APPENDIX A: Continuous-Time Signals and Systems and the Basic Tools for Their Analysis and Synthesis

- The purpose of this appendix is to give a short review of continuous-time signals and systems as well as the basic mathematical tools for studing these signals and systems.
- This review consists of the following topics:

  I. Classification of continuous-time signals into periodic, transient, and random signals.

  II. Frequency-domain representation of periodic signals: Fourier series.

  III. Frequency-domain representation of aperiodic signals: Fourier transform.

  IV. Generalized Fourier transform including Fourier series.

  V. Generalization of Fourier Transform: Laplace transform.

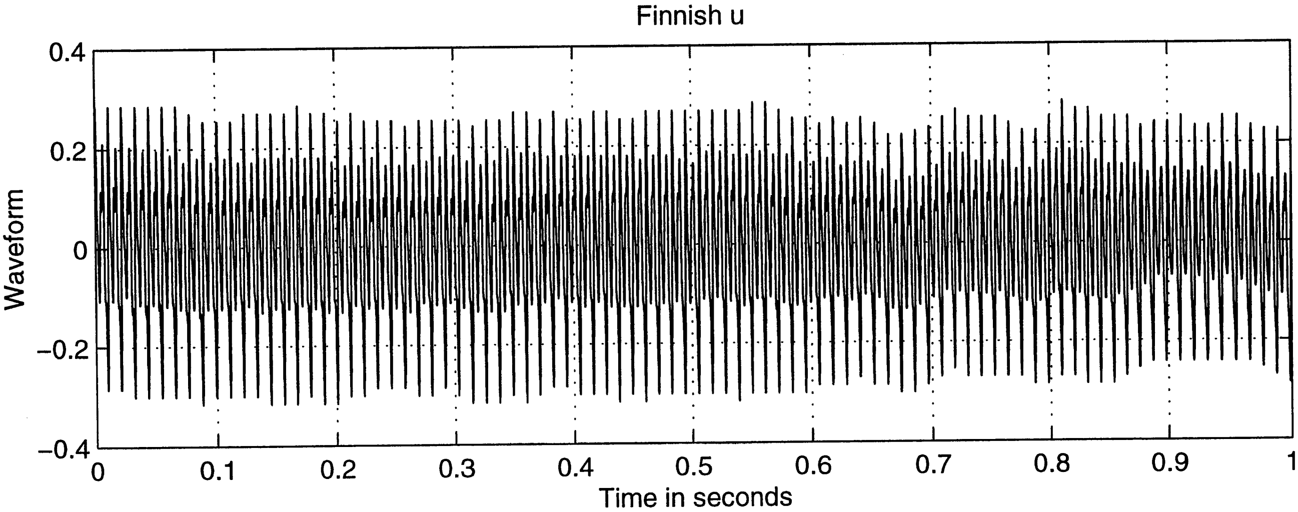
  VI. The use of the Laplace and Fourier transforms in studing the input-output relation of a linear time-invariant system.

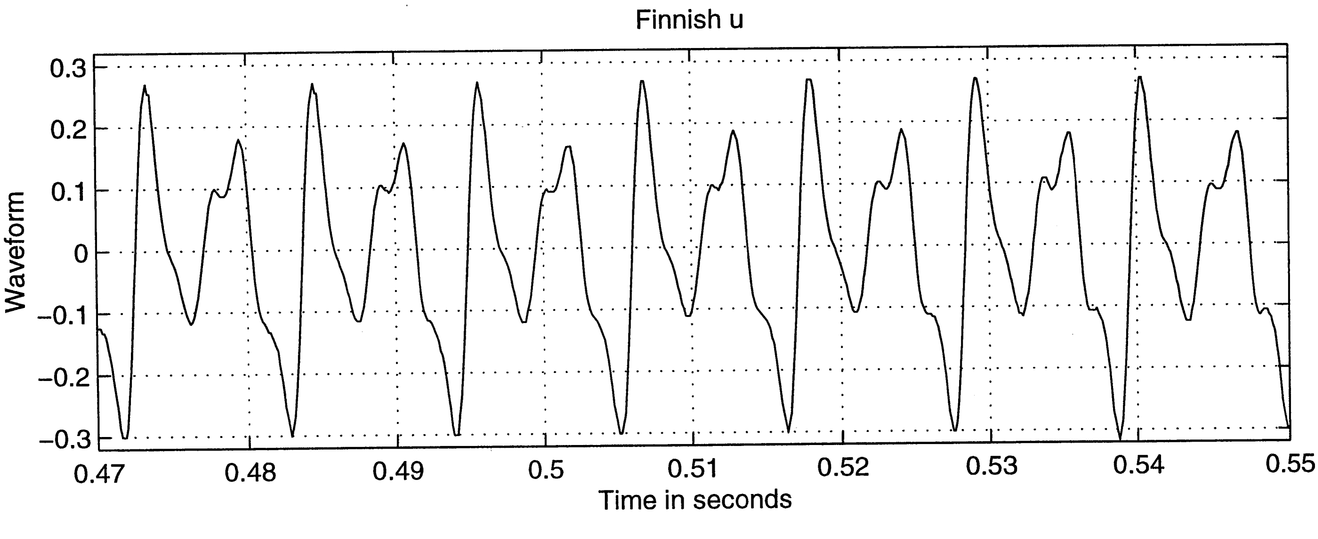
Classification of continuous-time signals

- The continuous-time signals, denoted by $x_a(t)$, can be classified into:

1. **Periodic signals:**
   - In theory, for $-\infty < t < \infty$ the signal satisfies $x_a(t + T_0) = x(t)$, where T_0 is the periodity of the signal.
   - In practice, this is true for $t_1 < t < t_2$, where $t_2 - t_1$ is very large compared to the period T_0. In addition, the signals are typically just nearly periodic.
   - Typical examples are waveforms of vowels and an eletrocardiogram as well as the waveform generated when playing one note using a piano, a guitar, or a harmonica. The actual waveform can be picked up by using a microphone.
   - Page 3 shows the waveform for the Finnish phoneme 'u', whereas Page 4 shows the waveform of the lower f played by a harmonica.

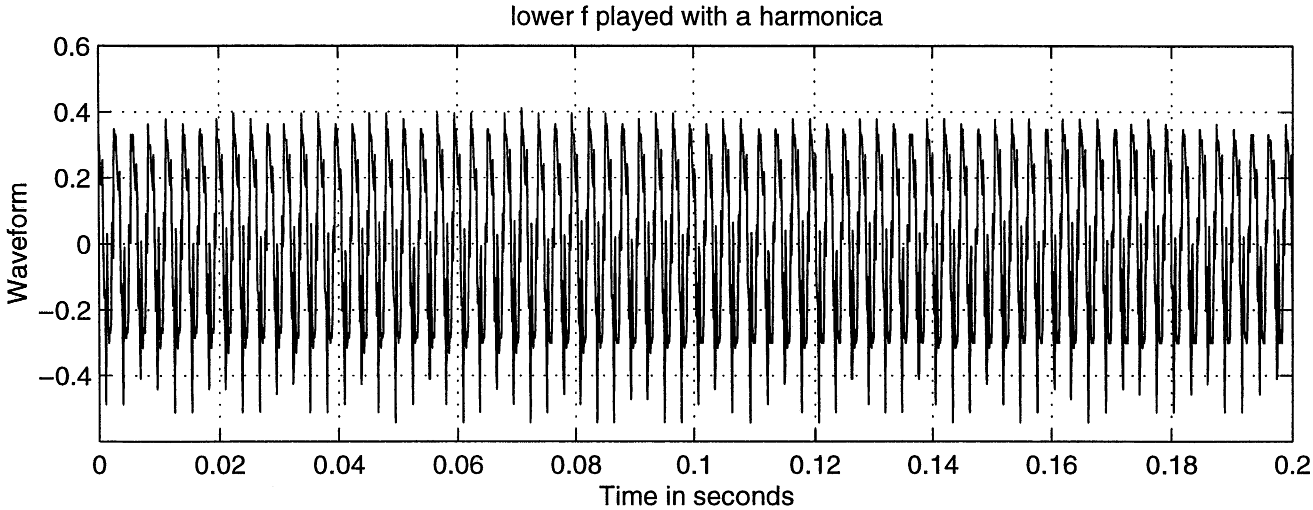
Waveform for the the Finnish phoneme 'u'

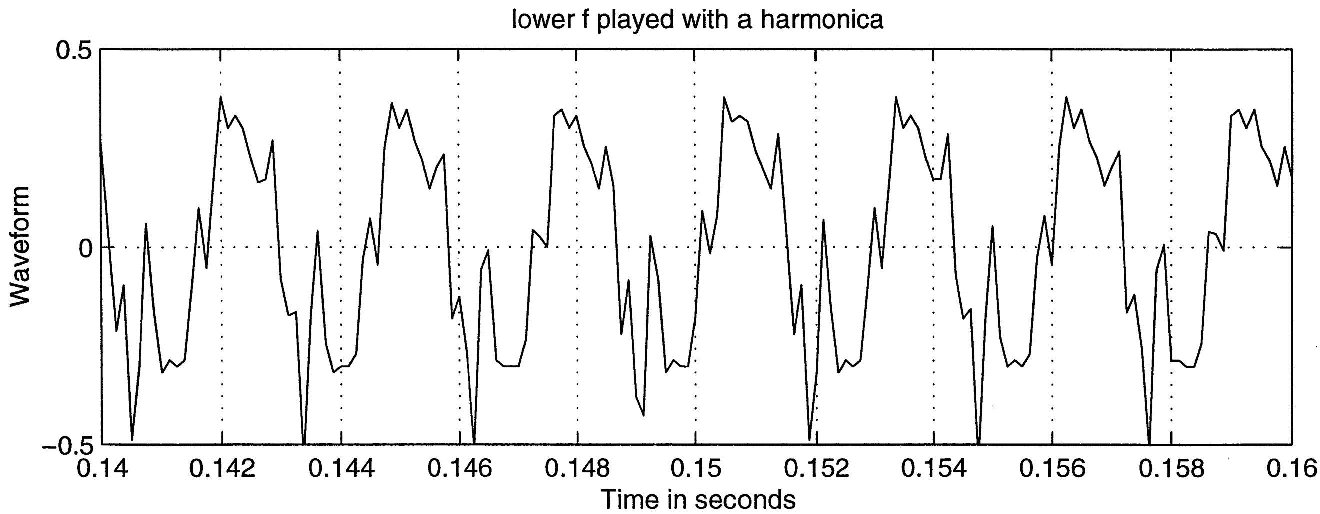




Waveform for the lower f played by a harmonica

- The fundamental frequency is approximately 350 Hz.





2. Transient signals:

- These are pulse-type signals of finite duration.
- Typical examples are waveforms of the phonemes 'k', 'p', and 't'.
- Also, when playing a drum, we generate transients.
- Page 6 shows the waveform for the Finnish phoneme 't'.

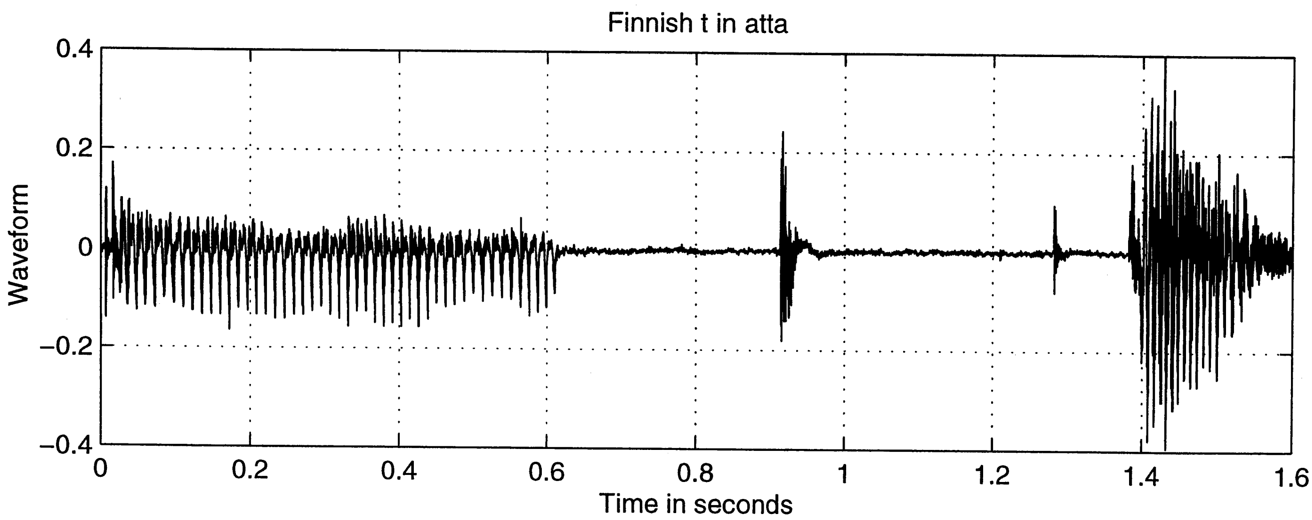
3. Random and weakly correlated signals:

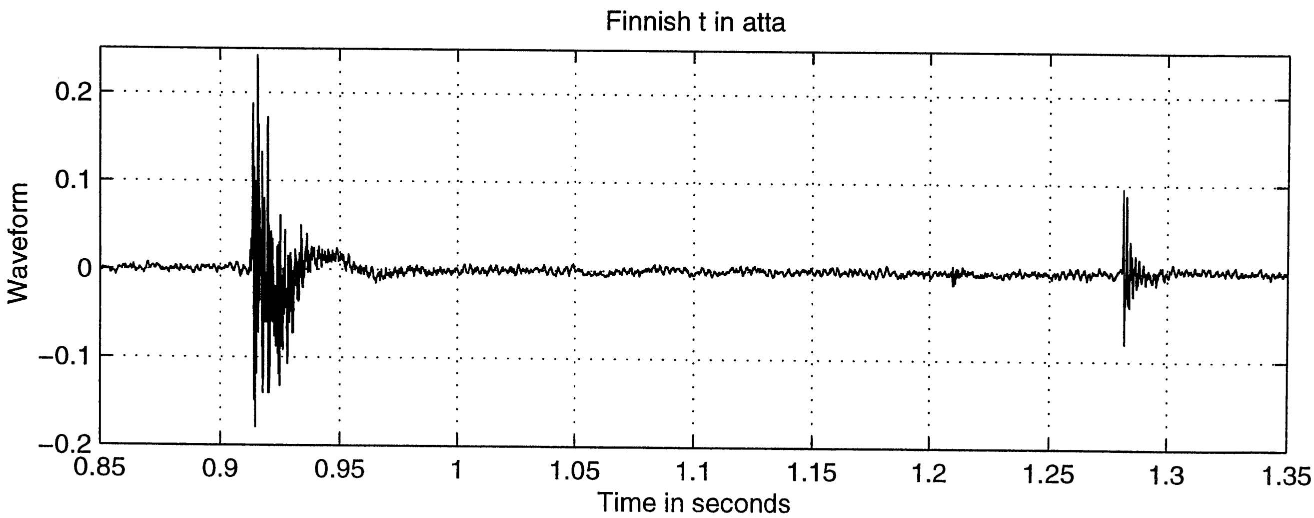
- Typical examples are Gaussian noise as well as the waveform of the Finnish phoneme 's'. See the next page.

- Page 8 shows shows the waveform for the Finnish word 'sieppo'.

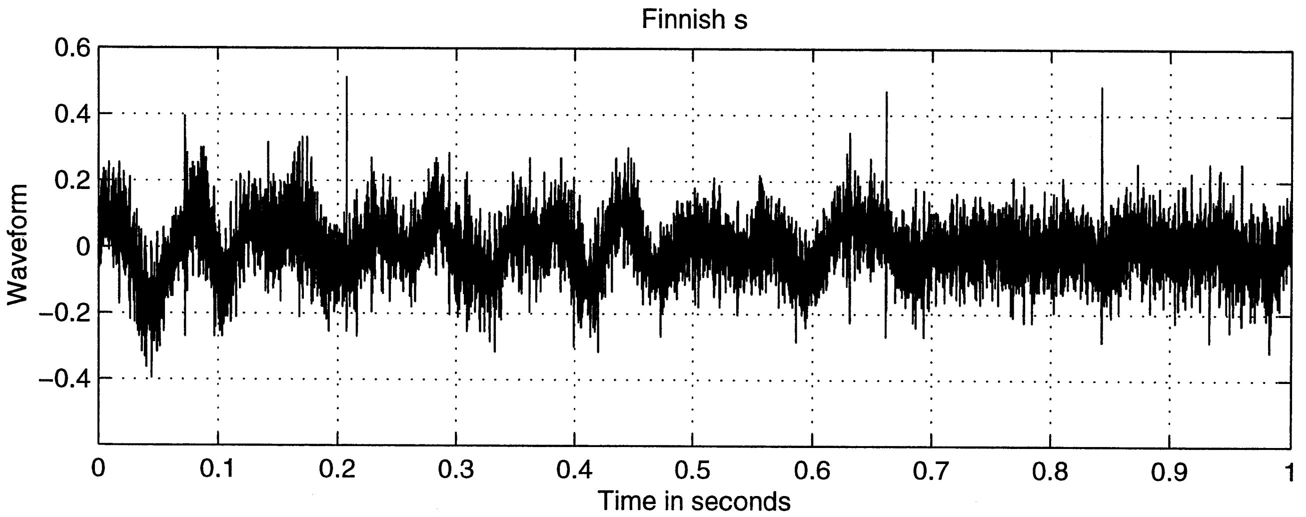
Waveform for the Finnish phoneme 't' in 'aatta'

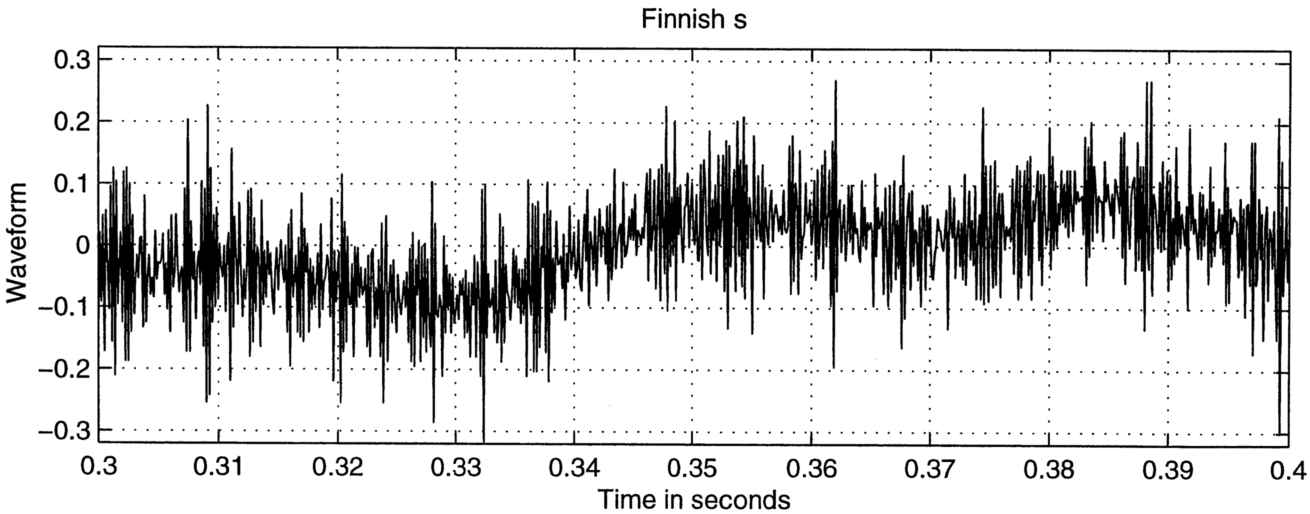
- The two transients within the silence correspond to two 't's. These two phonemes have been pronounced very strongly.





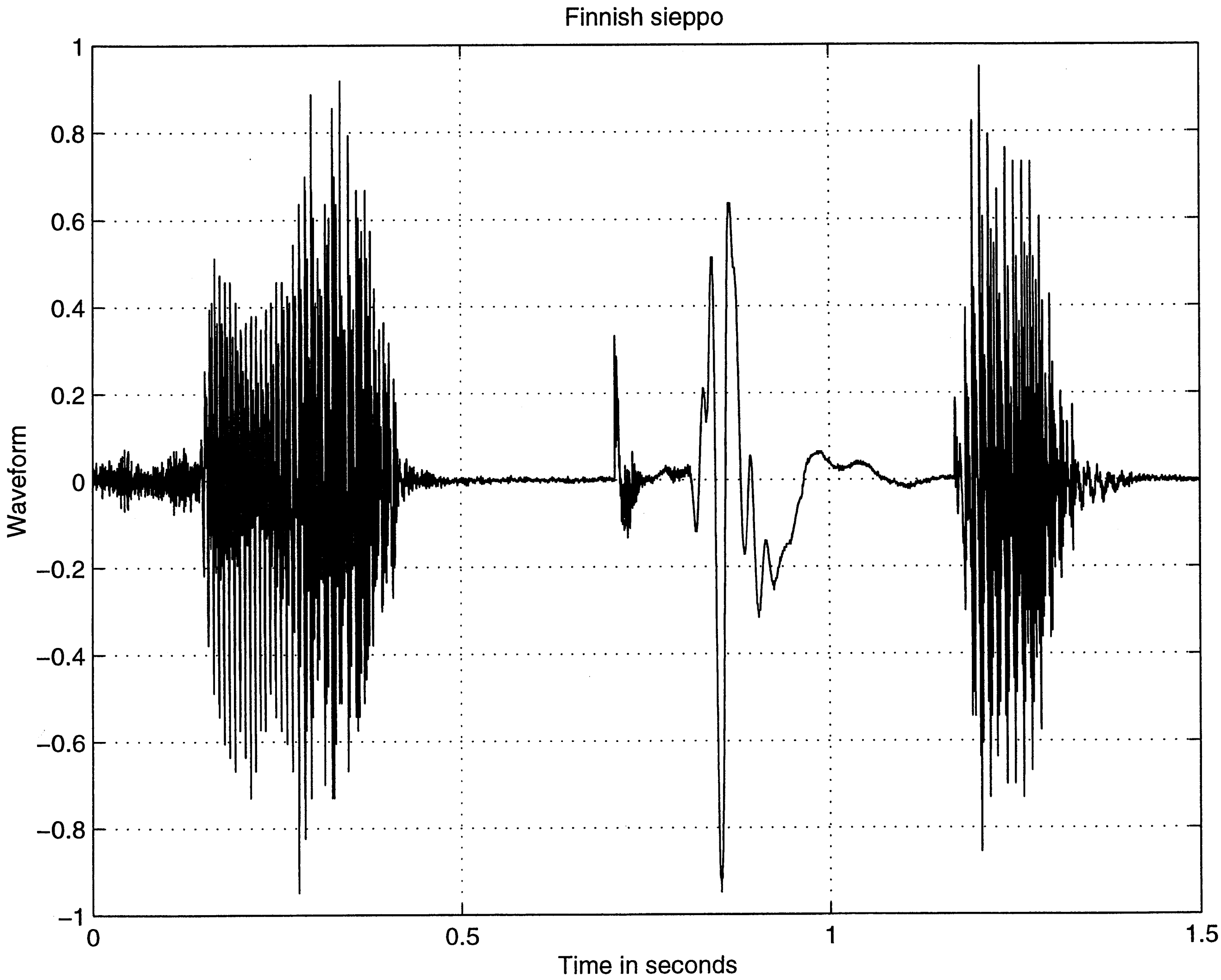
Waveform for the Finnish phoneme 's'





Waveform for the Finnish word 'sieppo'

- 'p's have been pronounced very strongly.



