth and the moments of a signal duration, as well as that of its spectrum.

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Estimating Two-Dimensional Frequencies by a Cumulant-Based FBLP Method

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***Abstract*—This correspondence presents a higher order statistics (HOS) based modified version of the two-dimensional forward-backward linear prediction (2-D FBLP) method. The modified method is suitable for estimating 2-D frequencies of a 2-D data set observed in white or colored Gaussian noise (W/CGN) of unknown covariance. Simulation results are provided to verify the effectiveness of the proposed method.**

I. Introduction

Estimating two-dimensional (2-D) frequencies from a 2-D data set is of interest in fields such as sonar, radar, and geophysics. In general, estimate of the 2-D frequencies are obtained by determining the locations of the spectral peaks in a 2-D space. Spectral estimation using the correlogram method, implemented via 2-D FFT, can be satisfactory if the data set is very large. If the data set is relatively small, the correlogram method is ineffective because of the resolution limit. To overcome the resolution limit problem, high-resolution techniques such as the 2-D maximum likelihood (2-D ML) method and the 2-D maximum entropy (2-D ME) have been developed from their 1-D versions. However, the 2-D ML and the 2-D ME methods each require a solution of a nonlinear optimization problem [1], [9]. The 2-D FBLP method for frequency estimation problem has been discussed in [3]. Although the 2-D FBLP method is a high-resolution method and has been applied to multiple 2-D sinusoids observed in Gaussian noise of unknown covariance, it is limited to the WGN case with a relatively high signal-to-noise ratio (SNR). In [2], an efficient method for estimating 2-D frequencies has been described. However, this method only applies to the case of single 2-D sinusoid observed in WGN.

In [6] and [7], estimation of 1-D harmonic frequencies via an HOS-based approach has been successfully addressed for respectively stochastic and deterministic harmonics buried in W/CGN of unknown covariance.

This correspondence presents a HOS-based version of the 2-D FBLP method. HOS has its own advantages in being tolerant to W/CGN. Therefore, the proposed method, termed the 2-D cumulant-based FBLP (2-D CBFBLP) method, is applicable to a 2-D data set observed in W/CGN. Section II reviews the 2-D FBLP method. In Section III, the proposed 2-D CBFBLP method is discussed. In Section IV, simulation results that concern the case of colored 2-D Gaussian noise are included to illustrate the effectiveness of the 2-D CBFBLP method. Finally, Section V conclues the correspondence.

II. Review of the 2-D FBLP Method

We assume that the noise free model for the 2-D frequency estimation problem has the following structure:

$$x(m,n) = \sum_{i=1}^{L} \alpha_i \,\mathrm{Exp}[j2\pi(f_{1i}m + f_{2i}n) + j\varphi_i] \quad (1)$$

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