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Shortened Lecture # 3

Polynomial-Based Interpolation for Digital Signal Processing (DSP) and Telecommunication Applications

- This pile of lecture notes is mainly based on the research work done by Dr. Jussi Vesma and the lecturer during the last five years.
- Later on, Djordje Babic and Prof. Markku Renfors have been provided contributions to this research.
- Many thanks to Vesma, Babic, and Renfors for their help in preparing this pile of lecture notes.

Contents of this Talk:

1. Interpolation Filters under Considerations and Applications
2. Statement of the Problem for Polynomial-Based Interpolators
3. Hybrid Analog/Digital Model to be Mimicked Digitally
4. Efficient Digital Implementation: Modified Farrow Structure
5. Optimization in the Frequency Domain
6. Application Examples:

a) Design of FIR filters with an adjustable fractional delay

b) Up-sampling between arbitrary sampling rates

c) Down-sampling between arbitrary sampling rates

d) Symbol time adjustment in all-digital receivers

e) Processing of continuous-time signals based on its discrete-time counterpart sequence

Applications for Interpolation Filters

- Timing adjustment in all-digital receivers (symbol synchronization)
- Time delay estimation
- Conversion between arbitrary sampling frequencies
- Echo cancellation
- Phased array antenna systems
- Speech coding and synthesis
- Derivative approximation of discrete-time signals
- Computer simulation of continuous-time systems
- ML symbol timing recovery in digital receivers

Interpolation Filters

- In many DSP and telecommunication applications there is a need to know the values of the signal also between the exiting discrete-time samples $x(n)$ as shown in Fig. 1.
- Special **interpolation filters** can be used to compute new sample values $y(l)=y_a(t_l)$ at **arbitrary** points $t_l=(n_l+\mu_l)T_{in}$ between the **existing samples** $x(n_l)$ and $x(n_l+1)$. Here, T_{in} is the sampling period.
- Here, $y_a(t)$ approximates either the original continuous-time signal $x_a(t)$ or the signal obtained with the aid of the existing discrete-time samples $x(n)$ using the sinc interpolation.
- The output sample time is determined by $\boldsymbol{n_l T_{in}}$, the location of the **preceding existing sample**, and the **fractional interval** $\boldsymbol{\mu_l}\in[0,1)$, the difference between t_l and $n_l T_{in}$ as a fraction of T_{in}.

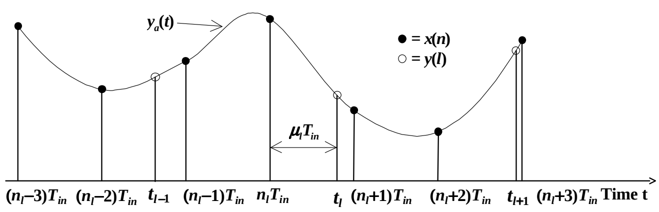


Fig. 1. Interpolation in the time domain.

Statement of Discrete-Time Interpolation Problem

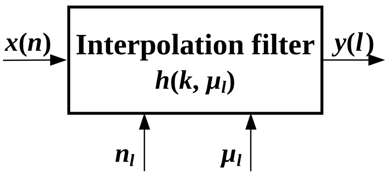


Fig. 2. Simplified block diagram for the interpolation filter.

Given the input sequence $x(n)$ as well as the time instant t_l of the lth output sample $y(l)=y_a(t_l)$,

Find the control parameters n_l and μ_l in Fig. 2 as $t_l=(n_l+\mu_l)T_{in}$, that is

$$n_l = \lfloor t_l / T_{in} \rfloor \text{ and } \mu_l = t_l / T_{in} - \lfloor t_l / T_{in} \rfloor \qquad (1)$$

and determine $y(l)$ according to the following convolution:

$$y(l) = \sum_{k=-N/2}^{N/2-1} x(n_l - k) h(k, \mu_l) \qquad (2)$$

where N (even) is the filter length and $h(k, \mu_l)$ is the discrete-time impulse response of the interpolation filter.

Comment: There are $N/2$ samples and before and after the time instant t_l and the impulse-response coefficients $h(k, \mu_l)$ depend on μ_l.

Statement of the Interpolation Problem

Given N, **find** the impulse-response coefficients $h(k, \mu_l)$ for $k=-N/2+1, -N/2+2,\cdots, N/2$ to meet the following two conditions:

1. Optimize them such that $y(l)=y_a((n_l+\mu_l)T_{in})$ for all values of $\mu_l\in[0,1)$, where $y_a(t)$ approximates according to some time-domain or frequency-domain criterion the signal

$$x_a(t) = \sum_{n=-\infty}^{\infty} x(n) \sin[\pi(t-nT_{in})/T_{in}]/[\pi(t-nT_{in})/T_{in}] .$$

2. The convolution

$$y(l) = \sum_{k=-N/2}^{N/2-1} x(n_l - k) h(k, \mu_l) \qquad (2)$$

can be implemented **digitally** using an efficient structure.

- In Condition 1, the frequency-domain criteria are usually preferred for DSP and telecommunication applications.

Various Approaches to Solve the Stated Problem

- There exist the following three approaches to solve our problem:

1. Fractional delay (FD) filter approach.
2. Use some classical interpolation method to calculate $y(l)$, e.g., Lagrange or B-spline interpolation (**time-domain approach**).
3. Utilize the analog model for the interpolation filter (**frequency-domain approach**).

Here, we concentrate on the last approach.

Hybrid Analog/Digital Model to be Mimicked

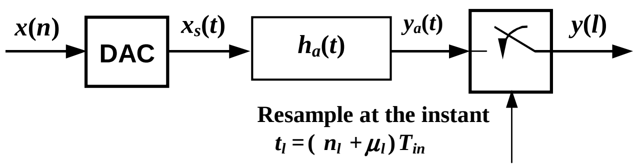


Fig. 3. Analog model for the interpolation filter.

- In this model,

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(n)\delta_a(t - nT_{in}) \qquad (3)$$

and

$$y_a(t) = \int_{-\infty}^{\infty} x_s(\tau) h_a(t-\tau) d\tau = \sum_{k=-\infty}^{\infty} x(k) h_a(t - kT_{in}). \qquad (4)$$

- Assuming that $h_a(t)$ is **zero outside the interval** $\boldsymbol{-NT_{in}/2 \le t < NT_{in}/2}$, $y(l)$ obtained by sampling $y_a(t)$ at t_l is given by

$$y(l) = y_a(t_l) = \sum_{k=-N/2}^{N/2-1} x(n_l - k) h_a((k + \mu_l)T_{in}). \qquad (5)$$

Hybrid Analog/Digital Model to be Mimicked

- By comparing Equations (2) and (5), that is,

$$y(l)=\sum_{k=-N/2}^{N/2-1}x(n_l-k)h(k,\mu_l) \quad (6)$$

and

$$y(l)=y_a(t_l)=\sum_{k=-N/2}^{N/2-1}x(n_l-k)h_a((k+\mu_l)T_{in}). \quad (7)$$

it can be seen that the impulse responses of the analog and discrete-time filters are related as follows:

$$h(k,\mu_l)=h_a((k+\mu_l)T_{in}) \quad (8)$$

for $k=-N/2, -N/2+1,\cdots,N/2-1$.

- In the causal case, $h_a(t)$ is delayed by $NT_{in}/2$, i.e., the impulse response is given by $h_a(t-NT_{in}/2)$.
- In this case, $y(l)$ obtained $NT_{in}/2$ time units later is given by

$$y(l)=\sum_{k=0}^{N-1}x(n_l+N/2-k)\,h_a((k+\mu_l-N/2)T_{in}). \quad (9)$$

- In the sequel, the non-causal $h_a(t)$ [Equation (7)] and the causal $h_a(t-NT_{in}/2)$ [Equation (9)] are used for the design and implementation purposes, respectively.

