DESIGN OF RECURSIVE FILTERS USING ALLPASS FILTERS AS BUILDING BLOCKS

1. How to design IIR filters as a parallel connection of two allpass sections?

2. Design of classical IIR filters.

3. Implementation of allpass sections.

4. Extended filter classes:

- Filters where one of the branches is a pure delay term.
- Filters whose passband phase response is approximately linear.
- Multiband filters.

5. Other applications:

- Phase equalization.
- Approximately linear-phase Hilbert transformers.
- Filters providing an arbitrary noninteger delay.

6. (Efficient filters for sampling rate alteration and for building filter banks.)

7. (Multiplier-free implementations.)

8. (Synthesis techniques for generalized filters.)

INTRODUCTION

- In recent years, a large number of digital filter structures have been developed.
- The main concern has been the performance of the structure in finite wordlength implementation on one hand and the computational complexity on the other.
- It has turned out that very useful digital filter structures for various applications can be constructed by using allpass filters as building blocks.
- Traditionally, allpass filters have been used as phase equalizers.
- Lattice wave digital filters, realized as a parallel connection of two allpass filters, are well-known among such filter classes.
- These structures can be used for realizing classical odd-order lowpass and highpass filters.
- In the bandpass and bandstop cases, lattice wave digital filters have been used for realizing filters which are obtained from an odd-order lowpass prototype via frequency tranformations.
- Techniques have also been advanced for designing similar filters and certain extended filter types directly in the z-domain.
- Special cases are filter where one of the branches is a pure delay term and filters providing approximately linear phase in the passband region.

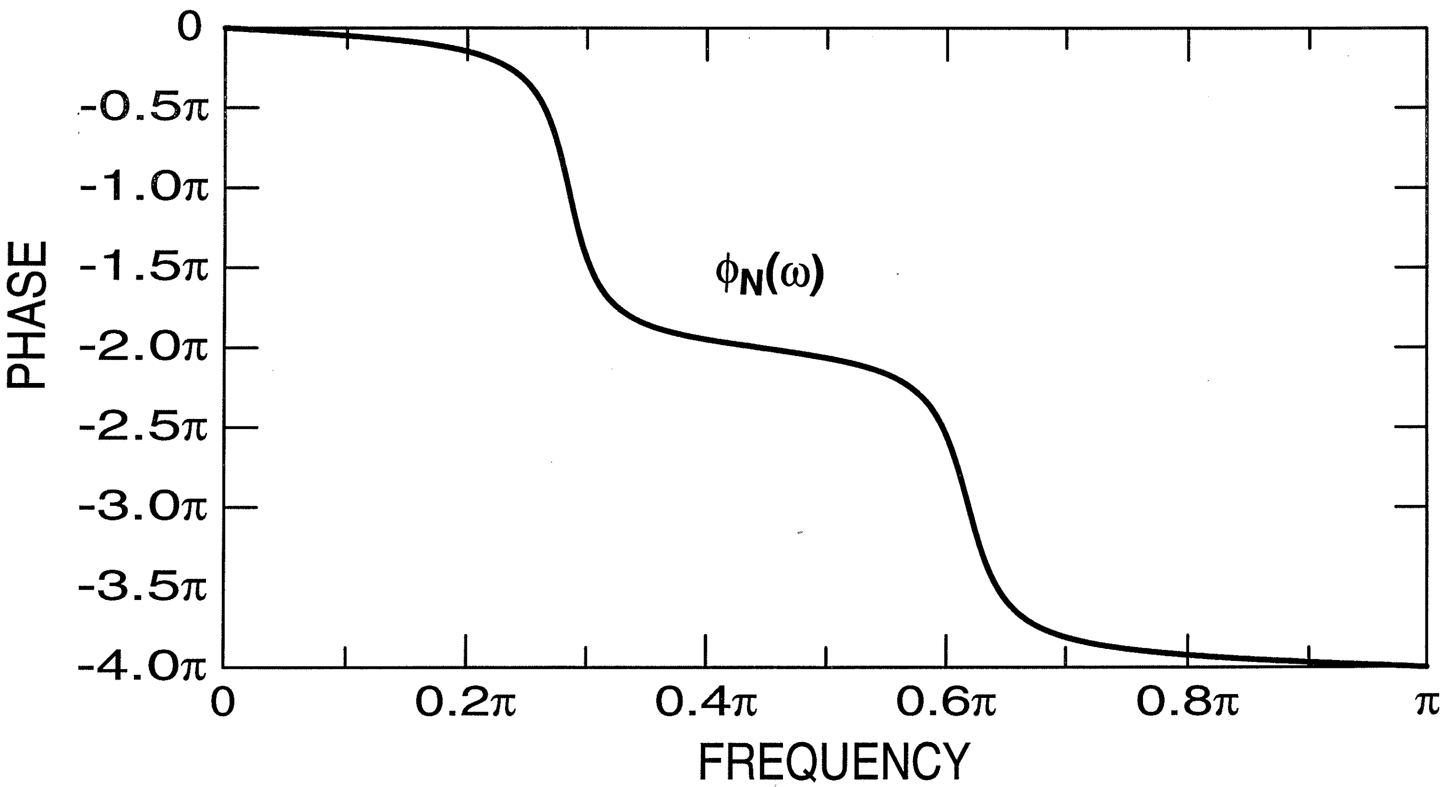
- One of the advantages of using a parallel connection of two allpass filters is that the complementary filter is obtained from the original one by simply changing the sign of one of the allpass sections.
- Therefore, a complementary filter pair (e.g., a lowpass/highpass or a bandstop/bandpass filter pair) can be implemented with the cost of a single filter.
- In addition, these filters can be realized using first- and second-order allpass sections as building blocks.
- The resulting filter structures are highly modular, making them suitable to signal processor and VLSI implementations.
- In addition to group delay equalization, allpass filters can also be used for phase or phase delay equalization.
- They are useful for synthesizing filters with arbitrary noninteger delay and filters whose phase differs by $\pi/2$ from a integer-valued contant, that is, special Hilbert transformers.
- They can be extended also for synthesizing generalized doubly complementary digital filters as well as multiband filters with arbitrary magnitude specifications.
- Special filters composed of allpass filter building blocks are extremely useful in designing decimators and iterpolators as well as filter banks.
- The above-mentioned filters can be designed based on amplitude or phase approximations.

ALLPASS FILTER

- Transfer function of order N

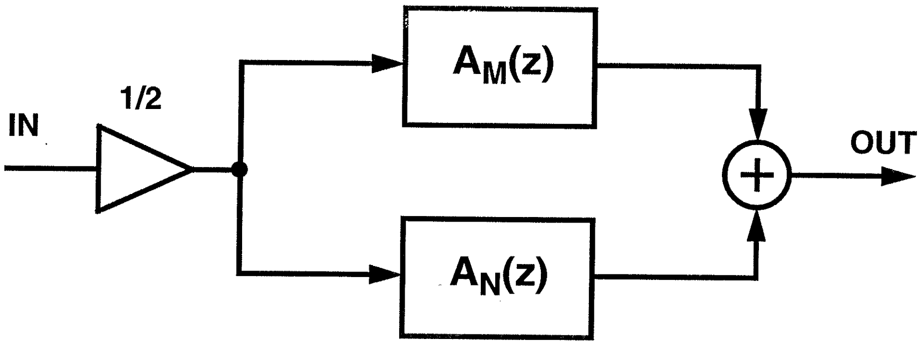
$$A_N(z) = \frac{z^{-N} D_N(z^{-1})}{D_N(z)}$$
$$= \frac{a_N + a_{N-1}z^{-1} \cdots a_1 z^{-(N-1)} + z^{-N}}{1 + a_1 z^{-1} + \cdots + a_{N-1} z^{-(N-1)} + a_N z^{-N}}$$

- For the stable filter, the poles [roots of $D_N(z)$] are inside the unit circle.
- $A_N(e^{j\omega}) = e^{j\phi_N(\omega)}$, $\phi_N(\omega) = \arg\, A_N(e^{j\omega})$.
- As ω varies from 0 to π, $\phi_N(\omega)$ decreases monotonically from 0 to $-N\pi$; $N = 4$:

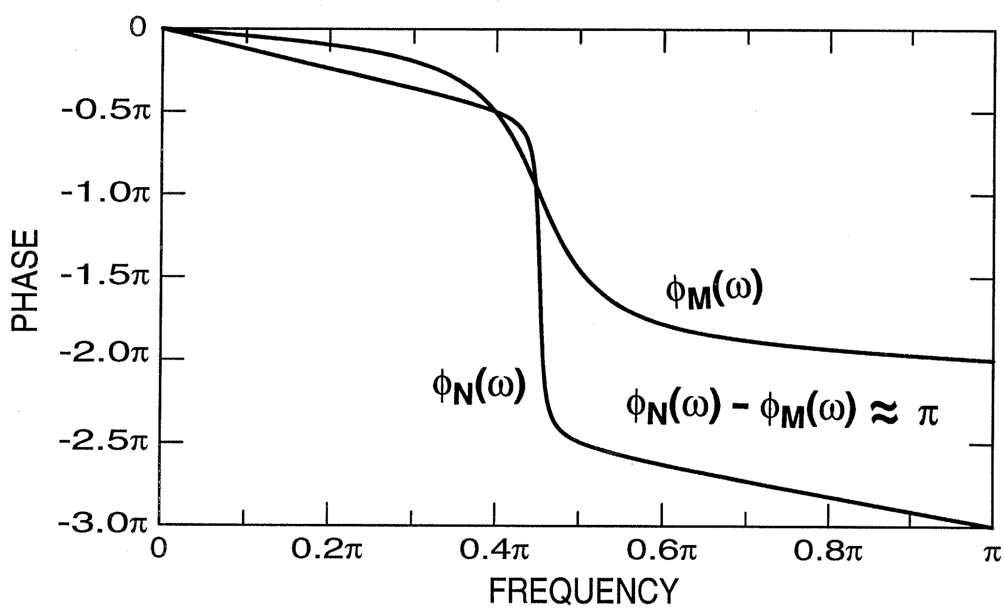


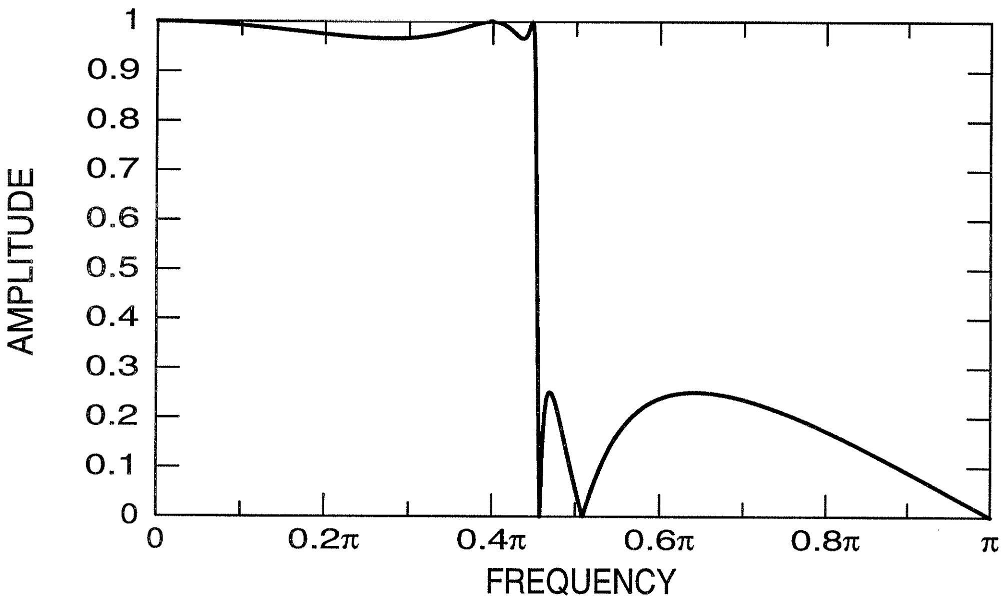
BUILDING A LOWPASS FILTER

- A parallel connection of $A_N(z)$ and $A_M(z)$ with **N=M+1**: $H_{\rm LP}(z) = [A_M(z) + A_N(z)]/2$.