Frequency-domain representation of periodic signals: Fourier series.

- From the theory of mathematics, it is well-known that any real-valued continuous-time signal $x_a(t)$ is expressible on the interval $[t_0 - T_0/2,\ t_0 - T_0/2]$ in the forms shown in the next transparency.
- The different expression forms are due to the facts $e^{j\omega} = \cos\omega + j\sin\omega$, $\cos\omega = [e^{j\omega} + e^{-j\omega}]/2$, and $\sin\omega = [e^{j\omega} + e^{-j\omega}]/(2j)$.
- Note also that if a complex number $a + jb$ is expressed in the polar form $a+jb = Re^{j\phi}(= R\cos\phi + jR\sin\phi)$, then $R = \sqrt{a^2 + b^2}$ and $\phi = \text{atan2}\ (b, a)$, where atan2 (b, a) is defined as shown on the next page. Note that atan2 (b, a) takes on values between $-\pi$ and π, whereas $\tan^{-1}(b/a)$ takes on values just between $-\pi/2$ and $\pi/2$, meaning that if the later one is used, then only positive values are allowed for a.
- If the interval under consideration, denoted by $[t_1,\ t_2]$ includes the interval $[t_0 - T_0/2,\ t_0 - T_0/2]$ and $x_a(t)$ satisfies on $[t_1,\ t_2]$

$$x_a(t + T_0) = x_a(t),$$

Expressions for a real-valued continuous-time signal $x_a(t)$ on the interval $[t_0 - T_0/2,\ t_0 - T_0/2]$

$$
\begin{aligned}
x_a(t) &= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)\,, \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} (A_n \cos n\omega_0 t + \phi_n)
\end{aligned}
$$

where

$$
\begin{aligned}
\omega_0 &= 2\pi/T_0, \\
c_n &= 1/T_0 \int_{t_0-T_0/2}^{t_0+T_0/2} x_a(t) e^{-jn\omega_0 t} dt, \\
a_n &= 2/T_0 \int_{t_0-T_0/2}^{t_0+T_0/2} x_a(t) \cos(n\omega_0 t) dt, \\
b_n &= 2/T_0 \int_{t_0-T_0/2}^{t_0+T_0/2} x_a(t) \sin(n\omega_0 t) dt, \\
c_0 = a_0, \quad c_n &= \frac{1}{2}(a_n - jb_n) = A_n e^{j\phi_n}, \\
c_{-n} &= \frac{1}{2}(a_n + jb_n) = A_n e^{-j\phi_n}, \\
a_n = c_n + c_{-n}, &\quad b_n = j(c_n - c_{-n}),
\end{aligned}
$$

and

$$
A_n = \sqrt{(a_n^2 + b_n^2)} = |F_n|, \quad \phi_n = -\text{atan2}(b_n, a_n)
$$

with

$$
\text{atan2}(b_n, a_n) = \begin{cases} \tan^{-1}(b_n/a_n), & a_n \geq 0 \\ \pi + \tan^{-1}(b_n/a_n), & a_n < 0 \text{ and } b_n \geq 0 \\ -\pi + \tan^{-1}(b_n/a_n), & a_n < 0 \text{ and } b_n < 0. \end{cases}
$$

that is, $x_a(t)$ is periodic on $[t_1,\ t_2]$ with periodicity equal to T_0, then the expressions shown in the previous page are valid on the overall interval $[t_1,\ t_2]$.

- For practically interpreting the formulas of the previous page, we consider a simple example of the next page.

Example: $x_a(t)$ satisfies for $-\infty < t < \infty$, (see the next page)

$$x_a(t + T_0) = x_a(t)$$

and on the interval $[t_0 - T_0/2,\ t_0 + T_0/2]$

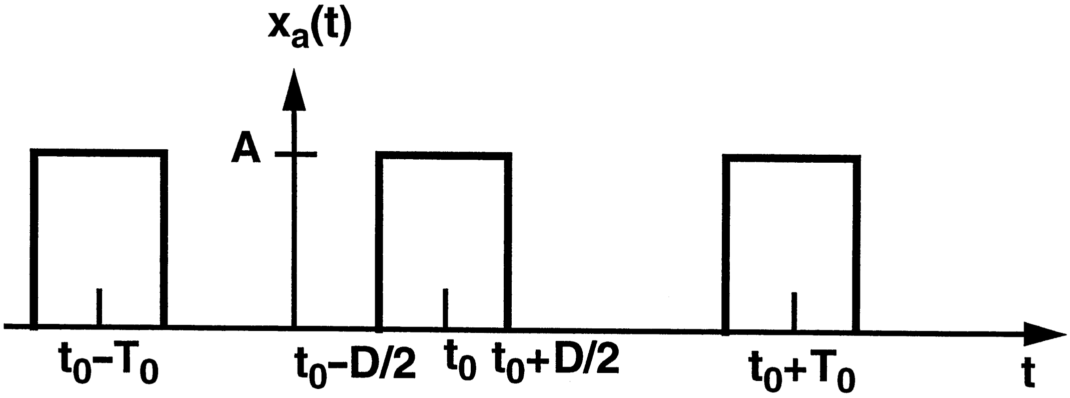
$$x_a(t) = \begin{cases} A, & \text{for } t \in [t_0 - D/2,\ t_0 + D/2] \\ 0, & \text{elsewhere.} \end{cases}$$

- The Fourier series for $x_a(t)$ can be determined by applying the equations of page 10. We obtain $(\omega_0 = 2\pi/T_0)$

$$\begin{aligned}
c_n &= 1/T_0 \int_{t_0-T_0/2}^{t_0+T_0/2} x_a(t) e^{-jn\omega_0 t} dt \\
&= 1/T_0 \int_{t_0-D/2}^{t_0+D/2} A e^{-jn\omega_0 t} dt \\
&= \frac{-A}{jn\omega_0 T_0} \Big/_{t_0-D/2}^{t_0+D/2} A e^{-jn\omega_0 t} \\
&= \frac{2A}{n\omega_0 T_0} \left[\frac{e^{jn\omega_0(-t_0+D/2)} - e^{jn\omega_0(-t_0-D/2)}}{2j}\right] \\
&= \frac{2A}{n\omega_0 T_0} e^{-jn\omega_0 t_0} \left[\frac{e^{jn\omega_0 D/2} - e^{-jn\omega_0 D/2}}{2j}\right] \\
&= \frac{AD}{T_0} e^{-jn\omega_0 t_0} [\sin(n\omega_0 D/2)/(n\omega_0 D/2)] \\
&= \frac{AD}{T_0} e^{-jn\omega_0 t_0} \operatorname{sinc}(n\omega_0 D/2),
\end{aligned}$$

where we define $\operatorname{sinc}(x) = (\sin x)/x$.

- The periodic signal under consideration:



- c_n is thus expressible as

$$c_n = A_n e^{j\phi_n}, \quad A_n = \frac{AD}{T_0}\text{sinc}\,(n\omega_0 D/2), \quad \phi_n = -n\omega_0 t_0.$$

- $c_{-n} = A_n e^{-j\phi_n}$ and $x_a(t)$ is expressible as

$$x_a(t) = A_0 + \sum_{n=1}^{\infty} x_a^{(n)}(t),$$

where

$$\begin{aligned} x_a^{(n)}(t) &= c_n e^{jn\omega_0 t} + c_{-n} e^{-jn\omega_0 t} \\ &= A_n e^{j(n\omega_0 t + \phi_n)} + A_n e^{-j(n\omega_0 t + \phi_n)} \\ &= 2A_n \cos(n\omega_0 t + \phi_n). \end{aligned}$$

Frequency-Domain Interpretation of $x_a^{(n)}(t) = 2A_n \cos(n\omega_0 t + \phi_n)$

- Since $x_a^{(n)}(t) = 2A_n \cos(n\omega_0 t + \phi_n)$ is expressible as

$$x_a^{(n)}(t) = A_n e^{j(n\omega_0 t + \phi_n)} + A_n e^{-j(n\omega_0 t + \phi_n)},$$

  it can be expressed as a sum of two phasors (see the next page)

$$\text{Phasor}^{(+)}(t) = A_n e^{j(n\omega_0 t + \phi_n)}$$
$$= A_n \cos(n\omega_0 t + \phi_n) + jA_n \sin(n\omega_0 t + \phi_n)$$

  and

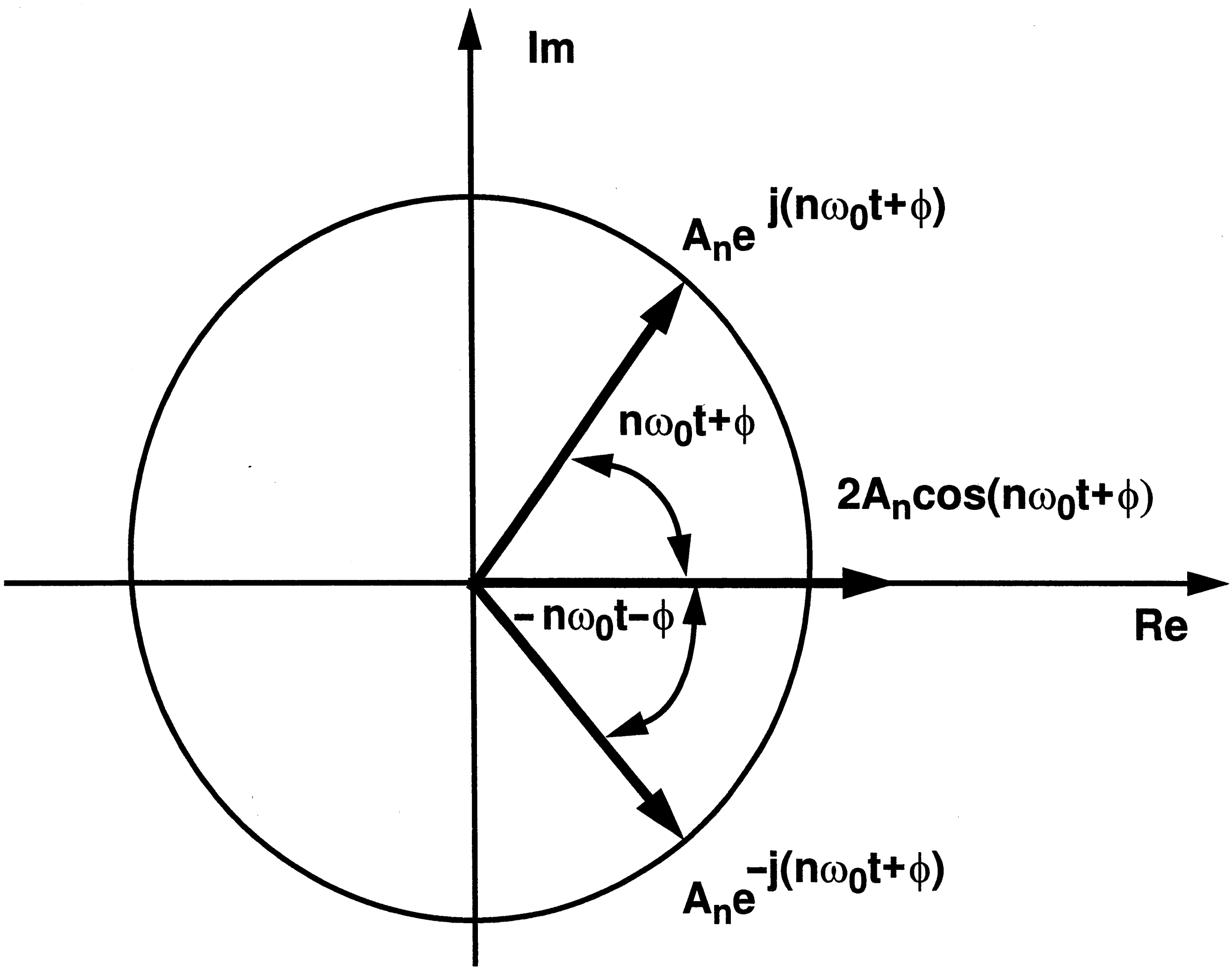
$$\text{Phasor}^{(-)}(t) = A_n e^{-j(n\omega_0 t + \phi_n)}$$
$$= A_n \cos(n\omega_0 t + \phi_n) - jA_n \sin(n\omega_0 t + \phi_n),$$

  which form a conjugate pair.

- $\text{Phasor}^{(+)}(t)$ and $\text{Phasor}^{(-)}(t)$ are rotating along the circle of radius A_n anticlockwise and clockwise with a speed of $n\omega_0$ radians per second such that their angles at the time instant t are $n\omega_0 t + \phi_n$ and $-(n\omega_0 t + \phi_n)$, respectively.

- The sum of these two phasors is two times the real part of both phasors.

Construction of $x_a^{(n)}(t) = 2A_n \cos(n\omega_0 t + \phi_n)$ using two phasors



- This sum achieves the same value after the time period of T_0/n, since the phasors have been rotated by by $\pm n\omega_0(T_0/n) = \pm\pi$ ($\omega_0 = 2\pi/T_0$) so that they are at the same position.
- Therefore, the periodicity of $x_a^{(n)}(t) = 2A_n \cos(n\omega_0 t + \phi_n)$ is T_0/n.
- The overall signal is expressible as

$$x_a(t) = A_0 + \sum_{n=1}^{\infty} 2A_n \cos(n\omega_0 t + \phi_n)$$

so that it consists of a constant term and cosine terms with the fundamental angular frequency ω_0 as well as its harmonics $n\omega_0$ for $n = 2, 3, ...$

- Any periodic signal is expressible in the above form. After knowing T_0, we know the fundamental harmonic frequency $\omega_0 = 2\pi/T_0$. Then the signal is completely determined after knowing the oscillation amplitudes A_n and the phase shifts ϕ_n of the cosine terms.
- The next page shows in our example case the A_n's and $n\omega_0$'s for various values of T_0 and D.

A_n's and $n\omega_0$'s for various values of $T_0 \equiv T$ and $D \equiv d$. A_0 is the constant occuring at $\omega = 0$, A_1 and A_{-1} are situated at $\omega = \pm\omega_0$ and so on

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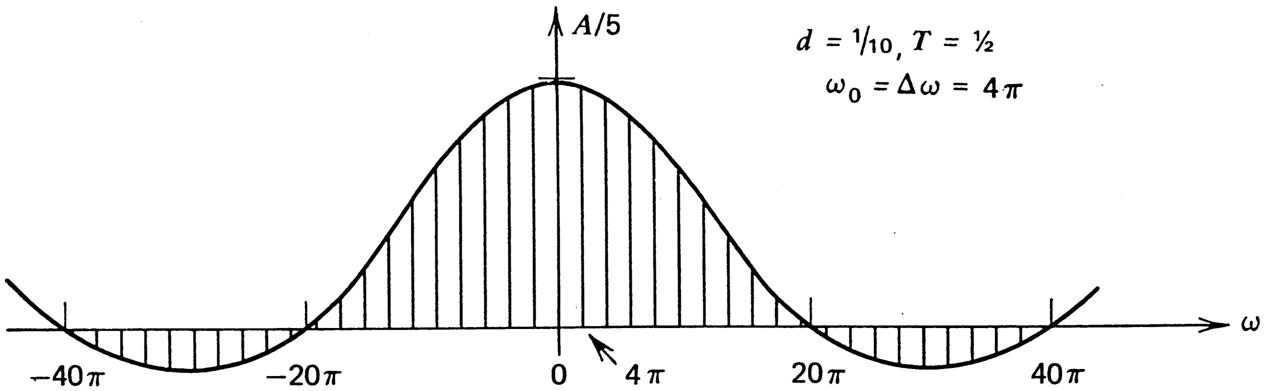
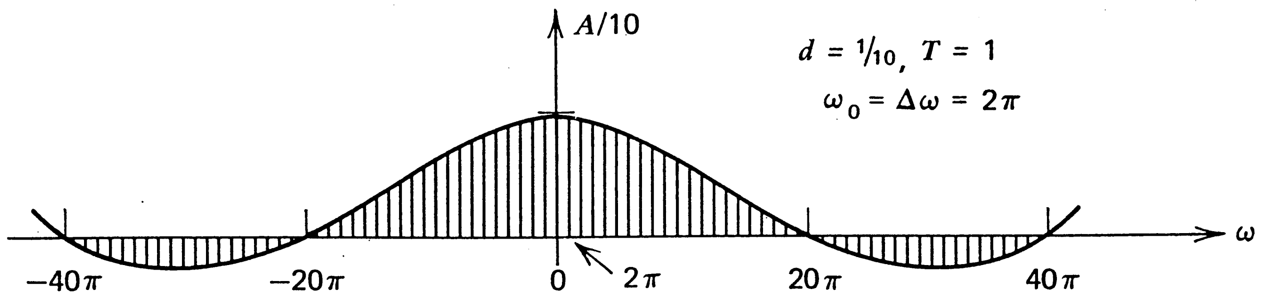
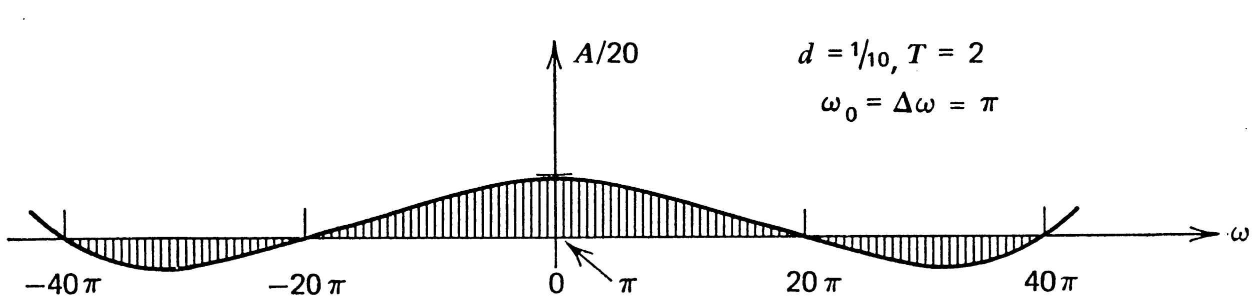






Periodic signals expressed in terms of the 'real' frequency

- Previously, we expressed our cosine terms using the angular frequencies which are measured by radians per second.
- In practice, the 'real frequency' f is measured in Hz and it is realated to ω trough

$$f = \omega/(2\pi).$$

- Using the substitution

$$n\omega_0 = 2\pi n f_0,$$

our periodic signal takes the form

$$x_a(t) = c_0 + \sum_{n=1}^{\infty} 2A_n \cos(n2\pi f_0 t + \phi_n),$$

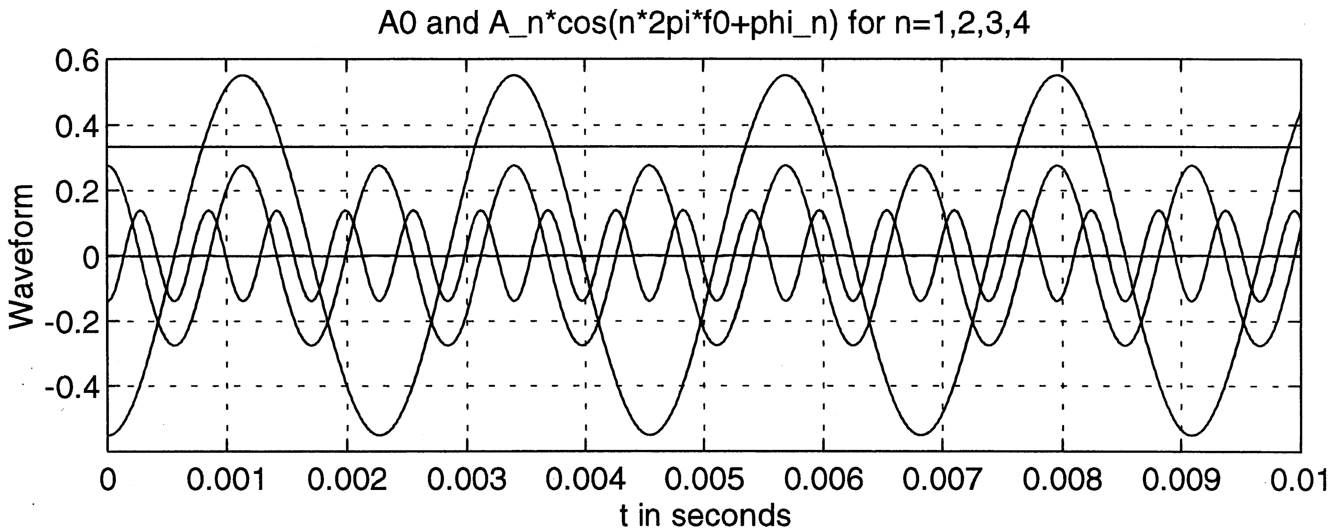
where

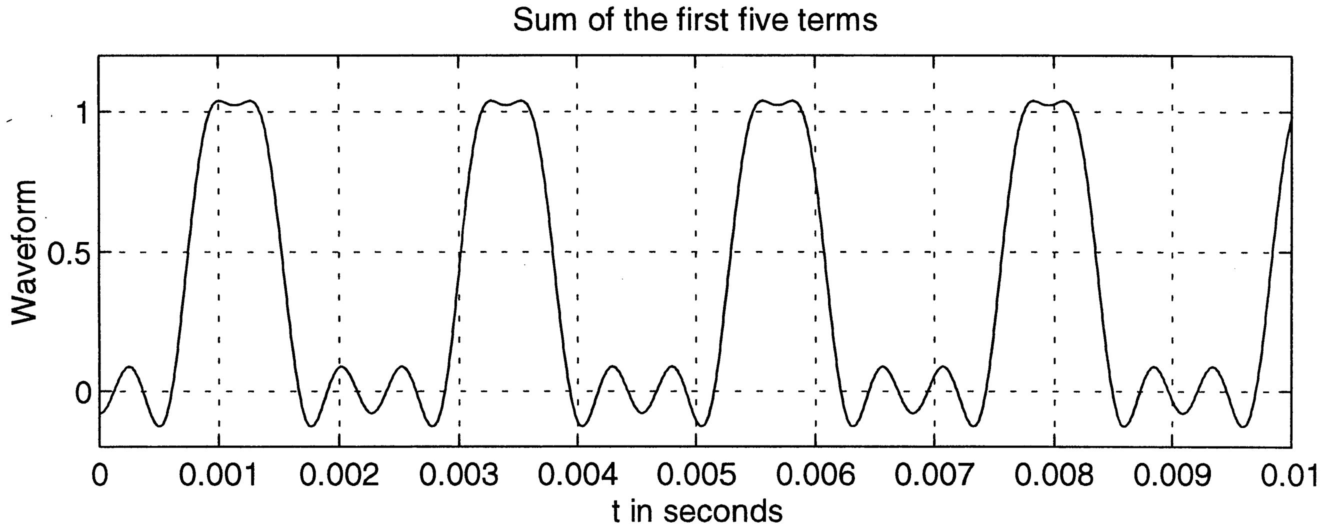
$$f_0 = 1/T_0$$

is the fundamental frequency.