Why to Use the Analog Model?

- Interpolation is generally considered as a time-domain problem of fitting polynomial through the existing samples, which is not very practical approach for DSP and telecommunication applications.
- These include the Lagrange and B-spline interpolations
- This is because the time-domain characteristics of the input sequence $x(n)$ are not usually known. What is usually known is the frequency-domain performance of the signal.
- It should be pointed out that recently Atanas Gotchev, Karen Egiazarian, and Tapio Saramäki have improved the performance of B-splines in interpolation problems, especially in the case of images, by using modified B-splines consisting of a weighted sum of odd-order B-splines. Contact: saram@vip.fi (home e-mail address of Saramäki).
- The main idea is to determine the weights in such a manner that the resulting filter effectively preserves the baseband of interest and attenuates the corresponding images.

Why to Use the Analog Model?

- The use of the analog model converts the interpolation problem from the time-domain to the frequency domain in a manner to be will be seen later on.

Synthesis problem for in general: Determine $h_a(t)$ such that

1. The overall system of Fig. 3 can be implemented **digitally** using an efficient structure.
2. It provides the desired filtering performances.

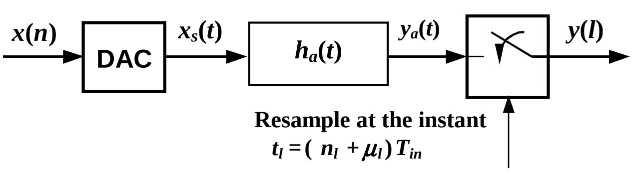


Fig. 3. Analog model for the interpolation filter.

Desired $h_a(t)$ leading to an Efficient Implementation

- Consider the following impulse response of an analog filter as

$$h_a(t)=\sum_{n=-N/2}^{N/2-1}\sum_{m=0}^{M}c_m(n)\,f(n,t-nT) \quad (10a)$$

where

$$c_m(n)=(-1)^m c_m(-n-1) \quad (10b)$$

for $m=0, 1, \cdots, M$ and $n=0, 1,\cdots, N/2-1$ are the unknowns and

$$f(m,t)=\left(\frac{2t-1}{T_{in}}\right)^m \quad (10c)$$

are the basis functions shown below.

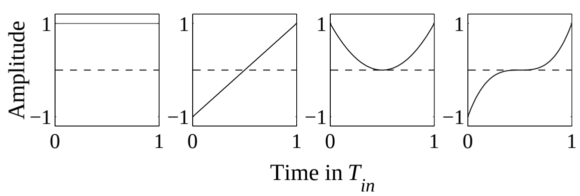


Fig. 4. Basis functions $f(m,t)$ for $m=0, 1, 2,$ and 3.

Desired $h_a(t)$ leading to an Efficient Implementation

- Alternatively, this impulse response can be expressed as

$$h_a(t) = \sum_{n=0}^{N/2-1} \sum_{m=0}^{M} c_m(n) g(n,m,t) \qquad (11a)$$

where $c_m(n)$'s are unknown coefficients and $g(n,m,t)$'s are the new basis functions given by

$$g(n,m,t) = \begin{cases} \left(\dfrac{2(t - nT_{in})}{T_{in}} - 1 \right)^m & \text{for } nT_{in} < t \le (n+1)T_{in} \\ (-1)^m \left(\dfrac{2(t + (n+1)T_{in})}{T_{in}} - 1 \right)^m & \text{for } -(n+1)T_{in} \le t < -nT_{in} \\ 0 & \text{otherwise} \end{cases} \qquad (11b)$$

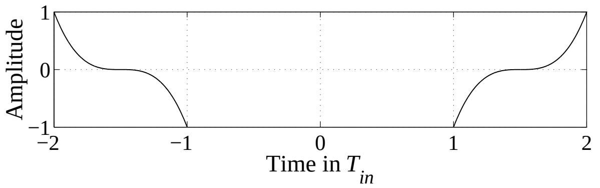


Fig. 5. The basis function $g(n,m,t)$ for $n=1$ and $m=3$.

Figure 5 shows an example basis function, whereas Fig. 6 shows how the overall impulse response can be constructed using weighted basis functions.

Example on how to construct $h_a(t)$ for $N=8$ and $M=3$.

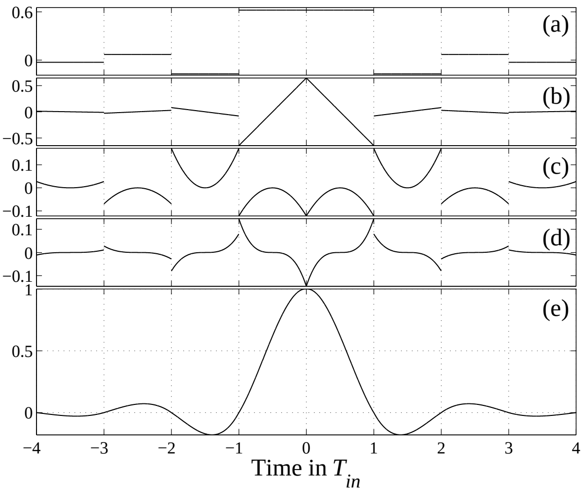


Fig. 6. Construction of the overall impulse response $h_a(t)$ for $N=8$ and $M=3$. The weighted basis functions $\sum_{n=0}^{N/2-1} c_m(n)g(n,m,t)$ for $m=0$, $m=1$, $m=2$, and $m=3$ are shown in (a), (b), (c), and (d). (e) The resulting impulse response $h_a(t)$ obtained as a sum of these responses.

Characteristics of the impulse response $h_a(t)$

- The resulting $h_a(t)$ is characterized by the following attractive properties:
  1) $h_a(t)$ is nonzero for $-NT_{in}/2 \le t < NT_{in}/2$.
  2) The length of the filter N is an even integer.
  3) $h_a(t)$ is a piecewise-polynomial of degree M in each interval $nT_{in} \le t < (n+1)T_{in}$ for $n = -N/2, -N/2+1, \cdots, N/2-1$.
  4) $h_a(t)$ is symmetrical, that is $h_a(-t) = h_a(t)$ except for the time instants $t = nT_{in}$ for $n = -N/2, -N/2+1, \cdots, N/2$.
- Properties 1, 2, and 3 guarantee the corresponding causal system with impulse response $h_a(t - NT_{in}/2)$ can be implemented using an efficient digital implementation.
- The role of Property 4 is twofold.
  1) For the causal system, the phase response is linear.
  2) In the modified Farrow to be described later, the fixed FIR filters have either a symmetrical or anti-symmetrical impulse responses. This enables us to utilize the coefficient symmetry, reducing the number of multipliers in the implementation compared to the original Farrow structure.

Modified Farrow Structure

- Substituting

$$h_a(t) = \sum_{n=0}^{N/2-1} \sum_{m=0}^{M} c_m(n) g(n,m,t) \qquad (11a)$$

where

$$g(n,m,t) = \begin{cases} \left(\dfrac{2(t - nT_{in})}{T_{in}} - 1 \right)^m & \text{for } nT_{in} < t \le (n+1)T_{in} \\ (-1)^m \left(\dfrac{2(t + (n+1)T_{in})}{T_{in}} - 1 \right)^m & \text{for } -(n+1)T_{in} \le t < -nT_{in} \\ 0 & \text{otherwise} \end{cases} \qquad (11b)$$

into

$$y(l) = \sum_{k=0}^{N-1} x(n_l + N/2 - k)\, h_a((k + \mu_l - N/2)T_{in}). \qquad (9)$$

gives, after some manipulations, the formula given in the following transparency.

