80505 DIGITAALINEN SUODATUS

SYKSY 1996

LUENTOJEN PITÄJÄ:

TAPIO SARAMÄKI , B322, **e-mail: ts@cs.tut.fi**

TAPAA PARHAITEN 10.00 – 16.40

LUENNOT TORSTAISIN JA PERJANTAISIN
12.15 – 14.00, HB116

HARJOITUKSET : Jussi Vesma, Konsta Koppinen ja Sari Siren

Harjoitusryhmiä on kaikkiaan 8 ja ne pidetään Signaalinkäsittelylaitoksen SUN-luokassa, kolmas kerros, B304. Harjoitusajat löytyvät luokan ovesta. Harjoitusten kesto on 2 kertaa 45 minuuttia.

KURSSIMATERIAALI

- **HARJOITUKSET:**
  - Löytyy valmis pruju, joitakin lisätehtäviä sekä Matlab-demoja
- **LUENNOT:**
  - LUENTOMONISTE
  - OPPIKIRJA: E. C. Ifeachor and B. W. Jervis: DIGITAL SIGNAL PROCESSING: A PRACTICAL APPROACH
  - Luentomoniste ja oppikirja poikkevat paljon toisistaan ja pyrkivät tarkatelemaan perusasioita eri kanteilta; toivottavasti tukevat toisiaan.

Mitäkö lukea kirjasta?

Luentomonistetta vastaava materiaali:

Luku 1: Miksi kannattaa käyttää digitaalista signaalinkäsittelyä ja mitä taphtuu aika- ja taajuustasoissa kun jatkuva-aikainen signaali reissaa diskreettiaikaisen järjestelmän lävitse ?

Luku 2: Mitä ovat DFT, IDFT, FFT ja IFFT?

Luku 3: Sellaisenaan.

Luku 5: Sellaisenaan paitsi ristikkorakenteet (lattice).

Luku 7: Sellaisenaan paitsi 7.6.16 ja 7.10. Klassisten IIR alipäästösuodattimien suunnittelun ja niiden muuntamisen toisiksi alipäästösuodattimiksi tai ylipäästö-, kaistanpäästö ja kaistanestosuodattimiksi voisi lukaista luentomateriaalista ja harjoituksista.

MITÄKÖ KURSSILLA PITÄISI OPPIA?

1) Jatkuva-aikaisen signaalin käsittely diskreettiaikaisen systeemin avulla

2) Perussignaalit ja -diskreettiaikaiset järjestelmät

3) Työkalut signaalinen ja järjestelmien tutkimiseen ja suunnitteluun

   - z-muunnos, taajuusvaste, nollat, navat . . .

4) Diskreettiaikaiset suodattimet: analysointi ja suunnittelu

5) Äärellisen laskentatarkuuden vaikutukset

   - Suodattimien skaalaus, kohina, erilaiset värähtelyt ja vaikutus suodattimen taajuusvasteeseen

6) Diskreetti Fourier muunnos (DFT) ja sen käyttö

LUENTOJEN LUONNE:

Luennoissa ei tulla tunnon tarkasti seuraamaan oppikirjaa.

Luennoissa keskitytään perusasioihin; yksityiskohdat löytyvät itse kirjasta, joka on kirjoitettu helposti luettavaan muotoon.

Tenttitehtävät laaditaan oppikirjan, luentoprujun ja harjoitusten pohjalta.

→ Ei tarvitse tehdä välttämättä luentomuistiinpanoja

Kirjekussilaisetkin pärjännevät

KOMMENTTI: Kyselkää mahdollisimman paljon, koska tämän kurssin luennoitsija on puuhastellut kyseisen aihepiirin parissa liki 20 vuotta. Tästä syystä on vaikea hahmottaa, mitkä asiat ovat opiskelijoille vaikeita tajuta, mitkä helppoja!!

LUENTOPRUJU

Pruju on laadittu englanniksi, koska sitä käytetään myös englanninkieliseen opetukseen. Se koostuu seuraavista osista:

Osa 1: Digitaalisen signaalinkäsittelyn perusteet:

1) Miten käsitellä jatkuva-aikaista signaalia diskreettiaikaisen järjestelmän avulla?
2) Perusjärjestelmät ja -signaalit
3) Työkalut diskreettiaikaisten järjestelmien analysointiin ja suunnitteluun:
   z-muunnos, taajuus-, amplitudi- ja vaihevasteet; nollat navat

Osa 2: Digitaalisten suodattimien suunnittelu ja toteuttaminen:

1) Perusrakenteet suodattiminen toteuttamiseksi
2) Vaatimusten antaminen suodattimille: amplitudi- ja vaihevasteet (ryhmä- ja vaiheviiveet)
3) Erilaiset approksimointikriteerit vaatimusten täyttämiseksi
4) Suodattimien jako FIR ja IIR suodattimiin (FIR==finite impulse response, IIR=infinite impulse response)

Osa 3: Esimerkkejä suodattimien tehokkuudesta erilaisten häiriöiden poistoon. Nollien ja napojen merkitys suodattimen vasteiden luontiin.

Osa 4: FIR-suodattimen suunnitelu

Osa 5: IIR-suodattimein suunnittelu

Osa 6: Äärellisen sanapituuden vaikutukset: pyöristyskohina, rajavärähtelyt, A/D-muuntimen aiheuttama kohina, keroinpyöristyksen vaikutus suodattimen vasteeseen

Osa 7: DFT, IDFT, FFT ja IFFT sekä niiden käyttö

MITÄ ON DIGITAALINEN SUODATUS?

Nykysuomen sanakirja: ei vastausta sanaan 'digitaalinen'; vastaus sanoihin 'suodattaa' ja 'suodatin'

diftongiutu a[4] *pass.v. kiel.* muuttua diftongiksi. | Pitkien vokaalien d:minen. **-maton**[57] *kielt.a.*
digamia[15] *s.* toiskertainen avioliitto.
digestio[3] *s.* ruoansulatus.
digitalis[64] [-ā-] *s.* sormustinkukka; sen lehdistä valmistettu sydänlääke.
dignitääri[4] *s.* arvohenkilö, ylimys.
diiv'a[10] *s.* **1.** juhlittu, suosittu laulajatar t. näyt-

suodat|in^{33*} *väl.; syn.* filtrumi, filtteri. 1. suodattamiseen käytetty huokoinen väline t. aine, tiheä (ja paksu) siivilä. | Kun neste painetaan s:timen läpi, jää sakka siihen. S:timena käytetään paperia, kangasta, hiiltä, posliinia jne. Juomaveden puhdistus s:timilla. — Kaasunaamarin s. sisältää aktivoitua hiiltä. — *Valok.* tasalaaka levy, joka pidättää osan lävitseen kulkevasta valosta ja suotaa lävitseen muun osan. |

Värillinen lasi s:timena. — *Yhd.* imu-, paine-, pikas.; hiekka-, hiili-, vanus.; ilman-, valon-, veden-, öljyns.; (valok.) harmo-, kelta-, kirkas-, puna-, sini-, viher-, väris. 2. *sähk. rad.* sähköpiiri, joka päästää lävitsensä vain määrätaajuisia vaihtovirtoja.

suodat|taa[2*] *v.* -us[64] *teonn.; syn.* filtrata. 1. antaa nesteen (t. kaasun) kulkea huokoisen väliseinän läpi kiinteiden ainesten, epäpuhtauksien tms. erottamiseksi siitä; suodatuslaitteista, aineista: pidättää t. erottaa kiinteät ainekset, epäpuhtaudet tms. lävitseen kulkevasta nesteestä (t. kaasusta); vrt seuloa, siivilöidä. | S. neste kankaan, vanun, paperin, hiilikerroksen läpi. Hiekkakerroksen läpi s:ettu vesi. Öljyn s:us. Rasvoista erotetaan epäpuhtaudet s:tamalla tai sentrifugoimalla. Kokoushuoneiden lämmitysilma s:etaan ennen kuin se puhalletaan sisään. S:tava maakerros. — Värillinen lasi s:taa lävitseen kulkevan valon 'pidättää osan ja läpäisee osan valonsäteistä'. 2. *sähk. rad.* saada aikaan vain tietyn jaksoalueen t. tiettyjen jaksoalueiden kulkeminen piirin läpi; suodattimesta: päästää määrätavalla lävitseen vain tietty jaksoalue t. tietyt jaksoalueet ja pidättää muut; *syn.* tyyntää.

suodattaja[16] *tek.* — Laitteista par. suodatin.

suodattamaton[57] *kielt.a.* S. liuos. S. valo.

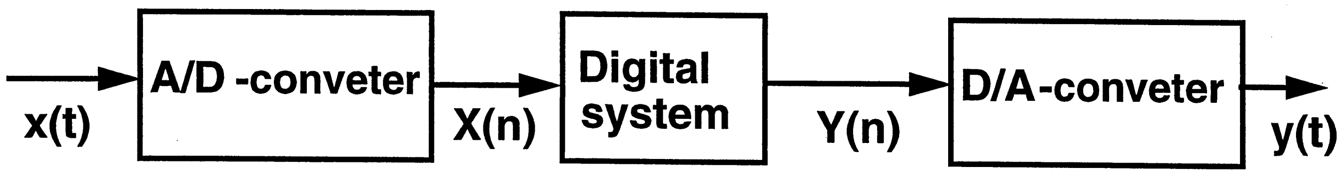
suodattu|a[1*] *pass.v.* < suodattaa. | Osa sadevettä

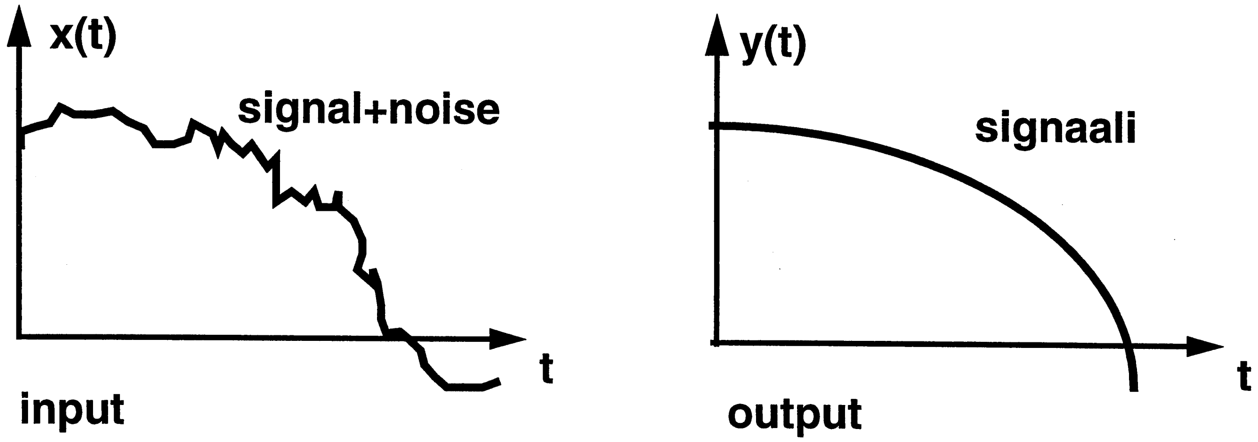
ENGLANNINKIELEN SANAKIRJA: löytyy sana 'digital':

digit 1. sormi, sormen leveys (= ¾"); 2. $^1/_{12}$ kuun tai auringon läpimitasta; 3. yksikkö (numerot 0—9), numero. ~ *absorption* sykäysten vaimennus *(tel)*. ~ *key* numeron anturi *(tel)*. ~ *key strip* painonappirima *(tel)*. ~ *selector* kirjaimen valitsija *(tel)*. **~al** sormi-, sormen muotoinen; numeerinen. **~alin** digitaliini *(kem, lääk)*. **~alis**, ~ *leaf* digitaliksen lehti. **~ate** haarautua sormimaisesti keskustasta päin; sormimainen. **~oxin** digitoksiini.
diglycerol diglyseroli *(kem)*.
digonal kaksikulmainen.

