DESIGN OF RECURSIVE FILTERS USING ALLPASS FILTERS AS BUILDING BLOCKS

1. How to design IIR filters as a parallel connection of two allpass sections?

2. Design of classical IIR filters.

3. Implementation of allpass sections.

4. Extended filter classes:
   - Filters where one of the branches is a pure delay term.
   - Filters whose passband phase response is approximately linear.
   - Multiband filters.

5. Other applications:
   - Phase equalization.
   - Approximately linear-phase Hilbert transformers.
   - Filters providing an arbitrary noninteger delay.

6. (Efficient filters for sampling rate alteration and for building filter banks.)

7. (Multiplier-free implementations.)

8. (Synthesis techniques for generalized filters.)
• In recent years, a large number of digital filter structures have been developed.

• The main concern has been the performance of the structure in finite wordlength implementation on one hand and the computational complexity on the other.

• It has turned out that very useful digital filter structures for various applications can be constructed by using allpass filters as building blocks.

• Traditionally, allpass filters have been used as phase equalizers.

• Lattice wave digital filters, realized as a parallel connection of two allpass filters, are well-known among such filter classes.

• These structures can be used for realizing classical odd-order lowpass and highpass filters.

• In the bandpass and bandstop cases, lattice wave digital filters have been used for realizing filters which are obtained from an odd-order lowpass prototype via frequency tranformations.

• Techniques have also been advanced for designing similar filters and certain extended filter types directly in the z-domain.

• Special cases are filter where one of the branches is a pure delay term and filters providing approximately linear phase in the passband region.
• One of the advantages of using a parallel connection of two allpass filters is that the complementary filter is obtained from the original one by simply changing the sign of one of the allpass sections.

• Therefore, a complementary filter pair (e.g., a low-pass/highpass or a bandstop/bandpass filter pair) can be implemented with the cost of a single filter.

• In addition, these filters can be realized using first- and second-order allpass sections as building blocks.

• The resulting filter structures are highly modular, making them suitable to signal processor and VLSI implementations.

• In addition to group delay equalization, allpass filters can also be used for phase or phase delay equalization.

• They are useful for synthesizing filters with arbitrary noninteger delay and filters whose phase differs by $\pi/2$ from an integer-valued constant, that is, special Hilbert transformers.

• They can be extended also for synthesizing generalized doubly complementary digital filters as well as multiband filters with arbitrary magnitude specifications.

• Special filters composed of allpass filter building blocks are extremely useful in designing decimators and interpolators as well as filter banks.

• The above-mentioned filters can be designed based on amplitude or phase approximations.
ALLPASS FILTER

- Transfer function of order $N$

$$A_N(z) = \frac{z^{-N}D_N(z^{-1})}{D_N(z)}$$

$$= \frac{a_N + a_{N-1}z^{-1} \cdots a_1z^{-(N-1)} + z^{-N}}{1 + a_1z^{-1} + \cdots + a_{N-1}z^{-(N-1)} + a_Nz^{-N}}$$

- For the stable filter, the poles [roots of $D_N(z)$] are inside the unit circle.

- $A_N(e^{j\omega}) = e^{j\phi_N(\omega)}$, $\phi_N(\omega) = \text{arg } A_N(e^{j\omega})$.

- As $\omega$ varies from 0 to $\pi$, $\phi_N(\omega)$ decreases monotonically from 0 to $-N\pi$; $N = 4$:

![Graph of $\phi_N(\omega)$ vs Frequency]

**Phase**

-4.0$\pi$ 0 0.2$\pi$ 0.4$\pi$ 0.6$\pi$ 0.8$\pi$ $\pi$

**Frequency**

-4.0$\pi$ -3.5$\pi$ -3.0$\pi$ -2.5$\pi$ -2.0$\pi$ -1.5$\pi$ -1.0$\pi$ -0.5$\pi$ 0
BUILDING A LOWPASS FILTER

- A parallel connection of $A_N(z)$ and $A_M(z)$ with $N = M + 1$: $H_{LP}(z) = [A_M(z) + A_N(z)]/2$.