Modified Farrow Structure

$$y(l)=\sum_{m=0}^{M} v_m(n_l)(2\mu_l-1)^m, \qquad (12a)$$

where

$$v_m(n_l)=\sum_{k=0}^{N-1} c_m(k-N/2)x(n_l+N/2-k). \qquad (12b)$$

- The resulting implementation form shown in Fig. 7 in the next transparency.
- This structure is characterized by the following properties:

  1) There exist M+1 **fixed** FIR filter transfer functions $C_m(z)=\sum_{k=0}^{N-1} c_m(k-N/2)z^{-k}$ for $m=0,1,\cdots,M$ with the following symmetry properties:

  a) For m is zero or even, $c_m(N/2-1+k)=c_m(-N/2-k)$ for k=0, 1,···,N/2–1.

  b) For m odd, $c_m(N/2-1+k)=-c_m(-N/2-k)$ for k=0, 1,···,N/2–1.

  2) The desired output sample value $y(l)$ at $t_l=(n_l+\mu_l)T_{in}$ is obtained by multiplying the output of the mth FIR filter output by $(2\mu_l-1)^m$ and adding the result.

  3) The last input sample is $x(n_l+N/2)$.

Modified Farrow Structure

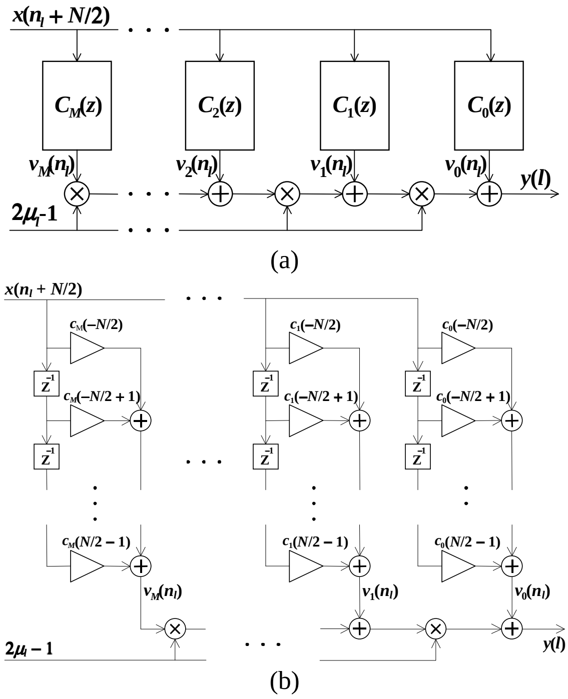


Fig. 7. Modified Farrow structure. (a) Basic structure. (b) Details.

Frequency-Domain Criteria

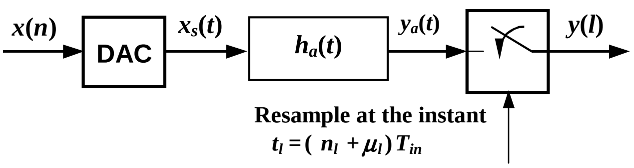


Fig. 3. Analog model for the interpolation filter.

- For the overall system of Fig. 3, the Fourier transform of $y_a(t)$ is related to that of the sequence $x(n)$ or equivalently to that of the signal $x_s(t)=\sum_{n=-\infty}^{\infty} x(n)\delta_a(t-nT_{in})$ through

$$Y_a(j2\pi f)=H_a(j2\pi f)X(e^{j2\pi f/F_{in}})= \\ =H_a(j2\pi f)F_{in}\sum_{k=-\infty}^{\infty} X_a(j2\pi(f-kF_{in})) \qquad (13)$$

where $F_{in}=1/T_{in}$ is the sampling rate of the input signal and $H_a(j2\pi f)$ is the Fourier transform of the reconstruction filter with impulse response $h_a(t)$.

- The last form of Equation (13) is for the case where $x(n)=x_a(nT_{in})$ are samples of a continuous-time signal $x_a(t)$ with $X_a(j2\pi f)$ being its Fourier transform.

Role of $h_a(t)$ in the Frequency Domain

As shown below, the role of the reconstruction filter with impulse response $h_a(t)$ is to attenuate the extra images of $x_s(t)=\sum_{n=-\infty}^{\infty} x(n)\delta_a(t-nT_{in})$ and to preserve the signal components only in the original baseband $[0,F_{in}/2]$.

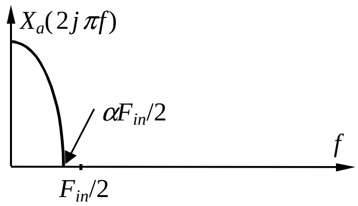


Fig. 8. The spectrum of the original continuous-time signal band-limited to $|f|\le\alpha F_{in}$. The sequence is formed as $x(n)=x_a(nT_{in})$.

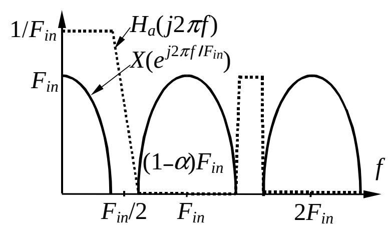


Fig. 9. The spectrum of $x_s(t)=\sum_{n=-\infty}^{\infty} x(n)\delta_a(t-nT_{in})$, denoted by $X(e^{j2\pi f/F_{in}})$ and the frequency response of the reconstruction filter with impulse response $h_a(t)$, denoted by $H_a(j2\pi f)$.

Criteria for the Uniform Sampling: Interpolation and Decimation

- If $y(l)$ is generated at the time instants $t_l = lT_{out}$, then

$$Y(e^{j2\pi f/F_{out}}) = F_{out}\sum_{k=-\infty}^{\infty} Y_a(j2\pi(f-kF_{out})), \qquad (14)$$

where $F_{out} = 1/T_{out}$ is the sampling rate of the output signal $y(l)$ and the baseband of interest is $[0, F_{out}/2]$.

- The case $\beta = F_{out}/F_{in} > 1$ corresponds to **the interpolation.**
- The case $\beta = F_{out}/F_{in} < 1$ corresponds to **the decimation.**
- In both cases, the ideal response for $H_a(j2\pi f)$ avoiding both imaging and aliasing is given by

$$D(f) = \begin{cases} 1/F_{in} & \text{for } 0 \le f \le F_C/2 \\ 0 & \text{for } f > F_C/2, \end{cases} \qquad (15a)$$

where

$$F_C = \min(F_{in}, F_{out}). \qquad (15b)$$

- Note that in the interpolation case, it is enough to attenuate the images of $X(e^{j2\pi f/Fin})$.

- In the decimation case, $X(e^{j2\pi f/Fin})$ should be band-limited into the range $[0, F_{out}/2]$, that is, the region $[F_{out}/2, F_{in}/2]$ should be attenuated by $H_a(j2\pi f)$ in order to avoid aliasing.