DISCRETE-TIME SIGNAL PROCESSING

PROCESSING OF CONTINUOUS-TIME SIGNALS:

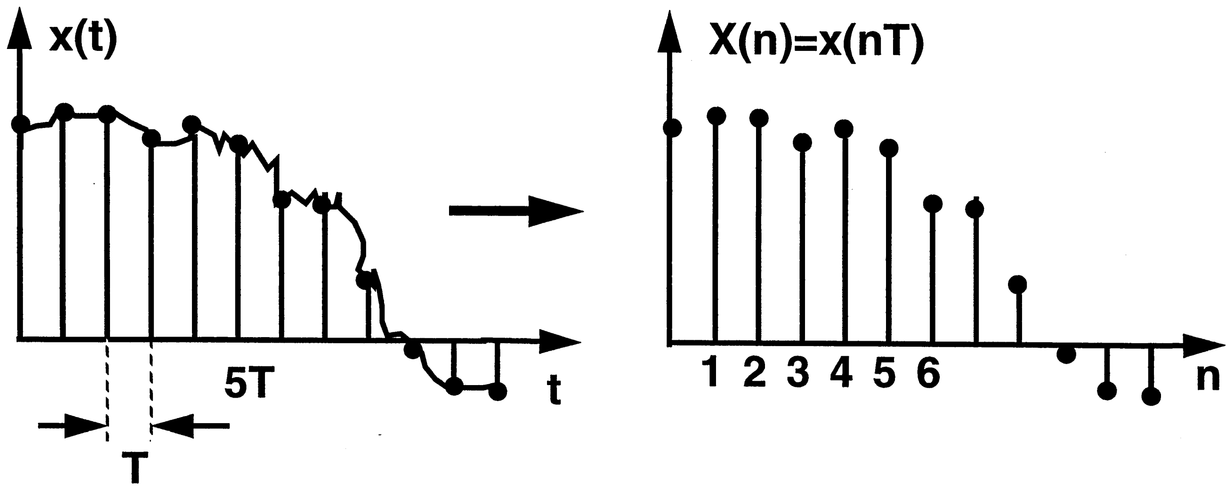




- The input-output relation is similar to that of a continuous-time (analog) system.

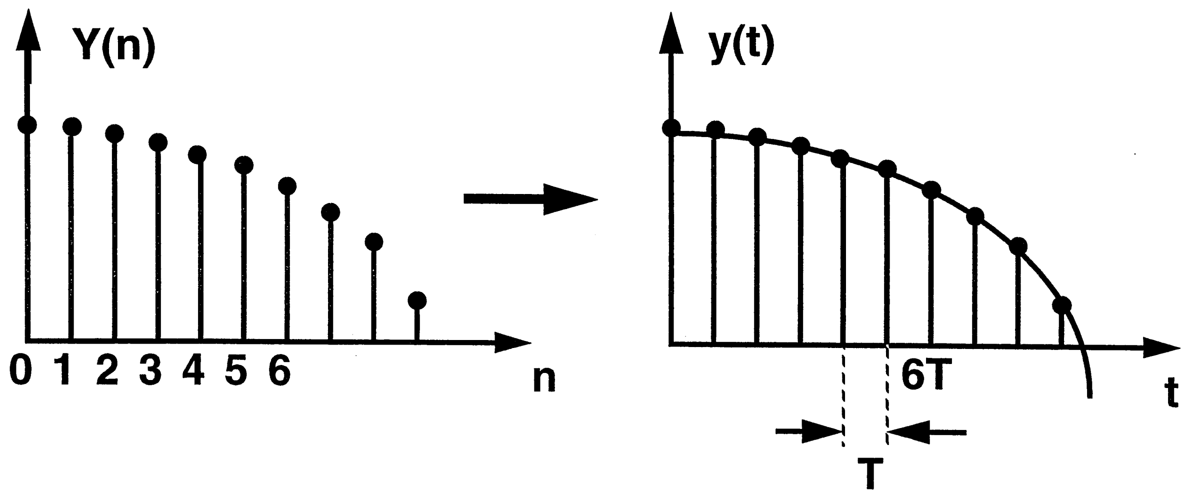
- What is inside our system ?

Step I: A continuous-time signal is sampled. Sampling interval = T. We end up with a sequence of numbers X(n):



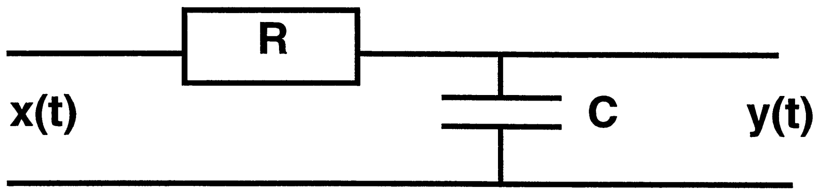
Step II: The sequence X(n) is converted into another sequence Y(n) using a discrete-time system:

Step III: The sequence Y(n) is converted into the analog form such that at t=nT y(t) achieves the value Y(n):

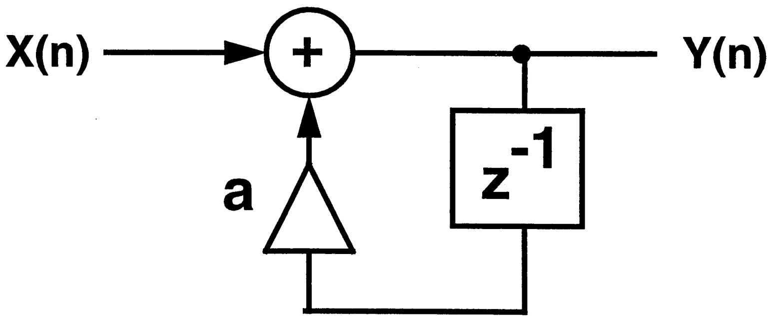


SIMPLE EXAMPLE: Mimicing an analog RC filter

Continuous-time filter:



Discrete-time filter:



$$Y(n) = aY(n-1) + X(n)$$

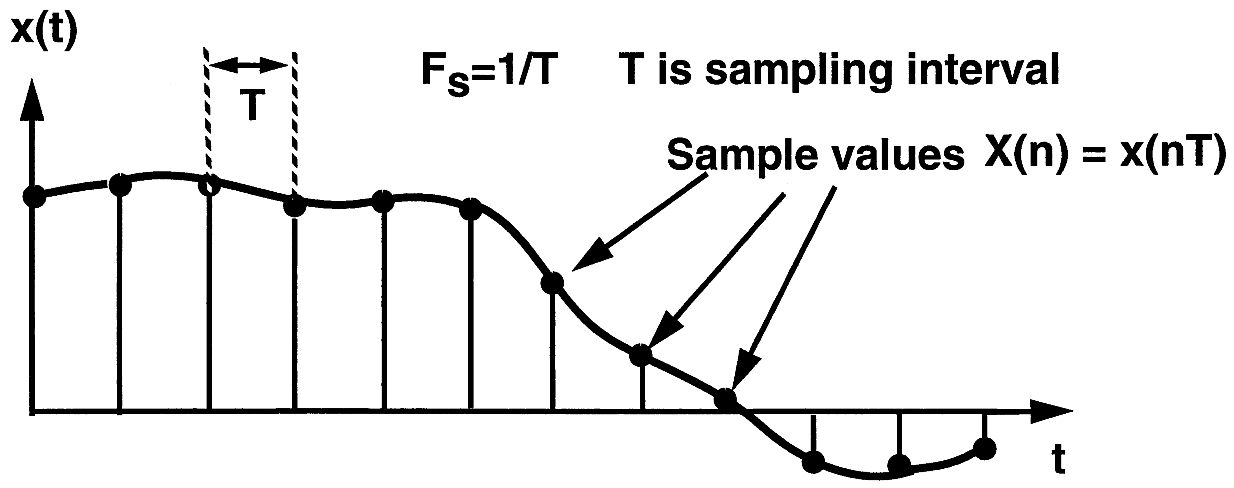
Output = a times the previous output plus

the present input

We can mimic an RC filter by properly selecting the value of a.

STARTING POINT FOR PROCESSING ANALOG SIGNALS USING DIGITAL SIGNAL PROCESSING

<u>Sampling Theorem:</u> A continuous-time signal can be reconstructed from its SAMPLE VALUES if the SAMPLING FREQUENCY F_S is at least two times the highest frequency compinent of the signal, that is, $X(j2\pi f) = 0,\ f > F_S/2$.

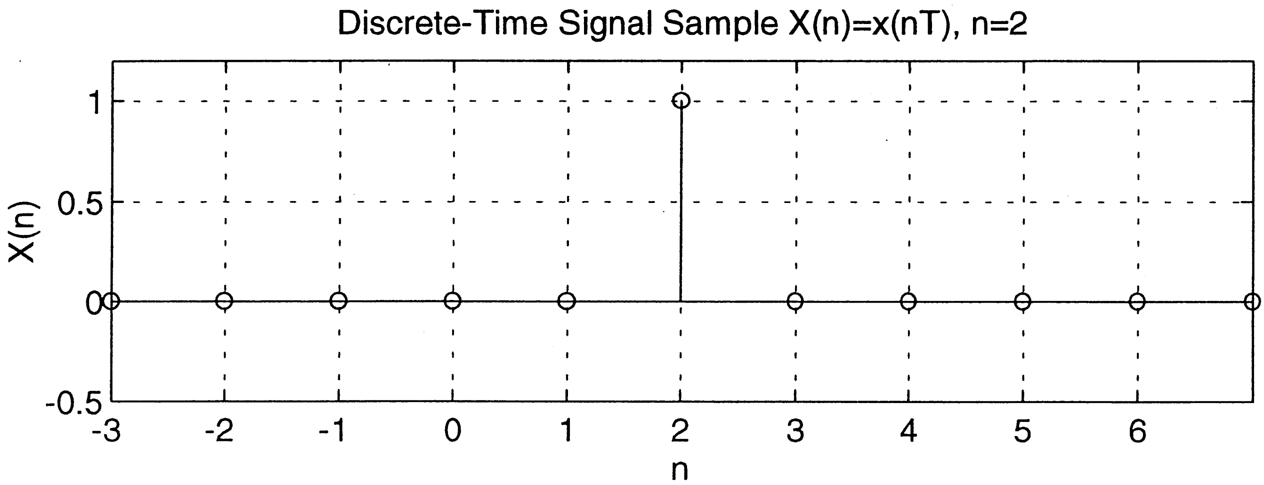


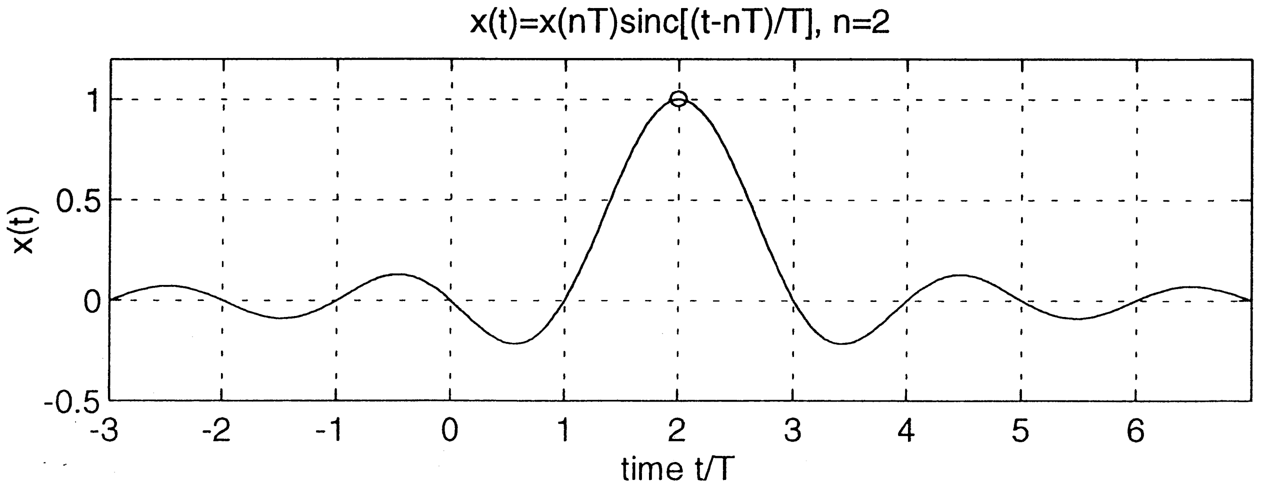
Interpolation using the sinc-function : **The analog signal can be reconstructed with the aid of the sample values x(nT) using the following formula**

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT) \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T} = \sum_{n=-\infty}^{\infty} x(nT)\,\mathrm{sinc}\,[(t-nT)/T]$$