- Desired responses: $|H_{LP}(e^{j\omega})| = \frac{1}{2}|e^{j\phi_M(\omega)} + e^{j\phi_N(\omega)}|$.
RESPONSES FOR SINUSOIDAL SIGNALS

INPUT SIGNAL: \( x(n) = C \sin(n\omega) \)

OUTPUT OF \( A_M(z)/2 \):
\[ y_1(n) = \frac{C}{2} \sin(n\omega + \phi_M(\omega)) \]

OUTPUT OF \( A_N(z)/2 \):
\[ y_2(n) = \frac{C}{2} \sin(n\omega + \phi_N(\omega)) \]

OVERALL OUTPUT
\[ y(n) = y_1(n) + y_2(n) \]

\( \phi_M(\omega) = \phi_N(\omega) \rightarrow y(n) = C \sin(n\omega + \phi_M(\omega)) \)
y_1(n) and y_2(n) are in phase

\( \phi_M(\omega) = \phi_N(\omega) + \pi \rightarrow y(n) = 0 \)
y_1(n) and y_2(n) are out of phase
FREQUENCY RESPONSE

- Transfer function:
  \[ H_{LP}(z) = \frac{[A_M(z) + A_N(z)]}{2}. \]

- Frequency response:
  \[ H_{LP}(e^{j\omega}) = e^{j[\phi_M(\omega)+\phi_N(\omega)]/2} \cos \frac{1}{2}[\phi_M(\omega) - \phi_N(\omega)]. \]
BUILDING A HIGHPASS FILTER

- A parallel connection of $A_M(z)$ and $-A_N(z)$ with $N = M + 1$: $H_{HP}(z) = [A_M(z) - A_N(z)]/2$.

- Desired responses: $|H_{HP}(e^{j\omega}) = \frac{1}{2}|e^{j\phi_M(\omega)} - e^{j\phi_N(\omega)}|$.
A LOWPASS-HIGHPASS FILTER PAIR

• Implementation:

![Block diagram of filter pair with symbols and connections]

• This pair is a power-complementary filter pair since

\[ |H_{LP}(e^{j\omega})|^2 + |H_{HP}(e^{j\omega})|^2 = 1. \]

• Responses:

![Graph showing squared magnitude against frequency]

- SQUARED MAGNITUDE
- FREQUENCY

- \[ 0 \quad 0.2\pi \quad 0.4\pi \quad 0.6\pi \quad 0.8\pi \quad \pi \]
• In this case, $N = M + 2$.

• Responses for the bandstop filter:
DESIGN OF CLASSICAL LOWPASS AND HIGHPASS FILTERS

• Any odd-order classical IIR filter (Butterworth, Chebyshev, inverse Chebyshev, elliptic filter) can be implemented in the desired form!!

• This was first observed in the case of wave digital lattice filters (allpass filters are implemented using special structures).

DESIGN PROCEDURE:

1. Determine a classical filter (or filter pair) meeting the given amplitude criteria. Let the filter order be $M + N$ (odd).

2. Include in $A_N(z)$ the outermost pole pair, the third outermost pole pair, the fifth outermost pole pair and so on (see the following transparency). The remaining pole pairs belong to $A_M(z)$. The real pole is included in $A_N(z)$ if $N + M$ is 5, 9, 13, 17 and so on. Otherwise, it belongs to $A_N(z)$.

3. Select the structures for the allpass filters and compute the coefficient values (to be considered later).
ALTERNATING DISTRIBUTION OF THE POLES TO THE ALLPASS SUBFILTERS
EXAMPLE: A lowpass-highpass filter pair with edges at $0.4\pi$ and $0.45\pi$ and at least 40-dB attenuations in the stopbands

- The minimum odd order is 9 to meet the criteria ($N = 5, M = 4$.)
- When the passband ripple of the lowpass filter is 0.00019788 dB, then the attenuations of both filters are 43.414 dB.
POLE AND ZERO LOCATIONS

• The poles of both filters are located at
  \[ z = 0.96546596 \exp(\pm j0.42471635\pi) \]
  \[ z = 0.87484735 \exp(\pm j0.42407724\pi) \]
  \[ z = 0.71070737 \exp(\pm j0.42024180\pi) \]
  \[ z = 0.42558157 \exp(\pm j0.39465517\pi) \]
  \[ z = 0.11873387. \]

• \( A_N(z) \) contains the first and third pole pairs as well as the real pole. \( A_M(z) \) contains the remaining poles.