Practical Criteria

- Like for conventional digital interpolators and decimators, the criteria can be stated as

$$1-\delta_p \le F_{in}|H_a(j2\pi f)| \le 1+\delta_p \quad \text{for} \quad f \in [0, f_p] \qquad (16a)$$

$$F_{in}|H_a(j2\pi f)| \le \delta_s \quad \text{for} \quad f \in \Omega_s, \qquad (16b)$$

where $f_p < F_C/2$ and

$$\Omega_s = \begin{cases} [F_C/2, \infty) & \text{for Type A} \\ [F_C - f_p, \infty) & \text{for Type B} \\ \bigcup_{k=1}^{\infty} [kF_C - f_p, kF_C + f_p] & \text{for Type C.} \end{cases} \qquad (16c)$$

- For Type A, no aliasing or imaging is allowed.
- For Type C decimation case, aliasing is allowed into the transition band $[f_p, F_{out}/2]$. For Type B, aliasing into this band is allowed only from band $[F_{out}/2, F_{out}-f_p]$.
- In the interpolation case, Types B and C are useful if most of the energy of the incoming signal is in the range $[0, f_p]$.

The frequency response for the analog filter with impulse response $h_a(t)$

- After some manipulations, the frequency response can be expressed as

$$H_a(j2\pi f) = \sum_{n=0}^{N/2-1}\sum_{m=0}^{M} c_m(n) G(n,m,f), \qquad (17)$$

where $G(n,m,f)$ is the frequency responses of the basis function $g(n,m,t)$, as given by Eq. (11b) in transparency 13.

- Since $g(n,m,t)$ is symmetrical around $t=0$, $G(n,m,f)$ is real and is given by

$$G(n,m,f) = \begin{cases} 2\cos(2\pi f T_{in}(n+\tfrac{1}{2}))\left[(-1)^{m/2} m!\Phi(m,f) + \dfrac{\sin(\pi f T_{in})}{\pi f T_{in}}\right] & \text{for } m \text{ even} \\ 2(-1)^{(m+1)/2} m! \sin(2\pi f T_{in}(n+\tfrac{1}{2}))\Phi(m,f) & \text{for } m \text{ odd,} \end{cases} \qquad (18a)$$

where

$$\Phi(m,f) = \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} (\pi f T_{in})^{2k-m} \frac{(-1)^k}{(2k)!}\left(\frac{\sin(\pi f T_{in})}{\pi f T_{in}} - \frac{\cos(\pi f T_{in})}{(2k+1)}\right). \qquad (18b)$$

Optimization Problems

- The very attractive property of the above $H_a(j2\pi f)$ is that it is **linear with with respect its unknowns $c_m(n)$**.
- These unknowns can be easily found to minimize

$$\delta_\infty = \max_{f \in X} \left| W(f)\left(\left|H_a(j2\pi f)\right| - D(f)\right)\right| \qquad (19)$$

or

$$\delta_2 = \int_X \left[W(f)\left(\left|H_a(j2\pi)\right| - D(f)\right)\right]^2 df \qquad (20)$$

subject to the following time-domain conditions of $h_a(t)$:

1) *Case A*: There are no time-domain conditions.

2) *Case B*: h_a(t) is continuous at $t = kT_{in}$ for $k=\pm1, \pm2,\cdots, \pm N/2-1$.

3) *Case C*: $h_a(0)=1$ and $h_a(kT_{in})=0$ for $k=\pm1, \pm2,\cdots, \pm N/2$.

4) *Case D*: The first derivative of $h_a(t)$ is continuous at $t = kT_{in}$ for $k=0$ and for $k=\pm1, \pm2,\cdots,\pm(N/2-1)$.

- The first and second criteria, as given by Eqs. (19) and (20) correspond to the optimization in the minimax and the least-mean-square sense subject to the given time-domain constraints, respectively.

- Here $X\subset[0,\infty)$ is a compact subset and $D(f)$ is an arbitrary desired function (continuous) and $W(f)$ is an arbitrary weighting function (positive).
- Here, the approximation region X consists of a set of passband and stopband regions.
- The actual optimization can be accomplished in a manner similar to the design of linear-phase FIR filters.
- Optimization algorithms have been implemented in Matlab. For minimax problem, linear programming can be used to optimize the filter coefficients.
- For both problems, the time domain conditions can be included in the problem in such a manner that they become unconstrained problems.
- This makes the overall optimization algorithms very fast.
- It should be pointed out that *Case C* time-domain condition guarantees that if the new sampling instant occurs at the instant of the existing sample, then the sample value is preserved.
- *Case D* time-domain condition is of importance when determining the derivative of a continuous-time signal with the aid of its discrete-time samples and a generalized modified Farrow structure.

Parameters for the optimization

- Design parameters for the optimization programs are the following:

1. Edge frequencies for passband(s) and stopband(s).
2. Desired amplitude and weight for every band.
3. N, the length of the filter.
4. M, the degree of the interpolation.
5. The number of grid points.
6. Time-domain constraints:

1) *Case A*: There are no time-domain conditions.

2) *Case B*: h_a(t) is continuous at $t=kT_{in}$ for $k=\pm1, \pm2,\cdots, \pm N/2-1$.

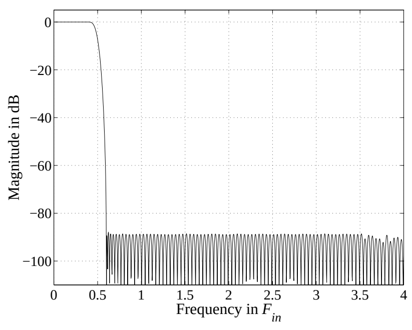
3) *Case C*: $h_a(0)=1$ and $h_a(kT_{in})=0$ for $k=\pm1, \pm2,\cdots, \pm N/2$.

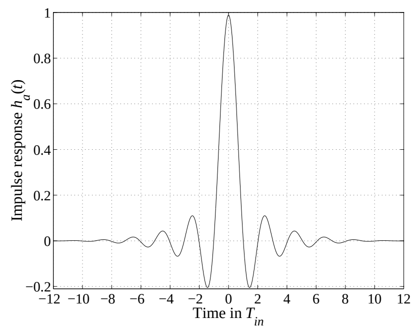
4) *Case D*: The first derivative of $h_a(t)$ is continuous at $t=kT_{in}$ for $k=0$ and for $k=\pm1, \pm2,\cdots,\pm(N/2-1)$.

- Before introducing the applications, two Case A design examples are considered.

Optimized Case A minimax design

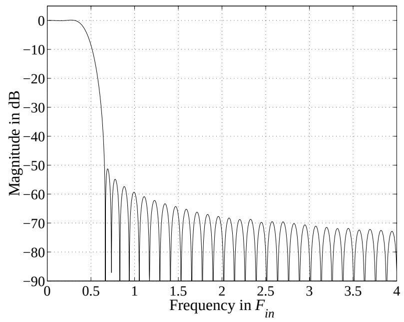
- $M=7, N=24$, $X=[0, 0.4F_{in}]\cup[0.6F_{in}, \infty)$, $D(f)$ is unity and zero on the first and second bands, whereas $W(f)$ is 0.002 and 1, respectively.

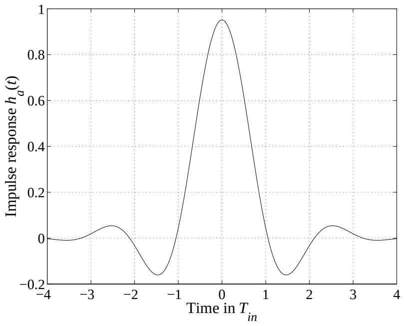




Optimized Case A least-squared design

- $M=5$, $N=8$, $X=[0, 0.35F_{in}]\cup[0.75F_{in}, \infty)$, $D(f)$ is unity and zero on the first and second bands, whereas $W(f)$ is 0.02 and 1, respectively.