- The next page shows A_0 and the four first terms $2A_n \cos(n2\pi f_0 t + \phi_n)$ in our example case when $T_0 = 1/440$ seconds, $t_0 = T_0/3$, $D = T_0/3$. In this case $f_0 = 440$ Hz corresponding the basic a in the piano.

A_0 and the four first terms $2A_n \cos(n2\pi f_0 t + \phi_n)$ for $k = 1, 2, 3, 4$ in the case where $T_0 = 1/440$ seconds, $A = 1$, $t_0 = T_0/2$, and $D = T_0/3$





Comments

- The perivious example was the ideal one.
- In practice, the signal is periodic only on a finite interval and the series is only approximately periodic.
- A typical example is the lower f note played using a harmonica. This was shown on page 4. In this case $f_0 \approx 350$ Hz so that $T_0 = 1/350$ s. Note that the amplitude level as well as the shape of the waveform vary slightly with time.

Frequency-Domain Representation of an Aperiodic Continuous-Time Signal: Fourier Transform

- Up to now, we have considered periodic signals having a Fourier series representation.
- This expression cannot, however, be used for aperiodic signals for which we use Fourier transform.
- As an introductory example, we consider the signal $x_a(t)$ defined for $-\infty < t < \infty$ and satisfying

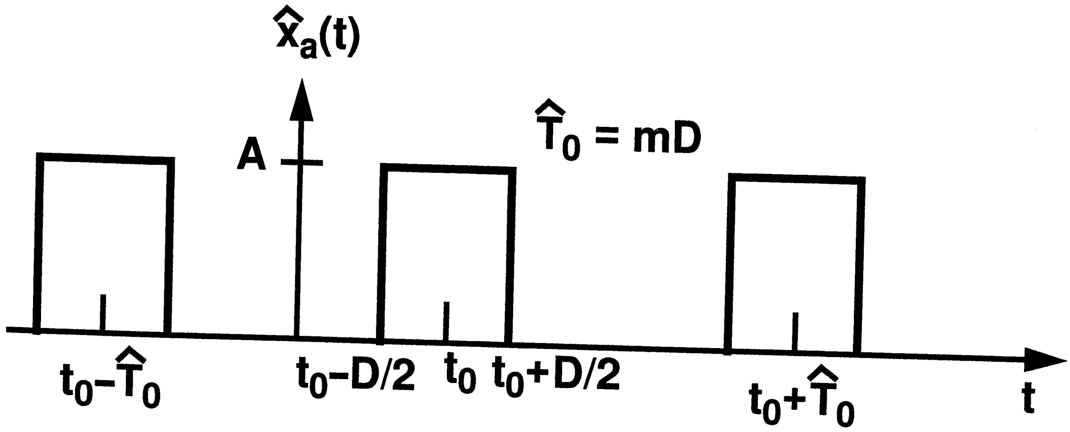
$$x_a(t) = \begin{cases} A, & \text{for } t \in [t_0 - D/2,\ t_0 + D/2] \\ 0, & \text{elsewhere.} \end{cases}$$

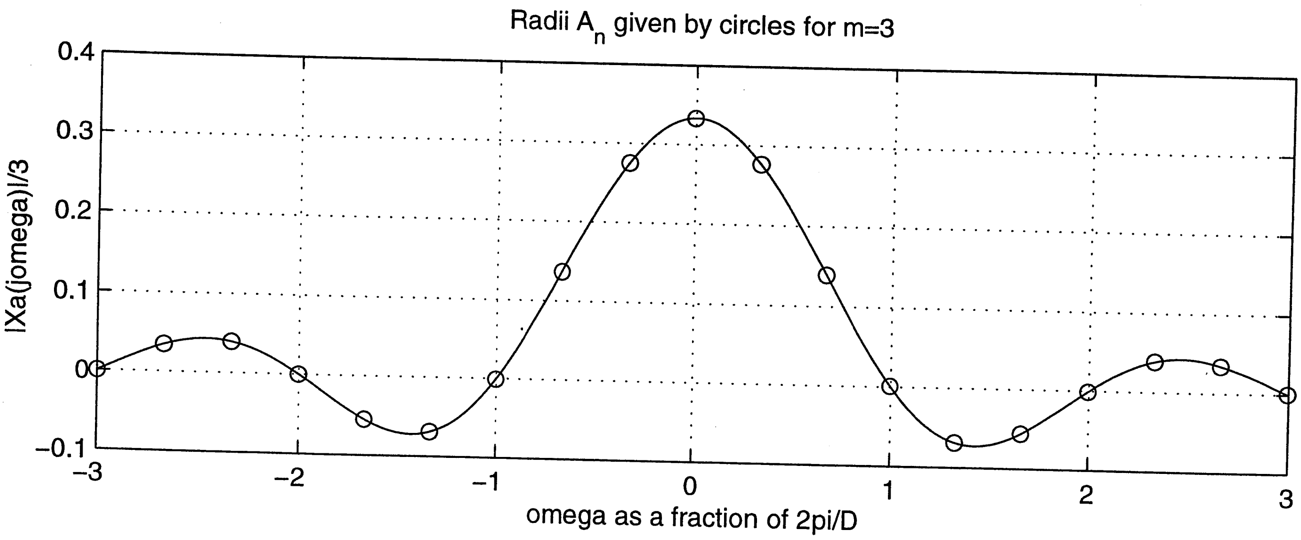
- In order to develop the Fourier transform for our signal, we consider the following strategy.
- First, we consider a signal $\widehat{x}_a(t)$ which satisfies for $-\infty < t < \infty$, $\widehat{x}_a(t+\widehat{T}_0) = \widehat{x}_a(t)$ and on the interval $[t_0 - \widehat{T}_0/2,\ t_0 + \widehat{T}_0/2]$

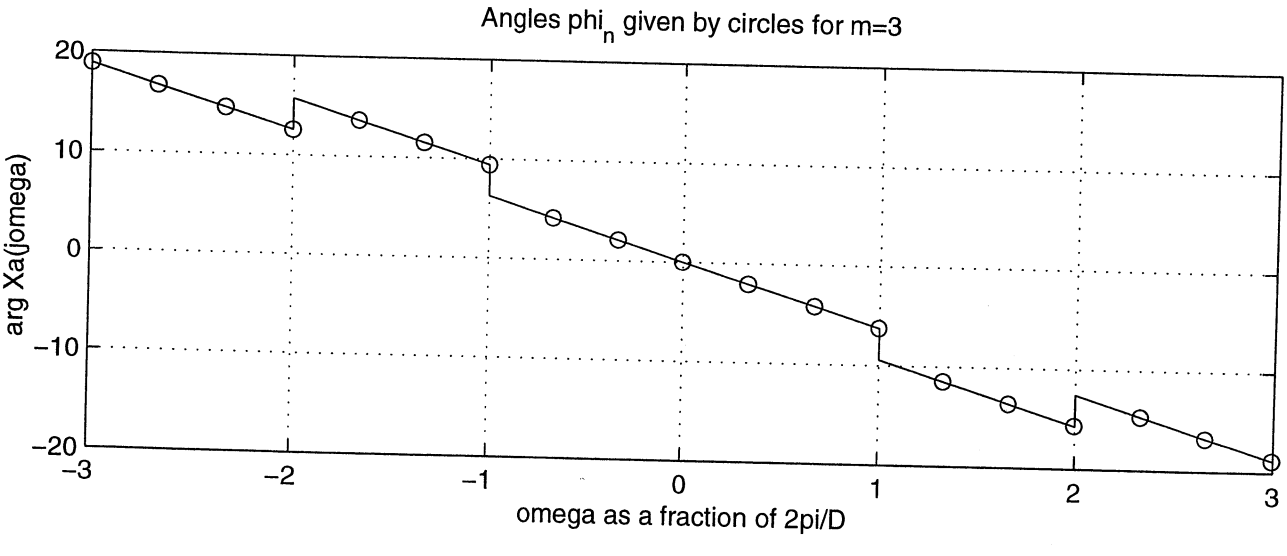
$$\widehat{x}_a(t) = \begin{cases} A, & \text{for } t \in [t_0 - D/2,\ t_0 + D/2] \\ 0, & \text{elsewhere.} \end{cases}$$

- This signal (see the next transparency) is made artifically periodic with period equal to $\widehat{T}_0$.

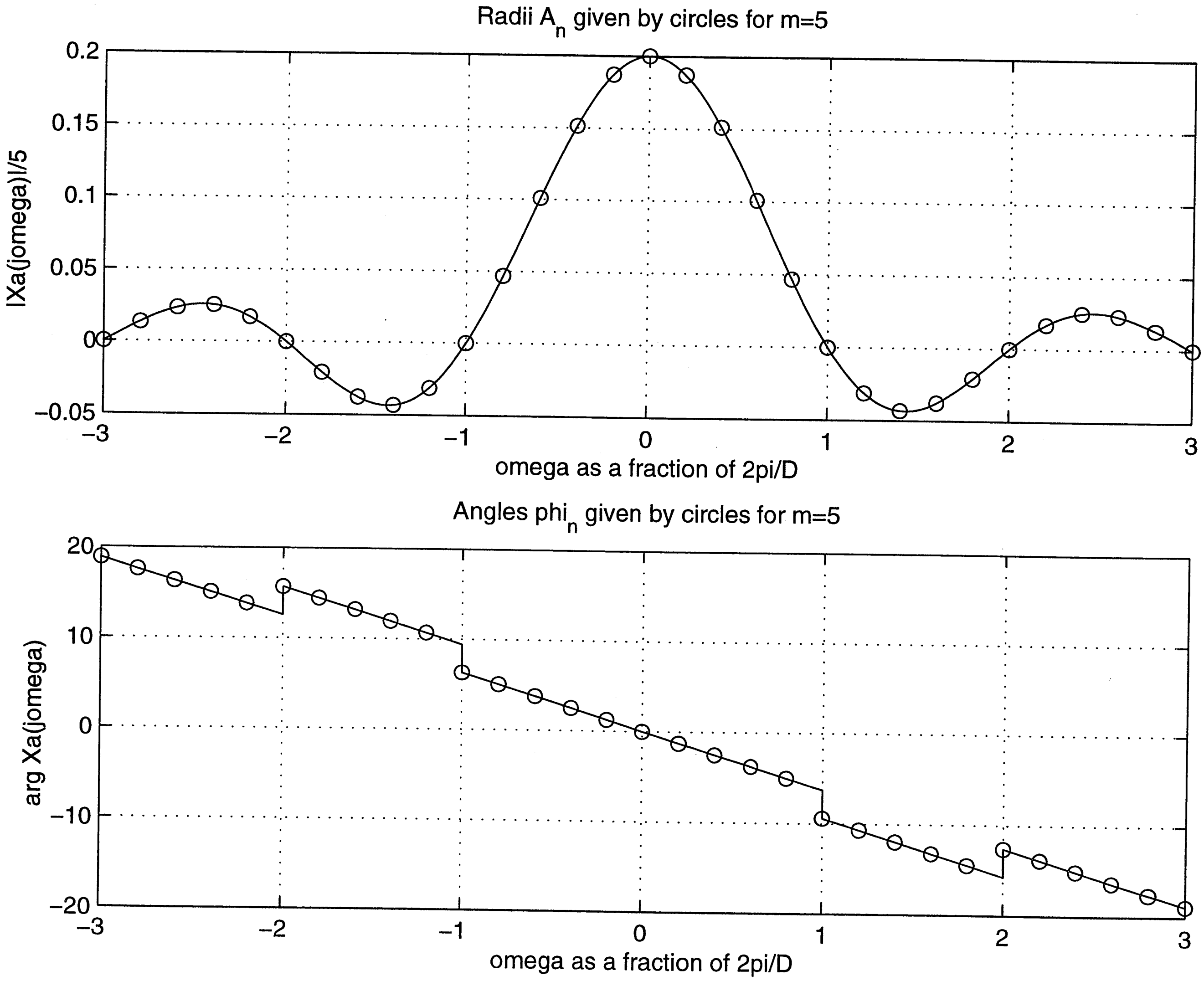
Development of the Fourier transform: a periodic signal $\widehat{x}_a(t)$ and its Fourier series coefficients for $m = 3$ and $m = 5$ ($\widehat{T}_0 = mD$). $A = 1$. **(see also the next page)**







Development of the Fourier transform: a periodic signal $\widehat{x}_a(t)$ and its Fourier series coefficients for $m = 3$ and $m = 5$ ($\widehat{T}_0 = mD$). $A = 1$



- The second step is to consider what happens when $\widehat{T}_0 \mapsto \infty$. In this case, $\widehat{x}_a(t) \mapsto x_a(t)$.
- According to the previous discussion, $\widehat{x}_a(t)$ is expressible as ($\widehat{\omega}_0 = 2\pi/\widehat{T}_0$)

$$\widehat{x}_a(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\widehat{\omega}_0},$$

where

$$c_n = 1/\widehat{T}_0 \int_{t_0-\widehat{T}_0}^{t_0+\widehat{T}_0} \widehat{x}_a(t) e^{-jn\widehat{\omega}_0 t} dt$$
$$= \frac{AD}{\widehat{T}_0} e^{-jn\widehat{\omega}_0 t_0} \text{sinc}\,(n\widehat{\omega}_0 D/2) = A_n e^{j\phi_n}$$

with

$$A_n = \frac{AD}{\widehat{T}_0} |\text{sinc}\,(n\widehat{\omega}_0 D/2)|$$

and

$$\phi_n = \begin{cases} -n\widehat{\omega}_0 t_0 & \text{for sinc}\,(-n\widehat{\omega}_0 D/2) \geq 0 \\ n\widehat{\omega}_0 t_0 - \pi & \text{for sinc}\,(n\widehat{\omega}_0 D/2) < 0. \end{cases}$$

- It is seen that the A_n's and ϕ_n's are obtainable from

$$X_a(j\omega) = ADe^{-j\omega t_0} \text{sinc}\,(D\omega/2) = |X_a(j\omega)| e^{j\arg\ X_a(j\omega)},$$

where

$$|X_a(j\omega)| = |AD\text{sinc}\,(D\omega/2)|$$

and

$$\arg\ X_a(j\omega) = \begin{cases} -t_0\omega & \text{for sinc}\,(D\omega/2) \geq 0 \\ -t_0\omega - \pi & \text{for sinc}\,(n\omega_0 D/2) < 0 \end{cases}$$

according to

$$A_n = \frac{1}{\widehat{T}_0} |X_a(jn\widehat{\omega}_0)|$$

and

$$\phi_n = \arg\ X_a(jn\widehat{\omega}_0).$$

- Hence, the A_n's and ϕ_n's are sampled versions of $|X_a(j\omega)|/\widehat{T}_0$ and $\arg\ X_a(j\omega)$ such that the sampling is performed at $\omega = n\widehat{\omega}_0$.
- In order to simplify the following discussion, we assume that $\widehat{T}_0 = mD$ giving

$$\widehat{\omega}_0 = 2\pi/\widehat{T}_0 = 2\pi/(mD),$$

$$A_n = \frac{A}{m} |\text{sinc}\,(n2\pi\frac{D}{2m})| = \frac{1}{m} |X_a(j2\pi\frac{n}{mD})|,$$

and

$$\phi_n = \begin{cases} -2\pi t_0 \frac{n}{mD} & \text{for sinc}\,(\pi\frac{nD}{m}) \geq 0 \\ -2\pi t_0 \frac{n}{mD} - \pi & \text{for sinc}\,(\pi\frac{nD}{m}) < 0 \end{cases}$$
$$= \arg\ X_a(j2\pi\frac{n}{mD}).$$

- In this case, the A_n's and ϕ_n's are the sampled versions of $|X_a(j\omega)|/m$ and $\arg\ X_a(j\omega)$ such that the sampling is performed at $\omega = 2\pi\frac{n}{mD}$.
- The figures on page 22 show the A_n's and ϕ_n's for $m = 3$ and $m = 5$.

- It is seen that as m (or $\widehat{T}_0$) increases, the fundamental angular frequency $\widehat{\omega}_0$ as well as the A_n's decrease (the envelope $|X_a(j\omega)|/m$ becomes smaller).
- Furthermore, the A_n's and ϕ_n's become more close spaced. Note that shapes of the envelopes $|X_a(j\omega)|/m$ and $\arg X_a(j\omega)$ remain the same, the samples are just taken more closely.
- As $m \mapsto \infty$, $\widehat{T}_0 \mapsto \infty$ and the additional pulses located around $t = t_0 - \widehat{T}_0$ and $t = t_0 + \widehat{T}_0$ approach $-\infty$ and $+\infty$, respectively.
- In this case the artificial periodic $\widehat{x}_a(t)$ approach the original aperiodic $x(t)$.
- Simultaneously, the A_n's and ϕ_n's become extremely closely spaced and the Fourier series has become the Fourier transform.
- The only problem left is the fact that A_n is the sample of $|X_a(j\omega)|/m$ at $\omega = 2\pi\frac{n}{mD}$. This means that $A_n \mapsto 0$ when $m \mapsto \infty$.
- However, the product $\widehat{T}_0 A_n$ is the sample $|X_a(j\omega)|$ and $\widehat{T}_0 c_n = \widehat{T}_0 A_n e^{j\phi_n}$ is the sample of $X_a(j\omega)$, that do not vanish as $m \mapsto \infty$ or $\widehat{T}_0 \mapsto \infty$.
- As $\widehat{T}_0 \mapsto \infty$, $\widehat{\omega}_0 \mapsto 0$, and the term $k\widehat{\omega}_0$ tends to

a continuous rather than a discrete variable, denoted by ω. In the similar manner, $\widehat{T}_0 c_n$ tends to a continuous variable, denoted by $X_a(j\omega)$.

- Hence, as $\widehat{T}_0 \mapsto \infty$

$$\widehat{T}_0 c_n = \int_{t_0-\widehat{T}_0}^{t_0+\widehat{T}_0} x_a(t) e^{-jn\widehat{\omega}_0 t} dt$$

$$\mapsto$$

$$X_a(j\omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\omega t} dt. \qquad (A)$$

- This is the desired Fourier transform of our signal $x_a(t)$.
- The inverse transform can be obtained by considering what is happening to

$$\widehat{x}_a(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\widehat{\omega}_0}$$

when $\widehat{T}_0 \mapsto \infty$.

- As $\widehat{T}_0 \mapsto \infty$,

$$c_n \mapsto X_a(j\omega)/\widehat{T}_0 = X_a(j\omega)\widehat{\omega}_0/2\pi.$$

- Furthermore, $n\widehat{\omega}_0 \mapsto \omega$ and the fundamental frequncy $\widehat{\omega}_0$ becomes vanishingly small and is written as $d\omega$.
- Finally, the summation in the expression of $x_a(t)$ above becomes an integration in the limit and we

obtain the following inverse transform:

$$x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(\omega)e^{jn\omega t}d\omega. \qquad (B)$$

- Equations (A) and (B) are known as the Fourier transform pair.
- We can interpret equation (A) as a decomposition of $x_a(t)$ in terms of the continuum of the elementary basis functions $e^{j\omega t}$. $X_a(j\omega)$ plays the same role as the c_n's in the Fourier series representation. $X_a(j\omega)/2\pi$ is the "coefficient" associated with $e^{j\omega t}$.
- $X_a(j\omega)$ is, in general, a complex function of ω and is expressible as

$$\begin{aligned} X_a(j\omega) &= \mathrm{Re}\left\{X_a(j\omega)\right\} + j\mathrm{Im}\left\{X_a(j\omega)\right\} \\ &= |X_a(j\omega)|e^{j\arg X_a(j\omega)} \end{aligned},$$

where

$$|X_a(j\omega)| = \sqrt{(\mathrm{Re}\left\{X_a(j\omega)\right\})^2 + (\mathrm{Im}\left\{X_a(j\omega)\right\})^2}$$

is the amplitude spectrum of $x_a(t)$ and

$$\arg\ X_a(j\omega) = \mathrm{atan2}\left(\mathrm{Im}\left\{X_a(j\omega)\right\}, \mathrm{Re}\left\{X_a(j\omega)\right\}\right)$$

is the phase spectrum spectrum of $x_a(t)$.