$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

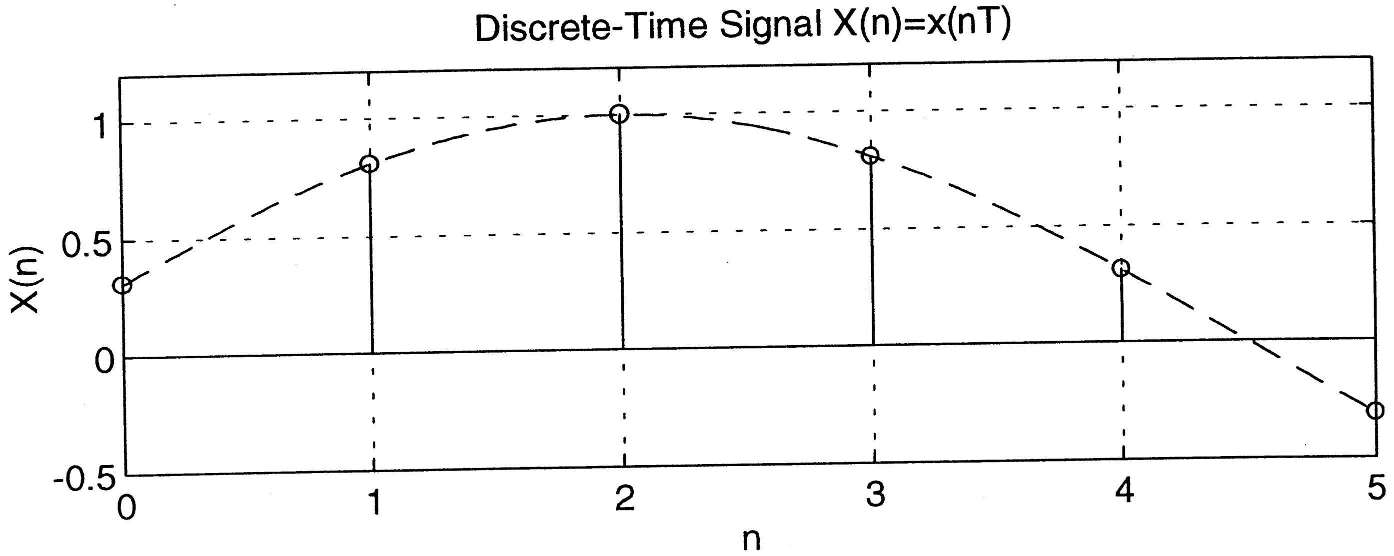
Term x(nT) sinc [(t-nT)/T]:

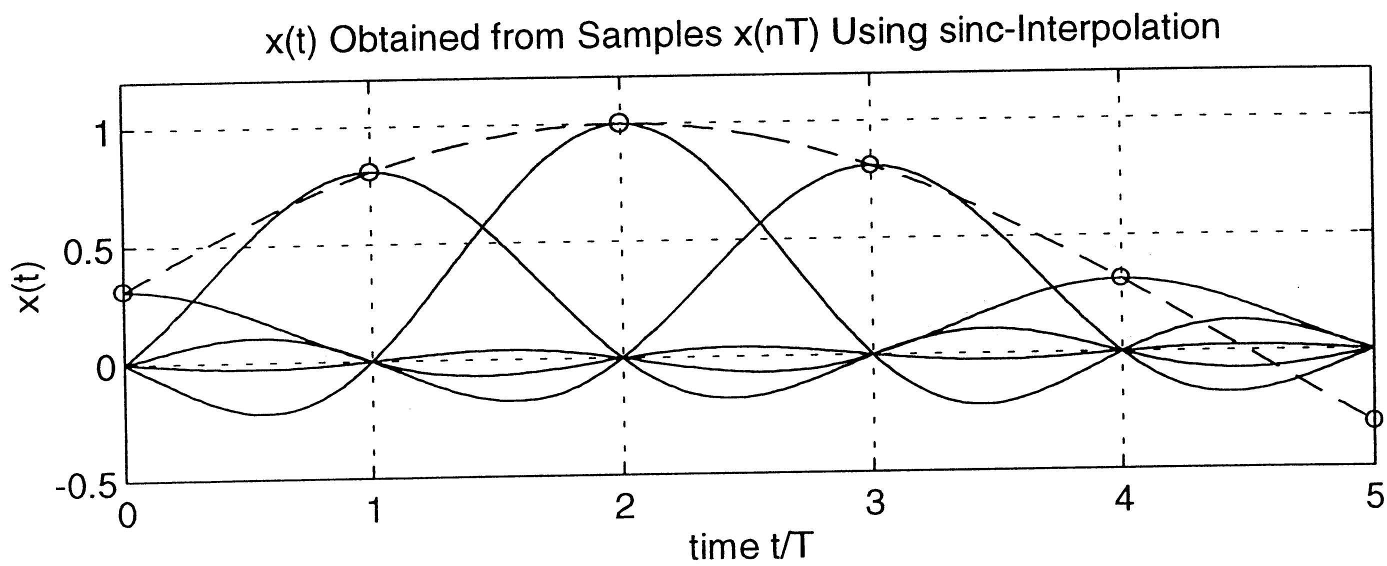




- achieves the value x(nT) at t = nT ja and 0 at t = kT ja k ≠ n.

Example on the reconstruction





Analog interpretation of the reconstruction

The above sinc-interpolation can be interpreted by an operration, where a continuous-time signal

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta_a(t-nT)$$

is filtered with an analog filter having the impulse response

$$h_a(t) = \mathrm{sinc}(t/T) = \frac{\sin(\pi t/T)}{\pi t/T} \; .$$

$\delta_a(t)$ is the analog impulse (see Appendix A).

$x_s(t)$ is an analog equivalent to the sequence $X(n)=x(nT)$.

It can be constructed from $x(t)$ as follows:

$$x_s(t) = x(t)s(t), \quad s(t) = \sum_{n=-\infty}^{\infty} \delta_a(t-nT) \, .$$

This suggests modelling the overall process by forming the sequence $X(n)$ from $x(t)$ and then reconstructing $x(t)$ back from the sequence $X(n)$ in a way shown in the following transparencies.

This model is then used for illustrating the overall filtering process with the aid of digital signal processing.

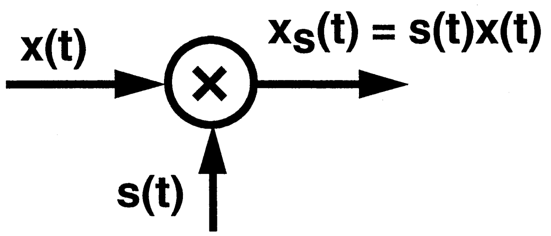
Time-domain model

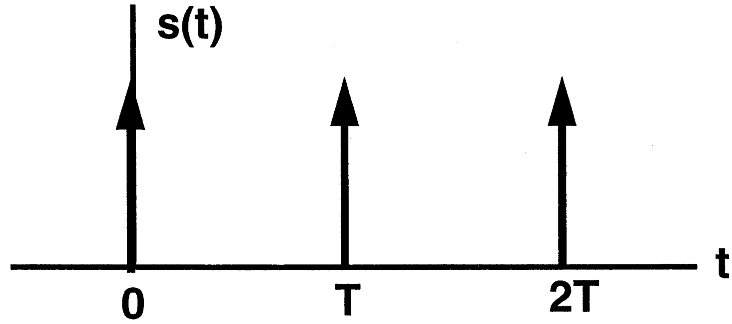
Step I: Multiply x(t) by

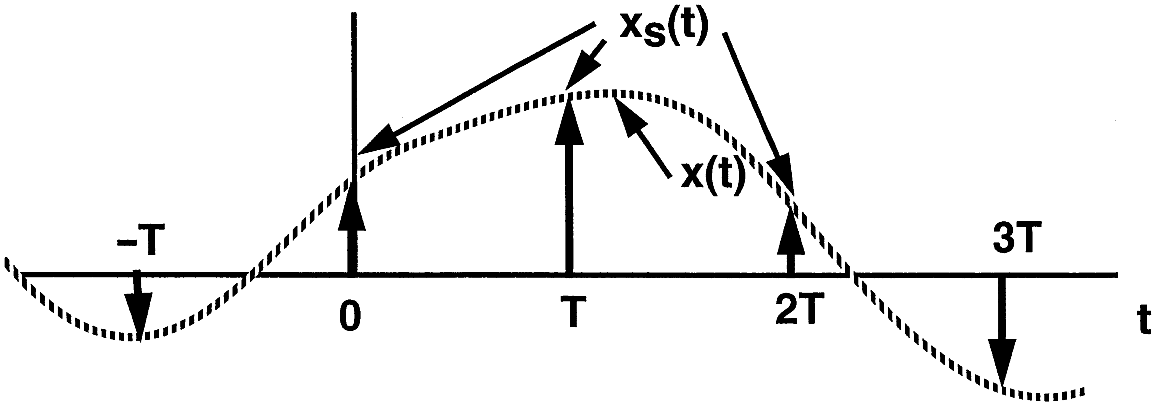
$$s(t) = \sum_{n=-\infty}^{\infty} \delta_a(t-nT),$$

giving the following weighted impulse train:

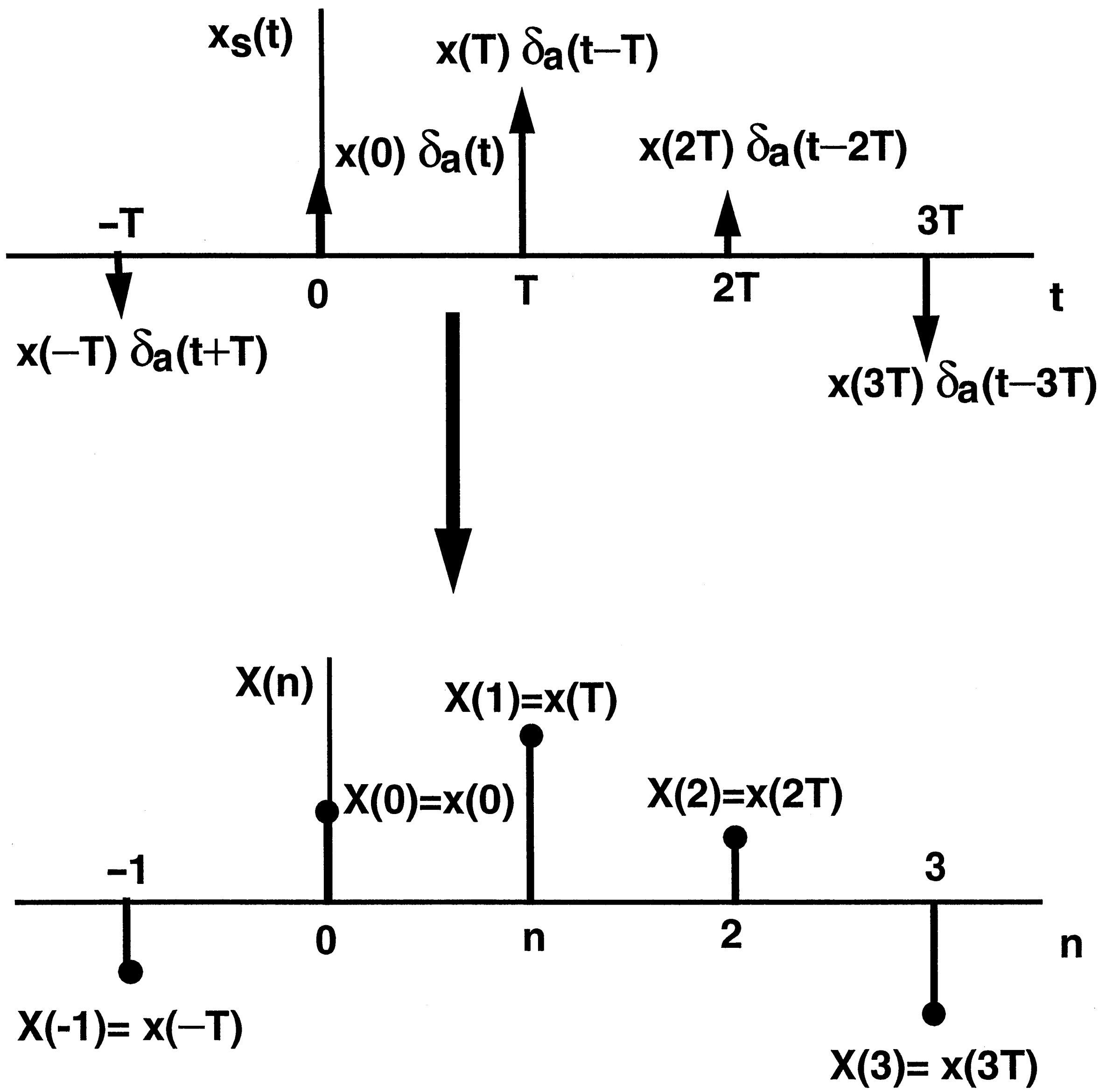
$$x_s(t) = s(t)\,x(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta_a(t-nT).$$