Application A: FIR Filters with an Adjustable Fractional Delay

- Using $\mu_l = 1-\lambda$, the delay of the modified Farrow structure of Fig.7 in transparency 18 becomes $D = N/2 - 1 + \lambda$, where $N/2-1$ is an integer delay and λ is a fractional delay with $0 \le \lambda < 1$.
- In this case, instead of $2\mu_l - 1$, $1-2\lambda$ is used.
- For this structure, the transfer function is expressible as

$$H(\Psi, z, \lambda) = \sum_{k=0}^{N/2-1}\left[\sum_{m=0}^{M} c_m(k)[1-2\lambda]^m\right] z^{-(N/2+k)} + \sum_{k=0}^{N/2-1}\left[\sum_{m=0}^{M} (-1)^m c_m(k)[1-2\lambda]^m\right] z^{-(N/2-1-k)} \quad (21)$$

where Ψ denotes adjustable parameters $c_m(k)$ for $k=0,1,\cdots, N/2-1$ and $m=0,1,\cdots,M$.

- Using a nonlinear optimization procedure, following problem can be solved: Given N, M, ε_a, and the passband region $\Omega_p = [0, \omega_p]$, $\omega_p < \pi$, find Ψ to minimize

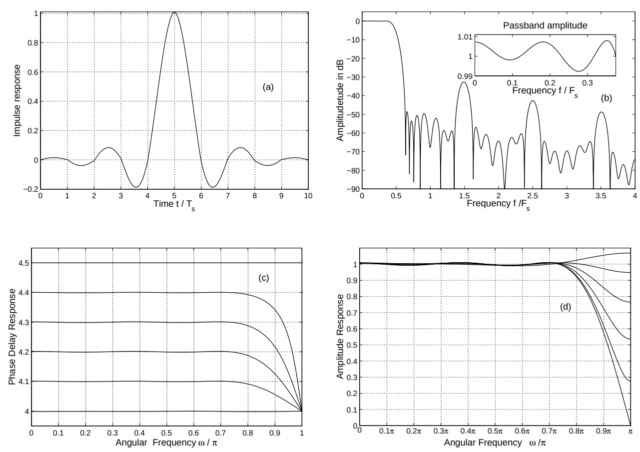
$$\Delta_p = \max_{0\le\lambda<1}\left[\max_{\omega\in\Omega_p}\left|-\arg H\left(\Psi, e^{j\omega}, \lambda\right)/\omega - (N/2-1+\lambda)\right|\right] \quad (22a)$$

subject to

$$\Delta_a = \max_{0\le\lambda<1}\left[\max_{\omega\in\Omega_p}\left|\,\left|H\left(\Psi, e^{j\omega}, \lambda\right)\right| - 1\right|\right] \le \varepsilon_a . \quad (22b)$$

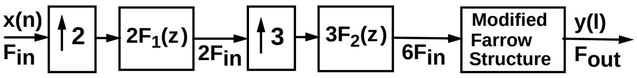
Example: $\omega_p = 0.75\pi$, $N = 10$, $M = 4$, and $\varepsilon_a = 0.01$. $\Delta_p = 0.0016$.

- Due to that fact that both the amplitude and phase delay errors in the passband are the same for λ and $1-\lambda$, Figs. (c) and (d) show the phase delay and amplitude responses only for $\lambda = 0$, 0.1, 0.2, 0.3, 0.4, and 0.5.



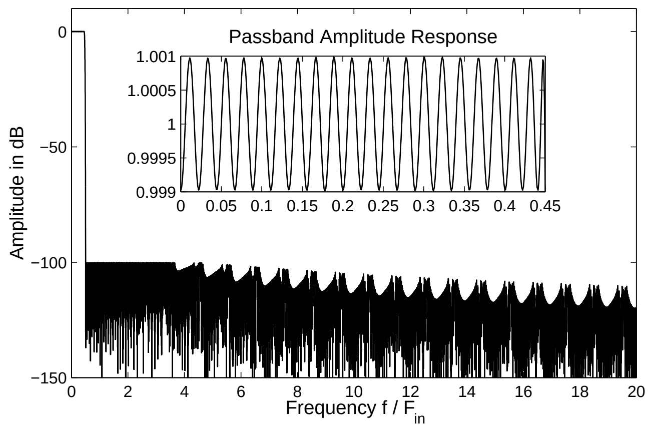
Application B: Up-Sampling Between Arbitrary Sampling Frequencies

- The Farrow structure can be directly used for providing the increse between an arbitrary input sampling rate F_{in} and an arbitrary output sampling rate F_{out}.
- It is desired that $H_a(j2\pi f)$ approximates unity for $0 \le f \le 0.45F_{in}$ with tolerance of 0.001 and zero for $f \ge 0.5F_{in}$ with tolerance of 0.00001 (100-dB attenuation).
- When using the minimax optimization, the given criteria are met by $\boldsymbol{N = 92}$ and $\boldsymbol{M = 6}$, as shown on Page 32. This implementation requires **7** fixed branch filters of length **92**.
- The implementation can be simplified using fixed linear-phase FIR interpolators before the Farrow structure, as proposed by Saramäki and Ritoniemi.
- $\boldsymbol{N = 4}$ and $\boldsymbol{M = 3}$ are required if the sampling rate is increased by a factor of six by using a two-stage fixed interpolator with interpolation factors of two and three and FIR filters of order 183 and 11, respectively. See Page 51.
- The block diagram for this multistage implementation is shown below.

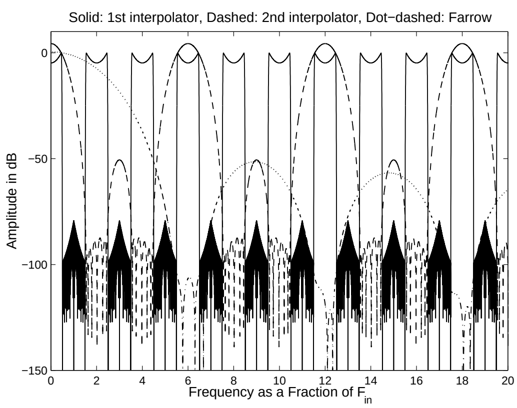


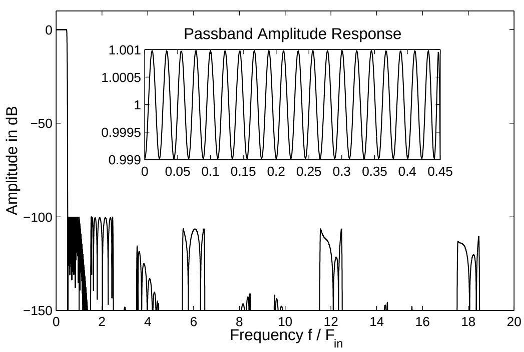
- Note that the same structure can be used for any $F_{out} > F_{in}$.

Direct Design



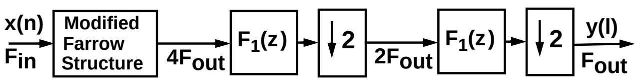
Design with fixed interpolators before the Farrow structure: Simultaneous optimization has been used.