- In many cases, we are interested in the 'real' frequency f. Using the substitutuion $\omega = 2\pi f$, the spectrum is expressible in terms of f as follows:

$$\begin{aligned} X_a(j2\pi f) &= \mathrm{Re}\left\{X_a(j2\pi f)\right\} + j\mathrm{Im}\left\{X_a(j2\pi f)\right\} \\ &= |X_a(j2\pi f)|e^{j\arg X_a(j2\pi f)}. \end{aligned}$$

Example: $x_a(t) = A$ for $t_0 - D/2 \leq t \leq t_0 + D/2$ and $x_a(t)$ is zero for other values of t

$$X_a(j\omega) = \int_{-\infty}^{\infty} x_a(t)e^{-j\omega t}dt = \int_{t_0-D/2}^{t_0+D/2} x_a(t)e^{-j\omega t}dt$$
$$= e^{-j\omega t_0} AD\mathrm{sinc}\,(D\omega/2) = |X_a(j\omega)|e^{j\arg\ X_a(j\omega)},$$

where

$$|X_a(j\omega)| = |AD\mathrm{sinc}\,(D\omega/2)|$$

and

$$\arg\ X_a(j\omega) = \begin{cases} -t_0\omega & \text{for } \mathrm{sinc}\,(D\omega/2) \geq 0 \\ -t_0\omega - \pi & \text{for } \mathrm{sinc}\,(\omega D/2) < 0. \end{cases}$$

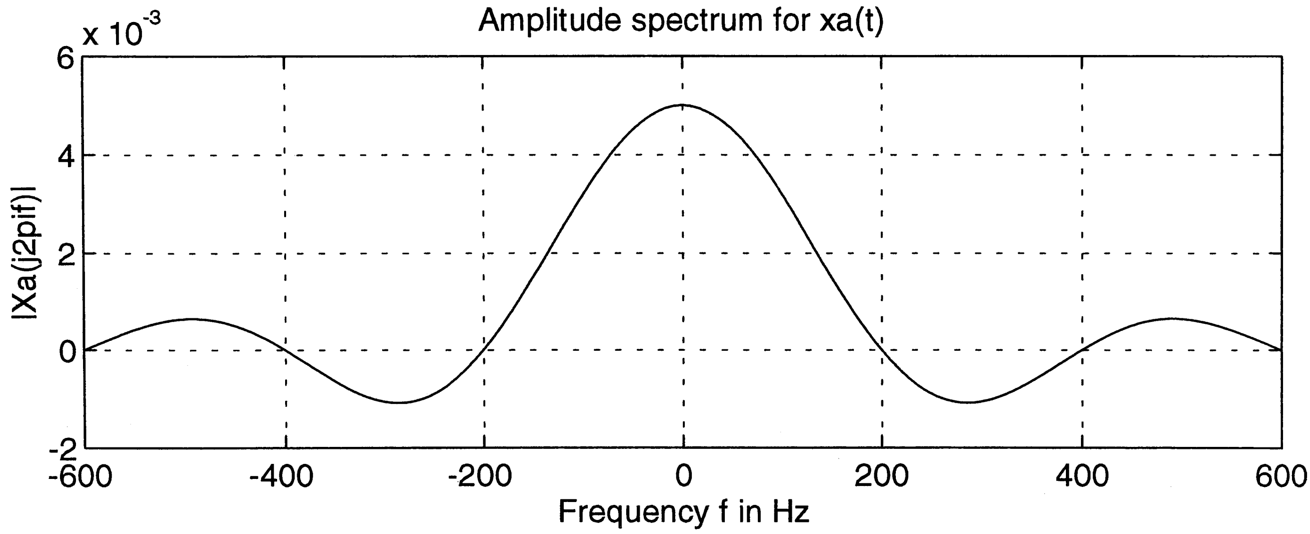
- In terms of the real frequency f, the amplitude and phase spectra become (see the next page)

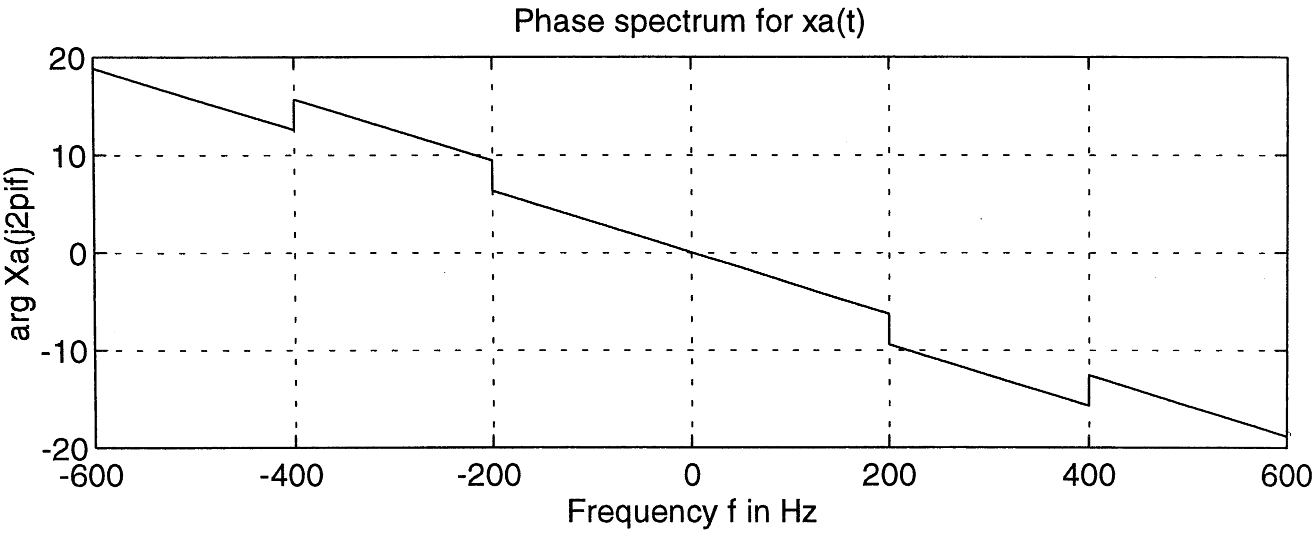
$$|X_a(j2\pi f)| = |AD\mathrm{sinc}\,(\pi D f)|$$

and

$$\arg\ X_a(j2\pi f) = \begin{cases} -2\pi t_0 f & \text{for } \mathrm{sinc}\,(\pi D f) \geq 0 \\ -2\pi t_0 f - \pi & \text{for } \mathrm{sinc}\,(\pi D f) < 0. \end{cases}$$

The amplitude and phase spectra of the signal of the previous page for $A = 1$, $D = 1/200$, and $t_0 = 1/200$





The Fourier transform for periodic signals

- Strictly speaking, the Fourier transformdoes not exist for signals

$$x_a(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0} \qquad (A1)$$

or

$$x_a(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0} u(t), \qquad (A2)$$

where

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0. \end{cases}$$

- This is because in these cases

$$X_a(j\omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\omega t} dt$$

becomes infinite.

- However, this problem can get around by using the continuous-time impulse $\delta_a(t)$ which is defined on page 33.

- By using this function, the Fourier transforms for the above functions are expressible as

$$X_a(j\omega) = 2\pi \sum_{n=-\infty}^{\infty} c_n \delta_a(\omega - n\omega_0) \qquad (B1)$$

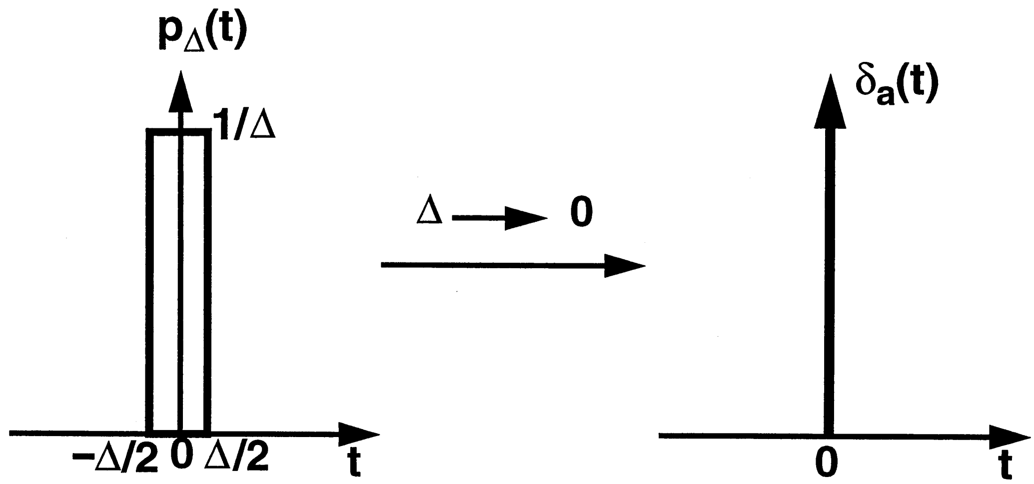
and

$$X_a(j\omega) = \sum_{n=-\infty}^{\infty} c_n (\pi \delta_a(\omega - n\omega_0) + \frac{c_n}{j(\omega - \omega_0)}). \quad (B2)$$

- It should be pointed out that these respresentation forms are not very important when filtering periodic signals using continuous-time filters. The output of a causal linear phase time invariant filter can be determined very easily without using these forms at all, as we will see later in connection of examples.

Continuous-time impulse function $\delta_a(t)$

- The continuous-time impulse function $\delta_a(t)$ is an abstact pulse which can be developed as follows.
- First, we generate a real pulse $p_\Delta(t)$ having the height of $1/\Delta$ and the width of Δ so that its area is 1 (see the figure shown below). The center of this pulse is at $t = 0$.



- $\delta_a(t)$ is then obtained by allowing Δ to approach zero, that is,

$$\delta_a(t) = \lim_{\Delta \mapsto 0} p_\Delta(t).$$

- The height of $\delta_a(t)$ is thus infinite and the width is zero such that the area is still unity.
- $\delta_a(t)$ possesses the following properties:

$$\delta_a(t) = 0 \quad \text{for} \quad t \neq 0,$$

$$\delta_a(t)\text{is indefined for } t = 0,$$

$$\int_{-\infty}^{\infty} \delta_a(t)dt = 1,$$

$$\int_{-\infty}^{\infty} f(t)\delta_a(t - t_0)dt = f(t_0),$$

and

$$f(t)\delta_a(t - t_0) = f(t - t_0).$$

- It should be pointed out that the abstract pulse $\delta_a(t)$ plays a very crucial role in considering continuou time signals and systems.

Fourier transform pairs and some properties of the Fourier transform

- The following two pages give some Fourier transform pairs and some properties of the Fourier transform. In this pages, instead of $x_a(t)$ and $X_a(j\omega)$, $f(t)$ and $F(\omega)$ is used. The notation $f(t) \leftrightarrow F(\omega)$ means that $f(t)$ and $F(\omega)$ are a Fourier transfor pair.
- Furthermore, $f_1(t) * f_2(t)$ means the following convolution

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau.$$

Fourier Transform Pairs

	Time Function, $f(t)$	Fourier Transform, $F(\omega)$
1.	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$
2.	$te^{-at}u(t)$	$\left(\dfrac{1}{a+j\omega}\right)^2$
3.	$g_T(t) = \begin{cases} 1, & \lvert t\rvert < \dfrac{T}{2} \\ 0, & \text{otherwise} \end{cases}$	$T\,\text{sinc}\left(\dfrac{\omega T}{2}\right)$
4.	$\begin{cases} A\left(1-\dfrac{\lvert t\rvert}{T}\right), & \lvert t\rvert < T \\ 0, & \lvert t\rvert > T \end{cases}$	$AT\,\text{sinc}^2\left(\dfrac{\omega T}{2}\right)$
5.	$e^{-a\lvert t\rvert}$	$\dfrac{2a}{a^2+\omega^2}$
6.	$e^{-at}\sin\omega_0 t\, u(t)$	$\dfrac{\omega_0}{(a+j\omega)^2+\omega_0^2}$
7.	$e^{-at}\cos\omega_0 t\, u(t)$	$\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$
8.	e^{-at^2}	$\dfrac{\pi}{a}e^{-\omega^2/4a}$
9.	$\dfrac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\dfrac{1}{(j\omega+a)^n}$
10.	$\dfrac{1}{a^2+t^2}$	$\dfrac{\pi}{a}e^{-a\lvert\omega\rvert}$
11.	$\dfrac{\cos bt}{a^2+t^2}$	$\dfrac{\pi}{2a}[e^{-a\lvert\omega-b\rvert}+e^{-a\lvert\omega+b\rvert}]$
12.	$\dfrac{\sin bt}{a^2+t^2}$	$\dfrac{\pi}{2aj}[e^{-a\lvert\omega-b\rvert}-e^{-a\lvert\omega+b\rvert}]$
13.	$\cos\omega_0 t\left[u\left(t+\dfrac{T}{2}\right)-u\left(t-\dfrac{T}{2}\right)\right]$	$\dfrac{T}{2}\left[\text{sinc}\left(\dfrac{(\omega-\omega_0)}{2}T\right)+\text{sinc}\left(\dfrac{(\omega+\omega_0)}{2}T\right)\right]$

Fourier Transform Pairs

Time Functions, $f(t)$	Fourier Transform, $F(\omega)$
1. $k\delta(t)$	k
2. k	$2\pi k\delta(\omega)$
3. $u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
4. $\operatorname{sgn}(t)$	$2/j\omega$
5. $\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
6. $\sin \omega_0 t$	$j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$
7. $e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
8. $t\,u(t)$	$j\pi\delta'(\omega) - \frac{1}{\omega^2}$
9. $\sum_{k=-\infty}^{\infty} \delta(t - kT)$	$\omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0), \quad \omega_0 = \frac{2\pi}{T}$
10. $\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$	$2\pi \sum_{n=-\infty}^{\infty} F_n \delta(\omega - n\omega_0)$
11. $\frac{d^n\delta(t)}{dt^n}$	$(j\omega)^n$
12. $\lvert t \rvert$	$\frac{-2}{\omega^2}$
13. t^n	$2\pi j^n \frac{d^n\delta(\omega)}{d\omega^n}$

Some Properties of the Fourier Transform

1.	Transformation	$f(t) \leftrightarrow F(\omega)$
2.	Linearity	$a_1 f_1(t) + a_2 f_2(t) \leftrightarrow a_1 F_1(\omega) + a_2 F_2(\omega)$
3.	Symmetry	$F(t) \leftrightarrow 2\pi f(-\omega)$
4.	Scaling	$f(at) \leftrightarrow \frac{1}{\lvert a \rvert} F\left(\frac{\omega}{a}\right)$
5.	Delay	$f(t - t_0) \leftrightarrow e^{-j\omega t_0} F(\omega)$
6.	Modulation	$e^{j\omega_0 t} f(t) \leftrightarrow F(\omega - \omega_0)$
7.	Convolution	$f_1(t) * f_2(t) \leftrightarrow F_1(\omega) F_2(\omega)$
8.	Multiplication	$f_1(t) f_2(t) \leftrightarrow \frac{1}{2\pi} F_1(\omega) * F_2(\omega)$
9.	Time Differentiation	$\frac{d^n}{dt^n} f(t) \leftrightarrow (j\omega)^n F(\omega)$
10.	Time Integration	$\int_{-\infty}^{t} f(\tau)\, d\tau \leftrightarrow \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$
11.	Frequency Differentiation	$-jt f(t) \leftrightarrow \frac{dF(\omega)}{d\omega}$
12.	Frequency Integration	$\frac{f(t)}{-jt} \leftrightarrow \int F(\omega')\, d\omega'$
13.	Reversal	$f(-t) \leftrightarrow F(-\omega)$

Laplace transform: Basic mathematical tool for studing continuous-time signals and systems

- The two-sided Laplace transform of a signal $x_a(t)$ is defined by
$$X_a(s) = \mathrm{L}_{\mathrm{two}}\{x_a(t)\} = \int_{-\infty}^{\infty} x_a(t)e^{-st}dt.$$
- The region of convergence (ROC) consists of those values of $s = \sigma$ for which
$$\int_{-\infty}^{\infty} |x_a(t)|e^{-\sigma t}dt$$
is finite. The ROC may consist of all values of s or it is of the form $a < s < b$, $s < b$, or $s > a$. There are also cases where there is no ROC at all.
- In practice, singnals are causal and we can concentrate usually on signals which are zero for $t < 0$. In this case, we can use the one-sided Laplace tranform defined by
$$X_a(s) = \mathrm{L}\{x_a(t)\} = \int_{0}^{\infty} x_a(t)e^{-st}dt.$$
- For the one-sided Laplace tranform, the ROC may consist of all values of s or it is of the form $s > a$.

- The inverse Laplace transform, that is, the signal $x_a(t)$ whose Laplace tranfrom is $X_a(s)$, could be evaluated as

$$x_a(t) = \mathrm{L}^{-1}\{X_a(s)\} = \int_{\sigma-j\infty}^{\sigma-j\infty} X_a(s)e^{st}ds,$$

where σ is constant and taken to be in the ROC.

- This is not worth doing. The simplest technique is to use simple techniques for developed for determing the inverse Laplace transform. One of them is to use the partial fraction expansion ($X_a(s)$ is a rational polynomial of s) and then to use tabulated transform pairs for simple functions.
- For most practical signals (except for periodic signals of infinite duration), the Fourier transform is obtained from its Laplace transform $X_a(s)$ using the substitution $s = j\omega$, giving $X_a(j\omega)$.
- This is very useful in studing the frequency-domain behavior of continuos-time filters.

Laplace transform pairs and some properties of the Laplace transform

- The following two pages give some Laplace transform pairs and some properties of the Laplace transform. In these pages, instead of $x_a(t)$ and $X_a(s)$, $f(t)$ and $F(s)$ are used. The notation $f(t) \leftrightarrow F(s)$ means that $f(t)$ and $F(s)$ are a Laplace transform pair.
- Furthermore, $f_1(t) * f_2(t)$ means the following convolution

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau.$$

Laplace Transform Pairs

	$f(t)$	$F(s)$	Convergence Region
1.	$e^{-at}u(t)$	$\frac{1}{s+a}$	$-\text{Re}\,(a) < \text{Re}\,(s)$
2.	$u(t)$	$\frac{1}{s}$	$0 < \text{Re}\,(s)$
3.	$tu(t)$	$\frac{1}{s^2}$	$0 < \text{Re}\,(s)$
4.	$t^n u(t)$	$n!/s^{n+1}$	$0 < \text{Re}\,(s)$
5.	$\delta(t)$	1	all s
6.	$\delta'(t)$	s	all s
7.	$\text{sgn}\, t$	$\frac{2}{s}$	$\text{Re}\,(s) = 0$
8.	$-u(-t)$	$\frac{1}{s}$	$\text{Re}\,(s) < 0$
9.	$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$-\text{Re}\,(a) < \text{Re}\,(s)$
10.	$t^n e^{-at}u(t)$	$\frac{n!}{(s+a)^{n+1}}$	$-\text{Re}\,(a) < \text{Re}\,(s)$
11.	$e^{-a\lvert t\rvert}$	$\frac{2a}{a^2 - s^2}$	$-\text{Re}\,(a) < \text{Re}\,(s) < \text{Re}\,(a)$
12.	$(1 - e^{-at})u(t)$	$\frac{a}{s(s+a)}$	$\max\,(0, -\text{Re}\,(a)) < \text{Re}\,(s)$
13.	$\cos \omega t\, u(t)$	$\frac{s}{s^2+\omega^2}$	$0 < \text{Re}\,(s)$
14.	$\sin \omega t\, u(t)$	$\frac{\omega}{s^2+\omega^2}$	$0 < \text{Re}\,(s)$
15.	$e^{-\sigma t}\cos \omega t\, u(t)$	$\frac{s+\sigma}{(s+\sigma)^2+\omega^2}$	$-\sigma < \text{Re}\,(s)$
16.	$e^{-\sigma t}\sin \omega t\, u(t)$	$\frac{\omega}{(s+\sigma)^2+\omega^2}$	$-\sigma < \text{Re}\,(s)$
17.	$\begin{cases} 1 - \lvert t\rvert, & \lvert t\rvert < 1 \\ 0, & \lvert t\rvert > 1 \end{cases}$	$\left(\frac{\sinh s/2}{s/2}\right)^2$	all s
18.	$\sum_{n=0}^{\infty} \delta(t - nT)$	$\frac{1}{1 - e^{-sT}}$	all s

Some Properties of the Laplace Transform

(1) Linearity

$$af_1(t) + bf_2(t) \leftrightarrow aF_1(s) + bF_2(s), \qquad \max(\alpha_1, \alpha_2) < \sigma < \min(\beta_1, \beta_2)$$

(2) Scaling

$$f(at) \leftrightarrow \frac{1}{|a|} F\left(\frac{s}{a}\right), \qquad |\alpha|\, a < \sigma < |a|\, \beta$$

(3) Time shift

$$f(t-\tau) \leftrightarrow F(s)e^{-\tau s}, \qquad \alpha < \sigma < \beta$$

(4) Frequency shift

$$e^{-at}f(t) \leftrightarrow F(s+a), \qquad \alpha - \mathrm{Re}\,(a) < \sigma < \beta - \mathrm{Re}\,(a)$$

(5) Time convolution

$$f_1(t) * f_2(t) \leftrightarrow F_1(s)F_2(s), \qquad \text{same as (1)}$$

(6) Frequency convolution

$$f_1(t)f_2(t) \leftrightarrow \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F_1(u) \times F_2(s-u)\,du, \qquad \begin{cases} \alpha_1 + \alpha_2 < \sigma < \beta_1 + \beta_2 \\ \alpha_1 < c < \beta_1 \end{cases}$$

(7) Time differentiation

$$\frac{df(t)}{dt} \leftrightarrow sF(s), \qquad \text{same as (3)}$$

(8) Time integration

$$\int_{-\infty}^{t} f(u)\,du \leftrightarrow \frac{F(s)}{s}, \qquad \max(\alpha, 0) < \sigma < \beta$$

$$\int_{t}^{\infty} f(u)\,du \leftrightarrow \frac{F(s)}{s}, \qquad \alpha < \sigma < \min(\beta, 0)$$

(9) Frequency differentiation

$$(-t)^n f(t) \leftrightarrow \frac{d^nF(s)}{ds^n}, \qquad \text{same as (3)}$$

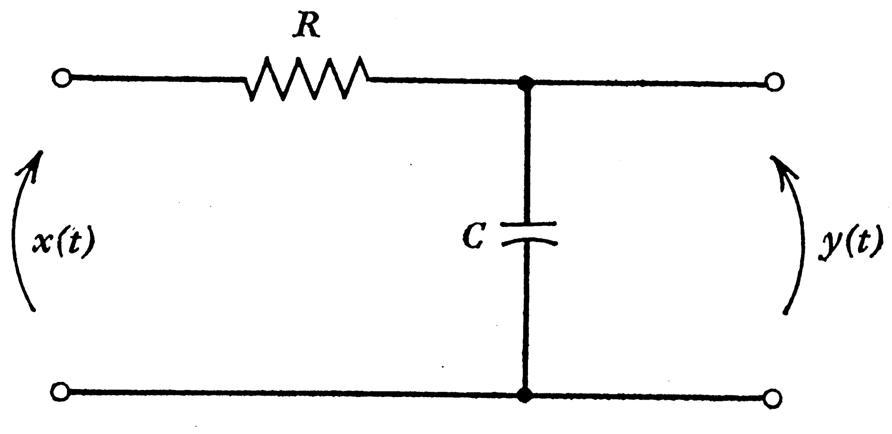
Importance of the Laplace and Fourier Transforms

- When using continuous-time filters (here we consider just linear causal time-invariant systems) for processing the the exitation $x_a(t)$ starting at time $t = 0$, then the output signal $y_a(t)$ can be obtained from the equation

$$y_a(t) = \int_{t=0}^{\infty} x_a(\tau)h_a(t-\tau)d\tau, \qquad (A)$$

where $h_a(t)$ is an impulse response of our filter, that is, the response to the exitation $x_a(t) = \delta_a(t)$.