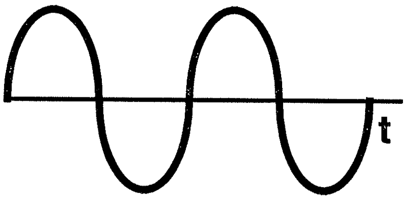
- Desired responses: $|H_{\rm LP}(e^{j\omega})| = \frac{1}{2}|e^{j\phi_M(\omega)} + e^{j\phi_N(\omega)}|$.





RESPONSES FOR SINUSOIDAL SIGNALS

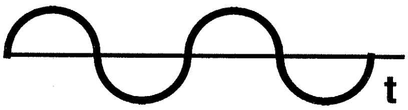
INPUT SIGNAL: $x(n) = C\sin(n\omega)$

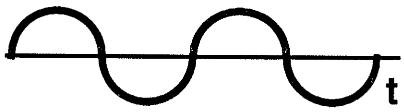


PASSBAND | **STOPBAND**

OUTPUT OF $A_M(z)/2$:

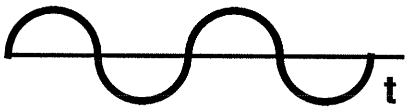
$$y_1(n) = \frac{C}{2}\sin(n\omega + \phi_M(\omega))$$

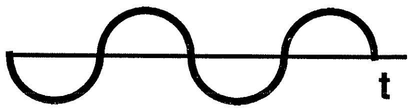




OUTPUT OF $A_N(z)/2$:

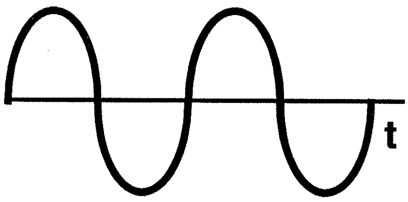
$$y_2(n) = \frac{C}{2}\sin(n\omega + \phi_N(\omega))$$





OVERALL OUTPUT

$$y(n) = y_1(n) + y_2(n)$$



$\phi_M(\omega) = \phi_N(\omega) \rightarrow$

$y(n) = C\sin(n\omega + \phi_M(\omega))$

$y_1(n)$ and $y_2(n)$ are in phase

$\phi_M(\omega) = \phi_N(\omega) + \pi \rightarrow$

$y(n) = 0$

$y_1(n)$ and $y_2(n)$ are out of phase

FREQUENCY RESPONSE

- Transfer function:

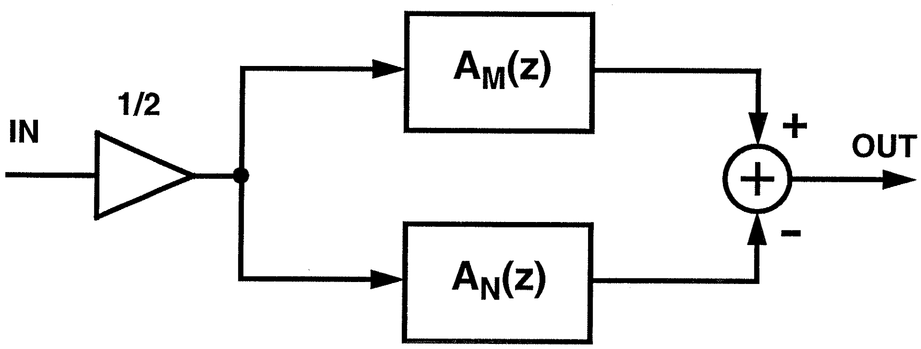
$$H_{LP}(z) = [A_M(z) + A_N(z)]/2.$$

- Frequency response:

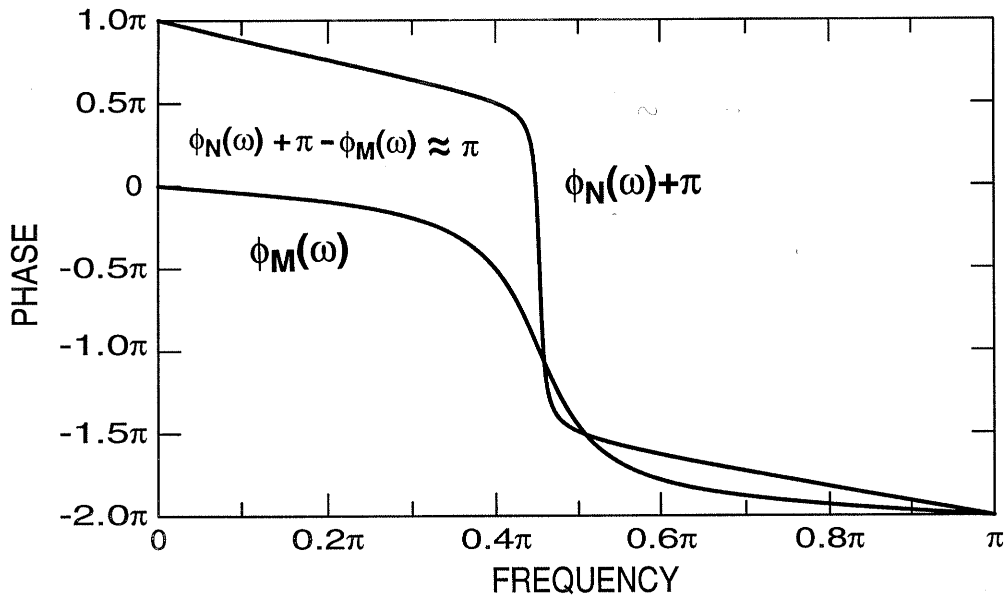
$$H_{LP}(e^{j\omega}) = e^{j[\phi_M(\omega)+\phi_N(\omega)]/2} \cos \frac{1}{2}[\phi_M(\omega) - \phi_N(\omega)].$$

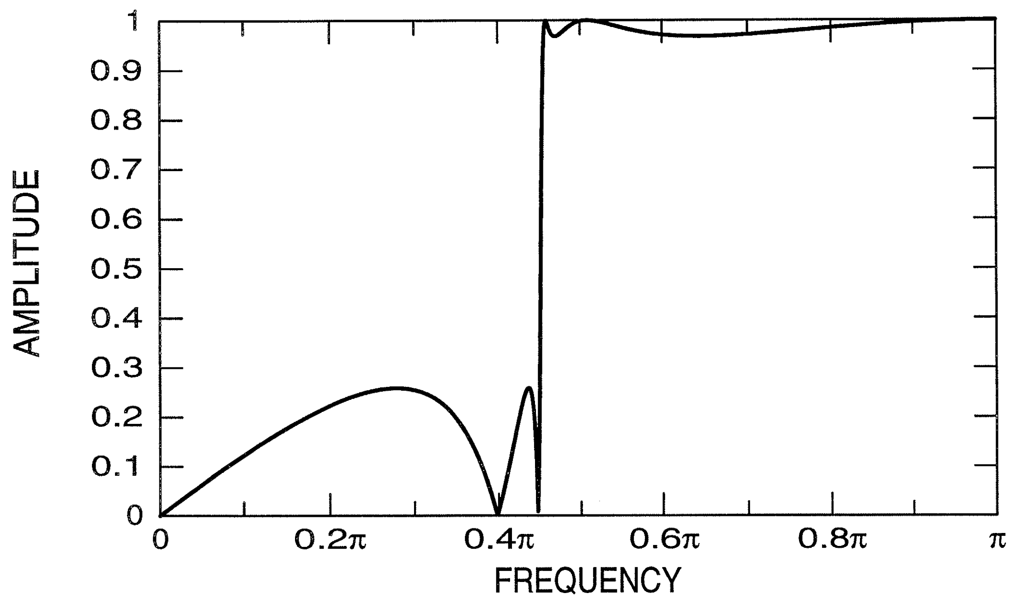
BUILDING A HIGHPASS FILTER

- A parallel connection of $A_M(z)$ and $-A_N(z)$ with **N=M+1**: $H_{\mathrm{HP}}(z) = [A_M(z) - A_N(z)]/2$.



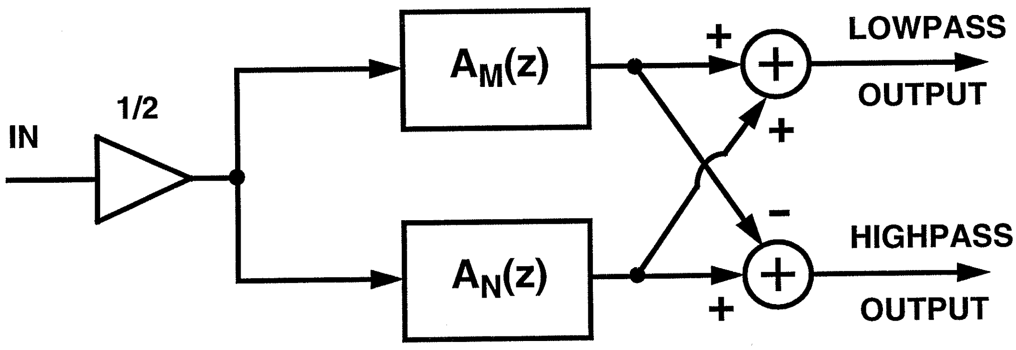
- Desired responses: $|H_{\mathrm{HP}}(e^{j\omega}) = \frac{1}{2}|e^{j\phi_M(\omega)} - e^{j\phi_N(\omega)}|$.