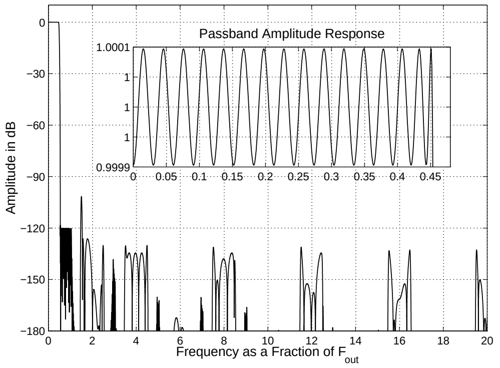
Application D: Down-Sampling Between Arbitrary Sampling Frequencies: First Alternative

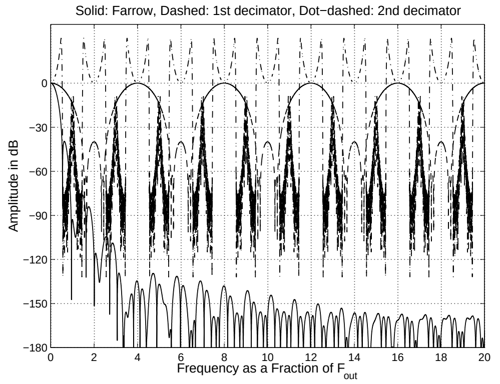
- There exist two alternatives to perform down-sampling.
- The first alternative is to increase the sampling rate to a multiple of the output sampling rate and then to decimate to the desired output sampling rate.
- As an example, consider sampling rate reduction from 48 kHz to 44.1 kHz using the a structure shown below



- The passband edge is 20 kHz and aliasing is allowed into the band between 20 kHz and 44.1/2 kHz.
- The passband ripple is 0.0001 and the minimum stopband attenuation is 120 dB.
- To meet these criteria M = 4 and N = 6 are required by the modified Farrow structure, whereas the orders of the first and second decimator are 4 and 126, respectively.
- The resulting responses are shown on the next page.

Three -stage Decimator using the Modified Farrow structure: Simultaneous optimization has been used.





Application D: Down-Sampling Between Arbitrary Sampling Frequencies: Second Alternative

- The second alternative is to use the transposed modified Farrow structure together with fixed decimators.
- The direct transposed modified Farrow structure is shown in the next transparency.
- Due to the lack of time, this alternative is not considered in details in this talk.
- For more information see

D. Babic, J. Vesma, T. Saramäki, and M. Renfors, "Implementation of the transposed Farrow structure," in *Proc. 2002 IEEE Int. Symp. Circuits and Systems*, Scotsdale, Arizona, USA, 2002, vol. 4, pp. 4–8.

D. Babic, T. Saramäki and M. Renfors, "Conversion between arbitrary sampling frequencies using polynomial-based interpolation filters," in *Proc. Int. Workshop on Spectral Methods and Multirate Signal Processing, SMMSP'02*, Toulouse, France, September 2002, pp. 57–64

- The main advantage of this structure is that the same structure can be used for any input sampling rate $F_{in} > F_{out}$.

Transposed Modified Farrow structure

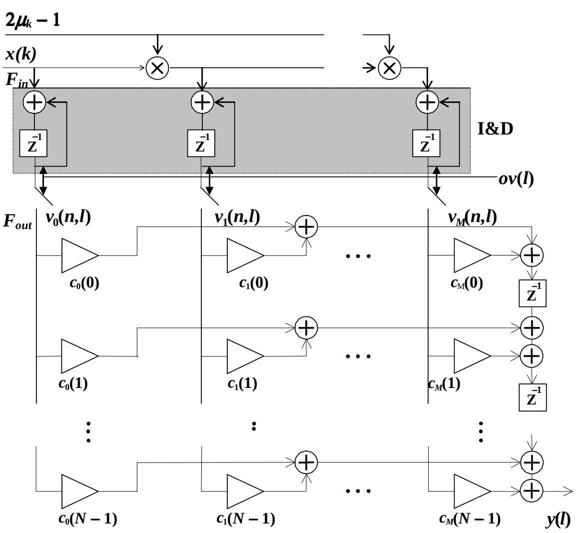


Fig.10. Transposed modified Farrow structure.

Application D: Continuous-Time Signal Processing

- The Farrow structure can be easily generalized to process digitally the reconstructed signal $y_a(t)$.
- These applications include, among others, determining the derivative or the integral of $y_a(t)$.
- The derivative is widely utilized, for example, in finding the location of the maximum or minimum of the signal.
- The integral can be used to calculate the energy of the signal over the given time interval.
- We concentrate on determining the derivative of $y_a(t)$.

Generalized Farrow Structure for Determining the Derivative of $y_a(t)$

- In the intervals $nT_{in} \le t < (n+1)T_{in}$ for $n=0, 1, 2, \cdot y_a(t)$ can be expressed as

$$y_a(t)\big|_{t=(n+\mu)T_{in}} = p(n,\mu) = \sum_{m=0}^{M} v_m(n)[2\mu-1]^m , \qquad (23)$$

where the $v_m(n)$'s are the output samples of the FIR branch filters in the modified Farrow structure.

- The derivative of $y_a(t)$ in the intervals is thus given by

$$\frac{dy_a(t)}{dt}\Big|_{t=(n+\mu)T_{in}} = \frac{dp(n,\mu)}{d\mu} = \sum_{m=0}^{M} v_m(n)2m[2\mu-1]^{m-1}. \qquad (24)$$

- The derivative of $y_a(t)$ at $t=(n+\mu)T_{in}$ can be determined by multiplying the $v_m(n)$'s by $2m(2\mu-1)^{m-1}$, instead of $(2\mu-1)^m$ in the modified Farrow structure.
- For estimating the derivative, it is desired that $H_a(j2\pi f)$ approximates $1/(2\pi f)$ in the passband with the weighting equal to $2\pi f$.
- In the stopband, it is desired to approximate zero with a constant weight.

Example on the Derivative Approximation

- It is desired to estimate the continuous-time derivative of a discrete-time ECG signal shown in Fig. 11(a).
- Figures 11(b) and 11(c) show the continuous-time interpolated signal and the derivative signal, respectively.
- For $h_a(t)$ used for determining the derivative signal, $N=8$, $M=5$, and the passband and stopband edges are located at $0.35F_s$ and $0.65F_s$, respectively.
- When using the minimax optimization criterion with weigting equal to 0.035 in the passband and equal to unity in the stopband, we end up with $h_a(t)$ with the amplitude and impulse responses shown in Fig. 12.

Continuous-time processing of an ECG signal

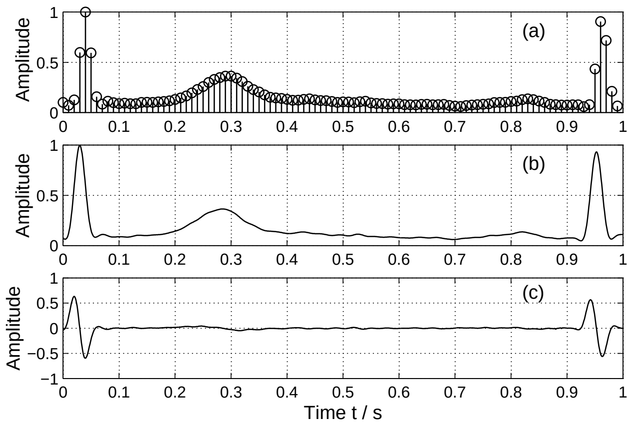
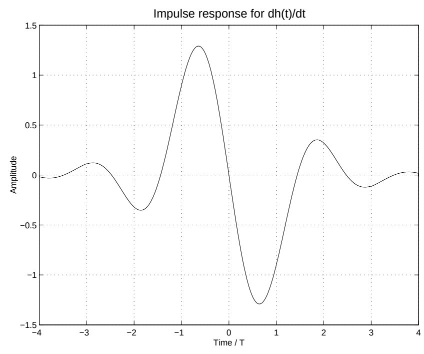
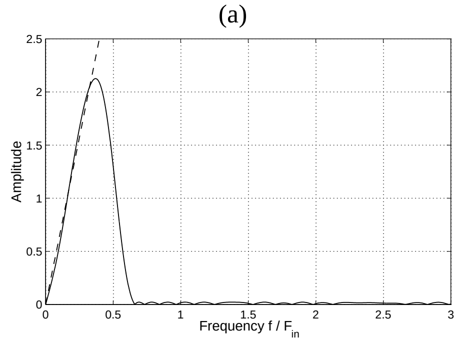


Fig.11. Continuous-time processing of an ECG signal. (a) Discrete-time ECG signal. (b) Interpolated continuous-time signal. (c) Continuous-time derivative.