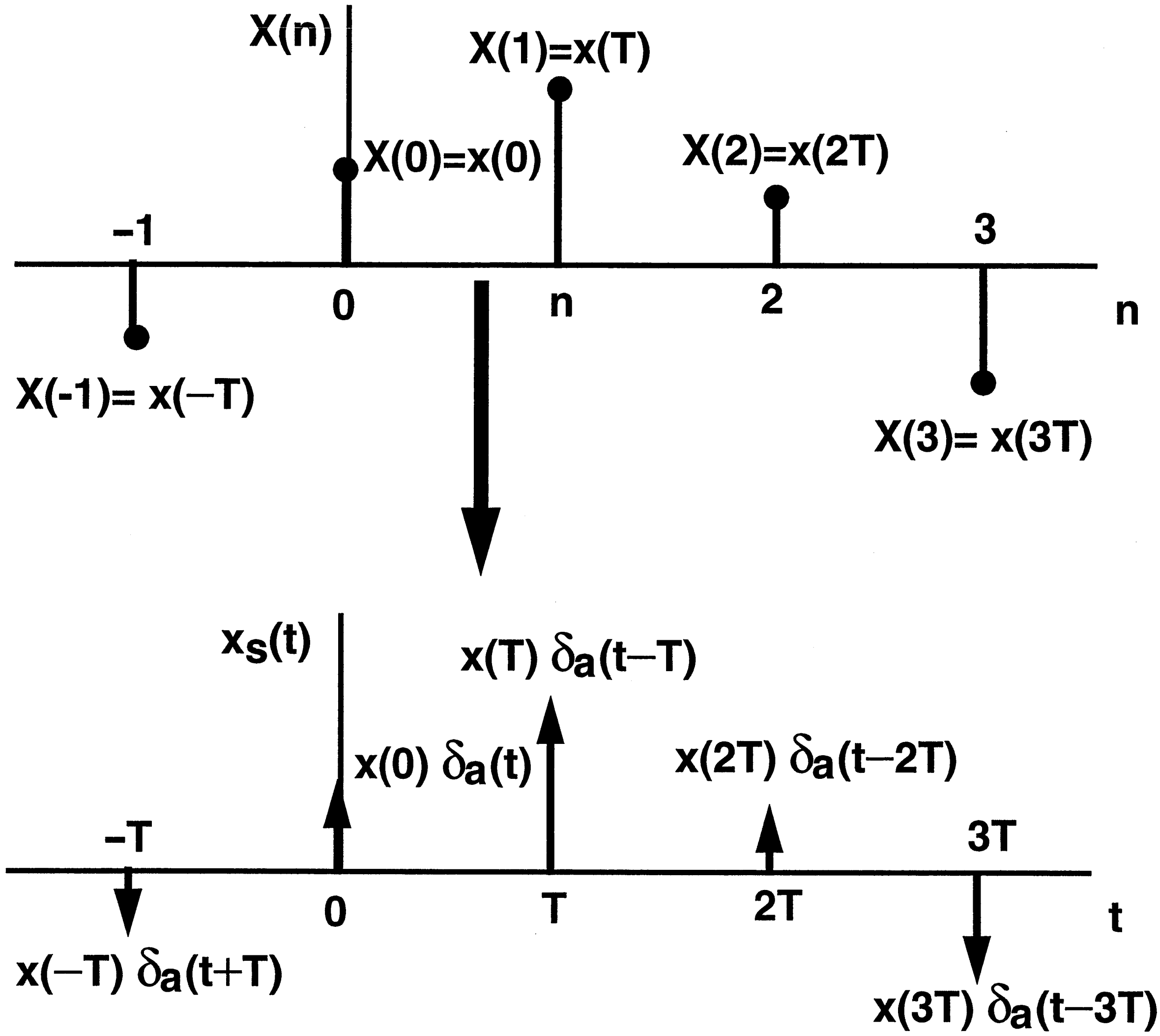


Step II: Form $X(n) = x(nT) = x_s(nT)$



<u>Step III</u>: Form $$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$$

such that $x(nT) = X(n)$



COMMENT: In practical signal processing, instead of X(n), we use the sequence Y(n) obtained from the sequence X(n) according to some filtering operation.

<u>Step IV</u>: Form the original signal x(t) by filtering

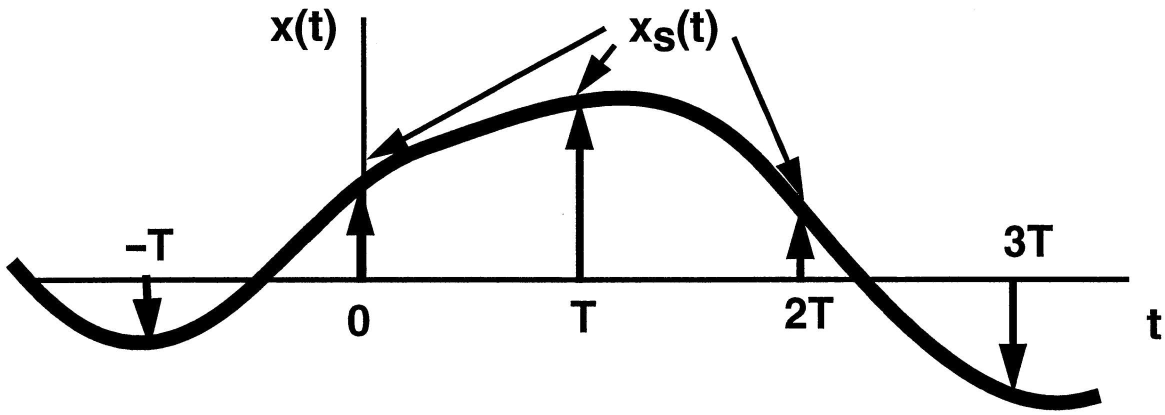
$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta_a(t-nT)$$

with an analog filter having the impulse response

$$h_a(t) = \mathrm{sinc}(t/T) = \frac{\sin(\pi t/T)}{\pi t/T}\,.$$

This gives the original signal

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T} = \sum_{n=-\infty}^{\infty} x(nT)\,\mathrm{sinc}\,[(t-nT)/T]\,.$$



Comments on sinc-interpolation

1) Not implementable in practice:

   - we need an infinite number of terms, not causal (starts at minus infinity and ends at infinity)

2) When allowing some delay, the sinc-interpolation can be approximated rather accurately.

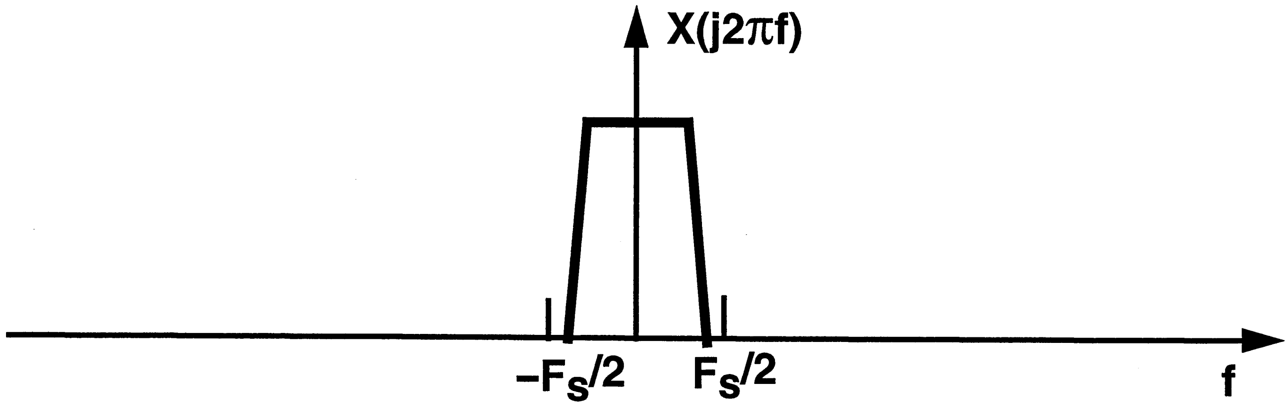
3) As we will see later in our course, the signal reconstruction can be done in practice rather easily.

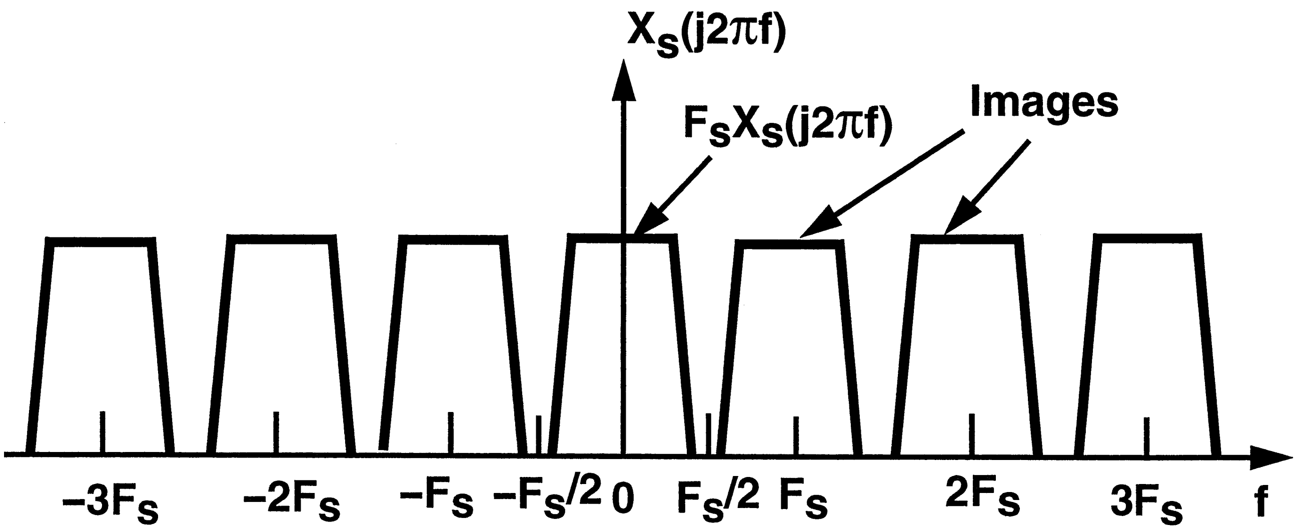
What is happening in the frequency domain?

Step I: If the Fourier transform of $x(t)$ is $X(j2\pi f)$, then the Fourier transform of $x_s(t)$ is given by

$$X_s(j2\pi f) = F_s \sum_{k=-\infty}^{\infty} X(j2\pi(f-kF_s))$$

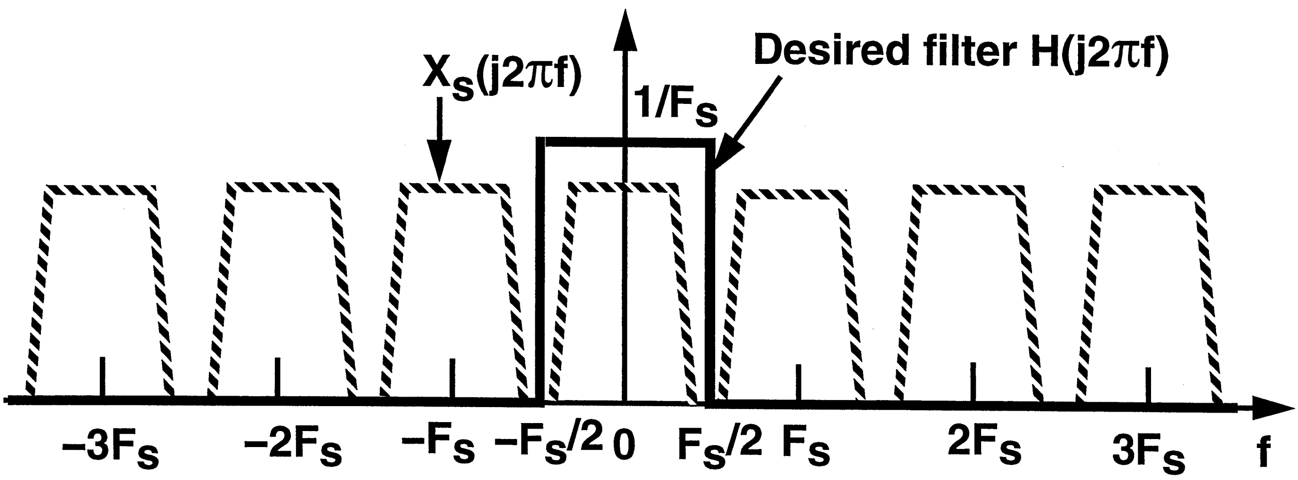
This fact is shown to be true in Appendix B.