A LOWPASS-HIGHPASS FILTER PAIR

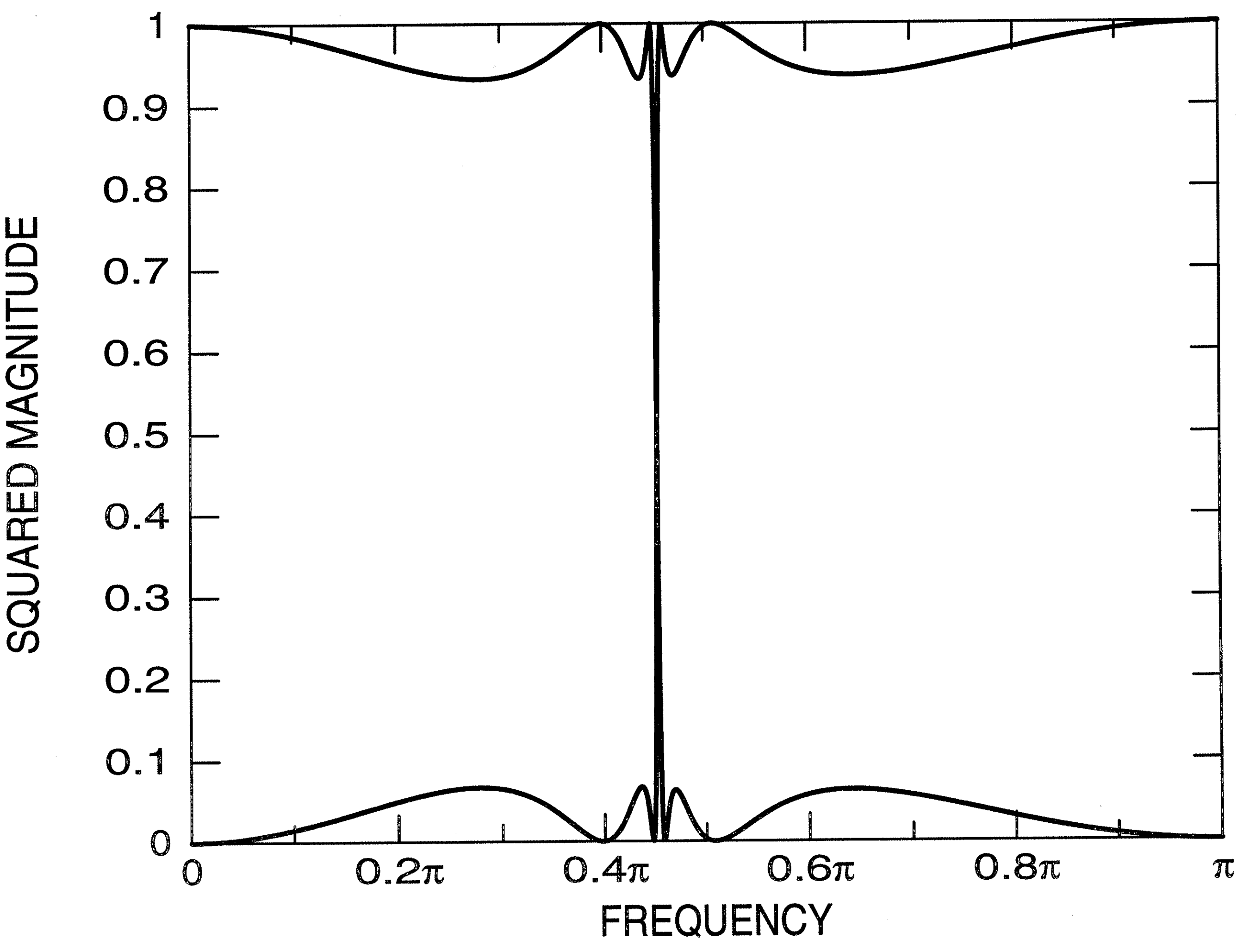
- Implementation:



- This pair is a *power-complementary* filter pair since

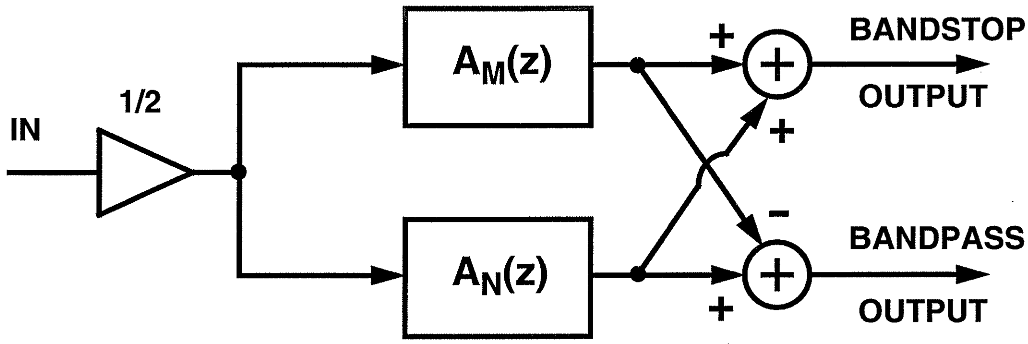
$$|H_{\mathrm{LP}}(e^{j\omega})|^2 + |H_{\mathrm{HP}}(e^{j\omega})|^2 = 1.$$

- Responses:

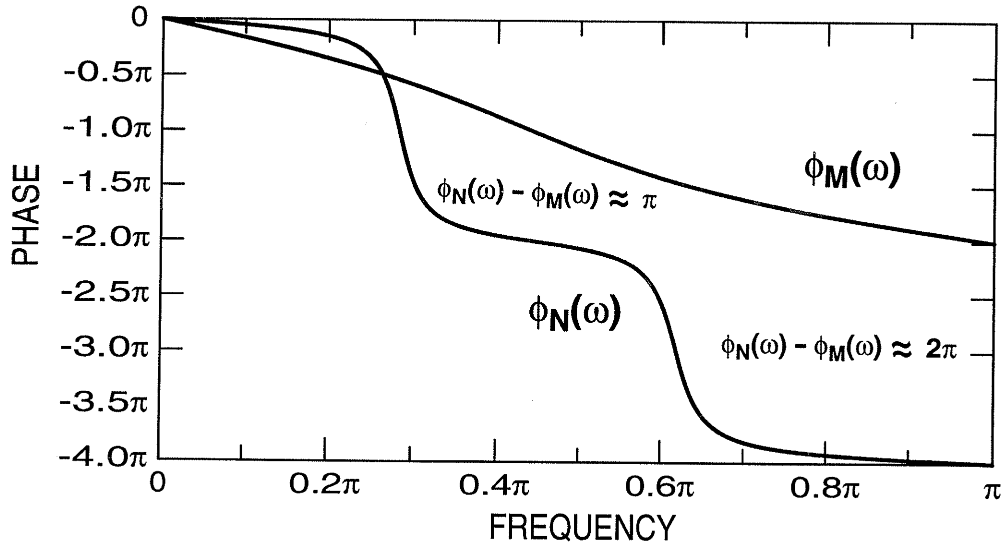


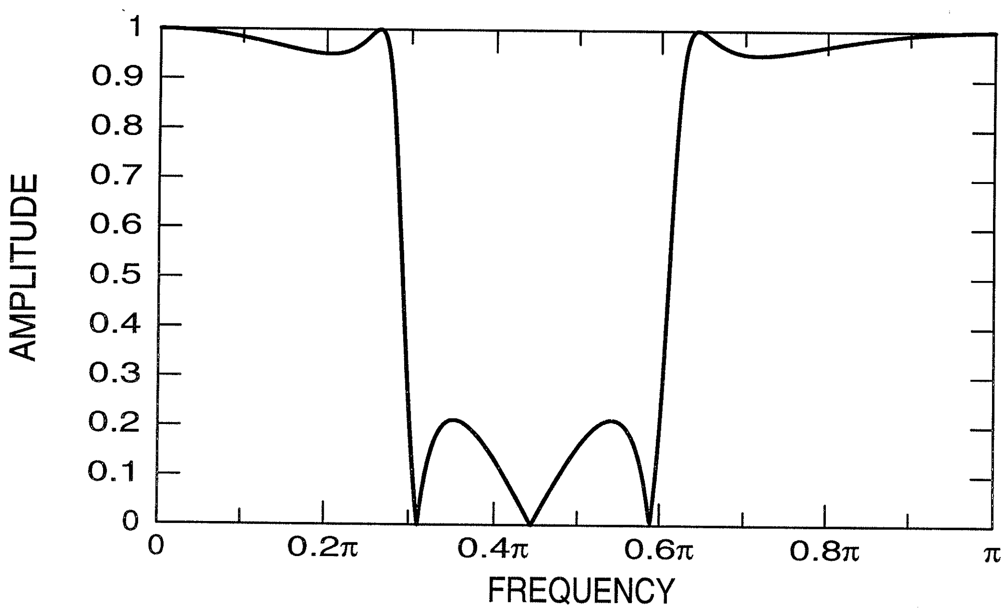
A BANDPASS-BANDSTOP FILTER PAIR

- In this case, **N=M+2**.



- Responses for the bandstop filter:





DESIGN OF CLASSICAL LOWPASS AND HIGHPASS FILTERS

- Any **odd-order classical** IIR filter (Butterworth, Chebyshev, inverse Chebyshev, elliptic filter) can be implemented in the desired form!!
- This was first observed in the case of wave digital lattice filters (allpass filters are implemented using special structures).

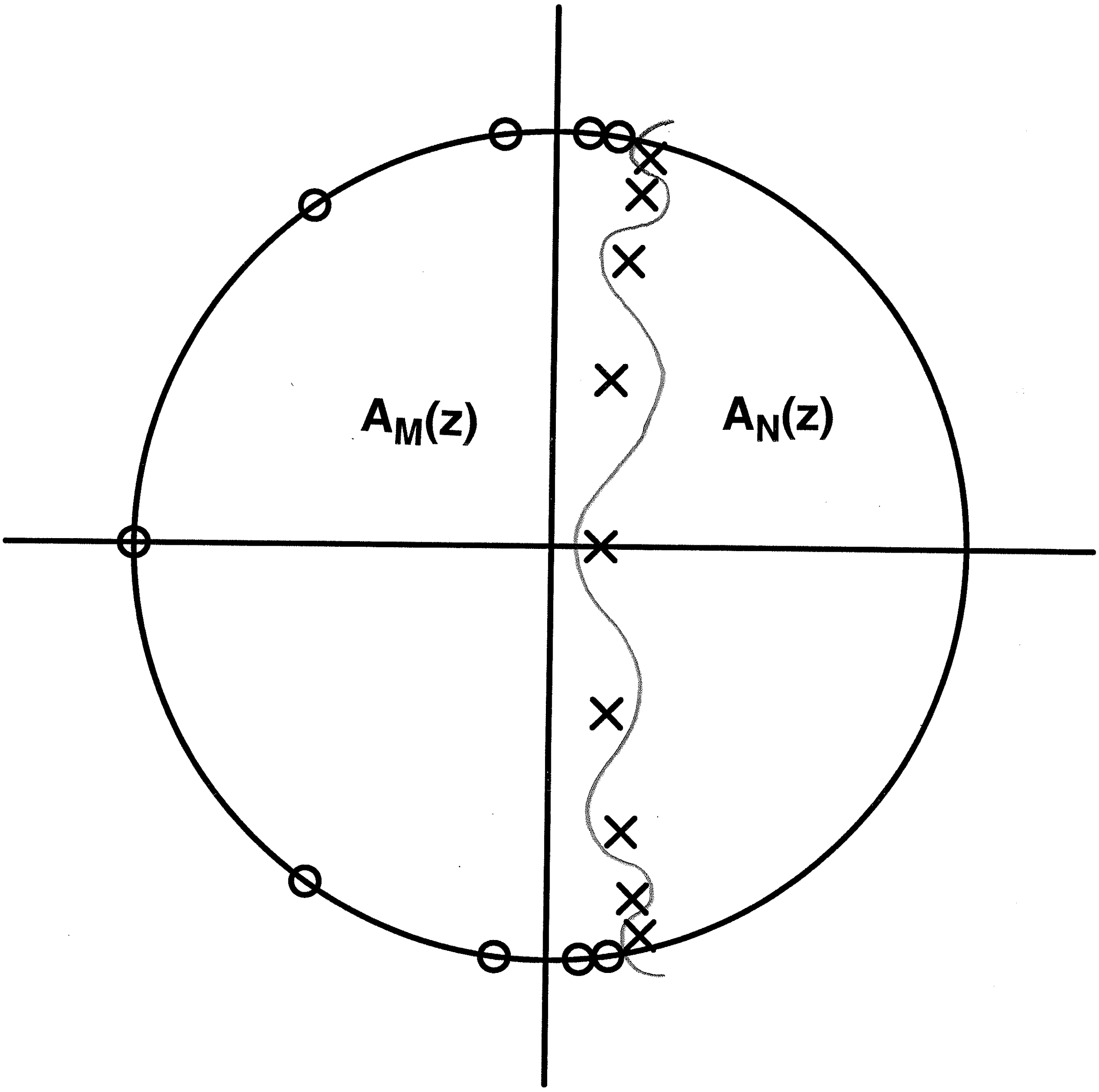
DESIGN PROCEDURE:

1. Determine a classical filter (or filter pair) meeting the given amplitude criteria. Let the filter order be $M+N$ (odd).

2. Include in $A_N(z)$ the outermost pole pair, the third outermost pole pair, the fifth otermost pole pair and so on (see the following transparency). The remaining pole pairs belong to $A_M(z)$. The real pole is included in $A_N(z)$ if $N+M$ is 5, 9, 13, 17 and so on. Otherwise, it belongs to $A_N(z)$.

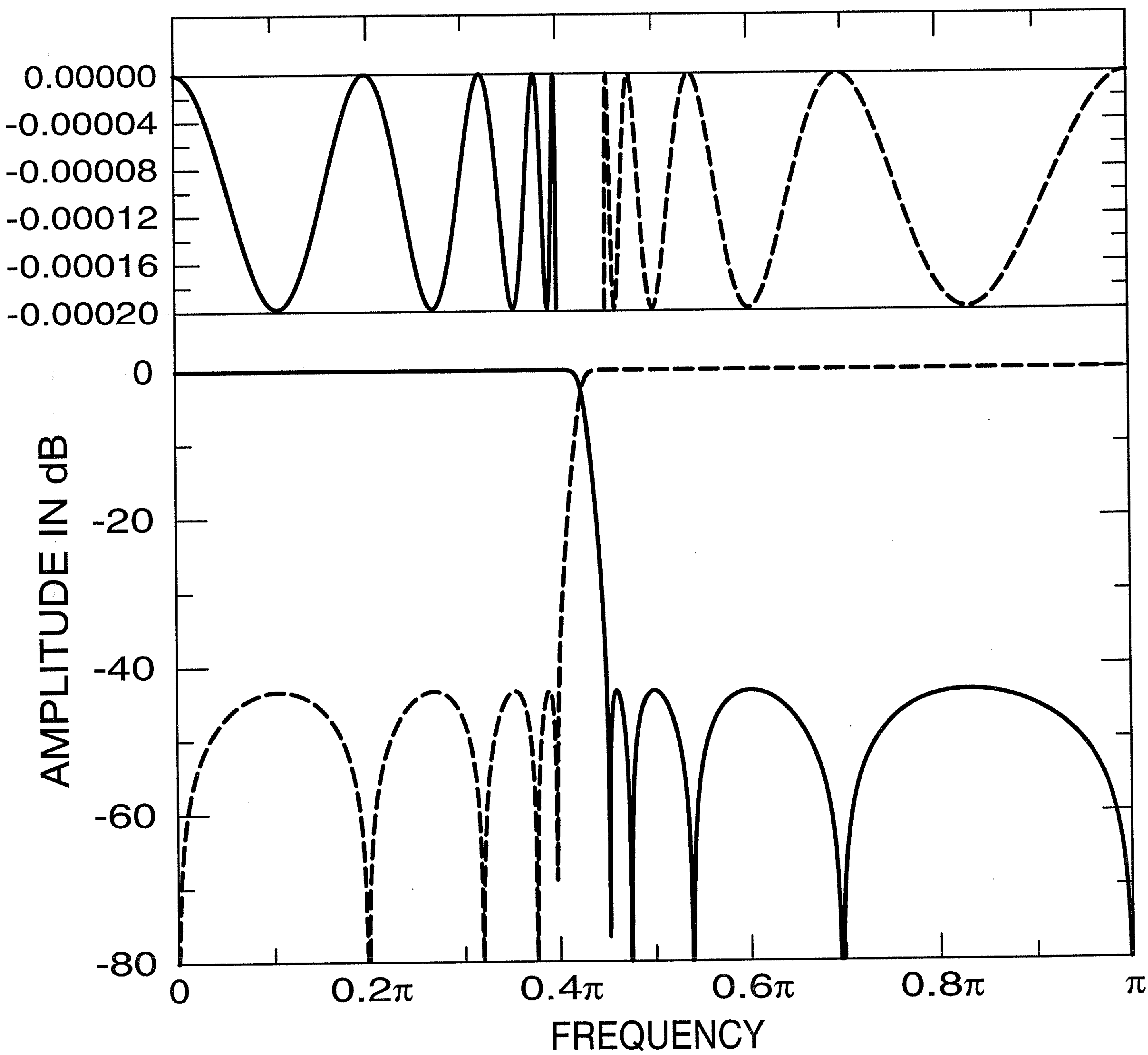
3. Select the structures for the allpass filters and compute the coefficient values (to be considered later).

ALTERNATING DISTRIBUTION OF THE POLES TO THE ALLPASS SUBFILTERS



EXAMPLE: A lowpass-highpass filter pair with edges at 0.4π and 0.45π and at least 40-dB attenuations in the stopbands

- The minimum odd order is 9 to meet the criteria ($N = 5$, $M = 4$.)
- When the passband ripple of the lowpass filter is 0.00019788 dB, then the attenuations of both filters are 43.414 dB.



POLE AND ZERO LOCATIONS

- The poles of both filters are located at

$$z = 0.96546596 \exp(\pm j 0.42471635\pi)$$
$$z = 0.87484735 \exp(\pm j 0.42407724\pi)$$
$$z = 0.71070737 \exp(\pm j 0.42024180\pi)$$
$$z = 0.42558157 \exp(\pm j 0.39465517\pi)$$
$$z = 0.11873387.$$

- $A_N(z)$ contains the first and third pole pairs as well as the real pole. $A_M(z)$ contains the remaining poles.

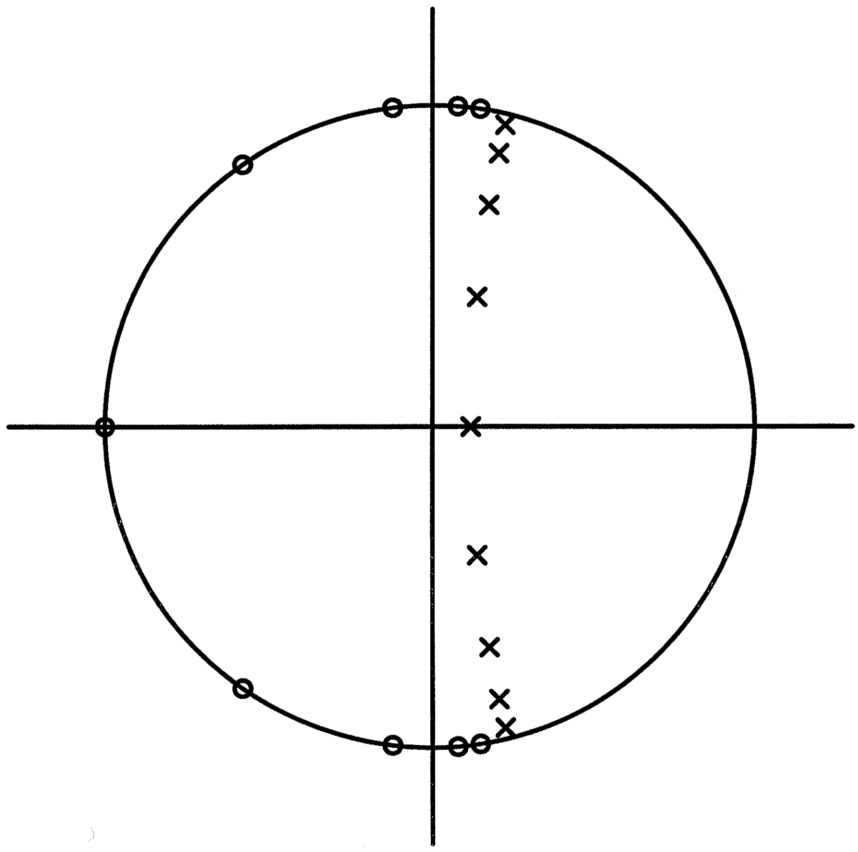
- The zeros of the lowpass filter are located at

$$z = \exp(\pm j 0.45243132\pi)$$
$$z = \exp(\pm j 0.47491693\pi)$$
$$z = \exp(\pm j 0.53905970\pi)$$
$$z = \exp(\pm j 0.69618255\pi)$$
$$z = -1.$$

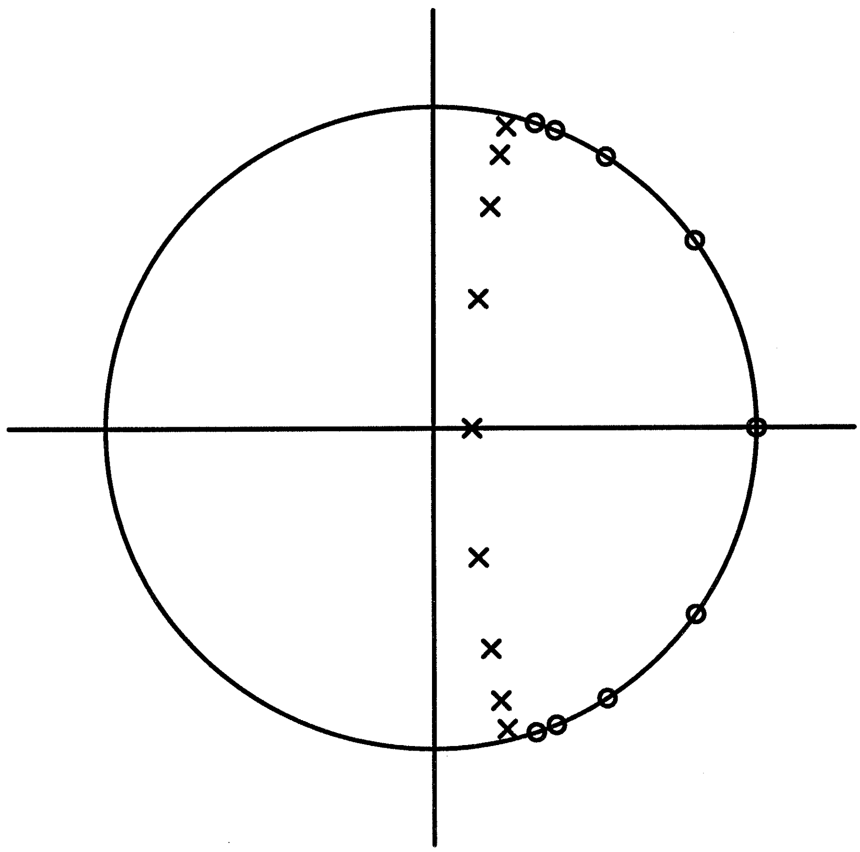
- The zeros of the highpass filter are located at

$$z = \exp(\pm j 0.39766309\pi)$$
$$z = \exp(\pm j 0.37644064\pi)$$
$$z = \exp(\pm j 0.31948331\pi)$$
$$z = \exp(\pm j 0.19766827\pi)$$
$$z = 1.$$