- As an example, we consider the simple RC-filter shown below.



- For this filter,

$$h_a(t) = \frac{1}{RC}e^{-t/RC}u(t),$$

where

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0. \end{cases}$$

- It should be pointed out that the RC-filter is the simplest filter. However, the evaluation of the integral of Eq. (A) is very difficult especially for a complicated input signal $x_a(t)$.
- Futhermore, we like to know the frequency-domain behaviors of both our filter and $y_a(t)$.
- We are capable of solving both of the above-mentioned problems by using the Laplace and Fourier transforms. In the case of the Laplace transform, the Laplace transform of the response $y_a(t)$ is simply given by

$$Y_a(s) = H_a(s)X_a(s),$$

where $H_a(s)$ is the Laplace tranform of the impulse response $h_a(t)$, called the filter transfer function, and $X_a(s)$ is the Laplace tranform of the exitation $x_a(t)$.
- In the case of the Fourier transform, the the Fourier transform of the output of a continuous-time system, denoted by $Y_a(j\omega)$ is directly the product of the Fourier transforms of the exitation and the impulse response, denoted by $X_a(j\omega)$ and $H_a(j\omega)$, that is,

$$Y_a(j\omega) = H_a(j\omega)X_a(j\omega).$$

Example: Determine the response of the RC-filter to the exitation $x_a(t) = \cos(\omega_0 t + \phi)u(t)$.

- The Laplace transforms of the impulse response $h_a(t) = \frac{1}{RC}e^{-t/RC}$ and the exitation are

$$H_a(s) = \frac{1}{1 + RCs}$$

and

$$X_a(s) = \frac{s\cos\phi - \omega_0 \sin\phi}{s^2 + \omega_0^2}.$$

- Therefore, the Laplace transform of the output is given by

$$Y_a(s) = H_a(s)X_a(s) = \frac{s\cos\phi - \omega_0 \sin\phi}{(1 + RCs)(s^2 + \omega_0^2)}$$

- After some manipulations, we obtain

$$Y_a(s) = |H_a(j\omega_0)| \times \frac{s\cos(\phi + \arg\ H_a(j\omega_0)) - \omega_0 \sin(\phi + \arg\ H_a(j\omega_0))}{s^2 + \omega_0^2} - \frac{RC|H_a(j\omega_0)|\cos(\phi + \arg\ H_a(j\omega_0))}{1 + RCs}.$$

- Here, $H_a(j\omega)$ is the frequency response of our RC-filter and is expressible as

$$H_a(j\omega) = \frac{1}{1 + jRC\omega} = |H_a(j\omega)|e^{j\arg H_a(j\omega)},$$

where

$$|H_a(j\omega)| = \frac{1}{\sqrt{1 + (RC\omega)^2}}$$

and

$$\arg\ H_a(j\omega) = -\tan^{-1}(RC\omega)$$

and the amplitude and phase responses of our RC-filter.

- By utilizing the facts that in the case of causal signals, the inverse Laplace transforms of $1/(s+a)$ and $(s\cos\beta - \alpha\sin\beta)/(s^2+\alpha^2)$ are $e^{-at}u(t)$ and $\cos(\alpha t+\beta)u(t)$, the output signal can be expressed as

$$y_a(t) = y_1(t) + y_2(t),$$

where

$$y_1(t) = |H_a(j\omega_0)|\cos(\omega_0 t + \phi + \arg\ H_a(j\omega_0))u(t)$$

and

$$y_2(t) = -|H_a(j\omega_0)|\cos(\phi + \arg\ H_a(j\omega_0))e^{-t/RC}u(t).$$

- Alternatively, using the substitution $\omega_0 = 2\pi f_0$ these components can be expressed in terms of the 'real' oscillation frequency f_0 as

$$y_1(t) = |H_a(j2\pi f_0)|\cos(2\pi f_0 t + \phi + \arg\ H_a(j2\pi f_0))u(t)$$

and

$$y_2(t) = -|H_a(j2\pi f_0)|\cos(\phi + \arg\ H_a(j2\pi f_0))e^{-t/RC}u(t).$$

- Here, $|H_a(j2\pi f_0)|$ and $\arg H_a(j2\pi f_0)$ are samples at $f = f_0$ taken from the frequency response of the filter expressed in terms of f, that is,

$$H_a(j2\pi f) = \frac{1}{1 + jRC2\pi f} = |H_a(j2\pi f)|e^{j\arg H_a(j2\pi f)},$$

where

$$|H_a(j2\pi f)| = \frac{1}{\sqrt{1 + (RC2\pi f)^2}}$$

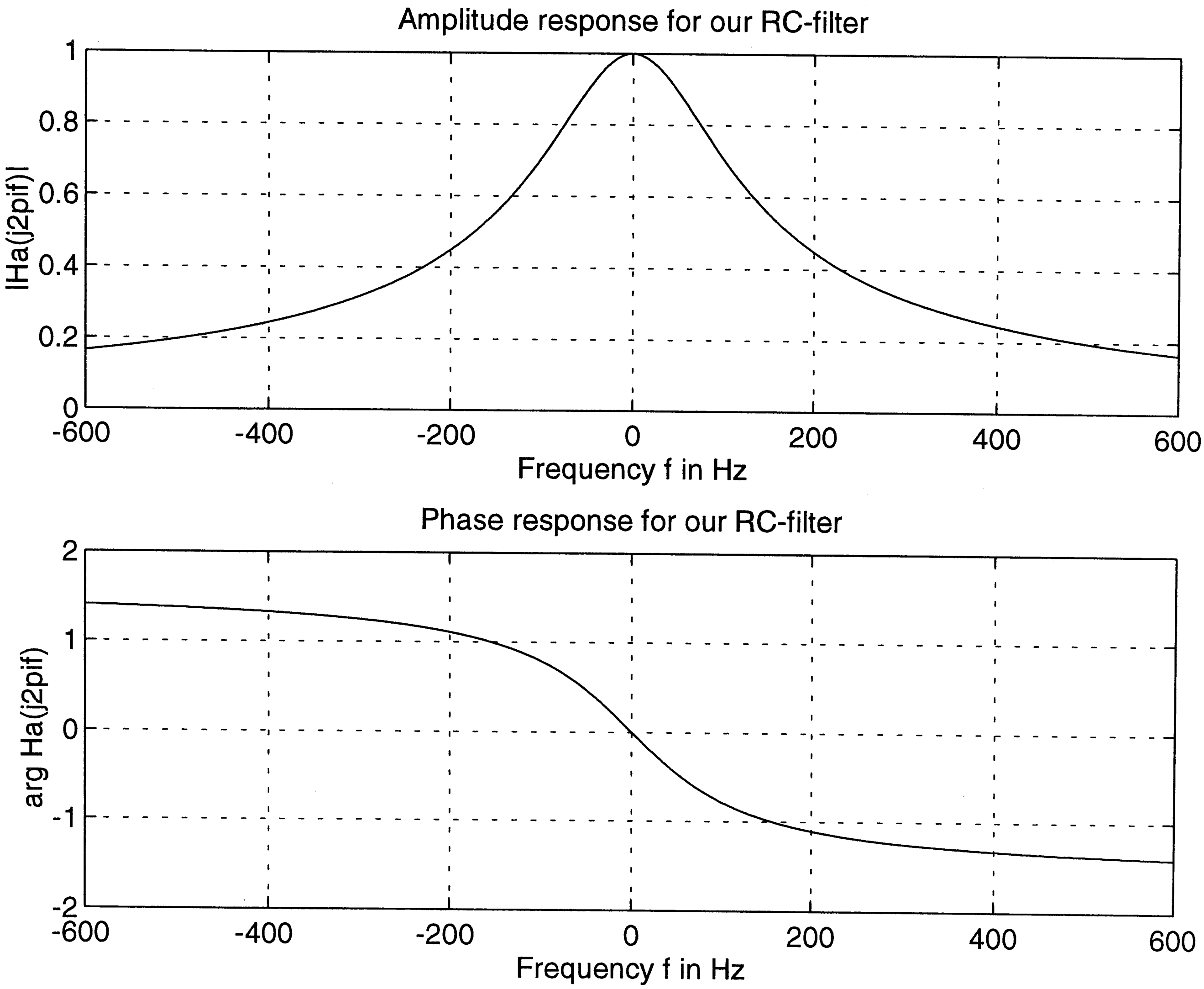
and

$$\arg H_a(j2\pi f) = -\tan^{-1}(RC2\pi f)$$

and the amplitude and phase responses of our RC-filter in terms of f.

- The following page gives the amplitude and phase responses of our RC filter for $RC = 1/(2\pi 100)$. In this case, $|H_a(j2\pi f)|$ achieves the value of $1/\sqrt{2}$ at $f = 100$ Hz. This is the so called 3-dB point in the filter theory.

Amplitude and Phase Responses of our RC Filter for $RC = 1/(2\pi 100)$



Interpretation of the terms $y_1(t)$ and $y_2(t)$

- According to the previous discussion, the overall input is expressible as

$$y_a(t) = y_1(t) + y_2(t),$$

where

$$y_1(t) = |H_a(j2\pi f_0)| \cos(2\pi f_0 t + \phi + \arg\ H_a(j2\pi f_0))u(t)$$

and

$$y_2(t) = -|H_a(j2\pi f_0)| \cos(\phi + \arg\ H_a(j2\pi f_0))e^{-t/RC}u(t).$$

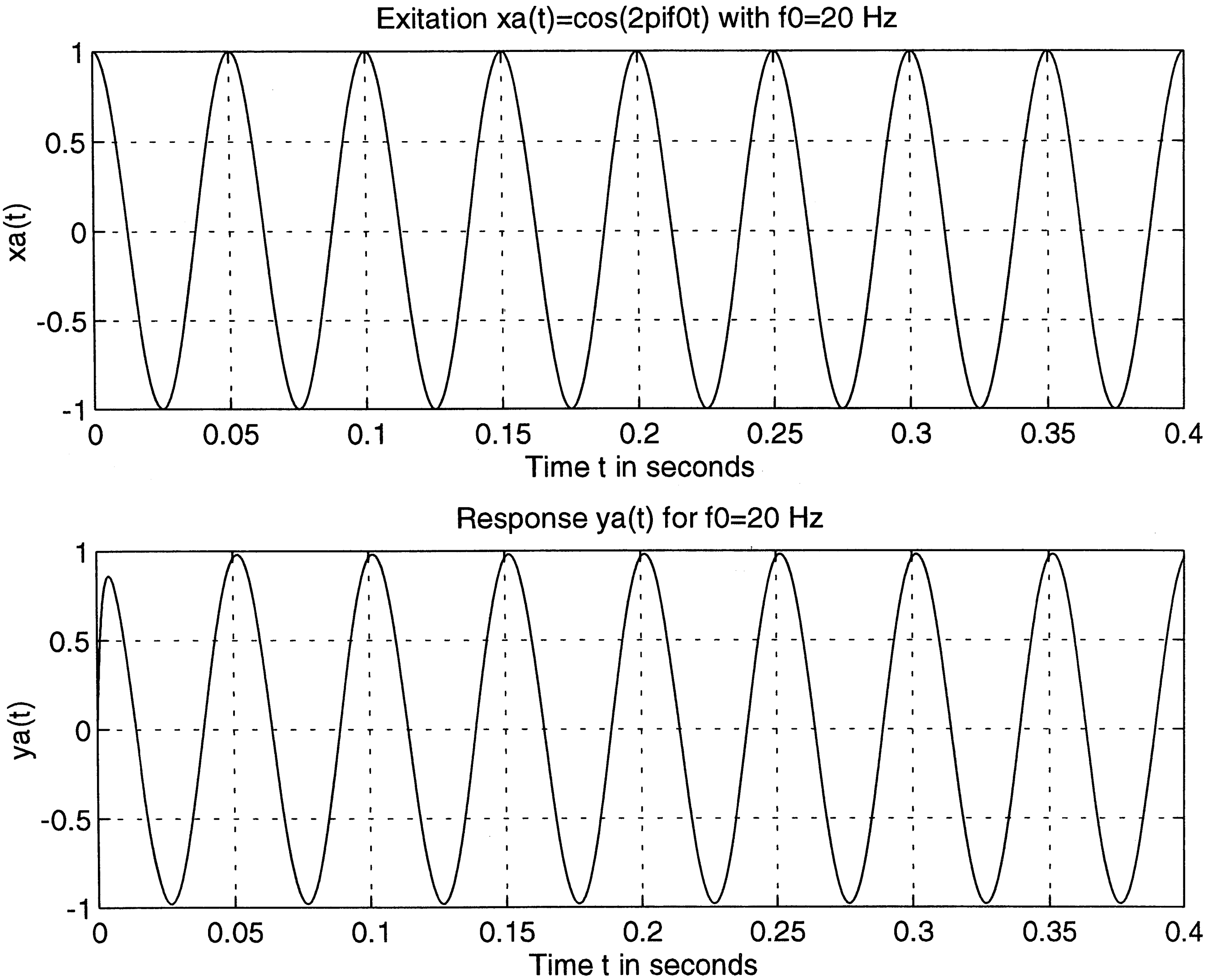
- $y_1(t)$ resemples the exitation

$$x_a(t) = \cos(2\pi f_0 t + \phi)u(t).$$

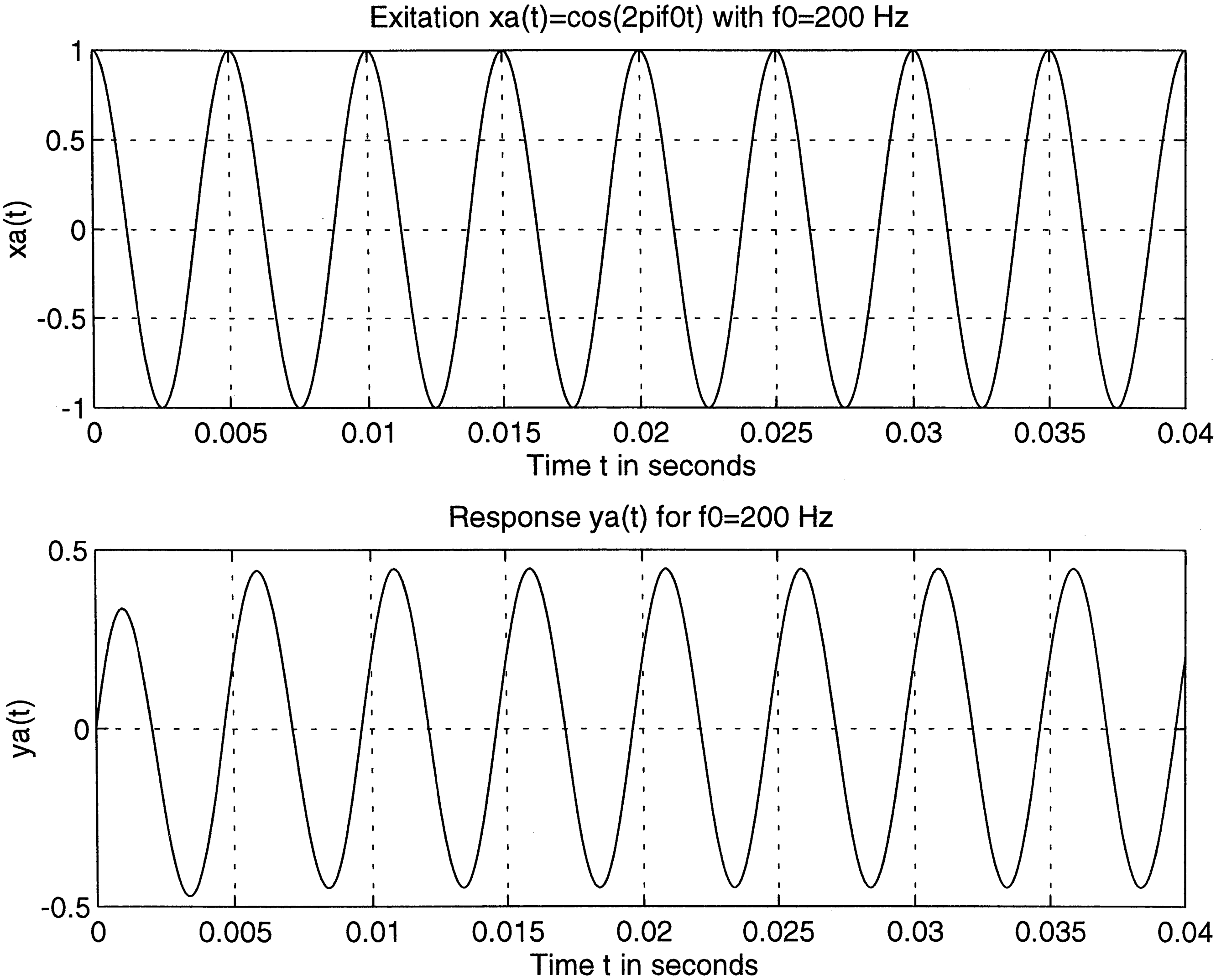
- The only difference is that the oscillation amplitude has changed from unity to $H_a(j2\pi f_0)|$ and there is an additional phase shift of arg $H_a(j2\pi f_0)$. Therefore, this term is called as a steady-state response.
- The second term is a short-duration transient part which affects in the very beginning.
- This is exemplified in the following two pages in two cases for the RC-circuit with $RC = 1/(2\pi 100)$. In these cases, $f_0 = 20$ Hz and $f_0 = 200$ Hz. In both cases, $\phi = 0$.

- As seen from these figure, in both cases, $y_a(t)$ becomes a sinusoidal signal with a different oscillation amplitude and a phase shift after the transient part.
- In the first case, $|H_a(j2\pi f_0)|$ is close to unity so that the oscillation amplitude is almost one.
- In the second case, $|H_a(j2\pi f_0)|$ is approximately equal to 0.45 so that the oscillation amplitude is also approximately equal to 0.45.

Exitation $x_a(t) = \cos(2\pi f_0)u(t)$ with $f_0 = 20$ Hz for the RC-filter with $RC = 1/(2\pi 100)$ and the response $y_a(t)$



Exitation $x_a(t) = \cos(2\pi f_0)u(t)$ with $f_0 = 200$ Hz for the RC-filter with $RC = 1/(2\pi 100)$ and the response $y_a(t)$



Example: Determine the response of the RC-filter to the exitation $x_a(t) = u(t-(t_0 - D/2)) - u(t-(t_0 + D/2))$

- This exitation is unity for $t_0 - D/2 \le t \le t_0 + D/2$ and zero otherwise. Its Laplace transform is

$$X_a(s) = [e^{-(t_0-D/2)s} - e^{-(t_0+D/2)s}]/s.$$

- Since $H_a(s) = 1/(1+RCs)$, the Laplace tranform of the response is

$$Y_a(s) = H_a(s)X_a(s) = \frac{e^{-(t_0-D/2)s} - e^{-(t_0+D/2)s}}{s(1+RCs)}.$$

- $Y_a(s)$ is expressible as

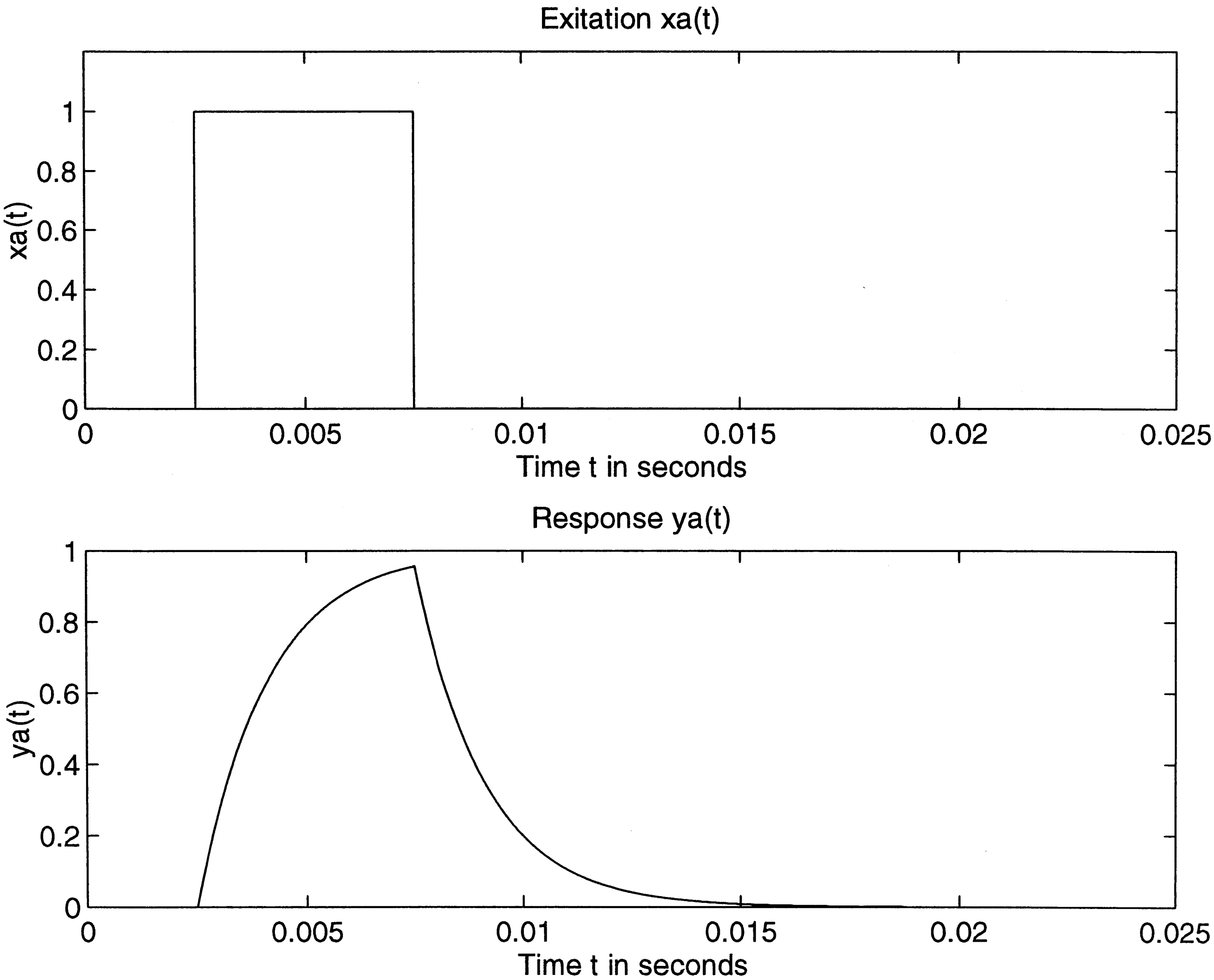
  $$Y_a(s) = [e^{-(t_0-D/2)s} - e^{-(t_0+D/2)s}][1/s - RC/(1+RCs)]$$

  so that the inverse transform is

  $$y_a(t) = (1-e^{-t/RC})[u(t-(t_0-D/2)) - u(t-(t_0+D/2))].$$

- The following page shows the exitation $x_a(t)$ for $D = 1/200$ and $t_0 = 1/200$ and the response of the RC filter with $RC = 1/(2\pi 100)$.