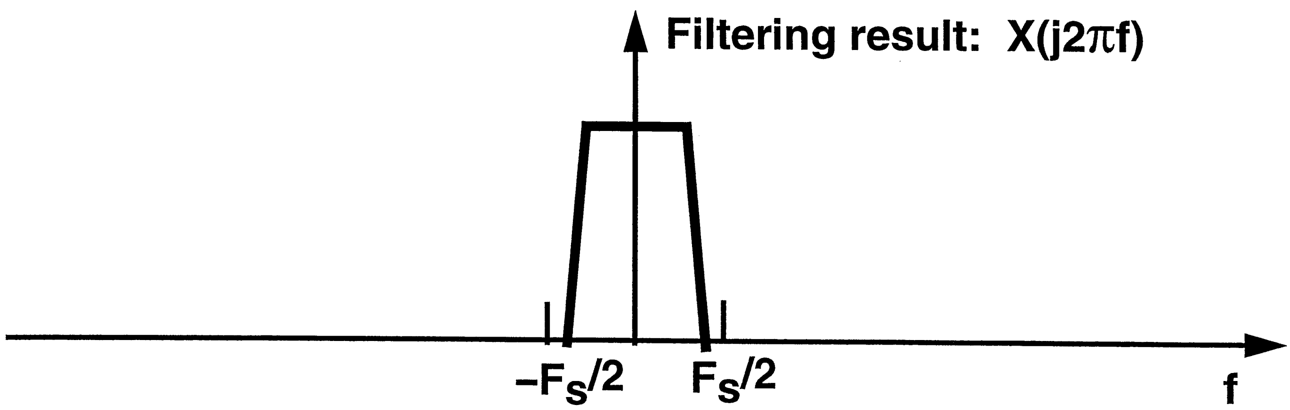




Step II: Reconstruction using the filter with impulse response $h(t)=sinc(t/T)$.

The corresponding amplitude response is shown below and it picks up the frequency range between $-F_S/2$ and $F_S/2$, as is desired. In this frequency range, the amplitude response of our filter has a constant value of $1/F_S$.





Simple example on filtering

1) The input continuous-time signal x(t) consists of three sinusoidal signals with frequencies $F_S/8$, $2F_S/8$ and $3F_S/8$. All these components start at t=0:

$$x(t)=[\sin\{2\pi(F_S/8)t\}+\sin\{2\pi(2F_S/8)t\}+\sin\{2\pi(3F_S/8)t\}]u(t).$$

- For F_S=20 kHz, the frequencies are 2.5, 5 ja 7.5 kHz.
- Sampling interval $T=1/F_S=0.05$ms.
- u(t) is the unit step which is 1 for $t \geq 0$ and 0 for $t<0$.

2) After sampling ($T=1/F_S$), we obtain

$$X(n)= x(nT)=[\sin\{(2\pi/8)n\}+\sin\{(4\pi/8)n\}+\sin\{(6\pi/8)n\}]u(n).$$

3) The discrete-time output is formed as follows:

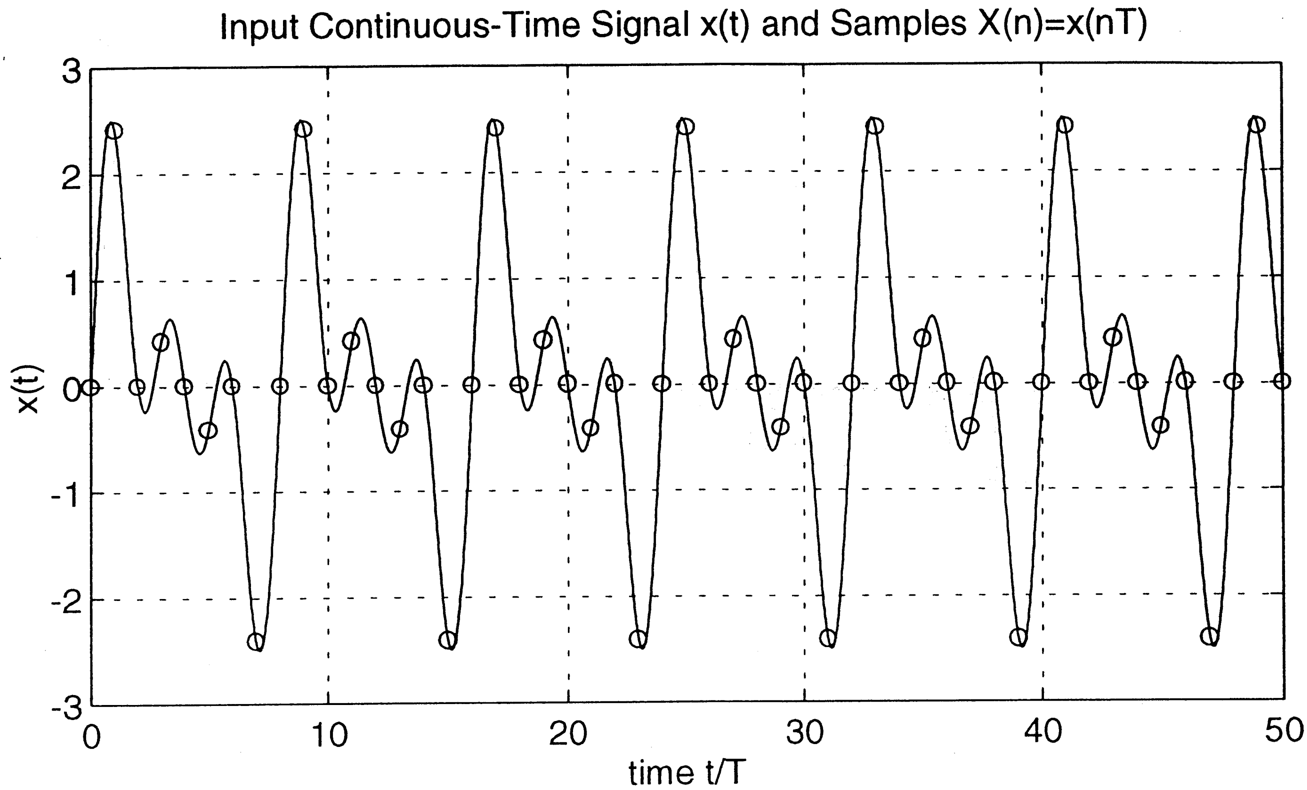
$$Y(n)=\{X(n)+X(n-4)\}/2.$$

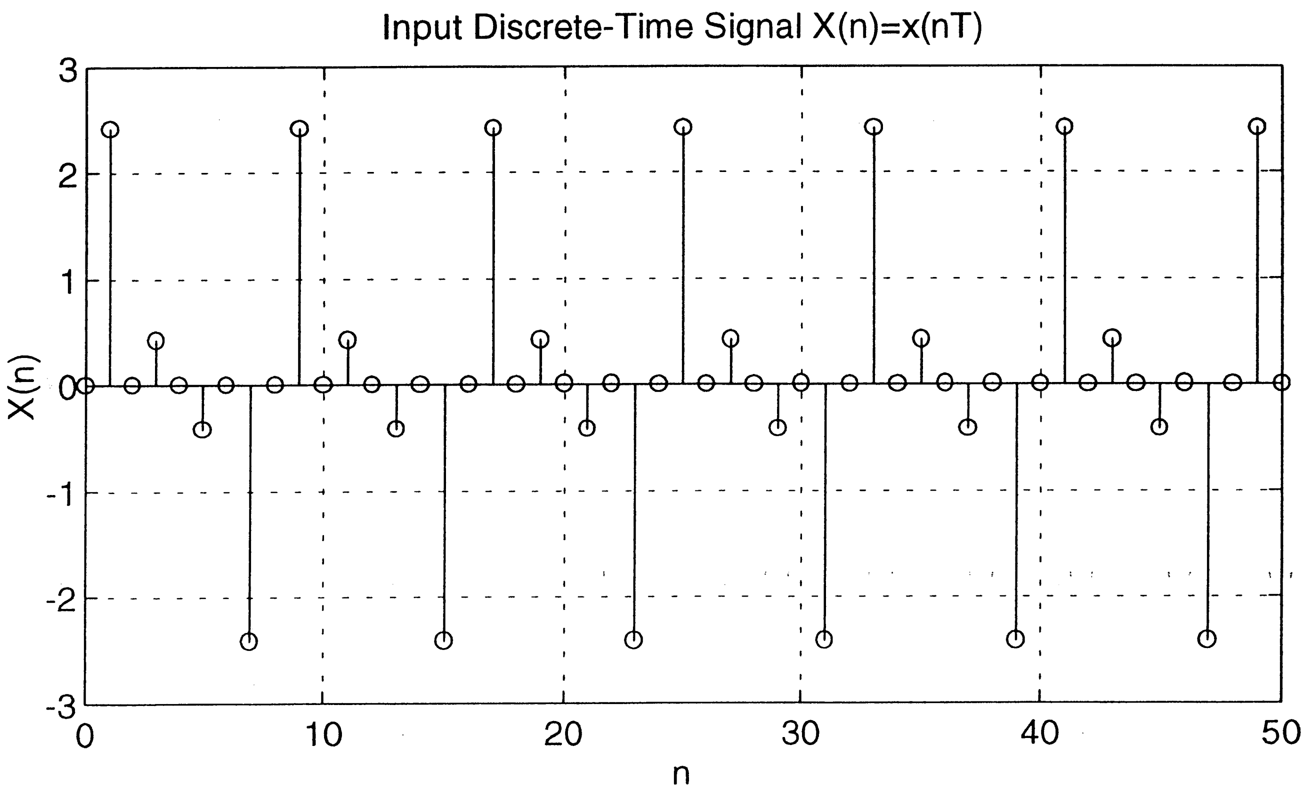
4) The sinc-interpolation is used for forming the continuous-time output signal (y(nT)=Y(n)):

$$y(t) = \sum_{n=0}^{\infty} y(nT) \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}.$$

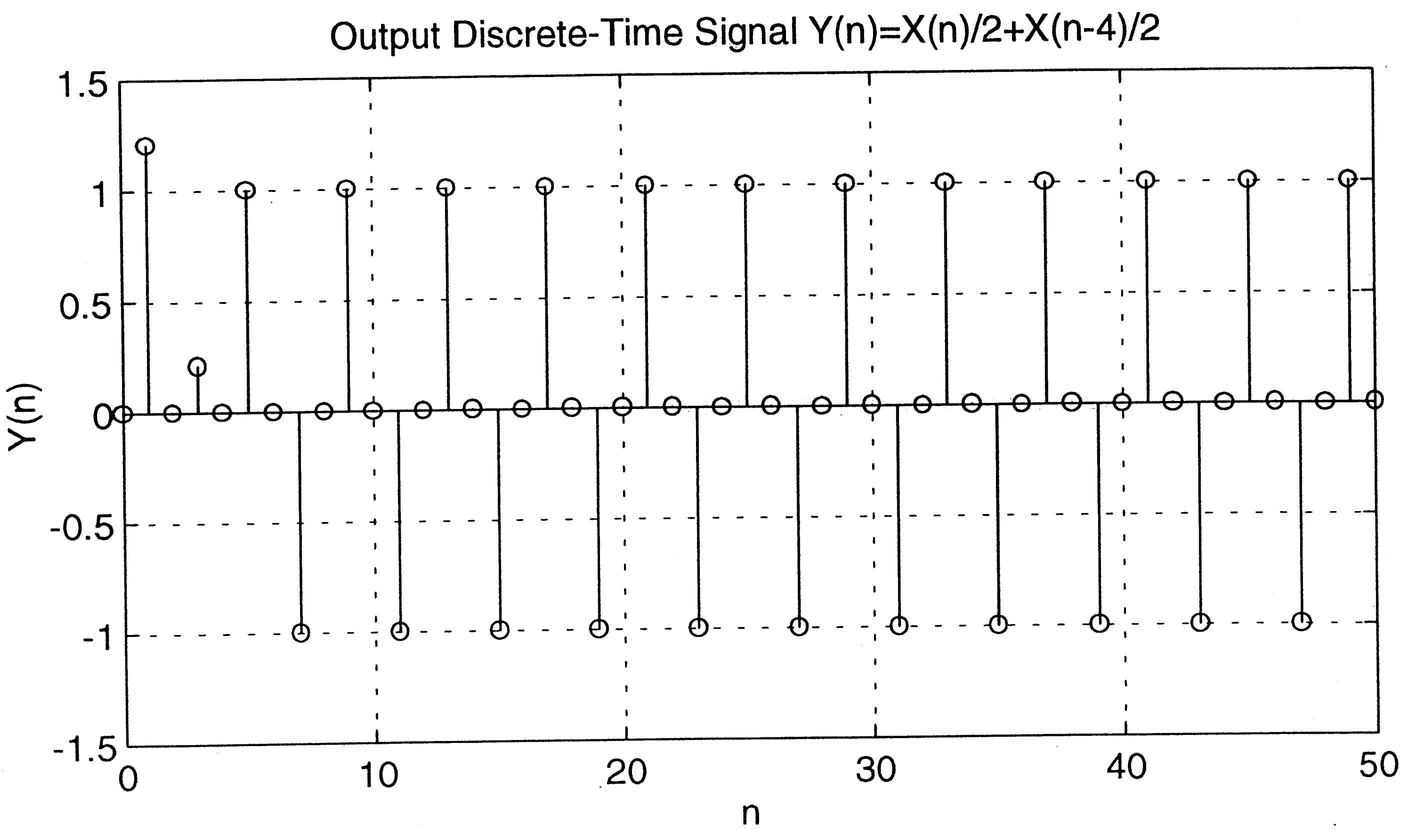
- The figures on the following two transparencies illustrate the overall process. As seen from the last figure, the output signal consists of a short-duration transient and a sinusoidal signal of frequency $2F_S/8$.