POLE-ZERO PLOTS



LOWPASS



HIGHPASS

DESIGN OF CLASSICAL BANDPASS AND BANDSTOP FILTERS

- The design can be accomplished by using a lowpass-bandpass transformation considered above. This transformation is of the the form

$$z^{-1} \mapsto -\frac{z^{-2} - [2\alpha k/(k+1)]z^{-1} + [(k-1)/(k+1)]}{[(k-1)/(k+1)]z^{-2} - [2\alpha k/(k+1)]z^{-1} + 1}.$$

DESIGN PROCEDURE:

1. Determine the criteria for the corresponding lowpass prototype filter and synthesize it as a parellel connection of $A_N(z)$ and $A_M(z)$.

2. Apply the lowpass-bandpass transformation to both $A_N(z)$ and $A_M(z)$. The resulting allpass sections are of the form $A_{2N}(z)$ and $A_{2M}(z)$, that is, the orders are doubled.

   - Since the order of the lowpass prototype is odd, the resulting overall filter orders are 2, 6, 10, or 14 and so on.

EXAMPLE: A bandpass-bandstop filter pair with edges at 0.2π, 0.225π, 0.775π, and 0.8π and at least 40-dB attenuations in the stopbands

- The desired filter bank is obtained by applying the substitution $z^{-1} \mapsto -z^{-2}$.
- $\Rightarrow A_{2N}(z)$ $(2N = 10)$ includes the following poles:

$$z = 0.96546596 \exp(\pm j0.42471635\pi)$$

$$z = 0.98258127 \exp(\pm j0.21235818\pi)$$

$$z = 0.98258127 \exp(\pm j0.78764182\pi)$$

$$z = 0.84303462 \exp(\pm j0.21012090\pi)$$

$$z = 0.84303462 \exp(\pm j0.78987910\pi)$$

$$z = \pm 0.34457781$$

- $\Rightarrow A_{2M}(z)$ $(2M = 8)$ includes the following poles:

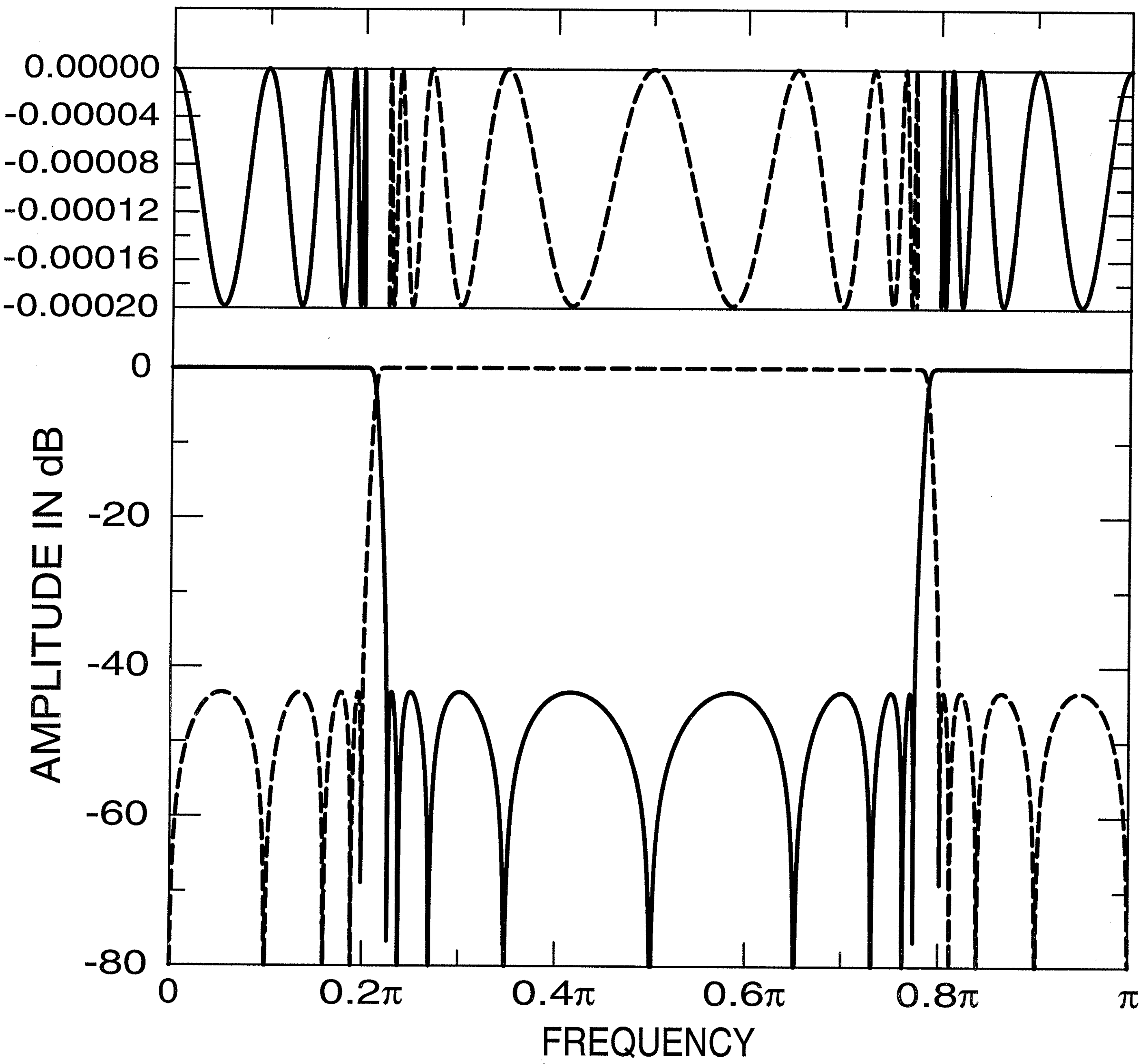
$$z = 0.93533275 \exp(\pm j0.21203862\pi)$$

$$z = 0.93533275 \exp(\pm j0.78796138\pi)$$

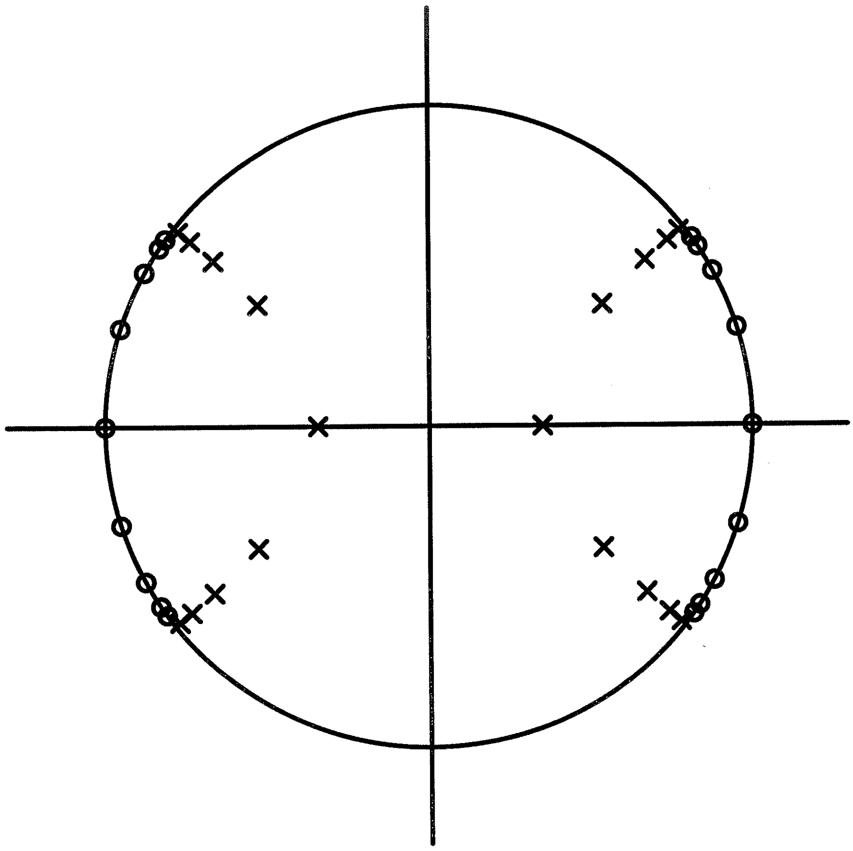
$$z = 0.65236613 \exp(\pm j0.19732758\pi)$$