Exitation $x_a(t)$ for $D = 1/200$ and $t_0 = 1/200$ and the Response of the RC Filter with $RC = 1/(2\pi 100)$



Relation of the Fourier Transform of the Output to Those of the Exitation and the Impulse response

- According to the previous discussion we know that the Fourier transform of the output signal $y_a(t)$ is given in terms of the real frequency f as

$$Y_a(j2\pi f) = H_a(j2\pi f)X_a(j2\pi f).$$

- Furthermore,

$$H_a(j2\pi f) = |H_a(j2\pi f)|e^{j\arg H_a(j2\pi f)},$$

where

$$|H_a(j2\pi f)| = \frac{1}{\sqrt{1+(RC2\pi f)^2}}$$

and

$$\arg\ H_a(j2\pi f) = -\tan^{-1}(RC2\pi f).$$

and

$$X_a(j2\pi f) = |X_a(j2\pi f)|e^{j\arg X_a(j2\pi f)},$$

where

$$|X_a(j2\pi f)| = |AD\mathrm{sinc}\,(\pi D f)|$$

and

$$\arg\ X_a(j2\pi f) = \begin{cases} -2\pi t_0 f & \text{for sinc}\,(\pi D f) \geq 0 \\ -2\pi t_0 f - \pi & \text{for sinc}\,(\pi D f) < 0. \end{cases}$$

- From this, it follows that

$$Y_a(j2\pi f) = |Y_a(j2\pi f)| e^{j \arg X_a(j2\pi f)},$$

where

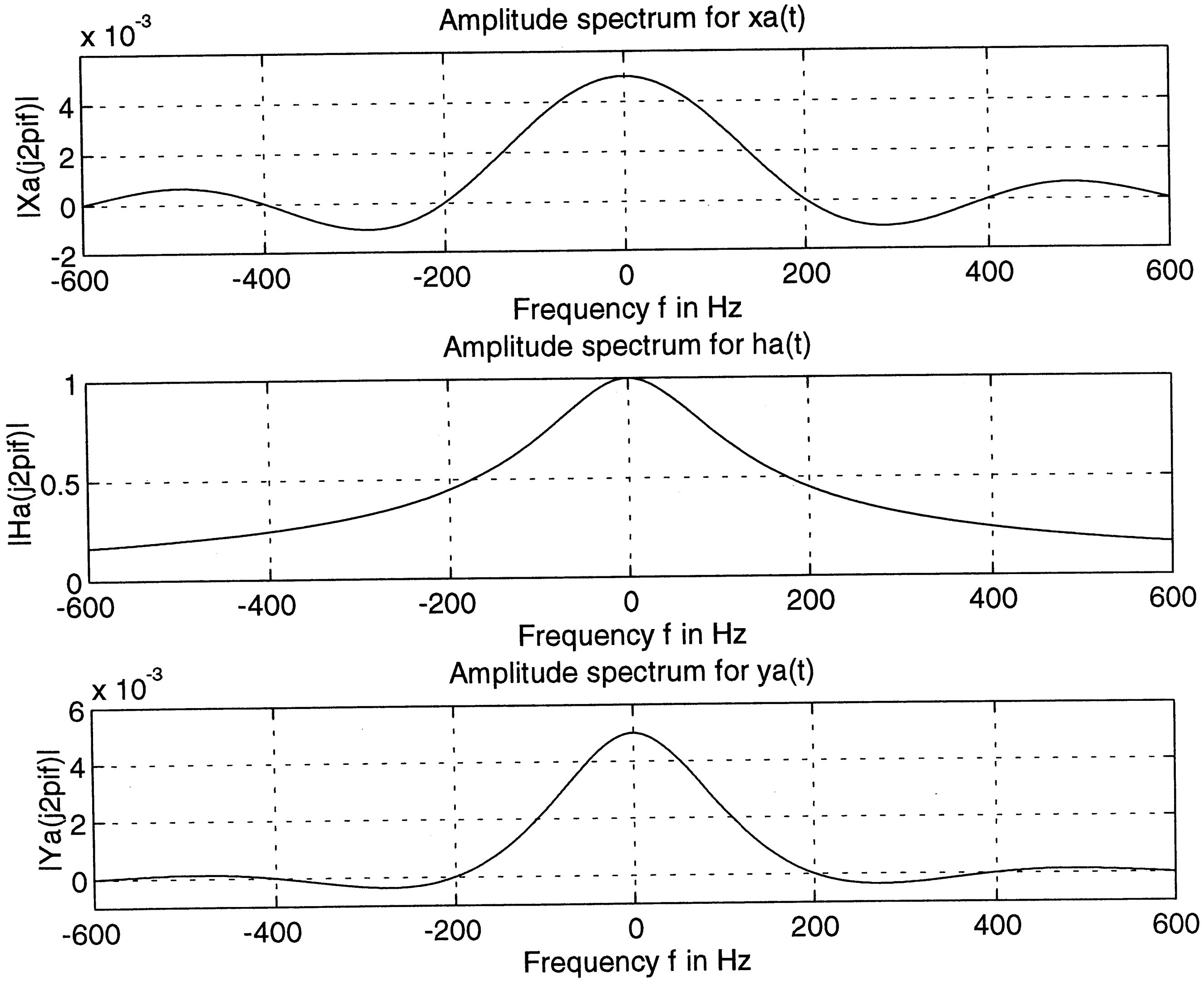
$$|Y_a(j2\pi f)| = |H_a(j2\pi f)||X_a(j2\pi f)|$$

and

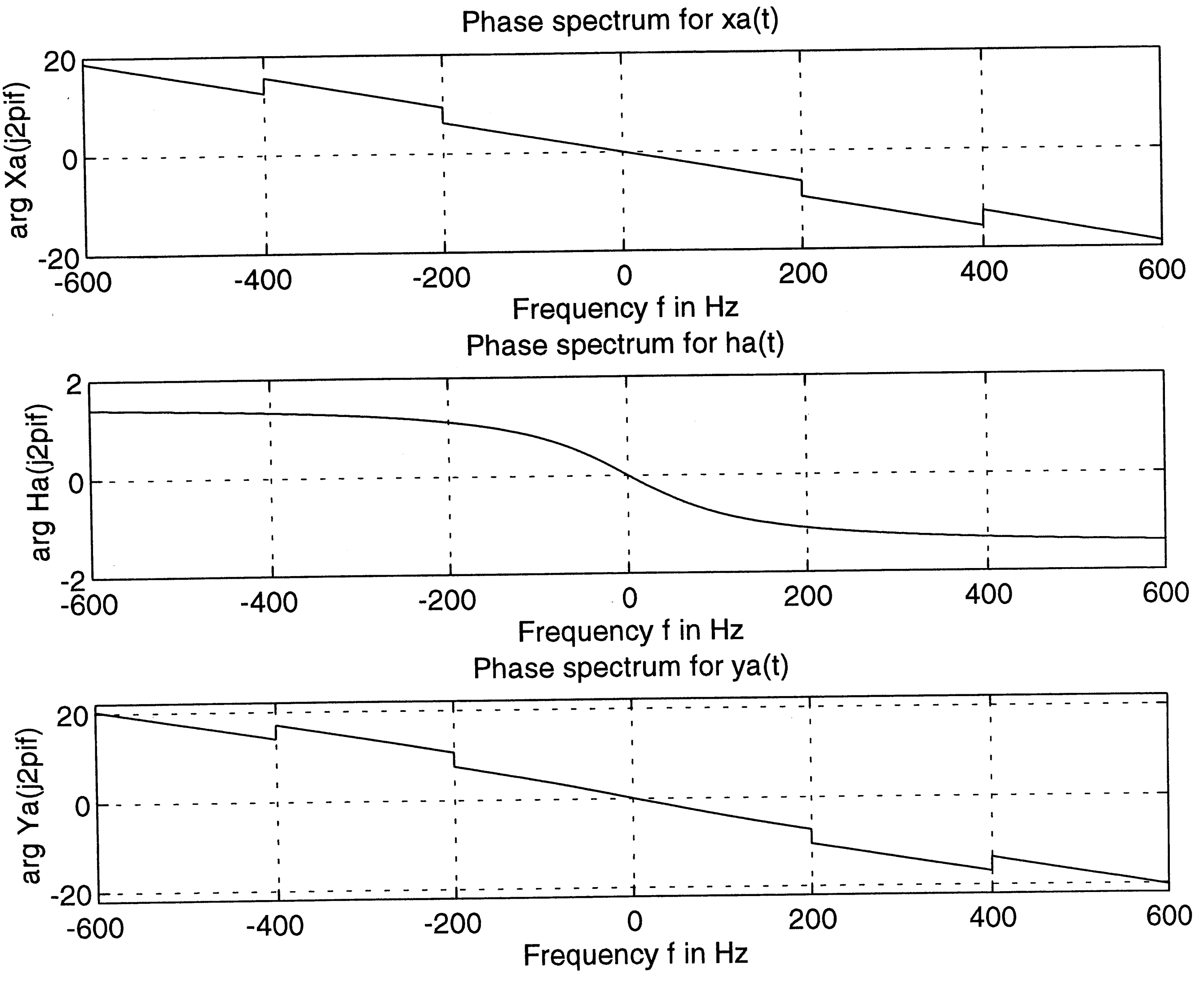
$$\arg\ Y_a(j2\pi f) = \arg\ H_a(j2\pi f) + \arg\ X_a(j2\pi f)$$

- These relations are exemplified in the following pages.

The Amplitude spectrum of the response of the exitation of $x_a(t) = u(t-(t_0-D/2)) - u(t-(t_0+D/2))$ in the case of a RC filter with $RC = 1/(2\pi 100)$

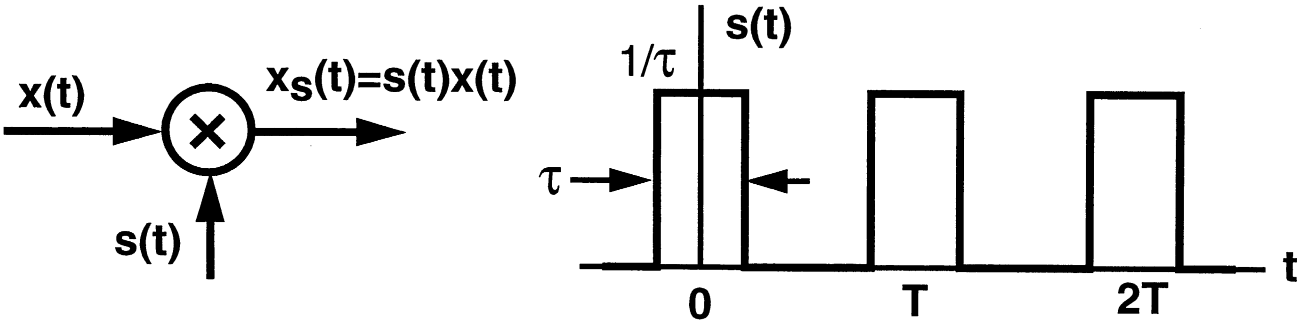


The Phase spectrum of the response of the exitation of $x_a(t) = u(t-(t_0-D/2)) - u(t-(t_0+D/2))$ in the case of a RC filter with $RC = 1/(2\pi 100)$

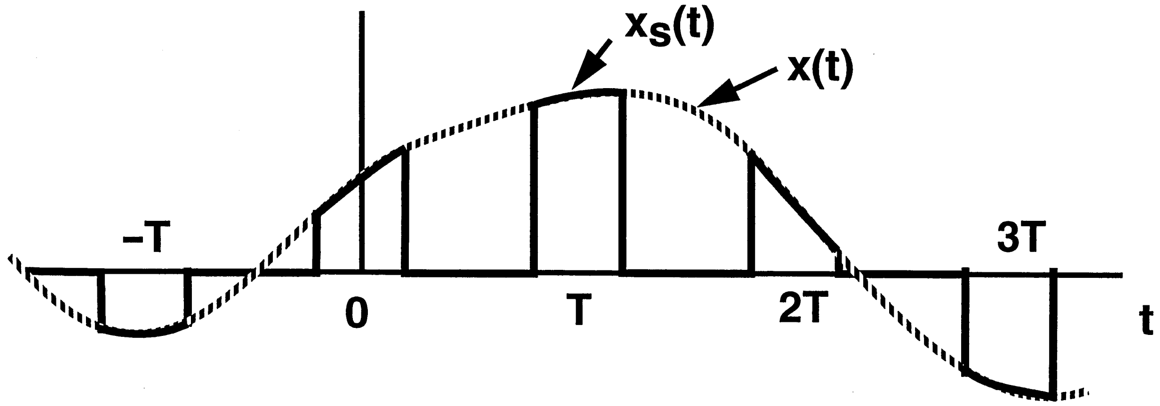


APPENDIX B: Why is the spectrum of a impulse train with sample values as weights periodic?

Consider the following system:



In this system, a continuous-time signal x(t) is multiplied by a pulse train s(t). The resulting output is $x_s(t)=x(t)s(t)$:



Since s(t) is periodic, it can be expressed in the following form assuming that the height of s(t) is $1/\tau$:

$$s(t) = c_0 + \sum_{n=1}^{\infty} 2c_k \cos\{n(2\pi F_s)t\},$$

where

$$c_k = F_s \operatorname{sinc}(nF_s\tau) .$$

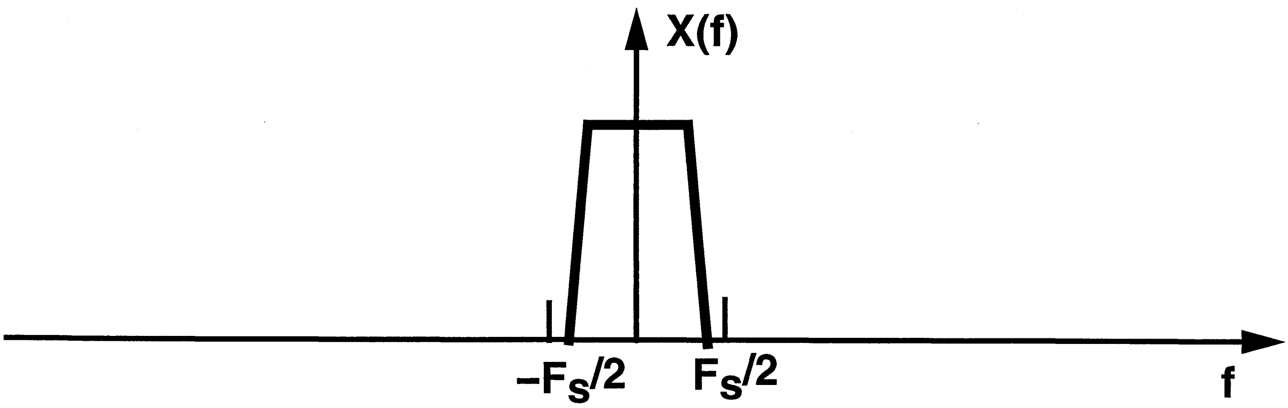
Hence, x(t) is expressible as

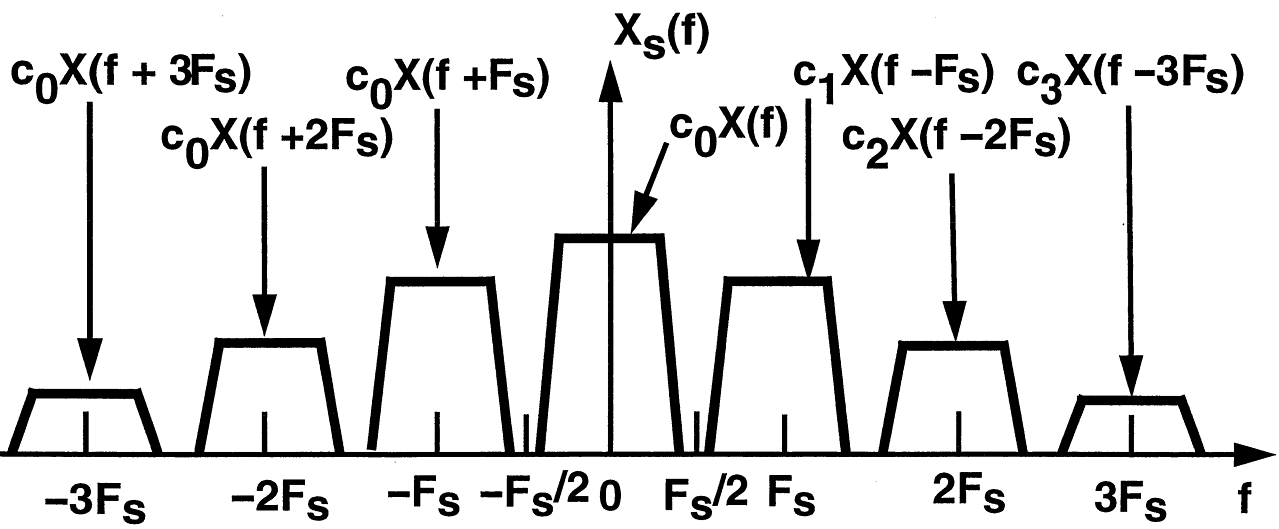
$$x_s(t) = c_0 + 2c_1\, x(t)\cos\{(2\pi F_s)t\} + 2c_2\, x(t)\cos\{2(2\pi F_s)t\} + \cdots .$$

From this it follows from the modulation theorem that the spectrum of $x_s(t)$ becomes

$$X_s(f) = c_0 X(f) + c_1[X(f-F_s) + X(f+F_s)] + c_2[X(f-2F_s) + X(f+2F_s)]$$
$$+ c_3[X(f-3F_s) + X(f+3F_s)] + c_4[X(f-4F_s) + X(f+4F_s)] + \cdots,$$

where X(f) is the spectrum of x(t). See the following figures.