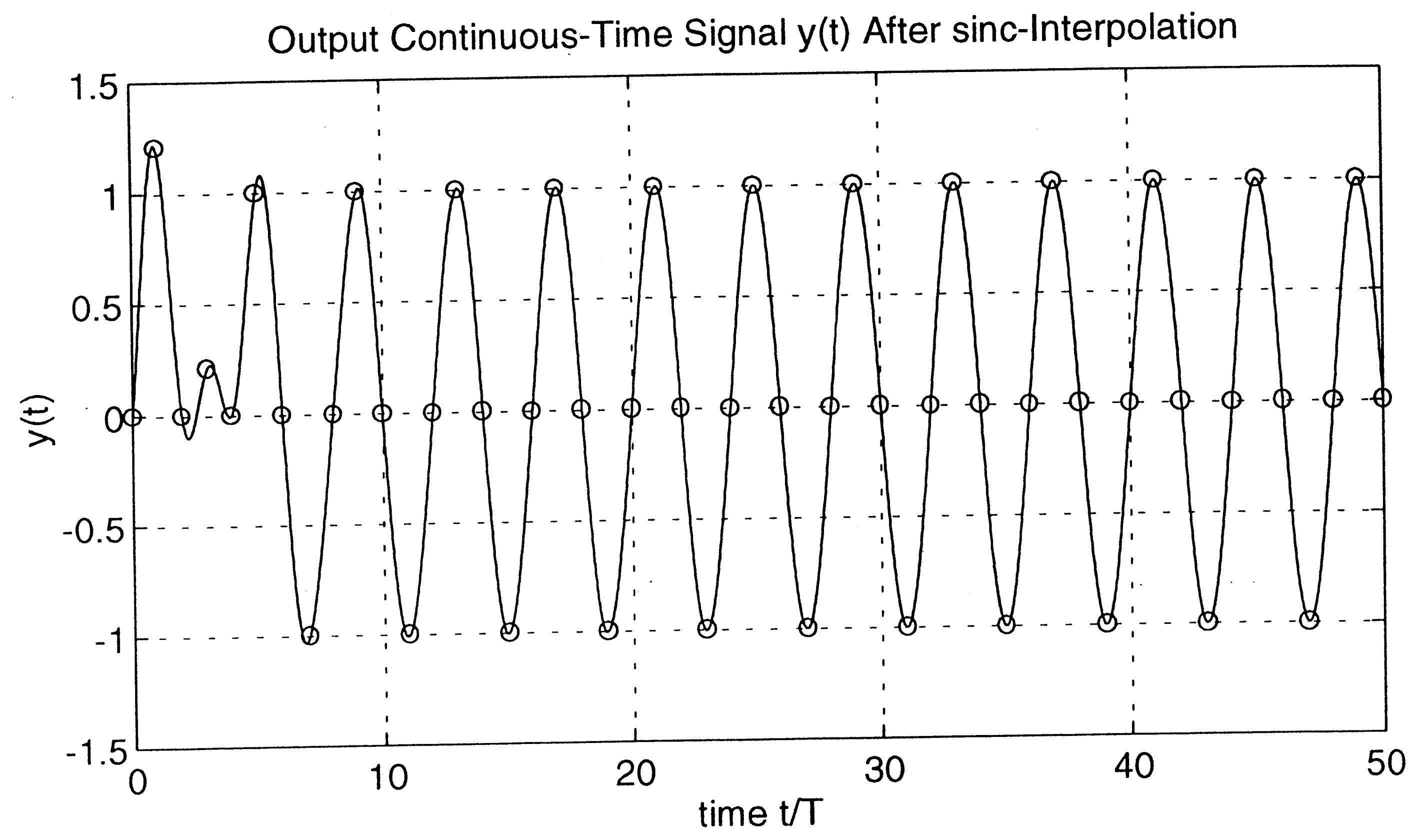
Continuous-time and discrete-time input signals





Discrete-time and continuous-time output signals



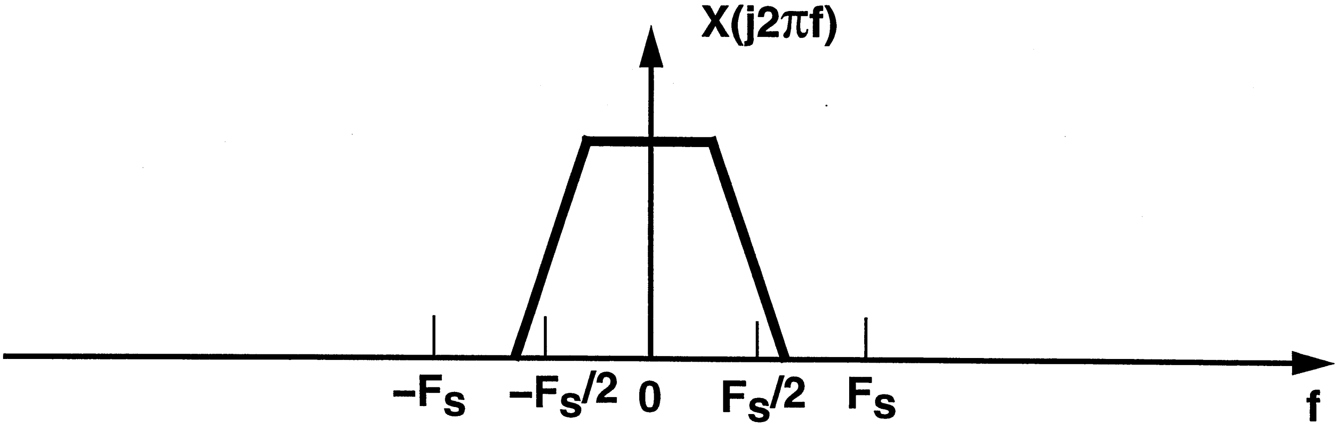


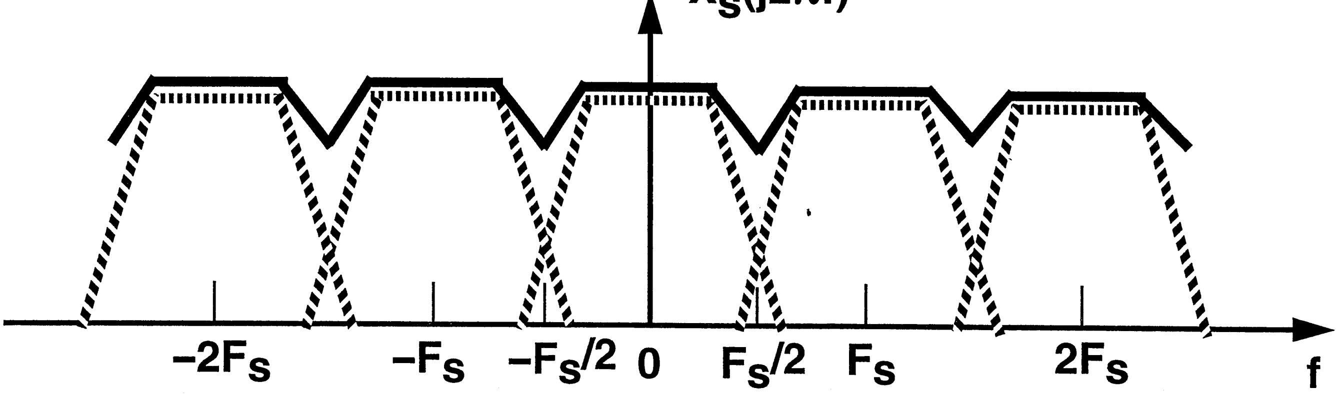
ALIASING

If the conditions of the sampling theorem are not satisfied, then the components $X(j2\pi(f-kF_s))$ ovelap, as can be seen in the the second figure shown below.

In this case, the baseband between the frequencies $-F_s/2$ and $F_s/2$ contains frequencies which do not originally belong to this band. This is called aliasing.

The original signal cannot be reconstructed!!





EXAMPLE

Two sinusoidal signals have the same sample values:

The frequency of the desired signal is $f_0 = 0.1F_S$: $F_S=1/T$:

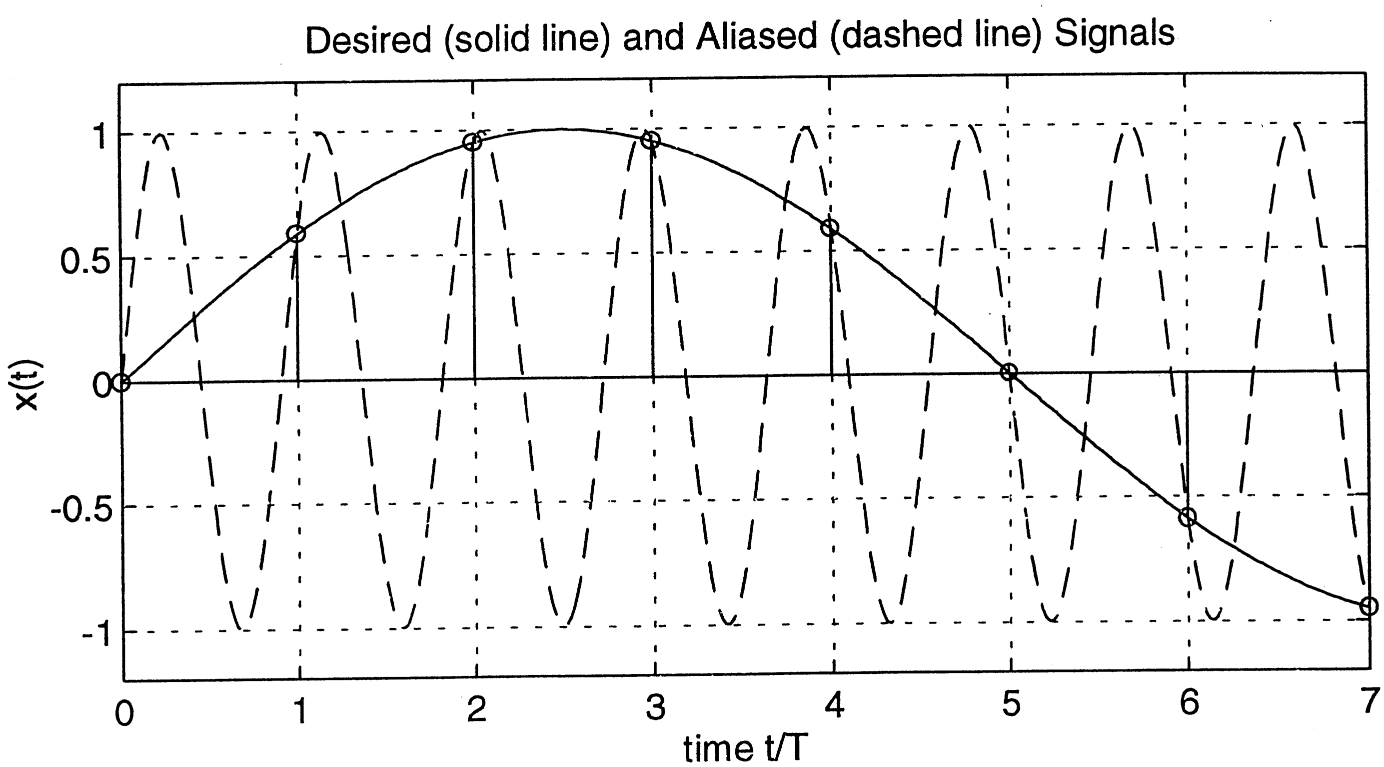
- $x(t) = \sin[2\pi f_0 t] = \sin[2\pi(F_S/10)t]$

The original frequency of the aliasing signal is $f_0+F_S=1.1F_S$:

- $y(t) = \sin[2\pi(f_0+F_S)t] = \sin[2\pi(F_S/10+F_S)t]$

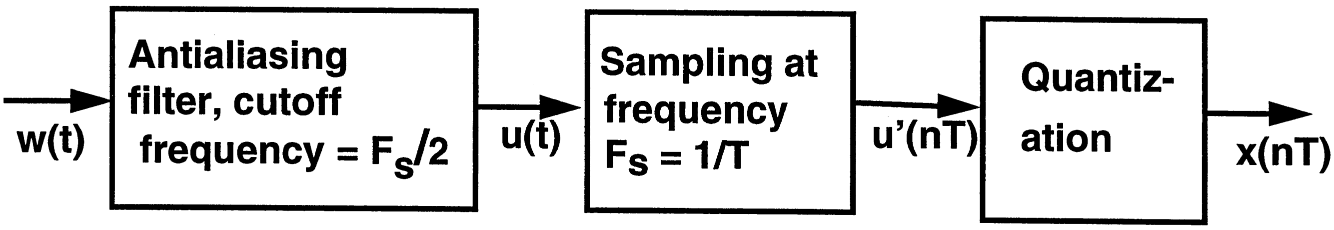
After sampling, we interpret y(t) to be x(t)!!