$$z = 0.65236613 \exp(\pm j0.80267242\pi)$$

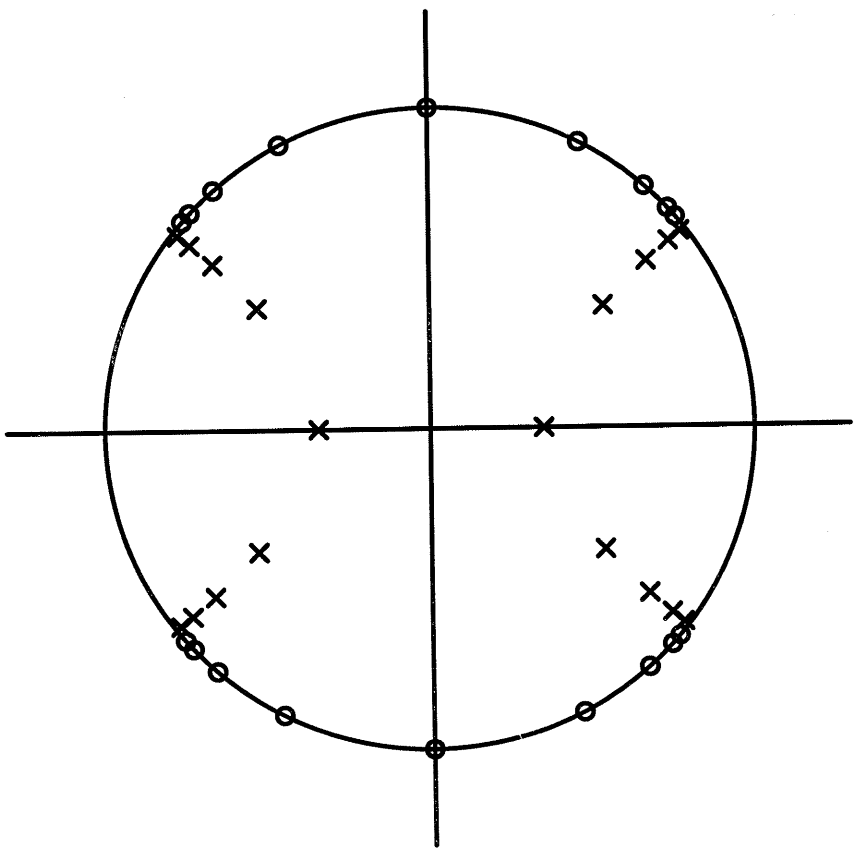
AMPLITUDE RESPONSES



POLE-ZERO PLOTS



BANDPASS



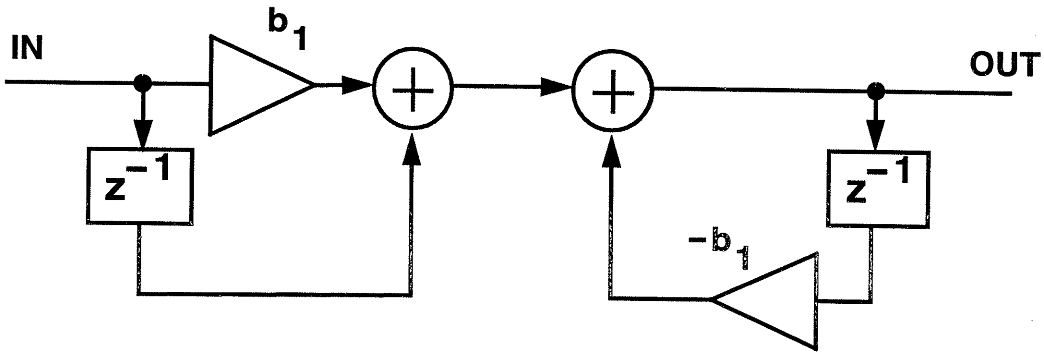
BANDSTOP

STRUCTURES FOR IMPLEMENTING ALLPASS FILTERS

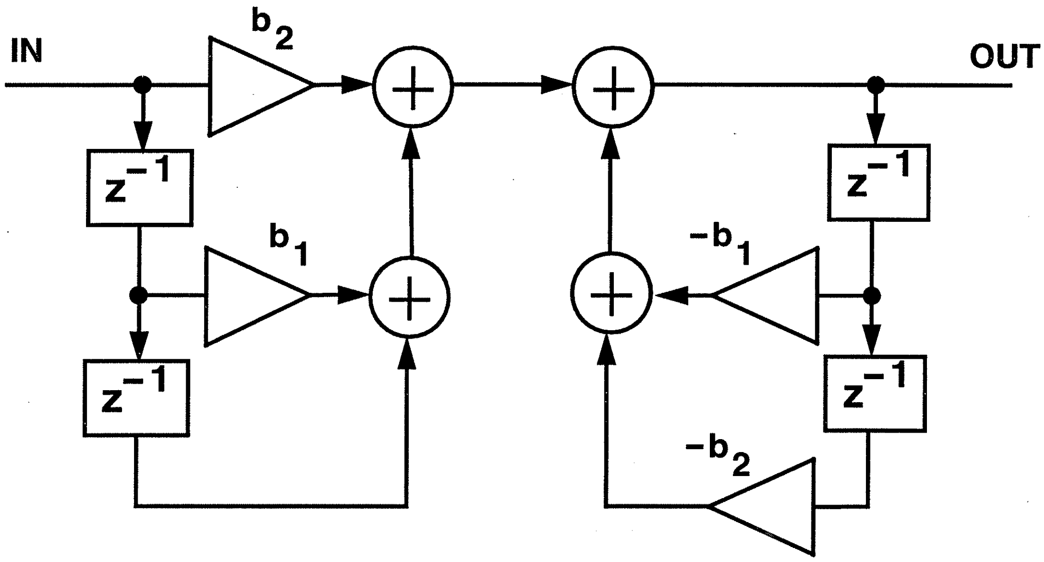
- The implementation becomes very modular if the allpass sections are realized as a cascade of first- and second-order allpass sections.
  - Easy to design, analyse, and scale.
- It is required that after coefficient quantization the building block is still an allpass filter.

DIRECT-FORM I STRUCTURES

- First-order: $H(z) = [b_1 + z^{-1}]/[1 + b_1 z^{-1}]$



- Second-order: $H(z) = [b_2 + b_1 z^{-1} + z^{-2}]/[1 + b_1 z^{-1} + b_2 z^{-2}]$

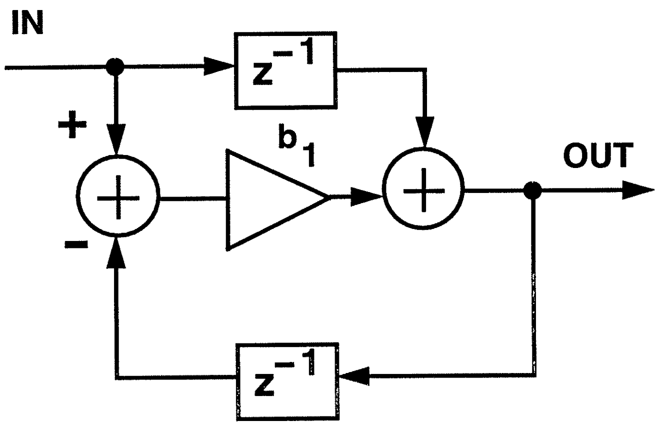


- Automatically scaled according to L_∞ norm.

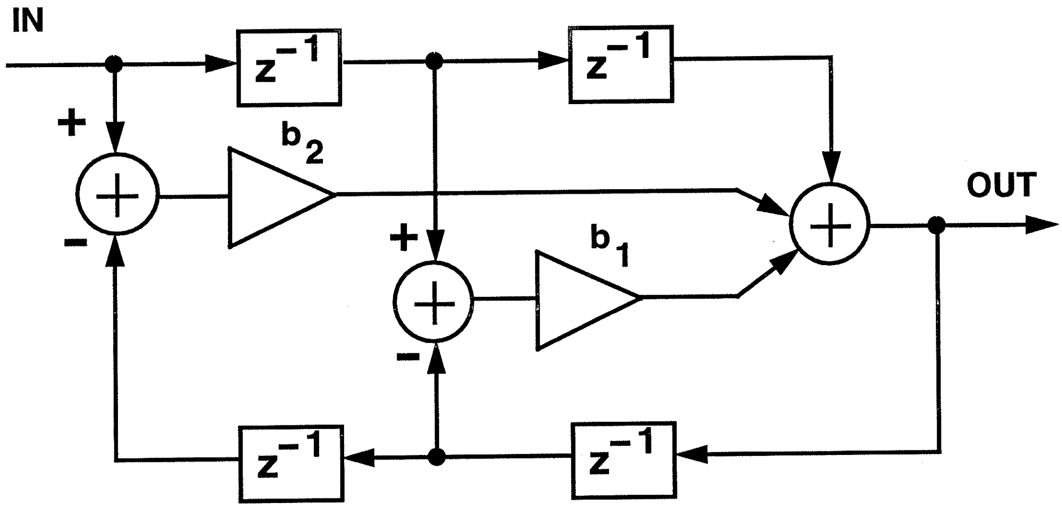
- If there are several blocks in cascde, delays can be shared.

SOME MITRA-HIRANO STRUCTURES

- First-order: Mitra-Hirano $1A_t$:



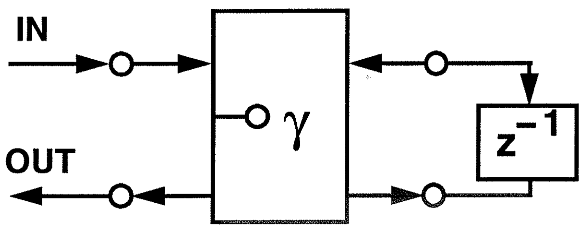
- Second-order: Mitra-Hirano $3D$:



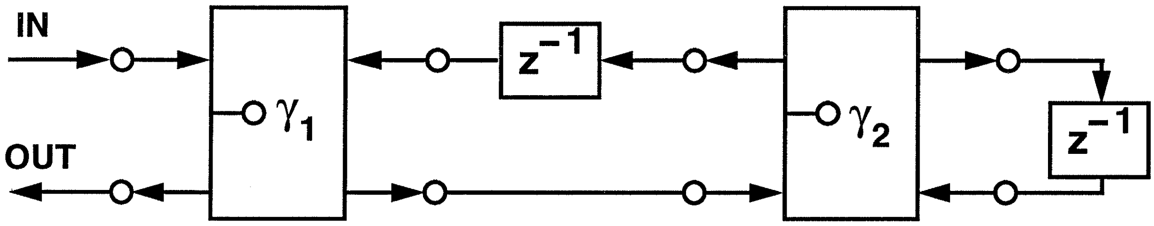
- The scaling constant of value of $1/2$ in needed at the input.

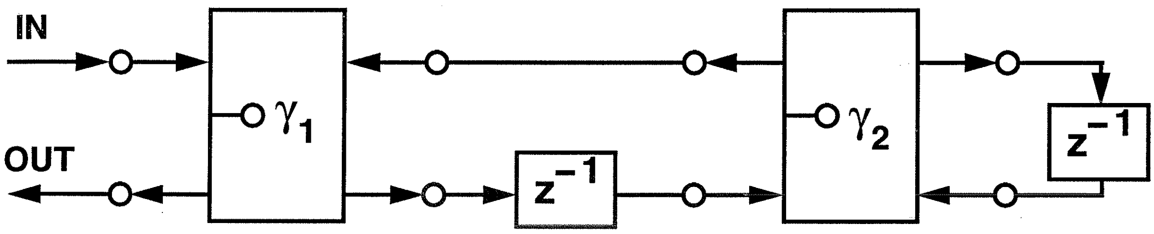
SOME WAVE DIGITAL FILTER STRUCTURES

- First-order: $H(z) = [-\gamma + z^{-1}]/[1 - \gamma z^{-1}]$



- Second-order: $H(z) = [-\gamma_1 + \gamma_2(\gamma_1 - 1)z^{-1} + z^{-2}]/[1 + \gamma_2(\gamma_1 - 1)z^{-1} - \gamma_1 z^{-2}]$





- The scaling constant of value of $1/2$ at the filter input guarantees the absense of overflows.