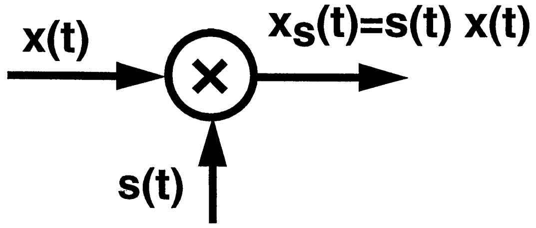


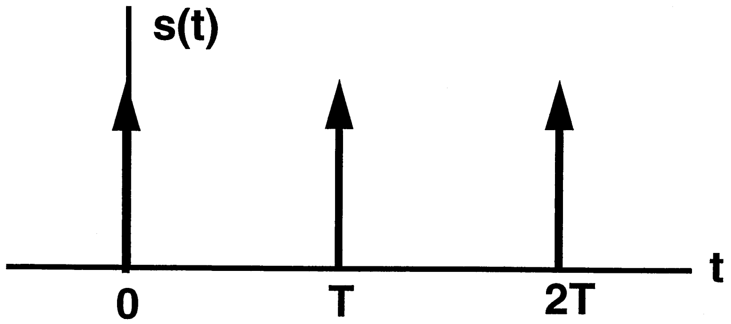


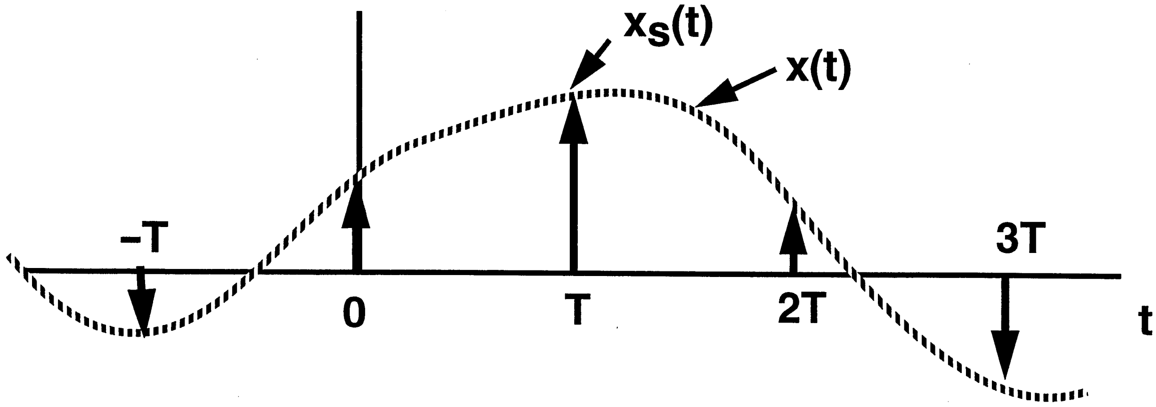
As $\tau \rightarrow 0$, s(t) and $x_s(t)$ approach the following impulse trains

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \quad x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\, \delta(t - nT).$$

according to the following figures:



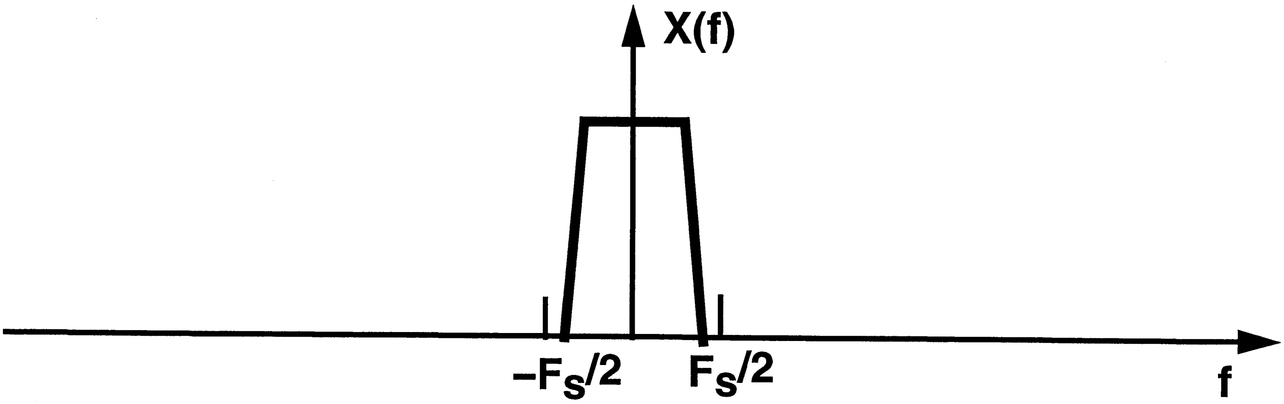


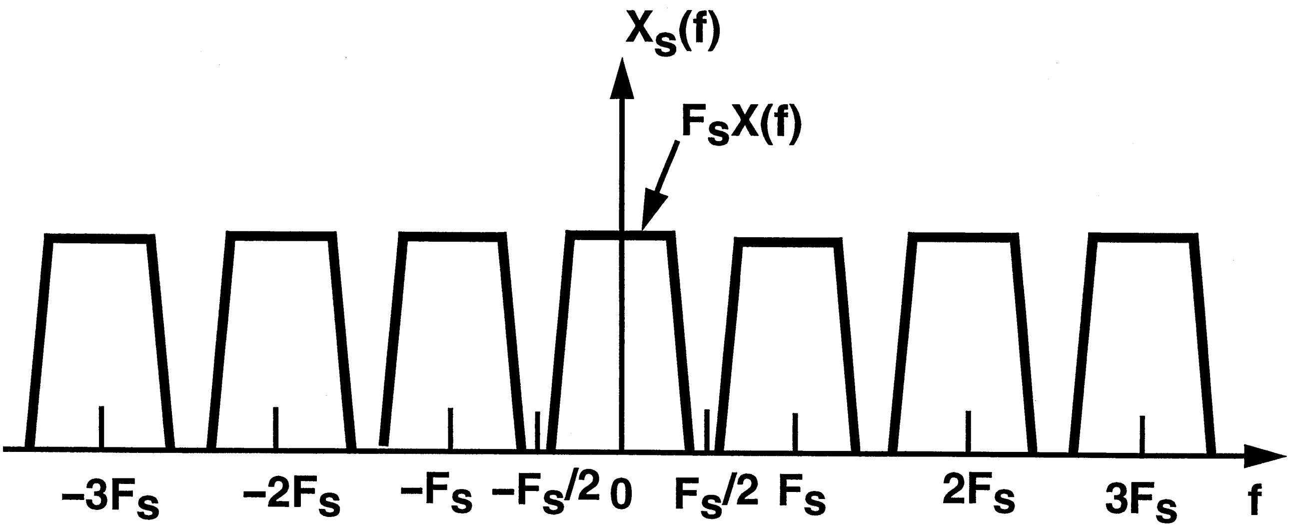


$\delta(t-nT)$ is an impulse which is generated by centering at t = nT a pulse with height of $1/\tau$ and width of τ and allowing τ to approach zero. The area of the resulting pulse is unity, the height is infinite, and the width is zero.

As $\tau \rightarrow 0$, then $c_k \rightarrow c_0$ = Fs ja the spectrum of $x_s(t)$ becomes perfectly periodic:

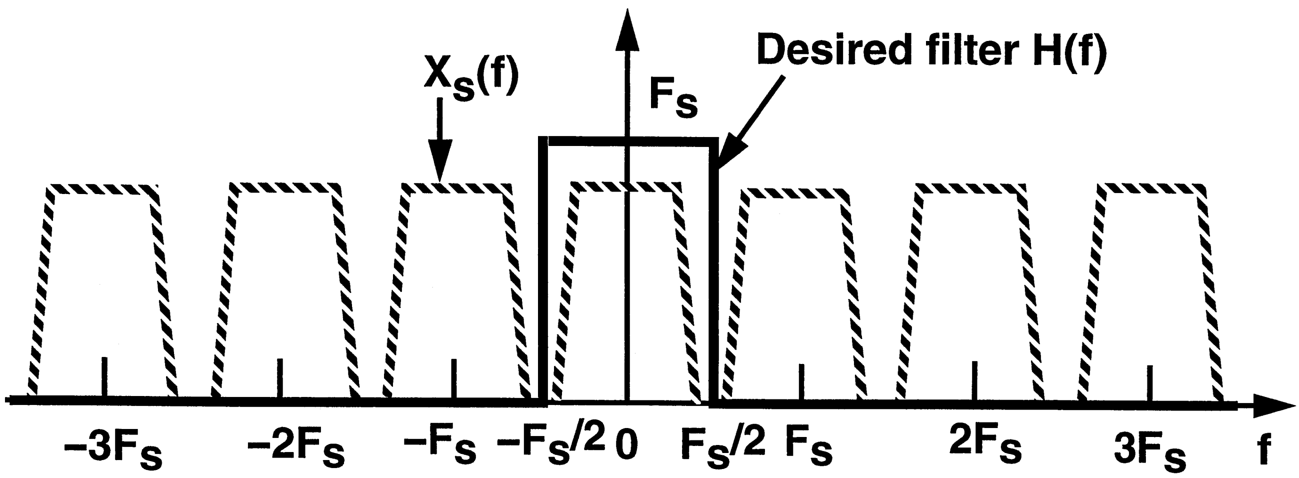
$$X_s(f) = F_s \sum_{n=-\infty}^{\infty} X(f - nF_s)$$

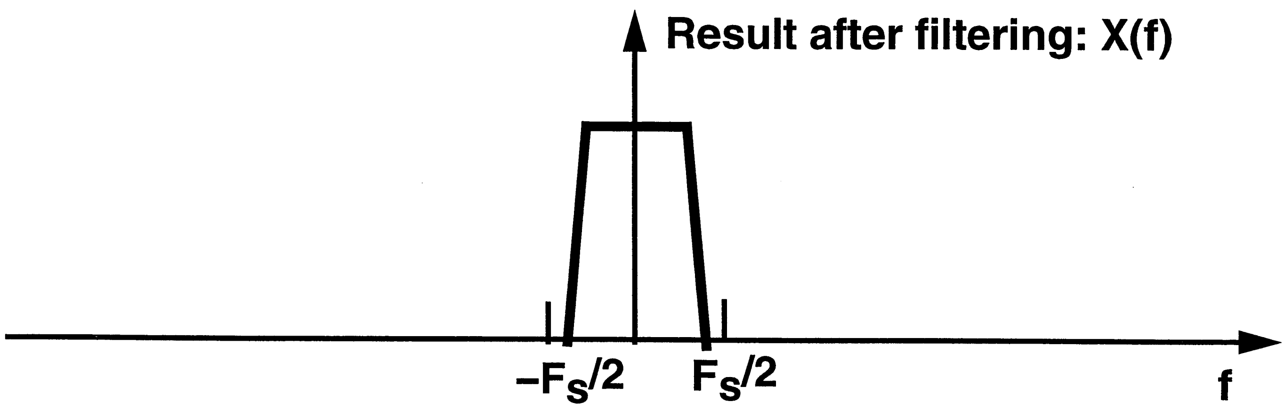




Reconstruction of the continuous-time signal

The original continuous-time signal can be reconstructed by filtering $x_S(t)$ with a continuous-time filter with amplitude equal to $1/F_S$ for $-F_S/2 < f < F_S/2$ and zero elsewhere :





Resulting continuous-time signal

The impulse response of our filter is given by

$h(t) = \mathrm{sinc}(t/T)$, $T = 1/F_S$.

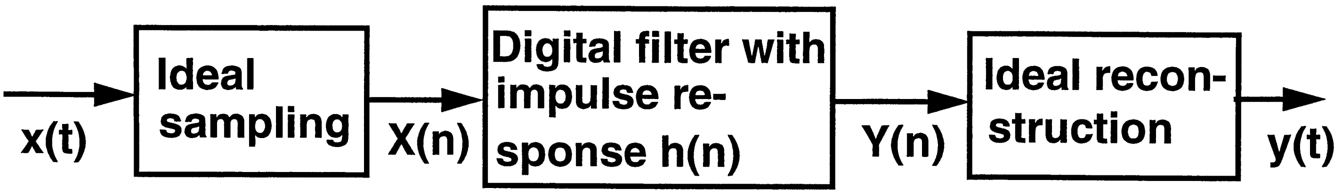
Since $x_S(t)$ is a sum of weighted (weights equal to the sample values $x(nT)$) and shifted impulses, the output is the following sum of weighted and shifted impulse responses:

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT)\, h(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\, \mathrm{sinc}[(t - nT)/T] = x(t).$$

Again, we ended up with the sinc-interpolation !!

APPENDIX C: Input-output relation for continuous-time signals in the case of the ideal sampling and reconstruction

Consider the following system:



This system is characterized by the equations (these become more clear later in this course):

$X(n) = x(nT)$, T is the sampling interval

$$Y(n) = \sum_{k=-\infty}^{\infty} h(k)\, X(n-k)$$

$$y(nT) = Y(n)$$

$$y(t) = \sum_{n=-\infty}^{\infty} y(nT)\, \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T} = \sum_{n=-\infty}^{\infty} y(nT)\ \mathrm{sinc}\,[(t-nT)/T]\ .$$

These are the time-domain relations between y(t) and x(t). The next transparency considers the frequency-domain relations.

On page B4 of Appendix B we noticed that if the Fourier transform of $x(t)$ is $X(j2\pi f)$, then the Fourier transform of

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$$

is given by

$$X_s(j2\pi f) = F_s \sum_{k=-\infty}^{\infty} X(j2\pi(f-kF_s)).$$

For the sequence $X(n) = x(nT)$ the z-transform is given by

$$\Xi(z) = \sum_{n=-\infty}^{\infty} X(n)\, z^{-n}$$

By substituting $z=e^{j2\pi f/F_s}$, $F_s=1/T$, we obtain

$$\Xi(e^{j2\pi f/Fs}) = X_s(j2\pi f) = F_s \sum_{k=-\infty}^{\infty} X(j2\pi(f-kF_s)).$$

This is the frequency-domain basic relation between the sequence $X(n)$ and t he sampled continuous-time signal $x(t)$.

For Y(n) satisfying

$$Y(n) = \sum_{k=-\infty}^{\infty} h(k)\, X(n-k)$$

the z-transform is

$$\Psi(z) = \sum_{n=-\infty}^{\infty} Y(n)\, z^{-n} = H(z)\Xi(z), \quad H(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n}$$

Here, H(z) is the transfer function of the digital filter.

The Fourier transform of

$$y_s(t) = \sum_{n=-\infty}^{\infty} y(nT)\, \delta(t-nT), \;\; y(nT) = Y(n)$$

is given by

$$Y_s(j2\pi f) = \Psi(e^{j2\pi f/F_s}) = H(e^{j2\pi f/Fs})\, \Xi(e^{j2\pi f/F_s})$$

$$= H(e^{j2\pi f/F_s})\, X_s(j2\pi f) = F_s\, H(e^{j2\pi f/Fs}) \sum_{k=-\infty}^{\infty} X(j2\pi(f-kF_s)).$$

The Fourier transform of

$$y(t) = \sum_{n=-\infty}^{\infty} y(nT)\, \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T} = \sum_{n=-\infty}^{\infty} y(nT)\, \mathrm{sinc}\,[(t-nT)/T]$$

is obtained from that of $y_S(t)$ using an analog filter with frequency response equal to $1/F_S$ in the frequency range from $-F_S/2$ to $F_S/2$ and zero elsewhere (analog filter with impulse response equal to h(t) = sinc (t/T)).

The Fourier transform of y(t) is thus related to that of x(t) according to (assuming that no aliasing occurs)

$$Y(j2\pi f) = G(j2\pi f)\, X(j2\pi f),$$

where

$$G(j2\pi f) = H(e^{j2\pi f/F_S}) \quad \text{for} \quad -F_S/2 \leq f \leq F_S/2$$

and zero elsewhere.

Hence, the input-output relation for y(t) and x(t) in the frequency range from $-F_S/2$ to $F_S/2$ is simply obtained by using the substitution $z=e^{j2\pi f/F_S}$ in the transfer function H(z) of the digital filter.

APPENDIX D: Fourier Tranform of Discrete-Time Signals

- The purpose of this appendix is to give a short review of the Fourier transform of discrete-time signals.
- The main emphasis is on the practical use of this transform on analysing discrete-time signals and systems.

FOURIER TRANSFORM

- The z-transform of a sequence $x[n]$ was defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}.$$

- The Fourier transform of this sequence is obtained by using the substitution $z = e^{j\omega}$ in the above equation, giving

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega}. \qquad (A)$$

- Alternatively, after knowing $X(e^{j\omega})$,

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{jn\omega}. \qquad (B)$$

- Equations (A) and (B) form a Fourier transform pair.
- Strictly speaking, $X(e^{j\omega})$ exists ($|X(e^{j\omega})| < \infty$) provided that $x[n]$ is absolutely summable, that is,

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty.$$

- In this case, the series of equation (A) converges uniformly to a continuous function of ω.

- **Example:** Let $x[n] = a^n u[n]$. Then,

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-jn\omega} = \sum_{n=-0}^{\infty} (ae^{-j\omega})^n$$
$$= 1/(1 - ae^{-j\omega}) \text{ if } |ae^{-j\omega}| < 1 \text{ or } |a| < 1.$$

- Clearly, if $|a| < 1$, then $x[n]$ is absolutely summable, that is,

$$\sum_{n=0}^{\infty} |a^n| = 1/(1 - |a|) < \infty.$$

- Some sequences that are square summable, that is,

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

can be represented by a Fourier transform if we relax the condition of uniform convergence of the infinite sum defining $X(e^{j\omega})$.

- **Example:** Let

$$X(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi. \end{cases}$$

- In this case,

$$x[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{jn\omega} = \frac{\sin(n\omega_c)}{n\pi}, \quad -\infty < n < \infty.$$

- This sequence is square summable, but not absolutely summable.

- By including only the terms $-M \leq n \leq M$ gives the partial sum

$$X_M(e^{j\omega}) = \sum_{n=-M}^{M} \frac{\sin(n\omega_c)}{n\pi} e^{-jn\omega}.$$

- It is well-known from the theory of mathematics, that as $M \mapsto \infty$, $X_M(e^{j\omega})$ has allways an overshoot of value 1.09 before the point $\omega = \omega_c$ and an undershoot of value -0.09 after this point independent of the value of M.
- This is the well-known Gibbs phenomenon and is illustrated in the following figure for several values of M.

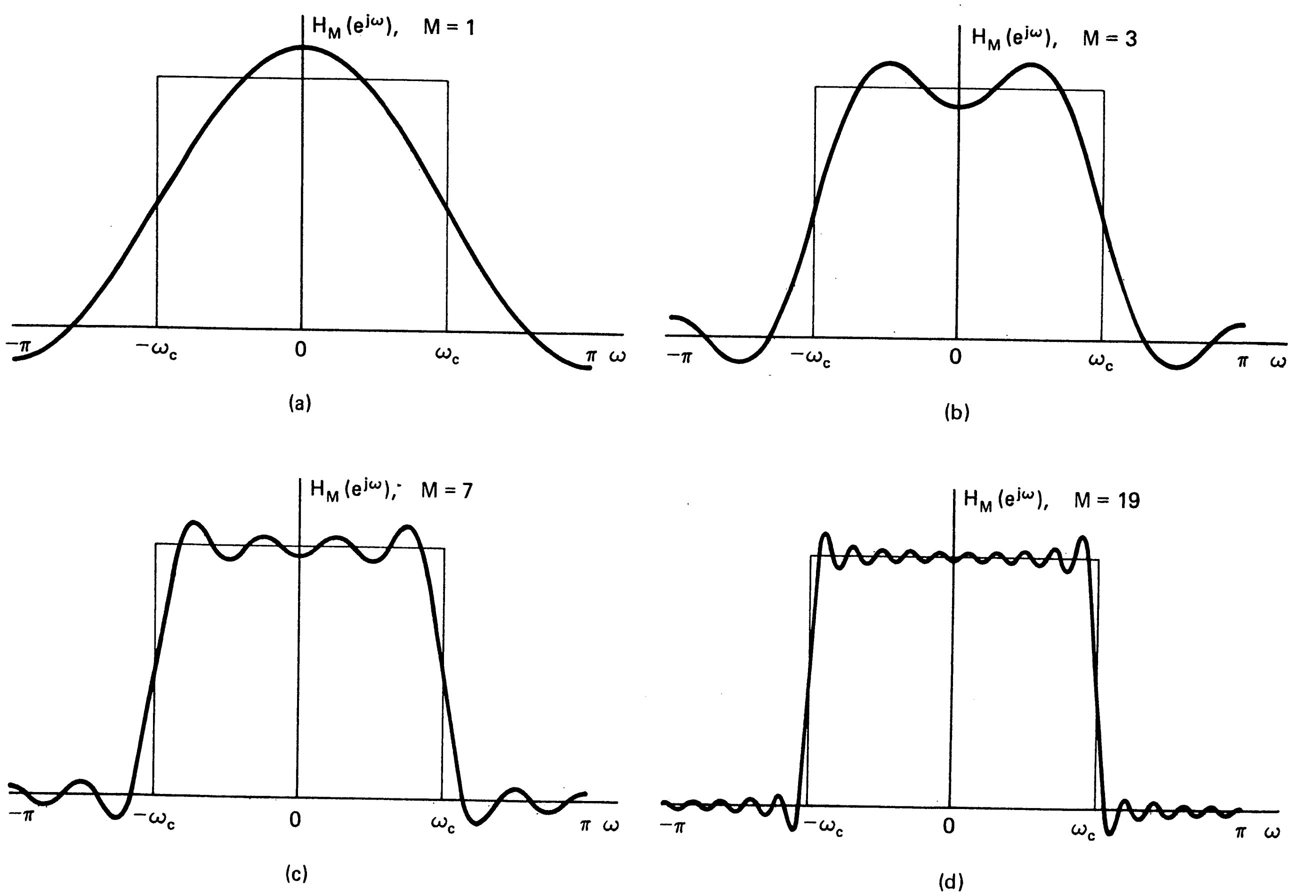


Figure 2.20 Convergence of the Fourier transform. The oscillatory behavior at $\omega = \omega_c$ is often called the Gibbs phenomenon.

- In this case, $X_M(e^{j\omega})$ converges to $X(e^{j\omega})$ as $M \mapsto \infty$ except for the discontinuity point $\omega = \omega_c$.
- The sequence

$$x[n] = A\cos(n\omega_0 + \phi) = A[e^{j(n\omega_0+\phi)} + e^{-j(n\omega_0+\phi)}]/2 \qquad (A)$$

does not have the Fourier transform in the normal manner.

- If desired, we can utilize the fact that the Fourier transform of $x[n] = e^{j(n\omega_0+\phi)}$ can be expressed as

$$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} e^{j\phi}\delta_a(\omega - \omega_0 + 2\pi k),$$

where $\delta_a(x)$ is the Dirac delta function defined on page 33 in Appendix A.

- Based on this, the Fourier transform of the sequnce given by equation (A) can be expressed as

$$\begin{aligned} X(e^{j\omega}) =& \pi \sum_{k=-\infty}^{\infty} e^{j\phi}\delta_a(\omega - \omega_0 + 2\pi k) \\ &+ \pi \sum_{k=-\infty}^{\infty} e^{-j\phi}\delta_a(\omega + \omega_0 + 2\pi k). \end{aligned}$$

- Fortunately, this formula is seldom used!! It is preferred to use the z-transform when treating sinusoisal signals and we have other ways to check what is happening to this signal when it is a passing a digital filter, as we shall see elsewhere in this course.

- **Basic Rule:** If you like to know the Fourier transform of an absolutely summable sequence (most practical signals are of this type), find first the z-transform $X(z)$ of this sequence. The Fourier transform $X(e^{j\omega})$ is then obtained from $X(z)$ by using the substitution $z = e^{j\omega}$.

FREQUNCY-DOMAIN INTERPRETATION OF THE FOURIER TRANSFORM

- We recall that the Fourier transform pair is defined by the following two equations:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega} \qquad (A)$$

and

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{jn\omega}. \qquad (B)$$

- Here, equation (A) is the analysis formula. Equation (B) is the synthesis formula, that is, it represents $x[n]$ a sum of infinitesimally small complex sidusoids of the form

$$\frac{1}{2\pi}X(e^{j\omega})e^{jn\omega},$$

with ω ranging over the interval $[-\pi,\ \pi]$ and $X(e^{j\omega})$ determining the relative amount of each complex sidusoidal component.

