

OHJ-2506 Program Verification: Exercises

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September 9, 2011

Accomplishment

- Exercises are obligatory and probably the grades will be determined by the exercises.
- That will be decided during the first lectures.
- The course is valid for post-graduate studies.

Exercises are given on the following way

- In the beginning of an exercise the instructor calls the names and asks who have done which problems, and selects the presenter.
- The precedence is usually given to him who have least presentations, so far. But there can be exceptions.
- If, while presentation, there reveals error(s) in the solution, then
 1. the presenter may try to fix it himself, with the instructors help if necessary,
 2. the problem is given to the presenter as a personal task for the next time, or
 3. another student can be asked to present his solution.
- Be prepared to present your answers. The grade is (or may be) given by the exercises. It is better to solve fewer problems carefully than many “poorly”.
- Although the grade is not determined by the number of the problems you have solved, but it is not possible to show what you have learned if you solve only few problems.

Exercise group 1

Preparatory problems

The idea is to show in which kind of programming problems the program verification techniques are useful. We are going to redo some of them after we have learned new means.

1. The following implementation of quicksort algorithm is used as an example in a C++ textbook:

```

1      void qsort (int *ia, int low, int high) {
2          if (low < high ) {
3              int lo = low;
4              int hi = high + 1;
5              int elem = ia [low]; // pivot
6              for (;;) {
7                  while (ia [++lo] < elem);
8                  while (ia [--hi] > elem);
9                  if (lo < hi)
10                     swap (ia, lo, hi);
11                     else break;
12             }
13             swap (ia, low, hi);
14             qsort (ia, low, hi - 1);
15             qsort (ia, hi + 1, high);
16         } // end, if (low < high)
17     }

```

- (a) The implementation has an error that the author seems not to have noticed. What?
 - (b) Estimate verbally the possibility to find the error by testing. Estimate also the possibility that the error never appears when the program is used.
2. Before beginning of the following binary search program for array $A[1 \dots n]$ it holds that $A[1] \leq A[2] \leq \dots \leq A[n]$. The program tries to find the greatest a such that $A[i] < key$ whenever $1 \leq i < a$.

```

1      a := 1; y := n;
2      while a < y do
3          v := (a + y) div 2;
4          if A[v] < key then a := v + 1;
5          else y := v
6          endif
7      endwhile

```

- (a) Find the error.
- (b) Estimate the probability that the error appears in m tests or production runs, if $A[1], \dots, A[n]$ all are non-equal and key is one of them.

(c) Like (b), but now all $A[1], \dots, A[n]$, and key are non-equal.

3. Does the following program find the longest proper ascending sequence $B[1 \dots k]$ that is created from $A[1 \dots n]$ by removing elements? Reason or give a counterexample.

```

1      k := 0;
2      for i := 1 to n do
3          h := 1; C[1] := A[i];
4          for j := i + 1 to n do
5              if A[j] > C[h] then h := h + 1; C[h] := A[j]
6              endif
7          endfor
8          if h > k then
9              k := h;
10             for j := 1 to k do B[j] := C[j] endfor
11         endif
12     endfor

```

4. Is it possible to find out the *height of vertex* u (= the longest length of a path from it to a vertex without any child-vertices) by using the following algorithms for (a) a tree (b) a directed loopless graph (c) a directed graph? The starting edges of the vertex are in linked list *outedges*. In the beginning all vertices have $m = -1$. Give a reasoning or a counterexample.

```

1      function Height1(u) is
2      if u↑.m = -1 then
3          u↑.m := 0; a := u↑.outedges;
4          while a ≠ Nil do
5              x := Height1(a↑.head); a := a↑.next;
6              if x ≥ u↑.m then u↑.m := x + 1 endif
7          endwhile
8      endif
9      return u↑.m

```

```

1      function Height2(u) is
2      if u↑.m = -1 then
3          k := 0; a := u↑.outedges;
4          while a ≠ Nil do
5              x := Height2(a↑.head); a := a↑.next;
6              if x ≥ k then k := x + 1 endif
7          endwhile
8          u↑.m := k;
9      endif
10     return u↑.m

```

Repetition of propositional logics

5. Building a binary operator \star requires creating such an expression $l(P, Q)$ that $P \star Q \Leftrightarrow l(P, Q)$ for all combinations of truth values of P and Q . This definition generalises also for other operators than just for binary operators.

In principle \vee and \neg suffice for the basic operators of propositional logic, since other operators `False`, `True`, \wedge , \rightarrow , and \leftrightarrow can be build using them if we have at least one propositional symbol. E.g. $P \wedge Q \Leftrightarrow \neg(\neg P \vee \neg Q)$, and `True` $\Leftrightarrow P \vee \neg P$.