EFFICIENT ADAPTOR IMPLEMENTATIONS

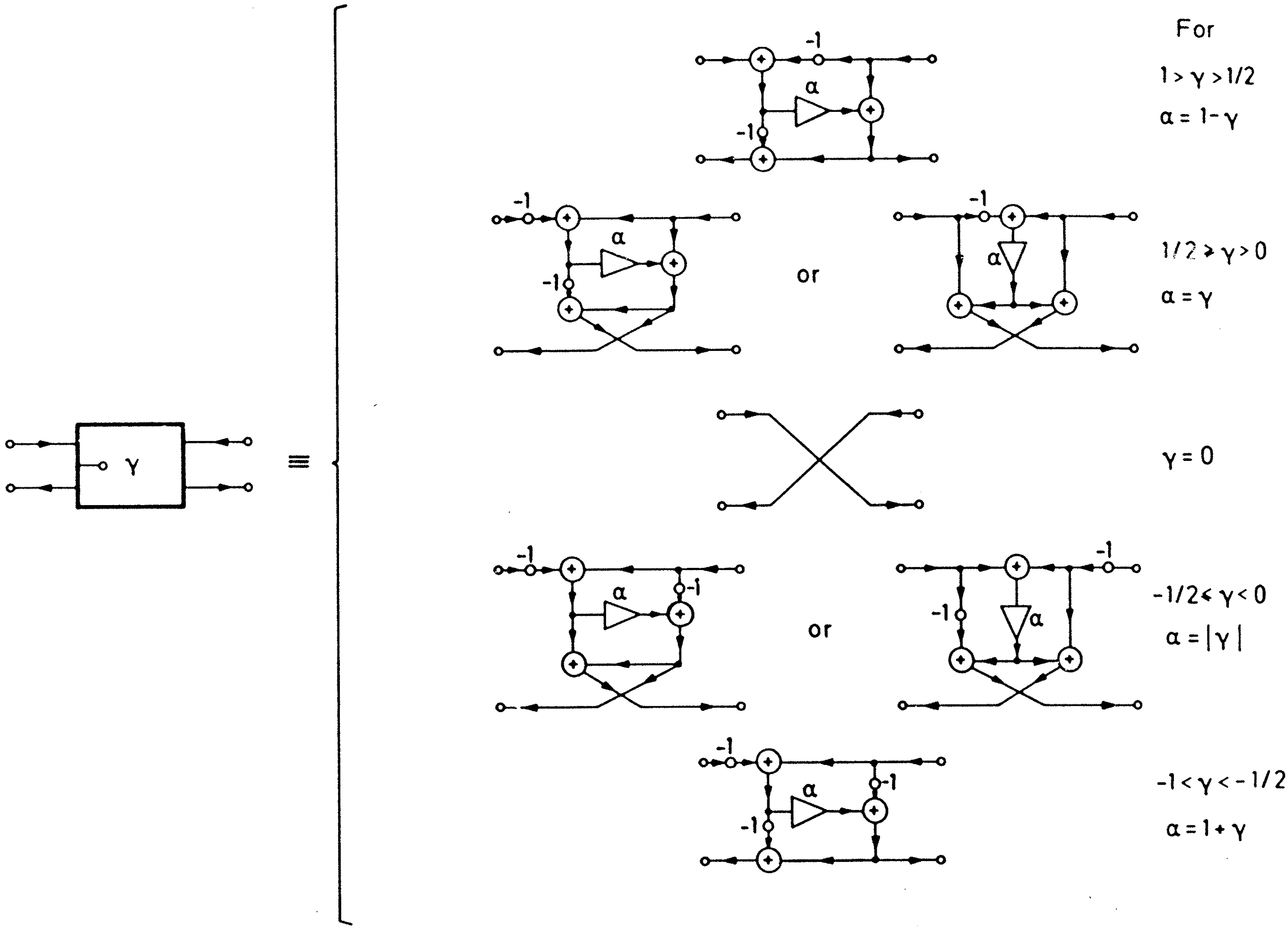
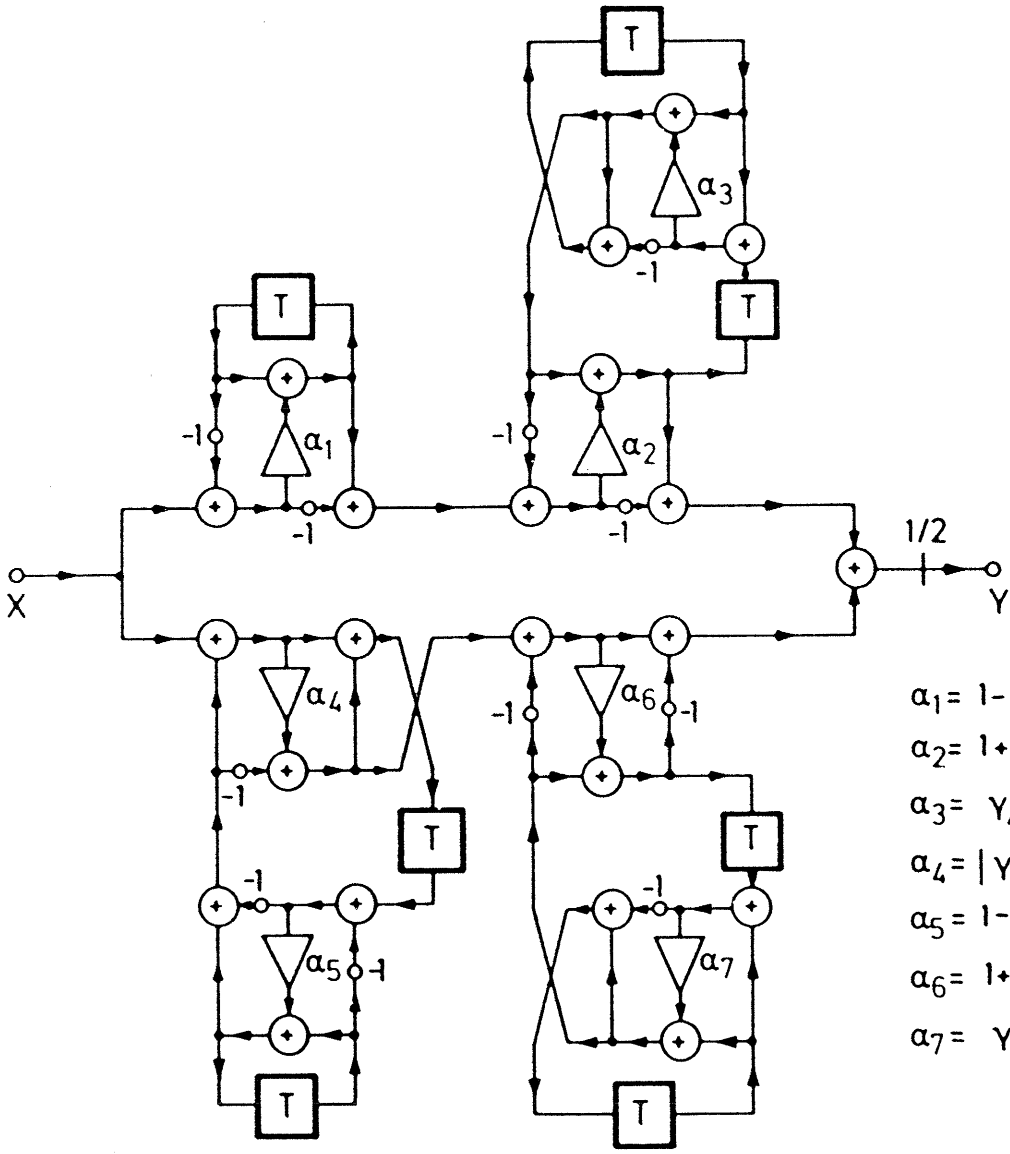


Fig. 9. Signal-flow diagrams of the two-port adaptor yielding optimal scaling for sinusoidal excitation. (Note that in the first diagram of the second last row, α should be replaced by $-\alpha$.)

EXAMPLE IMPLEMENTATION OF A SEVENTH-ORDER FILTER



$\alpha_1 = 1-\gamma_0 = 0.487\,1$

$\alpha_2 = 1+\gamma_3 = 0.331\,3$

$\alpha_3 = \gamma_4 = 0.334\,2$

$\alpha_4 = |\gamma_1| = 0.404\,4$

$\alpha_5 = 1-\gamma_2 = 0.392\,2$

$\alpha_6 = 1+\gamma_5 = 0.103\,8$

$\alpha_7 = \gamma_6 = 0.206\,7$

OTHER APPLICATIONS AND EXTENSIONS

- As mentioned in the first three transparencies, there are several other applications for filters using allpass filters as building blocks.

- Some of these are considered in the following.

- The main emphasis is on the results.

- All these filters can be designed very fast using Remez-type algorithms proposed by Markku Renfors and Tapio Saramäki.

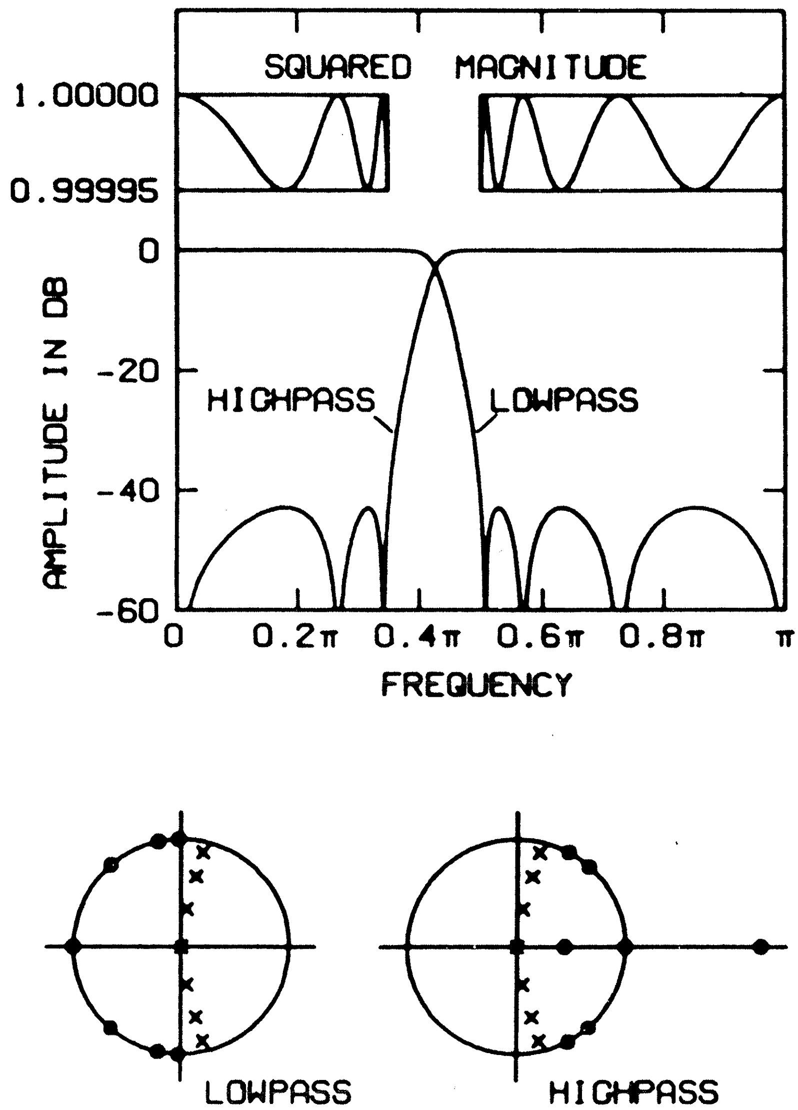
  - Some algorithms use amplitude approximations and some phase approximations.