- In other words, $X(e^{j\omega})$ tells us how the signal $x[n]$ is distributed in the frequency range $\omega \in [-\pi,\ \pi]$.

- Note that since $e^{j(k2\pi+\omega)} = e^{j\omega}$ for $k = \pm1, \pm2, \pm3, ...,$ $X(e^{j(k2\pi+\omega)}) = X(e^{j\omega})$ for $k = \pm1, \pm2, \pm3, ...$ This means that $X(e^{j\omega})$ is **periodic with periodicity**

equal to 2π. Therefore, we can concentrate on studing its performance only on the 'basic' interval $[-\pi, \ \pi]$.

- $X(e^{j\omega})$ is, in general, a complex function of ω and is expressible as

$$X(e^{j\omega}) = \mathrm{Re}\{X(e^{j\omega})\} + j\mathrm{Im}\{X(e^{j\omega}))\}$$
$$= |X(e^{j\omega})|e^{j\arg X(e^{j\omega})}$$

- Here,

$$|X(e^{j\omega})| = \sqrt{(\mathrm{Re}\{X(e^{j\omega})\})^2 + (\mathrm{Im}\{X(e^{j\omega})\})^2}$$

is the amplitude spectrum of $x[n]$ and

$$\arg\ X(e^{j\omega}) = \mathrm{atan2}\,(\mathrm{Im}\{X(e^{j\omega})\}, \mathrm{Re}\{X(e^{j\omega})\})$$

is the phase spectrum spectrum of $x[n]$.

- It should be pointed out that the above definition of $\arg\ X(e^{j\omega})$ forces it to stay between the limits $\pm\pi$.
- However, we can any integer multiple of 2π to it without affecting the result of the complex exponential. In many cases, it is desired to make $\arg\ X(e^{j\omega})$ continuous.
- This is possible at all points except for those where $|X(e^{j\omega})|$ achieves the value of zero. At this

points, arg $X(e^{j\omega})$ has always a jump of π or $-\pi$, as we shall see later in this course.

- Now it is time for an example.

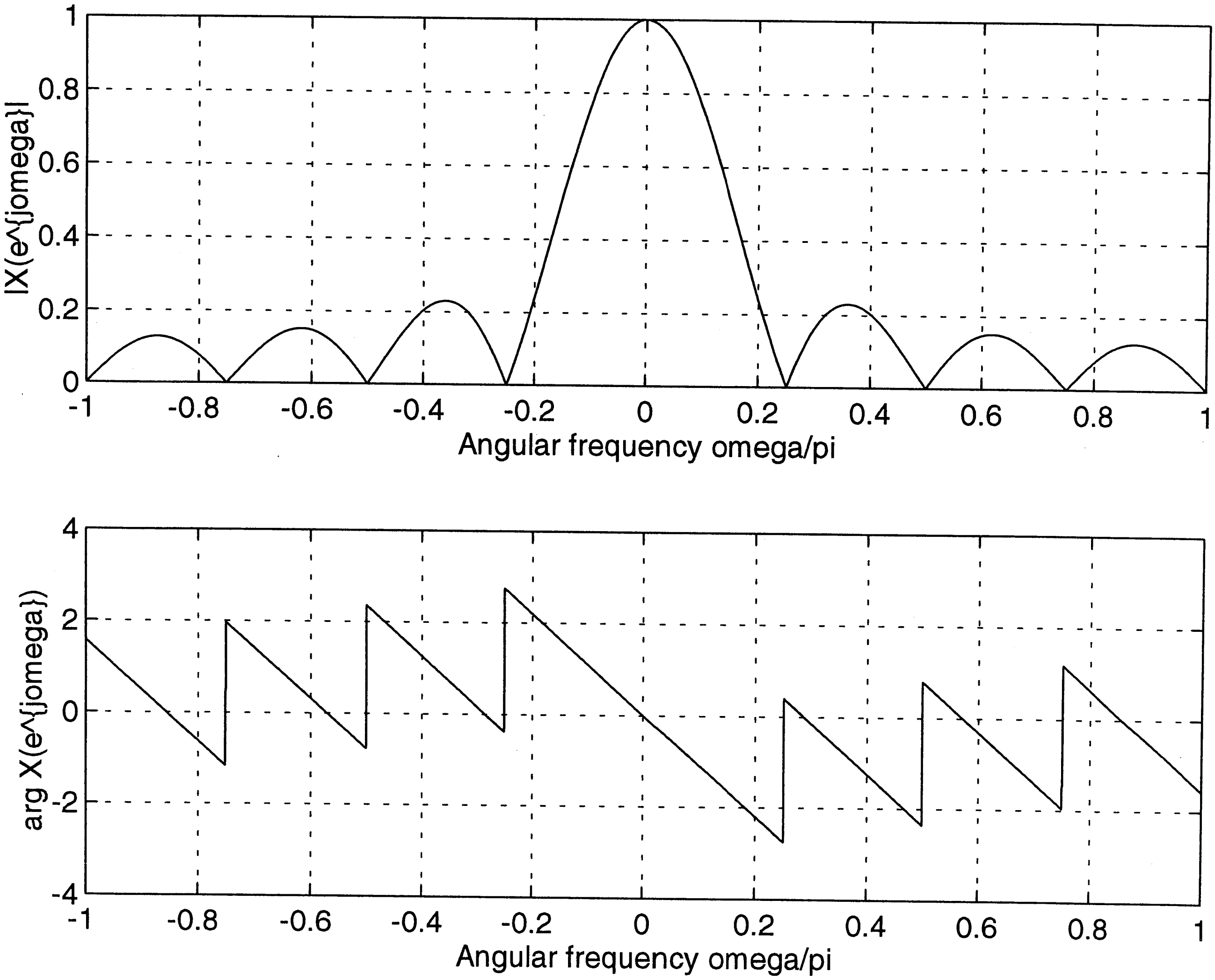
EXAMPLE

- Consider a signal $x[n]$ that has the value of $1/8$ for $0 \leq n \leq 7$ and is zero elsewhere.
- The Fourier transform of this signal is then

$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=0}^{7} \frac{1}{8} e^{-jn\omega} = \frac{1}{8} \frac{1 - e^{-j8\omega}}{1 - e^{j\omega}} \\
&= \frac{1}{8} e^{-j3.5\omega} \frac{e^{j4\omega} - e^{-j4\omega}}{e^{j\omega/2} - e^{-j\omega/2}} = \frac{1}{8} e^{-j3.5\omega} \frac{\sin(4\omega)}{\sin(\omega/2)}.
\end{aligned}
$$

- The following transparency shows the amplitude and phase spectra for this signal.

AMPLITUDE AND PHASE SPECTRA FOR OUR EXAMPLE SEQUENCE



FOURIER TRANSFORM OF A FILTERED SIGNAL

- If the z-transforms of the input signal $x[n]$ and the impulse response $h[n]$ of the filter (also called the transfer function) are $X(z)$ and $H(z)$, respectively, the z-transform of the output signal is

$$Y(z) = H(z)X(z).$$

- Using the substitution $z = e^{j\omega}$ in the above equation gives

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}),$$

 that is, the Fourier transform of the output signal is the product of the Fourier transforms of the input signal and the impulse response of the filter.

- $H(e^{j\omega})$ is called the frequency response of the filter and is considered in more details elsewhere in this course.

- Based on the above equation, we can express $Y(e^{j\omega})$ as

$$Y(e^{j\omega}) = |Y(e^{j\omega})|e^{j\arg Y(e^{j\omega})},$$

 where

$$|Y(e^{j\omega})| = |H(e^{j\omega})||X(e^{j\omega})|$$

and

$$\arg Y(e^{j\omega}) = \arg X(e^{j\omega}) + \arg H(e^{j\omega}).$$

- In other words, the amplitude spectrum of the output signal is obtained from that of the excitation by multiplying it by $|H(e^{j\omega})|$ that is called the amplitude response of the filter.
- The phase spectrum of the output signal is obtained from that of the excitation by adding to it $\arg H(e^{j\omega})$ that is called the phase response of the filter.
- It is again time to consider an example in order to make the above relations more clear.

EXAMPLE

- Consider a signal $x[n]$ that has the value of $1/8$ for $0 \leq n \leq 7$ and is zero elsewhere. As shown previously,

$$X(e^{j\omega}) = \frac{1}{8}e^{-j3.5\omega}\frac{\sin(4\omega)}{\sin(\omega/2)}.$$

- The impulse response of our filter is $h[n] = 0.1(0.9)^n u[n]$. The z transform of this signal is $H(z) = 0.1/(1 - 0.9z^{-1})$ so that the Fourier transform is

$$H(e^{j\omega}) = \frac{0.1}{1 - 0.9e^{-j\omega}}.$$

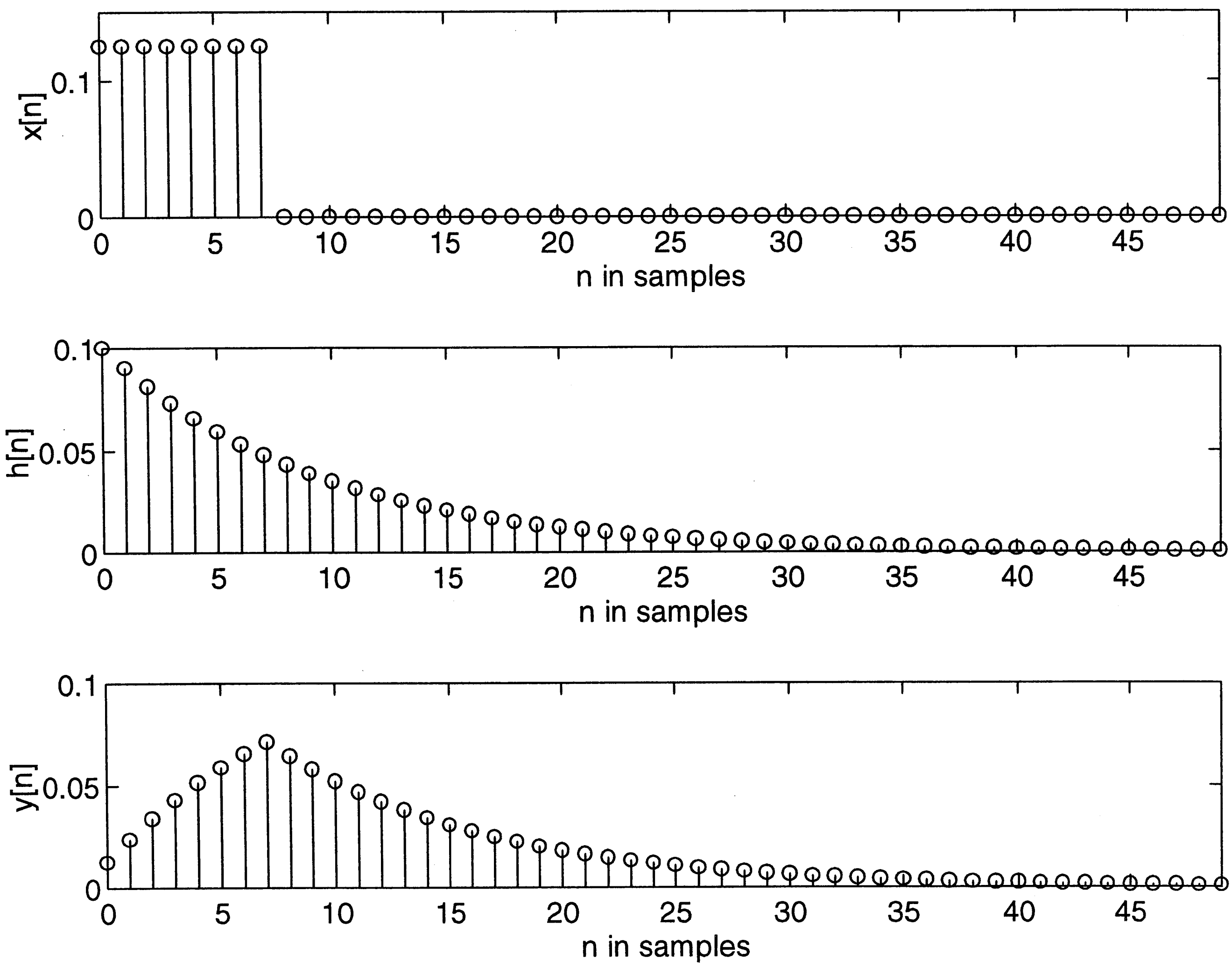
- In the time domain, the output sequence is given by

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k].$$

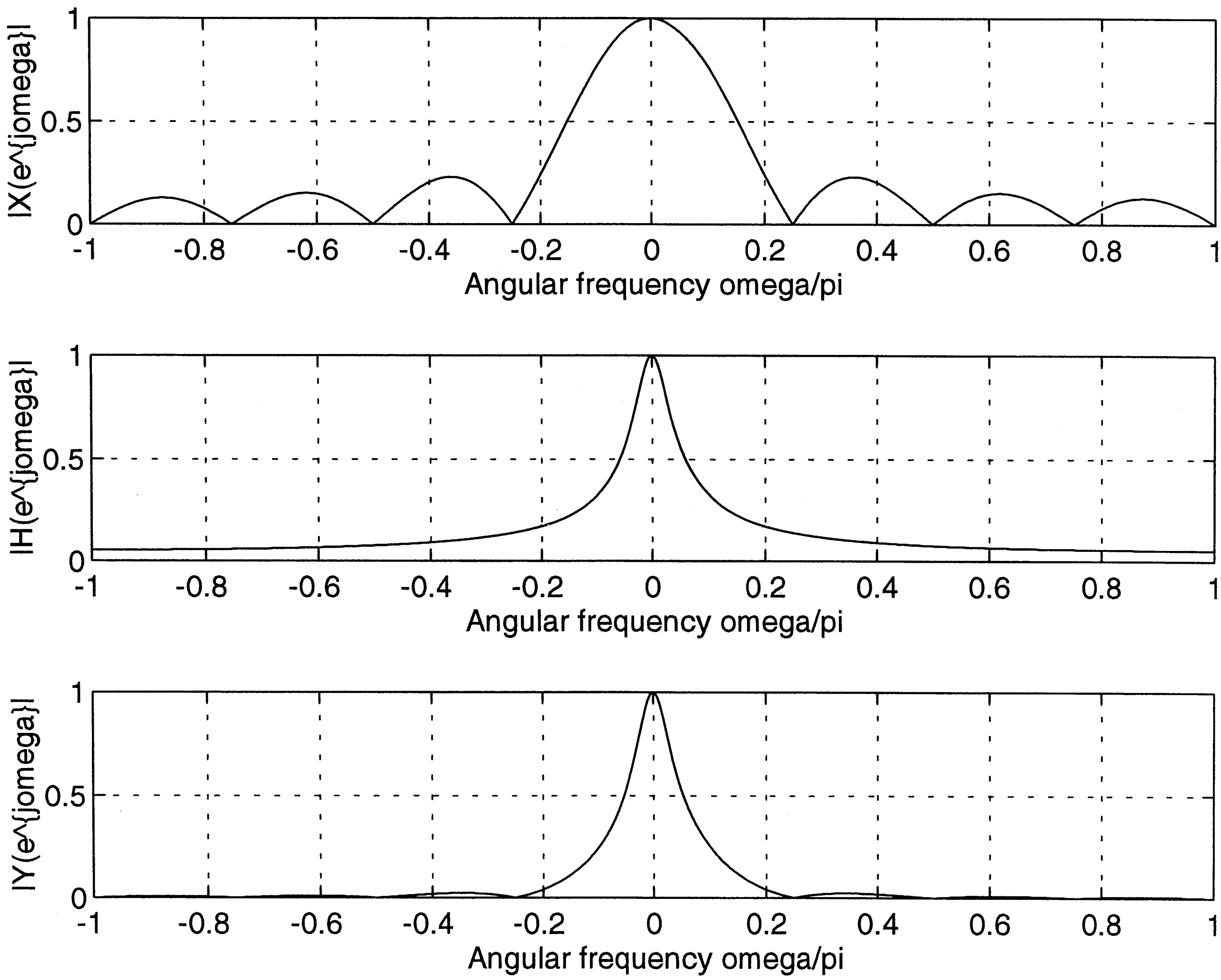
- In the following, the first, second, and third transparencies show the relations between $y[n]$ and $x[n]$; between $|Y(e^{j\omega})|$ and $|X(e^{j\omega})|$; and between $\arg Y(e^{j\omega})$ and $\arg X(e^{j\omega})$.

- It is seen that our filter preserves the amplitude spectrum near the zero frequency, whereas it attenuates this spectrum more and more as ω approaches the value of π or $-\pi$.

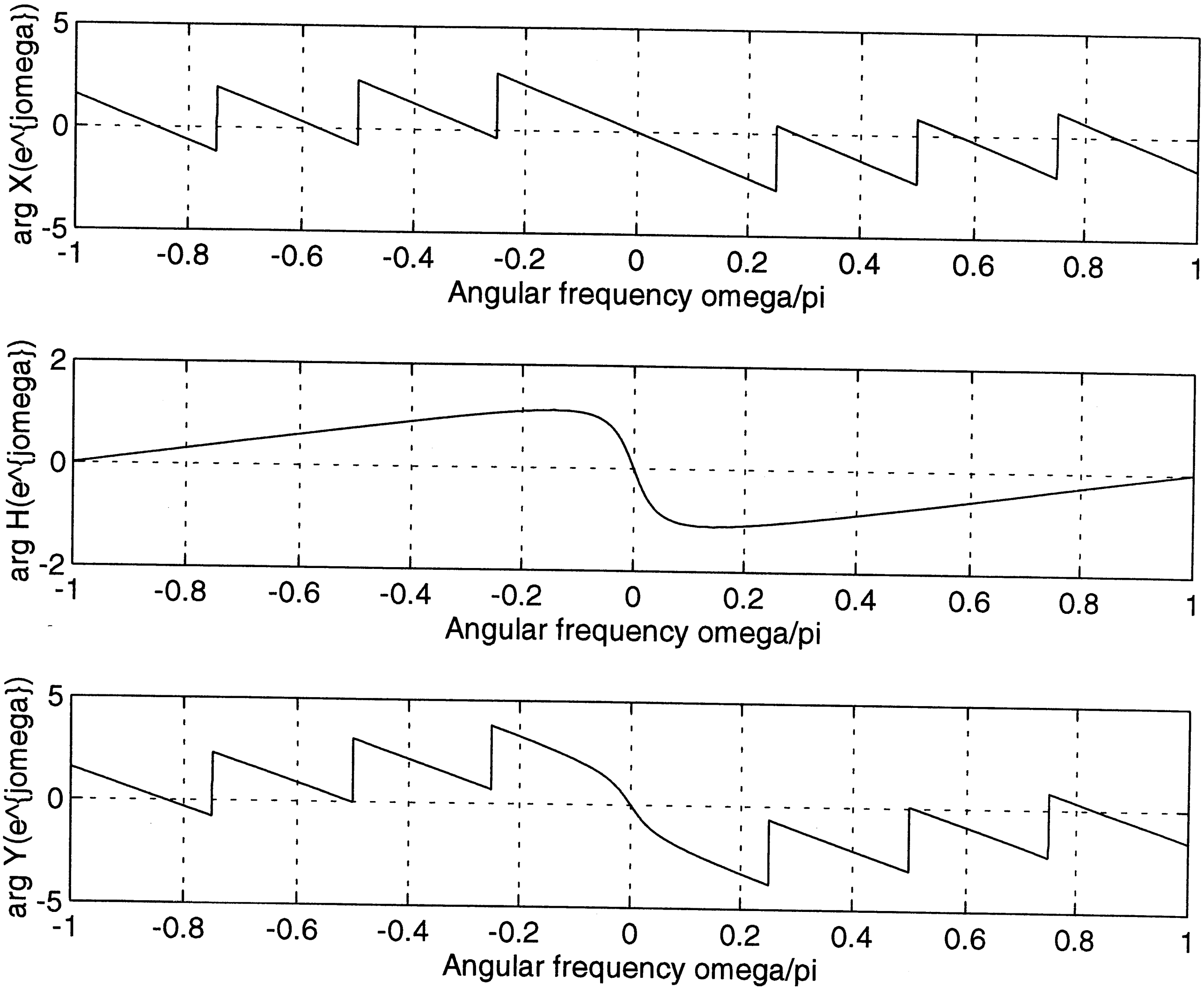
RELATIONS BETWEEN $y[n]$ AND $x[n]$ IN OUR EXAMPLE



RELATIONS BETWEEN $|Y(e^{j\omega})|$ AND $|X(e^{j\omega})|$ IN OUR EXAMPLE



RELATIONS BETWEEN $\arg Y(e^{j\omega})$ AND $\arg X(e^{j\omega})$ IN OUR EXAMPLE



OTHER PROPERTIES OF THE FOURIER TRANSFORM

- In the following, we use the notation

$$x[n] \Longleftrightarrow X(e^{j\omega})$$

to indicate that $x[n]$ and $X(e^{j\omega})$ form a Fourier transform pair.

Linearity: If

$$x_1[n] \Longleftrightarrow X_1(e^{j\omega})$$

and

$$x_2[n] \Longleftrightarrow X_2(e^{j\omega}),$$

then

$$ax_1[n] + bx_2[n] \Longleftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega}).$$

Time and frequency shiftings: If

$$x[n] \Longleftrightarrow X(e^{j\omega}),$$

then

$$x[n - n_0] \Longleftrightarrow e^{-jn_0\omega}X(e^{j\omega})$$

and

$$e^{-jn\omega_0}x[n] \Longleftrightarrow X(e^{j(\omega-\omega_0)}).$$

Time reversal: If

$$x[n] \Longleftrightarrow X(e^{j\omega}),$$

then

$$x[-n] \iff X(e^{-j\omega}).$$

Differentiation in frequency: If

$$x[n] \iff X(e^{j\omega},$$

then

$$nx[n] \iff j\frac{dX(e^{j\omega})}{d\omega}.$$

Parseval's Teorem: If

$$x[n] \iff X(e^{j\omega}),$$

then

$$E = \sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega.$$

- The function $|X(e^{j\omega})|^2$ is called the **energy density spectrum** since it determines how the energy is distributed in frequency.

The Windowing Teorem: If

$$x[n] \iff X(e^{j\omega}),$$
$$w[n] \iff W(e^{j\omega}),$$

and

$$y[n] = x[n]w[n],$$

then

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})d\theta.$$