- (a) We have at least one propositional symbol, and operators \wedge and \neg . Show how operators `False`, `True`, \vee , \rightarrow , and \leftrightarrow can be created using them.
- (b) Sometimes – with propositional symbols – just one operator suffices to create the other operators. Let us denote by ∇ the operator that is defined as:

∇	False	True
False	True	False
True	False	False

Show how by using ∇ the common operators `False`, `True`, \wedge , \neg , \vee , \rightarrow , and \leftrightarrow can be created.

(Hint: think first how ∇ is created using the common operators and first create \neg .)

6. Let us denote by \triangle the operator that is defined as:

\triangle	False	True
False	True	True
True	True	False

How can you build ∇ using \triangle ? And how \triangle using ∇ ?

Some notifications considering this course:

- Use of truth tables is usually not strictly denied (unless when it **is** explicitly denied), but since in practical calculations they are quite useless or at least inefficient, it is strongly recommended to avoid them. (Of course you can use them for verifying your answers.)
- For truth values we use symbols `False` and `True`, which can be abbreviated as F and T , correspondingly. It is not allowed to use values 0 and 1 to express logical values since (1) even they are used in digital techniques, they denote *bits* not *logical values*, and (2) we do have need to use both integers and logical expressions in same formulas and therefore it is possible that values `False` and 0, and/or `True` and 1 exist in same formulas.
- Also, remember to write a sufficient amount of intermediate phases so that your solution is easy to follow.

Exercise group 2

7. Reduce the followings. Do not use truth tables.

- (a) $P \rightarrow P \rightarrow P \rightarrow P$
- (b) $P \rightarrow (P \rightarrow P) \rightarrow P$
- (c) $P \rightarrow ((P \rightarrow P) \rightarrow P)$
- (d) $P \rightarrow (P \rightarrow (P \rightarrow P))$
- (e) $(P \rightarrow P) \rightarrow (P \rightarrow P)$

8. Do the following claims hold or not? Reason your answers. Do not use truth tables.

- (a) $P \rightarrow (P \wedge Q) \Rightarrow P \rightarrow (P \vee Q)$.
- (b) $P \rightarrow (Q \wedge R) \Rightarrow P \rightarrow (Q \vee R)$.
- (c) $(P \wedge Q) \rightarrow R \Rightarrow (P \vee Q) \rightarrow R$.

Prove the following absorption laws. That is, without using themselves (and neither their correspondings for two propositional symbols). Do not use truth tables.

- (d) $P \wedge (P \vee Q \vee R) \Leftrightarrow P$.
- (e) $P \vee (P \wedge Q \wedge R) \Leftrightarrow P$.

9. Prove the followings. Do not use truth tables.

- (a) Constructive Dilemma $(P \rightarrow Q) \wedge (R \rightarrow S) \wedge (P \vee R) \Rightarrow Q \vee S$.
- (b) Destructive Dilemma $(P \rightarrow Q) \wedge (R \rightarrow S) \wedge (\neg Q \vee \neg S) \Rightarrow \neg P \vee \neg R$.
- (c) Bidirectional Dilemma $(P \rightarrow Q) \wedge (R \rightarrow S) \wedge (P \vee \neg S) \Rightarrow Q \vee \neg R$.

Introduction to problems 10–12: In a country there are two kind of people, *citizens* who always speak truth and *politicians* who always lie. We assume that everyone in that country is either citizen or politician, unless something else is told in the problem. Solve the following problems by (1) formalising the problem, (2) solving it using means of propositional logic, and (3) interpret your solution.

An example problem and its solution:

The problem: someone asks from a person A (who is either citizen or politician): “Are you citizen?”. Where he answers: “If I am citizen, I’ll eat my hat”. Prove that A must eat his hat.

Solution: we introduce the following abbreviations:

- X X is citizen (where X is A, B, C, ...)
- L The truth value of L, the sentence spoken by X.
- H A eats his hat.

Now, if X is citizen then the sentence spoken by him, L, is true, in other words, $X \rightarrow L$. Correspondingly, if X is politician, it holds that $\neg X$, and we know that L is not true, in other words, $\neg X \rightarrow \neg L$. These are also the only alternatives. In other words, the premisses of the problem are $X \rightarrow L$ and $\neg X \rightarrow \neg L$ which can be written in a shorter form $X \leftrightarrow L$. This means that X is citizen if and only if the sentence spoken by him is true.