Characteristics of the differentiator



(a)



(b)

Fig.12. Optimized differentiator. (a) Impulse response. (b) Amplitude response

Application C: Symbol Synchronization in Digital Receivers

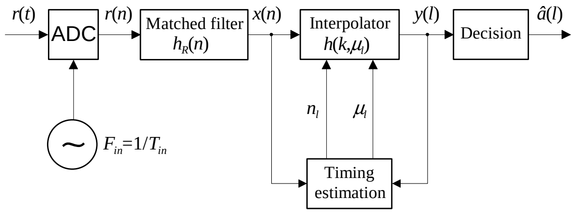


Fig. 13. Digital receiver with non-synchronized sampling.

- The sampling of the received signal is performed by a fixed sampling clock, and thus, sampling is not synchronized to the received symbols.
  - ⇒ timing adjustment must be done by digital methods after sampling.
- Can be done by using interpolation filter.
- Advantages of nonsynchronized sampling:
  - separates the analog and digital parts
  - easy to change the sampling rate
  - sampling frequency does not have to be a multiple of the symbol frequency (only high enough to avoid aliasing)
  - no need for complex PLL circuit.

Practical Case

- When deriving the frequency-domain specifications for the anti-imaging filter $h_a(t)$, it is assumed that
  1) The pulse shape of the transmitted signal has the excess bandwidth of α and the ratio between the sampling rate F_{in} and the symbol rate is R.
  2) In order to avoid aliasing, it is required that $R \geq (1+\alpha)$.
- Based on these assumptions, the input signal of the interpolator $x(n)$ contains the desired component in the frequency range $[0, \beta F_{in}]$, where $\beta = (1+\alpha)/R/2$ and undesired images in the bands $[(k-\beta)F_{in}, (k+\beta)F_{in}]$ for $k = 1, 2, \cdots$.
- Therefore, the desired function for $H_a(j2\pi f)$ is specified by

$$D(f) = \begin{cases} 1/F_{in} & \text{for } 0 \leq f \leq \beta F_{in} \\ 0 & \text{for } f \in \Omega_s, \end{cases} \qquad (25)$$

where the stopband region, denoted by Ω_s, can be selected as

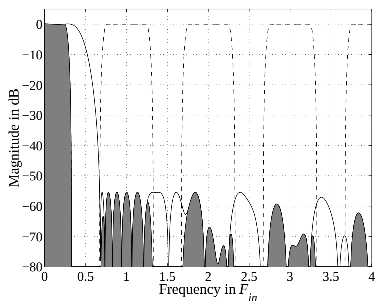
$$\Omega_s = [(1-\beta)F_{in}, \infty) \qquad (26a)$$

or

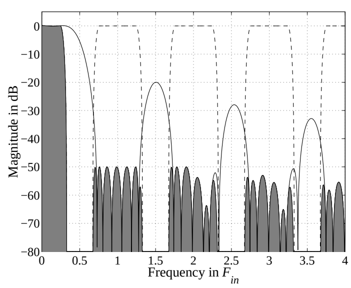
$$\Omega_s = \bigcup_{k=1}^{\infty} [(k-\beta)F_{in}, (k+\beta)F_{in}] \qquad (26b)$$

- Two polynomial-based interpolators have been designed in the minimax sense to illustrate the use of the above-mentioned specifications.
- It is assumed that the received signal has a raised cosine pulse shape with the excess bandwidth of $\alpha = 0.15$ and the oversampling ratio is $R = 1.75$.
- The passband edge for both filters is $f_p = \beta F_{in} = 23/70 F_{in}$ ($\approx 0.33 F_{in}$)
- Furthermore, it is required that the passband distortion is less than $\delta_p = 0.01$ and the minimum stopband attenuation is $A_s = 50$dB.
- The first filter has a uniform stopband .In order to meet the specifications, $\boldsymbol{N=8}$ and $\boldsymbol{M=3}$ are required, as shown in Figure 21(a) on the next page
- The second filter has a non-uniform stopband as given by Eq. (31b) and the spectrum of the raised cosine pulse shape is used as a weighting function. In this case $\boldsymbol{N=6}$ and $\boldsymbol{M=3}$ meets the requirements giving $A_s = 50.0$dB and $\delta_p = 0.01$, as shown in Figure 21(b) on the next page.

Fig. 14. The magnitude response of the interpolation filter (solid line), the spectrum for the raised cosine pulse (dashed line) and for the reconstructed signal $y_a(t)$ (dark area). (a)With uniform stopband. (b) With non-uniform stopbands having the raised cosine weighting.



(a)



(b)

Properties of Minimax Case A designs

- Case A: The minimum even value of N can be estimated by

$$N = 2\left\lceil \frac{-20\log_{10}\left(\sqrt{\delta_p \delta_s}\right) - 8.4}{7.6\left(f_s - f_p\right)/F_{in}} \right\rceil \qquad (27)$$

where δ_p and δ_s are the maximum deviations of the amplitude response from unity for $f \in [0, f_p]$ and the maximum deviation from zero for $f \in [f_s, \infty)$.

- Here, $\lceil x \rceil$ stands for the smallest integer that is larger or equal to x.
- It has been observed that in most cases the above estimation formula is rather accurate with only a 2 % error.
- The next problem is to find the minimum value of M to meet the criteria.
- To illustrate this the following specifications are considered:

  Specifications 1: The passband and stopband edges are at $f_p = 0.25F_{in}$ and at $f_s = 0.75F_{in}$.
  Specifications 2: The passband and stopband edges are at $f_p = 0.25F_{in}$ and at $f_s = 0.5F_{in}$.
  Specifications 3: The passband and stopband edges are at $f_p = 0.375F_{in}$ and at $f_s = 0.675F_{in}$.
  Specifications 4: The passband and stopband edges are at $f_p = 0.375F_{in}$ and at $f_s = 0.5F_{in}$.