  - An interested reader may read the enclosed articles and try to understand how the algorithms work.

EXTENSION 1: ONE OF THE SUBSECTIONS CONTAINS A PURE DELAY TERM

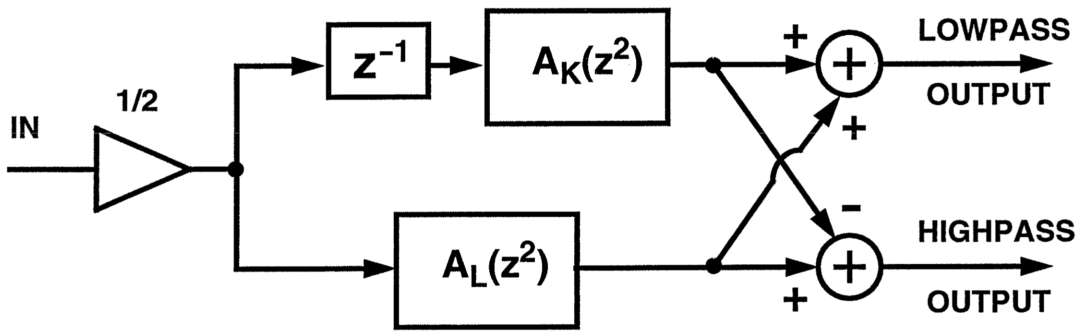
- The real pole include in one of the allpass sections contains a pole at the origin. $\Rightarrow$ The first-order allpass section reduced to a pure delay term z^{-1}, that is, no multiplier is needed.

- Example:



SPECIAL CASE: HALF-BAND IIR FILTERS

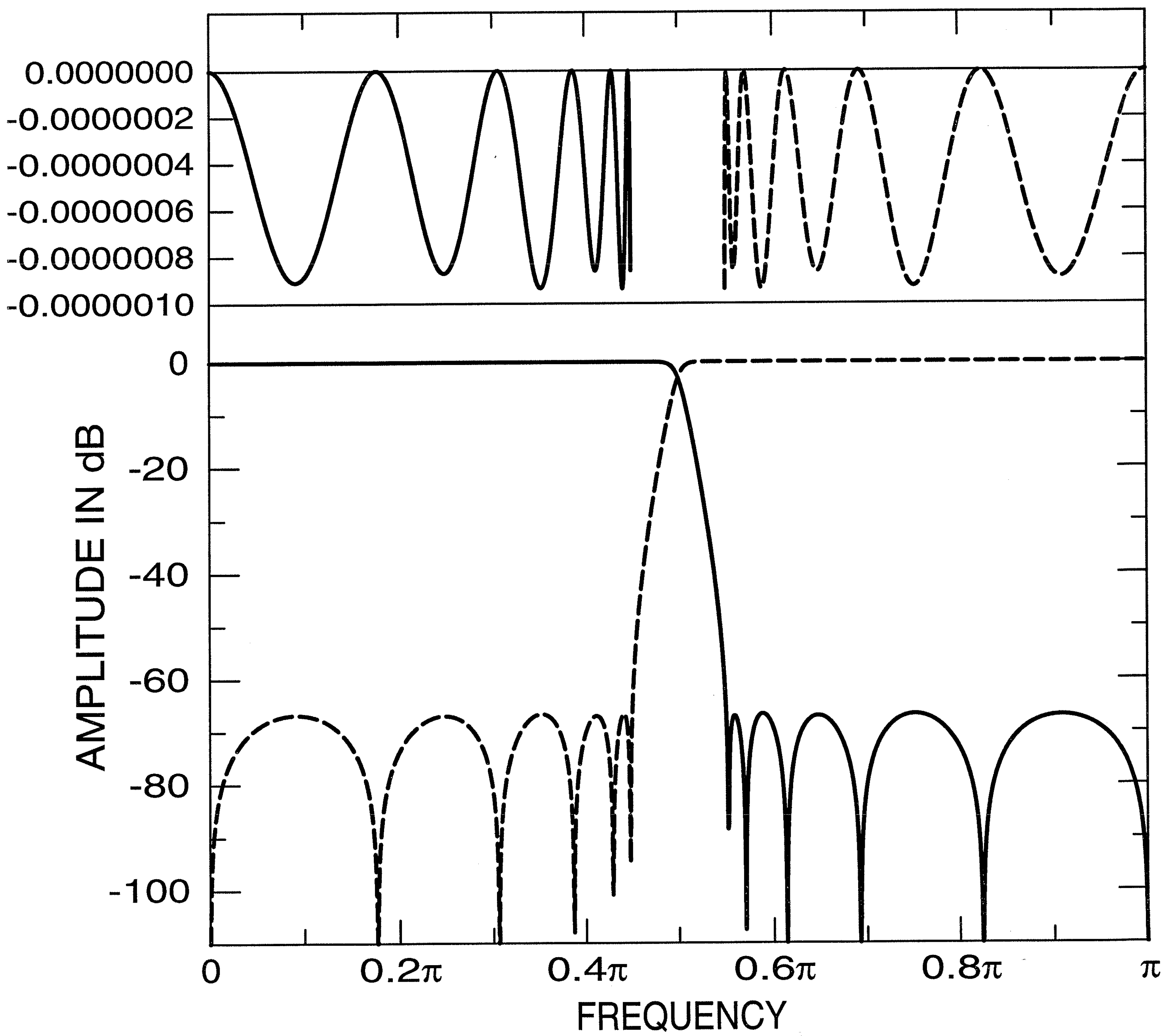
- For these filters, the first order section is a pure delay term z^{-1} and all the poles on the imaginary axis.
- This means that the second-order allpass section reduces into the form $A_2(z) = [b_2 + z^{-2}]/1 + b_2 z^{-2}]$, that is, b_1 becomes zero.
- The implementation of the first-order section requires no multipliers and the implementation of the second order section only one multiplier.
- The second-order section can, in fact, obtained from a first-order section $A_1(z) = [b_2 + z^{-1}]/1 + b_2 z^{-1}]$ by replacing a unit delay by a double delay.
- Implementation of a lowpass-highpass filter pair:



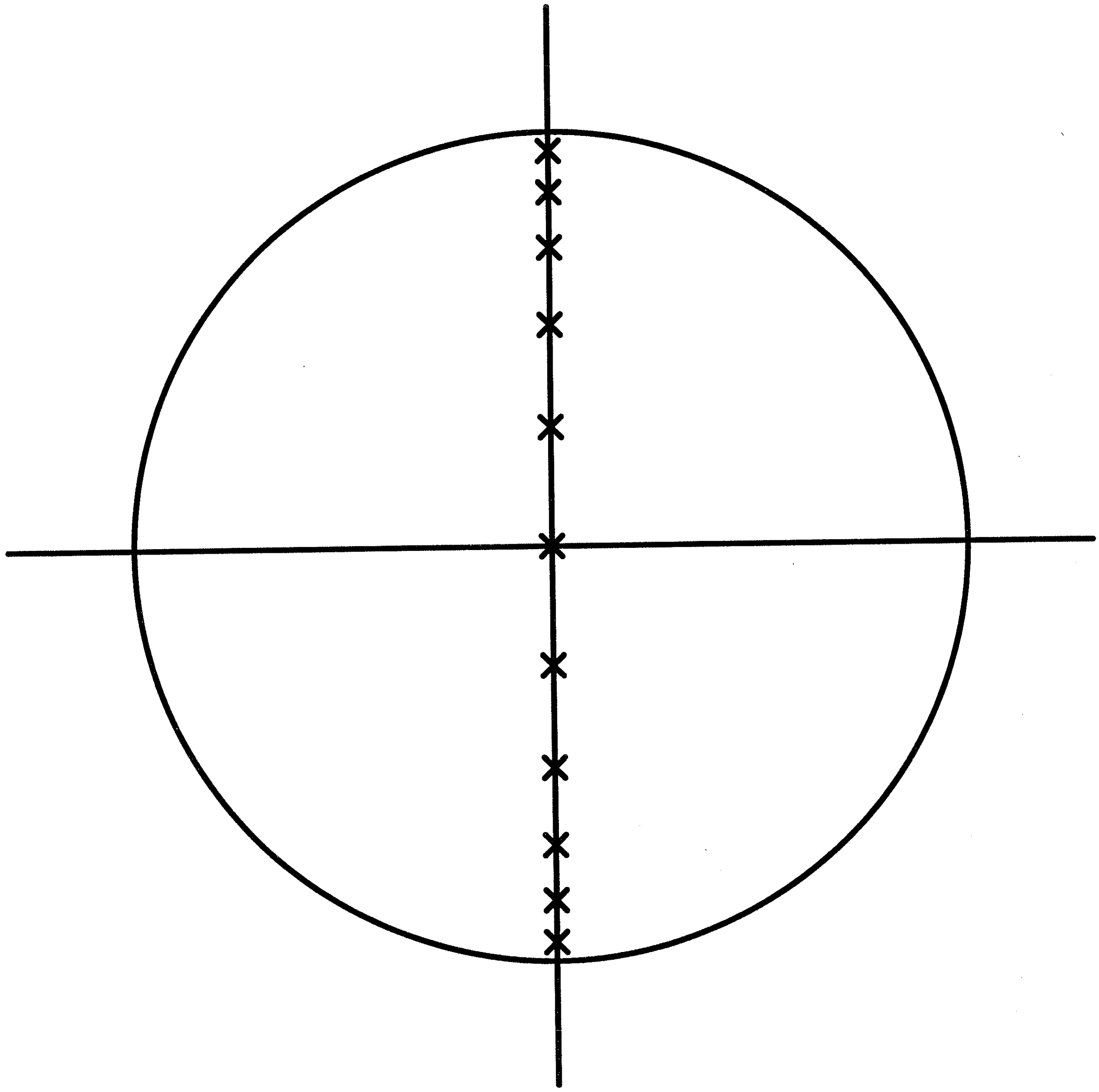
PROPERTIES

- For the lowpass filter, $\omega_s = \pi - \omega_p$.
- The stopband attenuations of both the lowpass and highpass filters are the same.
- These filters are very useful in sampling rate conversion applications and in building filter banks.

A ELEVENTH-ORDER ELLIPTIC HALF-BAND FILTER PAIR REQUIRING ONLY FIVE MULTIPLIERS AND HAVING EDGES AT 0.45π AND 0.55π



POLE LOCATIONS



MORE APPLICATIONS

- In the following there are altogether four sets of transparencies for conference talks.
- Please look at the results. Synthesis techniques are not so important to learn.
- In the case you are interested in the actual papers, please do not hesitate to contact the lecturer of this course.
- Also a short article on the use of allpass filters as basic building blocks, written by Prof. Markku Renfors, is included.