Person A’s answer or sentence can be written in form $A \rightarrow H$, meaning if I am citizen (A), then I shall eat my hat (H). The problem can now be presented in a form where we start from the premiss $A \leftrightarrow (A \rightarrow H)$ and we must derive H using means of logic.

$$\begin{array}{ll}
 A \leftrightarrow (A \rightarrow H) & \Leftrightarrow \text{(remove implication)} \\
 A \leftrightarrow (\neg A \vee H) & \Leftrightarrow \text{(remove equivalence)} \\
 A \wedge (\neg A \vee H) \vee \neg A \wedge \neg(\neg A \vee H) & \Leftrightarrow \text{(distributive law and DeMorgan)} \\
 A \wedge \neg A \vee A \wedge H \vee \neg A \wedge \neg\neg A \wedge \neg H & \Leftrightarrow (2 \times \text{miscellaneous law}) \\
 \text{False} \vee A \wedge H \vee \text{False} \wedge \neg H & \Leftrightarrow \text{(commutativity) and “}\varphi \wedge \text{F} \Leftrightarrow \text{F”)} \\
 \text{False} \vee A \wedge H \vee \text{False} & \Leftrightarrow \text{(commutativity and } 2 \times \text{“}\varphi \vee \text{F} \Leftrightarrow \varphi\text{”)} \\
 A \wedge H & \Rightarrow \text{(miscellaneous law)} \\
 H &
 \end{array}$$

Especially elimination of operator \leftrightarrow can easily lead to long expressions. Often a problem can be solved easier by eliminating some variable, that is, by setting its value to True and False, in turn. The previous problem can be solved by elimination e.g. in the following way:

$$\text{If } A \Leftrightarrow \text{True then } A \leftrightarrow (A \rightarrow H) \Leftrightarrow \text{True} \leftrightarrow (\text{True} \rightarrow H) \Leftrightarrow \text{True} \leftrightarrow H \Leftrightarrow H.$$

$$\text{But, if } A \Leftrightarrow \text{False then } A \leftrightarrow (A \rightarrow H) \Leftrightarrow \text{False} \leftrightarrow (\text{False} \rightarrow H) \Leftrightarrow \text{False} \leftrightarrow \text{True} \Leftrightarrow \text{False}.$$

By setting True to A the value of H became True, but setting False to A causes a contradiction. Therefore the only possible solution is that both A and H are true. That implies H, in other words, “A eats his hat” is true.

10. We are now in the country of citizens and politicians. Formalize the following problems and solve them using means of propositional logic.
- I met two people, A and B, of that country. B said that if A is politician then also he is. What type A and B are?
 - X and Y are prosecuted for involvement of a robbery. A and B saw the case. A says that if X is guilty then also Y is. B claims that **either** X is not guilty **or** Y is guilty (notice the exclusive or!). Are A and B same type? (Are they both either citizens or politicians.)
 - Once again, there had been a robbery. A, B, and C were brought to police station for interrogation. The police found out that
 - no one else than A, B, or C has been involved in the case,
 - A will never do a heist alone, and
 - that C is not guilty
 Is B guilty or not guilty?

11. We move to the next country where half of its people are citizens and half are politicians. Also in this country the citizens tell always true and the politicians always lie. But in this country half of the people are crazy having delusions and therefore they believe that the true sentences are untrue and the false sentences are true. Instead of that half of the citizens are sane and they know which claims are true and which are untrue.

Therefore the people of this country are of four different types:

- sane citizens* who speak always true because they know what is true and what is not,
- crazy citizens* who always lie but because they believe that lies are true,
- sane politicians* who always lie even they know what is true and what is untrue, and
- crazy politicians* who speak always true but because they believe that they are lying.

Let us investigate a person X of that other country.

- Let us denote that *he (=X) is sane* by $S(X)$, and *he (=X) is citizen* by $C(X)$ (if we have only one person we can use only symbols S and C). Formalize the four types of people of that country.
 - Let us denote (*the truth value of*) *the sentence that X says* by $L(X)$. Formalize the problem: "Person X says sentence L. What type he is?" (C.f the introduction for citizen/politician-problems.)
 - Formalize the following problem and solve it using means of propositional logic: I once met a guy from that country and he said that he is sane or citizen. Which type he is? What about if he had said that he is **either** sane **or** citizen?
12. We are now in the other country. Formalize the following problems and solve them using means of propositional logic.
- A native claimed that he is not sane citizen. What type he is?
 - Another native told me that he is crazy citizen. What type he is?