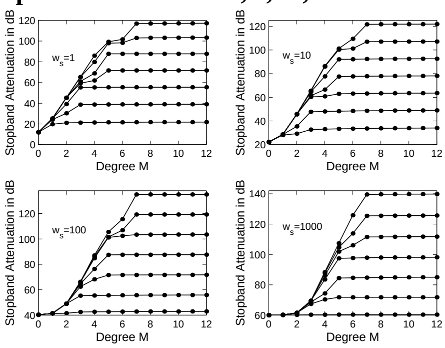
Properties of Minimax Case A designs

- In each case, several filters have been designed in the minimax sense with the passband weighting equal to unity and stopband weightings of $W_s = 1$, $W_s = 10$, $W_s = 100$, and $W_s = 1000$.
- M, the degree of the polynomial in each subinterval, varies from 0 to 12. N, the number of intervals varies from 2 to the smallest integer for which the stopband ripple for the amplitude response is less than or equal to 0.0001 (100 dB) for $W_s = 1$.
- For Specifications 1, 2, 3, and 4,, the corresponding smallest values of N are 12, 24, 24, and 48, respectively. Recall that N is an even integer.
- The following two pages give the results for Case A.
- For other cases, N is either the same or should be incrased by two.
- For Case C the passband and stopband edges satisfy

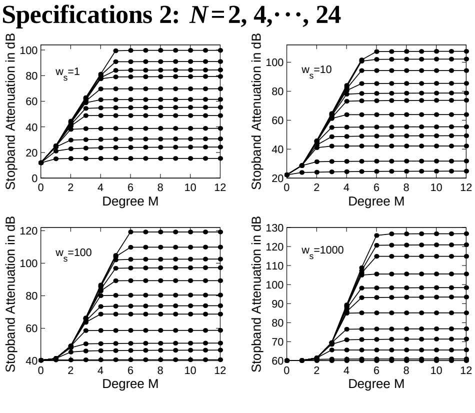
$$f_p = (1-\rho)F_{in}/2, \quad f_s = (1+\rho)F_{in}/2.$$

Specifications 1 and 2:

Specifications 1: $N = 2, 4, \cdots, 14$

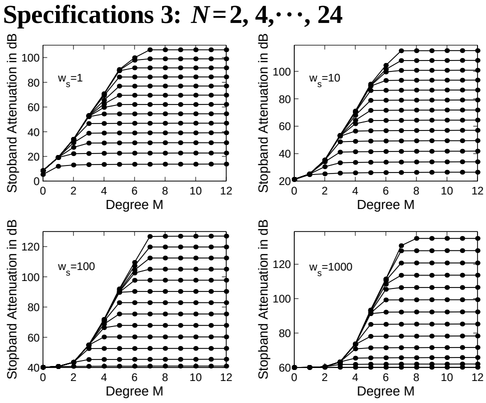


Specifications 2: $N = 2, 4, \cdots, 24$

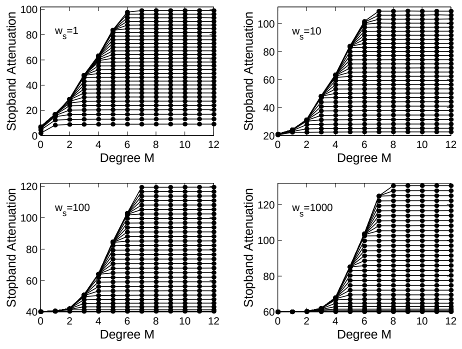


Specifications 3 and 4:

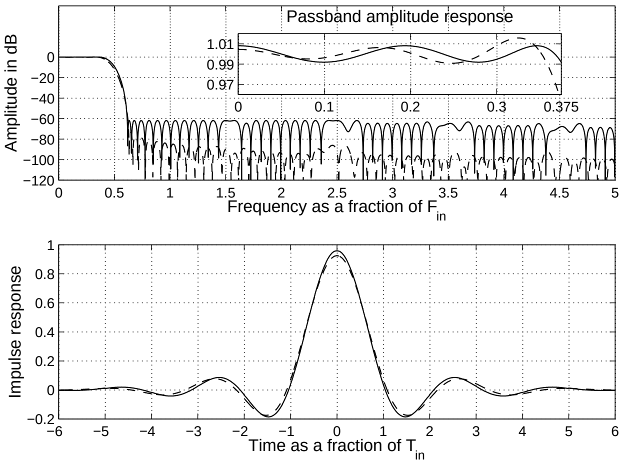
Specifications 3: $N = 2, 4, \cdots, 24$



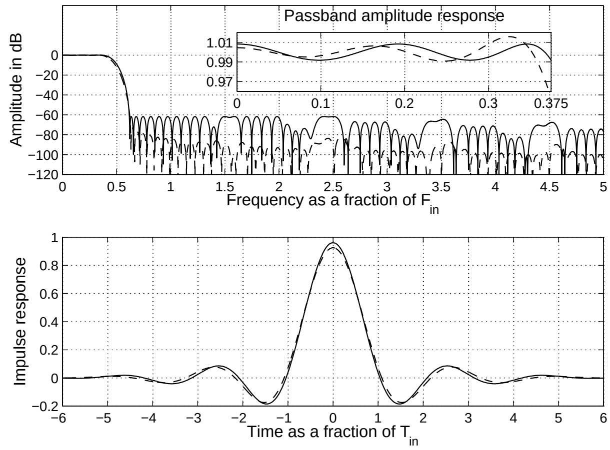
Specifications 4: $N = 2, 4, \cdots, 48$



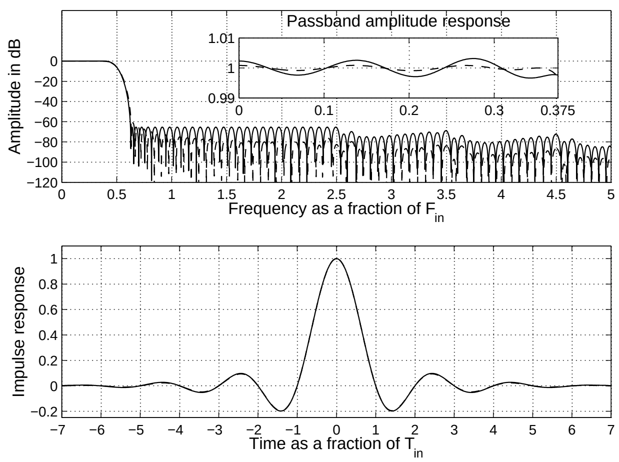
Case A: $f_s = 0.625F_{in}$, $f_p = 0.375F_{in}$, $\delta_p = 0.01$, $\delta_s = 0.001$: N=12; Minimax (solid line): M= 4; Least-squared (dashed line): M= 5



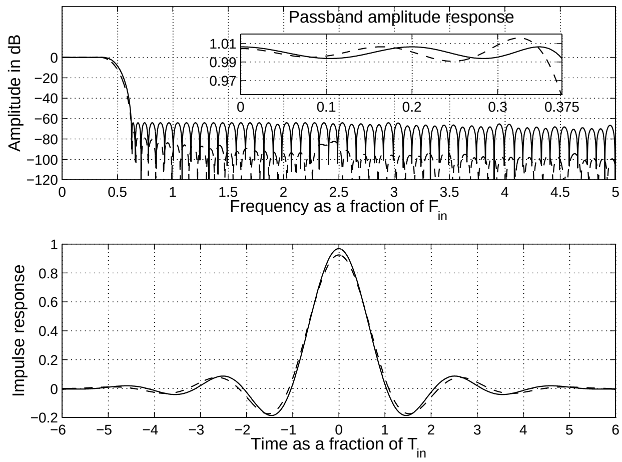
Case B: $f_s = 0.625F_{in}$, $f_p = 0.375F_{in}$, $\delta_p = 0.01$, $\delta_s = 0.001$: N=12; Minimax (solid line): M= 4; Least-squared (dashed line): M= 5



Case C: $f_s = 0.625F_{in}$, $f_p = 0.375F_{in}$, $\delta_p = 0.01$, $\delta_s = 0.001$: N=14; Minimax (solid line): M= 5; Least-squared (dashed line): M= 5



Case D: $f_s = 0.625F_{in}$, $f_p = 0.375F_{in}$, $\delta_p = 0.01$, $\delta_s = 0.001$: N=12; Minimax (solid line): M= 5; Least-squared (dashed line): M= 5



Conclusion

- An efficient approach has been described for interpolating new sample values between the existing discrete-time samples.
- This approach has the following advantages:
    - Design directly in the frequency domain is straightforward.
    - The overall system has an efficient implementation form.
    - The analysis of the system performance is easy.
    - There exist several DSP applications.

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