• The zeros of the lowpass filter are located at
  \[ z = \exp(\pm j0.45243132\pi) \]
  \[ z = \exp(\pm j0.47491693\pi) \]
  \[ z = \exp(\pm j0.53905970\pi) \]
  \[ z = \exp(\pm j0.69618255\pi) \]
  \[ z = -1. \]

• The zeros of the highpass filter are located at
  \[ z = \exp(\pm j0.39766309\pi) \]
  \[ z = \exp(\pm j0.37644064\pi) \]
  \[ z = \exp(\pm j0.31948331\pi) \]
  \[ z = \exp(\pm j0.19766827\pi) \]
  \[ z = 1. \]
POLE-ZERO PLOTS

LOWPASS

HIGHPASS
DESIGN OF CLASSICAL BANDPASS AND BANDSTOP FILTERS

- The design can be accomplished by using a lowpass-bandpass transformation considered above. This transformation is of the form
  \[ z^{-1} \leftrightarrow -\frac{z^{-2} - [2\alpha k/(k + 1)]z^{-1} + [(k - 1)/(k + 1)]}{[(k - 1)/(k + 1)]z^{-2} - [2\alpha k/(k + 1)]z^{-1} + 1}. \]

DESIGN PROCEDURE:

1. Determine the criteria for the corresponding lowpass prototype filter and synthesize it as a parallel connection of \( A_N(z) \) and \( A_M(z) \).

2. Apply the lowpass-bandpass transformation to both \( A_N(z) \) and \( A_M(z) \). The resulting allpass sections are of the form \( A_{2N}(z) \) and \( A_{2M}(z) \), that is, the orders are doubled.

- Since the order of the lowpass prototype is odd, the resulting overall filter orders are 2, 6, 10, or 14 and so on.
EXAMPLE: A bandpass-bandstop filter pair with edges at $0.2\pi$, $0.225\pi$, $0.775\pi$, and $0.8\pi$ and at least 40-dB attenuations in the stopbands

- The desired filter bank is obtained by applying the substitution $z^{-1} \rightarrow -z^{-2}$.

- $A_{2N}(z)$ ($2N = 10$) includes the following poles:
  
  $z = 0.96546596 \exp(\pm j0.42471635\pi)$

  $z = 0.98258127 \exp(\pm j0.21235818\pi)$

  $z = 0.98258127 \exp(\pm j0.78764182\pi)$

  $z = 0.84303462 \exp(\pm j0.21012090\pi)$

  $z = 0.84303462 \exp(\pm j0.78987910\pi)$

  $z = \pm 0.34457781$

- $A_{2M}(z)$ ($2M = 8$) includes the following poles:

  $z = 0.93533275 \exp(\pm j0.21203862\pi)$

  $z = 0.93533275 \exp(\pm j0.78796138\pi)$

  $z = 0.65236613 \exp(\pm j0.19732758\pi)$

  $z = 0.65236613 \exp(\pm j0.80267242\pi)$
AMPLITUDE RESPONSES
POLE-ZERO PLOTS

BANDPASS

BANDSTOP
STRUCTURES FOR IMPLEMENTING ALL-PASS FILTERS

- The implementation becomes very modular if the all-pass sections are realized as a cascade of first- and second-order allpass sections.
  - Easy to design, analyse, and scale.
- It is required that after coefficient quantization the building block is still an allpass filter.
• First-order: $H(z) = [b_1 + z^{-1}]/[1 + b_1 z^{-1}]$

• Second-order: $H(z) = [b_2 + b_1 z^{-1} + z^{-2}]/[1 + b_1 z^{-1} + b_2 z^{-2}]$

• Automatically scaled according to $L_\infty$ norm.

• If there are several blocks in cascade, delays can be shared.
SOME MITRA-HIRANO STRUCTURES

- First-order: Mitra-Hirano $1A_t$:

- Second-order: Mitra-Hirano $3D$:

- The scaling constant of value of $1/2$ in needed at the input.
• First-order: \( H(z) = [-\gamma + z^{-1}]/[1 - \gamma z^{-1}] \)

• Second-order: \( H(z) = [-\gamma_1 + \gamma_2(\gamma_1 - 1)z^{-1} + z^{-2}]/[1 + \gamma_2(\gamma_1 - 1)z^{-1} - \gamma_1 z^{-2}] \)

• The scaling constant of value of 1/2 at the filter input guarantees the absence of overflows.
Fig. 9. Signal-flow diagrams of the two-port adaptor yielding optimal scaling for sinusoidal excitation. (Note that in the first diagram of the second last row, \( \alpha \) should be replaced by \(-\alpha\).)
EXAMPLE IMPLEMENTATION OF A SEVENTH-ORDER FILTER

\[ \alpha_1 = 1 - \gamma_0 = 0.4871 \]
\[ \alpha_2 = 1 + \gamma_3 = 0.3313 \]
\[ \alpha_3 = \gamma_4 = 0.3342 \]
\[ \alpha_4 = |\gamma_1| = 0.4044 \]
\[ \alpha_5 = 1 - \gamma_2 = 0.3922 \]
\[ \alpha_6 = 1 + \gamma_5 = 0.1038 \]
\[ \alpha_7 = \gamma_6 = 0.2067 \]
OTHER APPLICATIONS AND EXTENSIONS

- As mentioned in the first three transparencies, there are several other applications for filters using allpass filters as building blocks.

- Some of these are considered in the following.

- The main emphasis is on the results.

- All these filters can be designed very fast using Remez-type algorithms proposed by Markku Renfors and Tapio Saramäki.
  
  - Some algorithms use amplitude approximations and some phase approximations.

  - An interested reader may read the enclosed articles and try to understand how the algorithms work.
EXTENSION 1: ONE OF THE SUBSECTIONS CONTAINS A PURE DELAY TERM

- The real pole include in one of the allpass sections contains a pole at the origin. ⇒ The first-order allpass section reduced to a pure delay term $z^{-1}$, that is, no multiplier is needed.

- Example:
SPECIAL CASE: HALF-BAND IIR FILTERS

- For these filters, the first order section is a pure delay term $z^{-1}$ and all the poles on the imaginary axis.

- This means that the second-order allpass section reduces into the form $A_2(z) = \frac{b_2 + z^{-2}}{1 + b_2 z^{-2}}$, that is, $b_1$ becomes zero.

- The implementation of the first-order section requires no multipliers and the implementation of the second order section only one multiplier.

- The second-order section can, in fact, obtained from a first-order section $A_1(z) = \frac{b_2 + z^{-1}}{1 + b_2 z^{-1}}$ by replacing a unit delay by a double delay.

- Implementation of a lowpass-highpass filter pair:
PROPERTIES

- For the lowpass filter, $\omega_s = \pi - \omega_p$.

- The stopband attenuations of both the lowpass and highpass filters are the same.

- These filters are very useful in sampling rate conversion applications and in building filter banks.
A ELEVENTH-ORDER ELLIPTIC HALF-BAND FILTER PAIR REQUIRING ONLY FIVE MULTIPLIERS AND HAVING EDGES AT $0.45\pi$ AND $0.55\pi$
MORE APPLICATIONS

• In the following there are altogether four sets of transparencies for conference talks.

• Please look at the results. Synthesis techniques are not so important to learn.

• In the case you are interested in the actual papers, please do not hesitate to contact the lecturer of this course.

• Also a short article on the use of allpass filters as basic building blocks, written by Prof. Markku Renfors, is included.