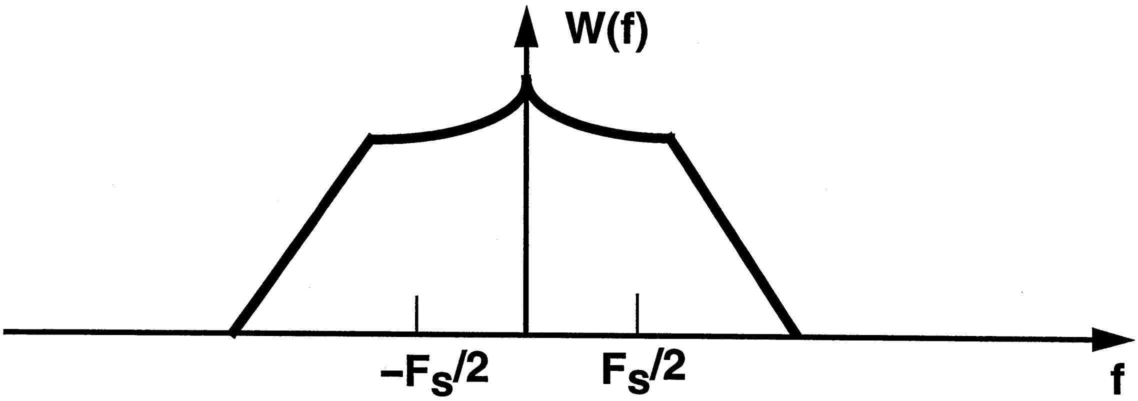
- T= 0.1ms ⟶ F_S = 10 kHz, $0.1F_S$ = 1 kHz, $1.1F_S$ = 11 kHz.

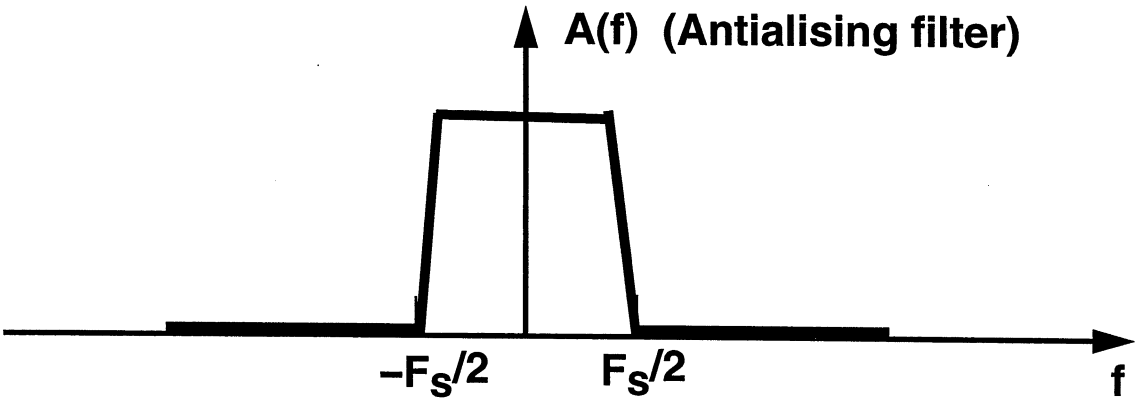


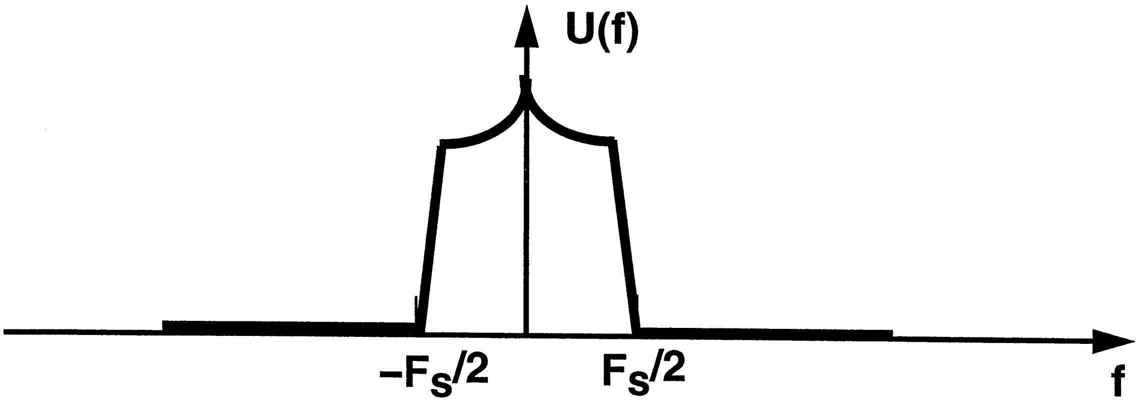
PRACTICAL SYSTEM

Step I: Conversion of an analog signal into the digital form:

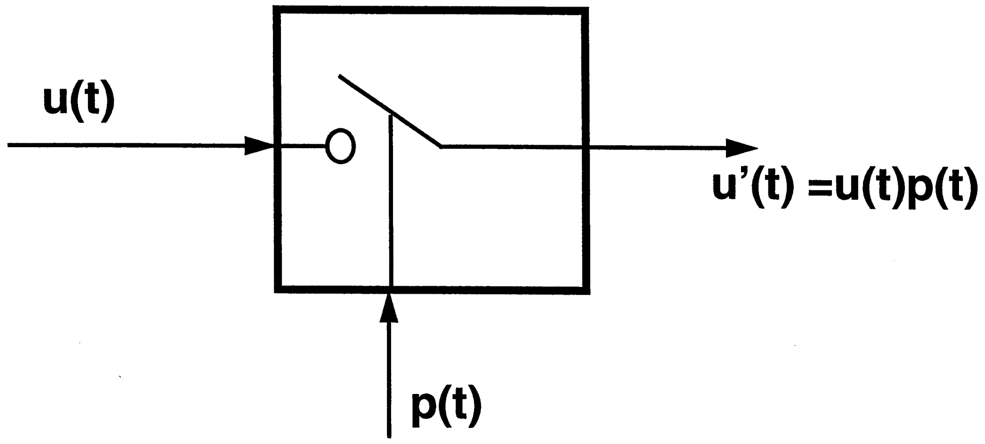


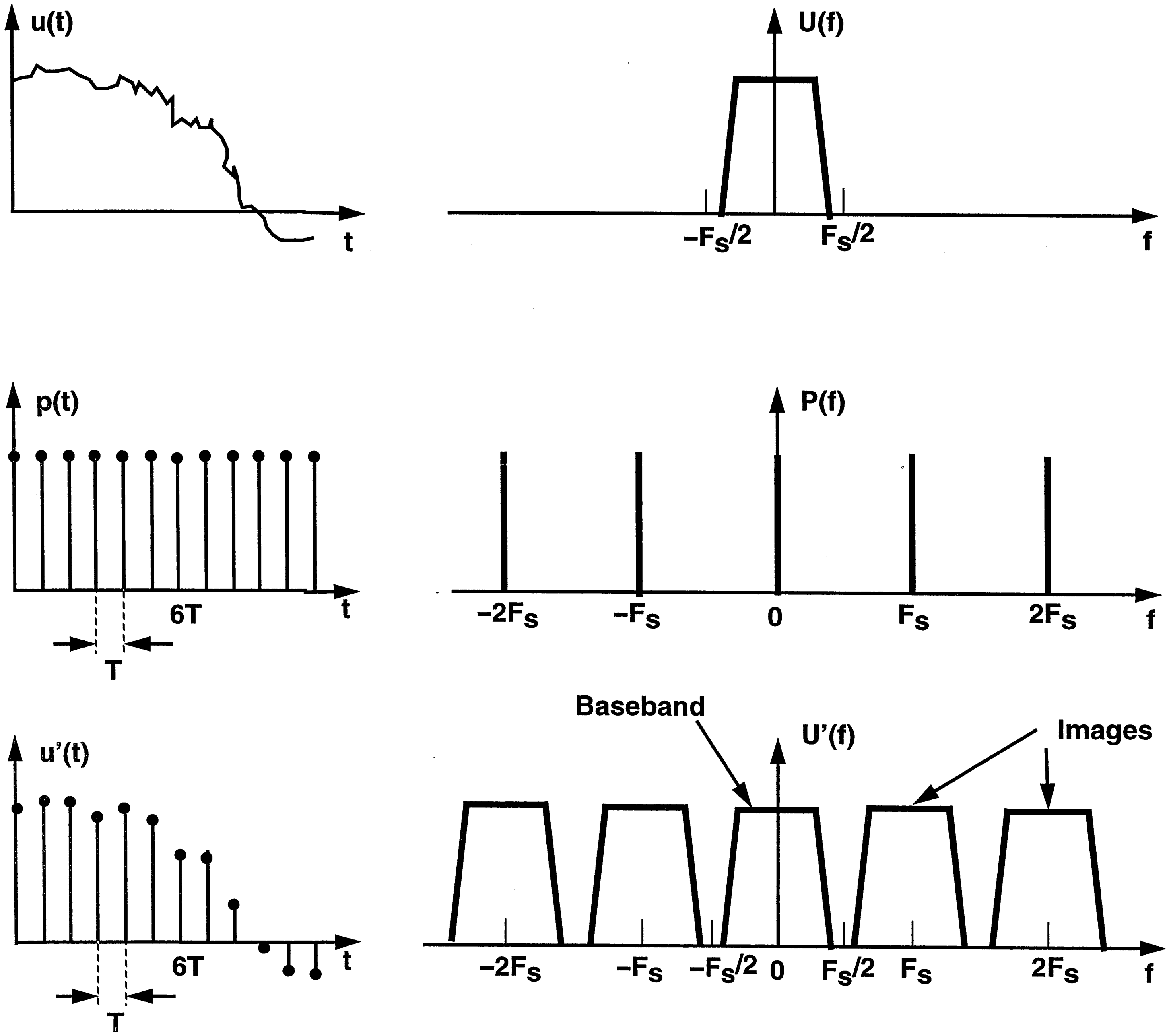




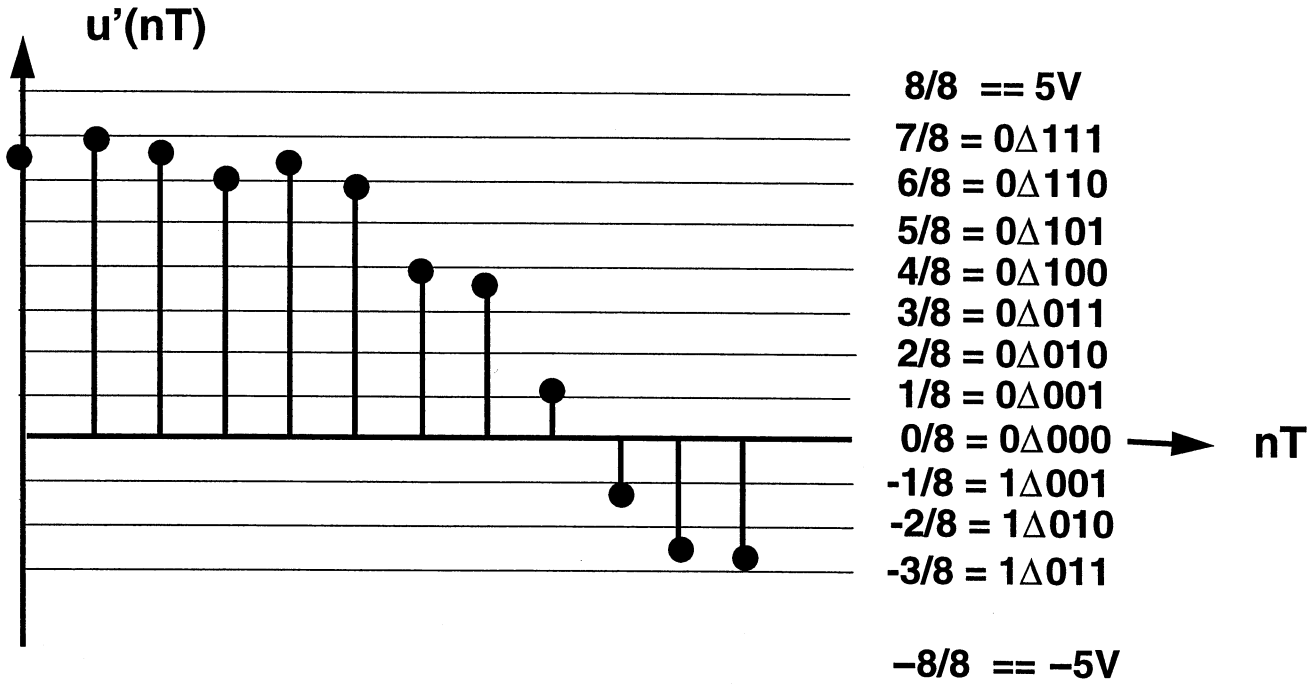


SAMPLING AT FREQUNENCY F_S= 1/T

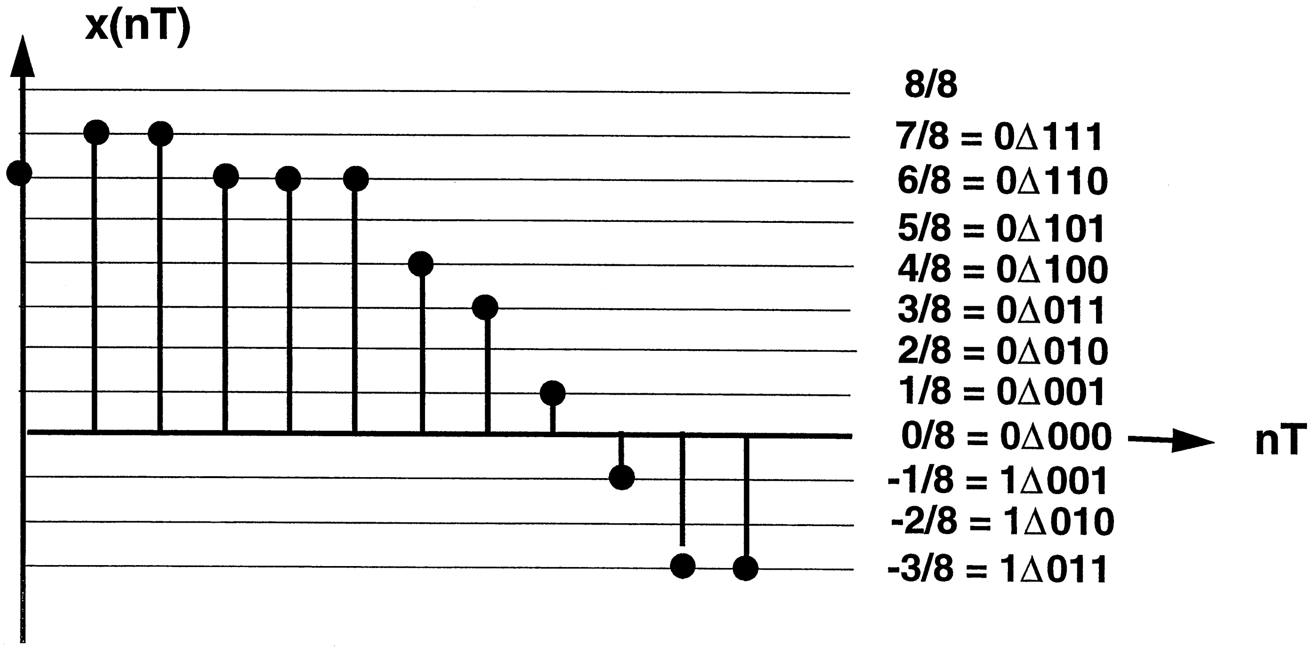


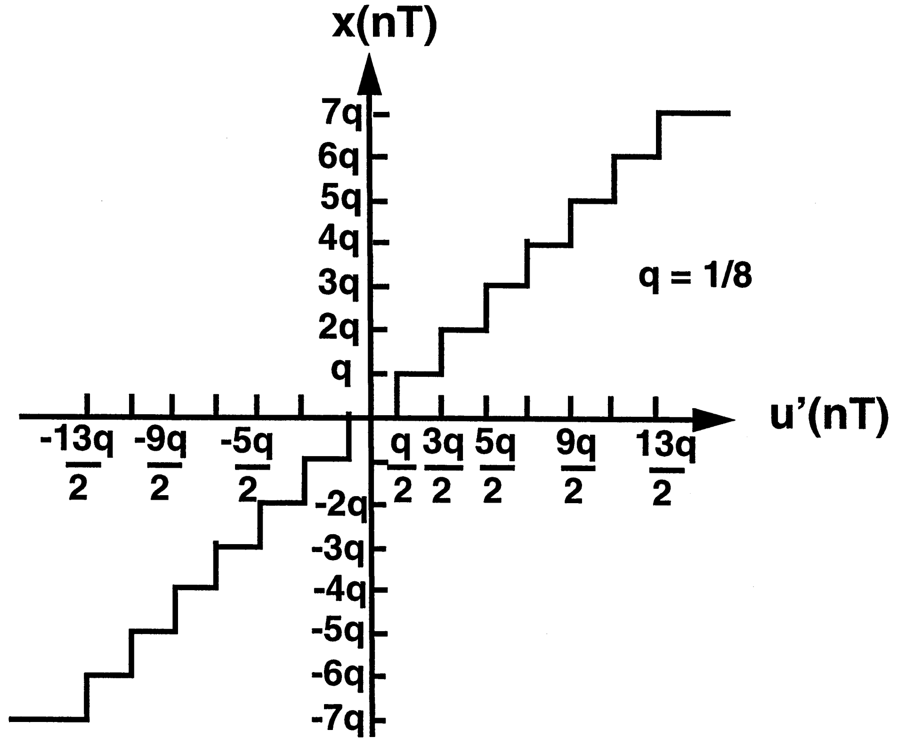


QUANTIZATION

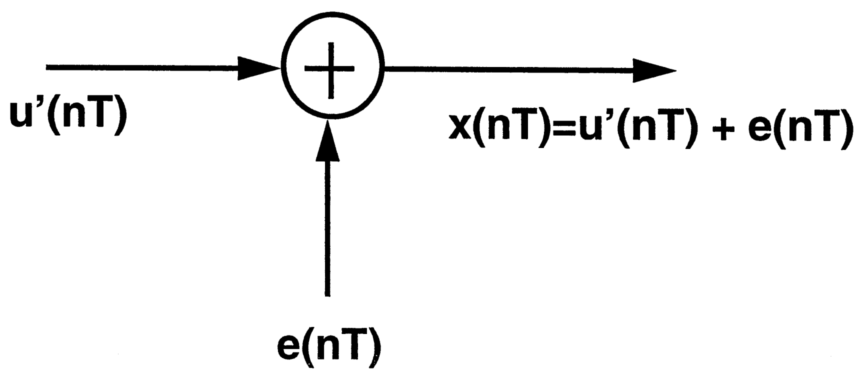


Rounding to the nearest number

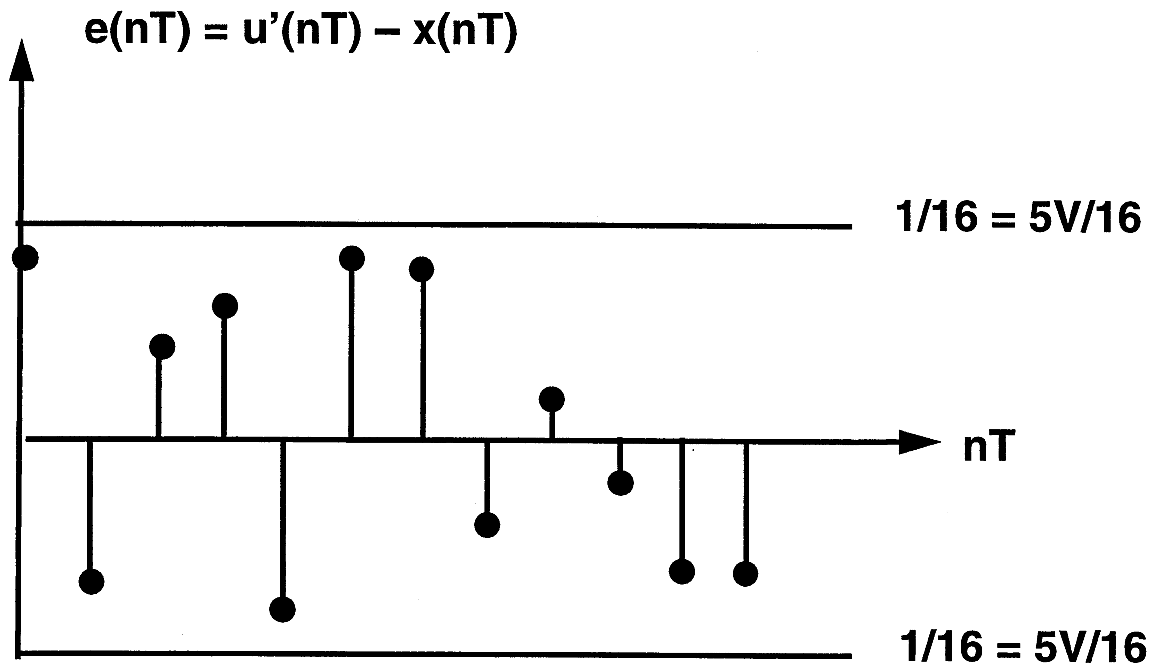




Modelling (to be considered later)



Quantization error



Quantization step is q=1/8==5V/8.

In our example we use one sign bit + 3 fractional bits.

If the number of fractional bits is B, then $q=1/(2^B) = 5V/(2^B)$.

The maximum error is half, that is, q/2.

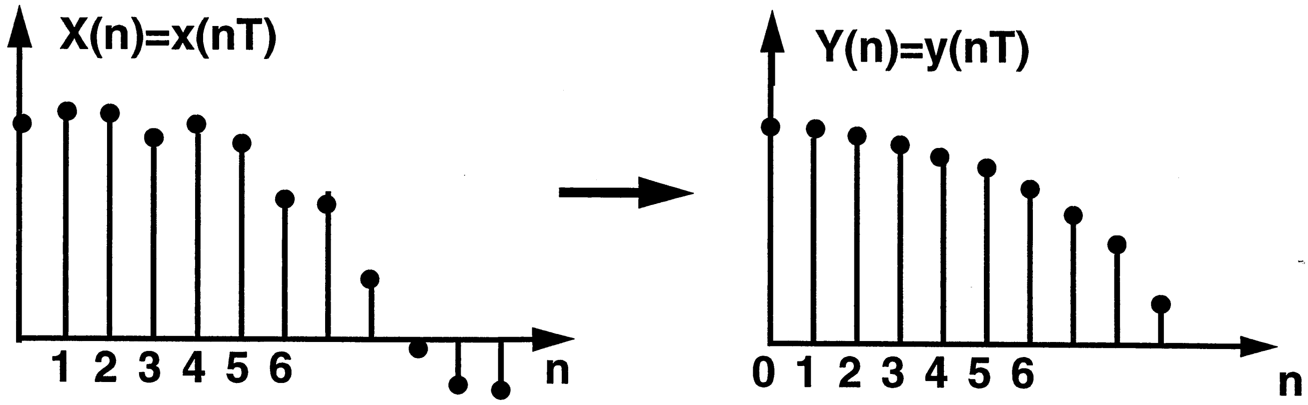
The error is uniformly distributed in the range [–q/2 , q/2].

The noise power due to the quantization is $q^2/16$.

For a sinusoidal signal oscillating between -5V and 5V ,

the signal-to-quantizatio noise ratio is (6.02 B+1.76) dB.

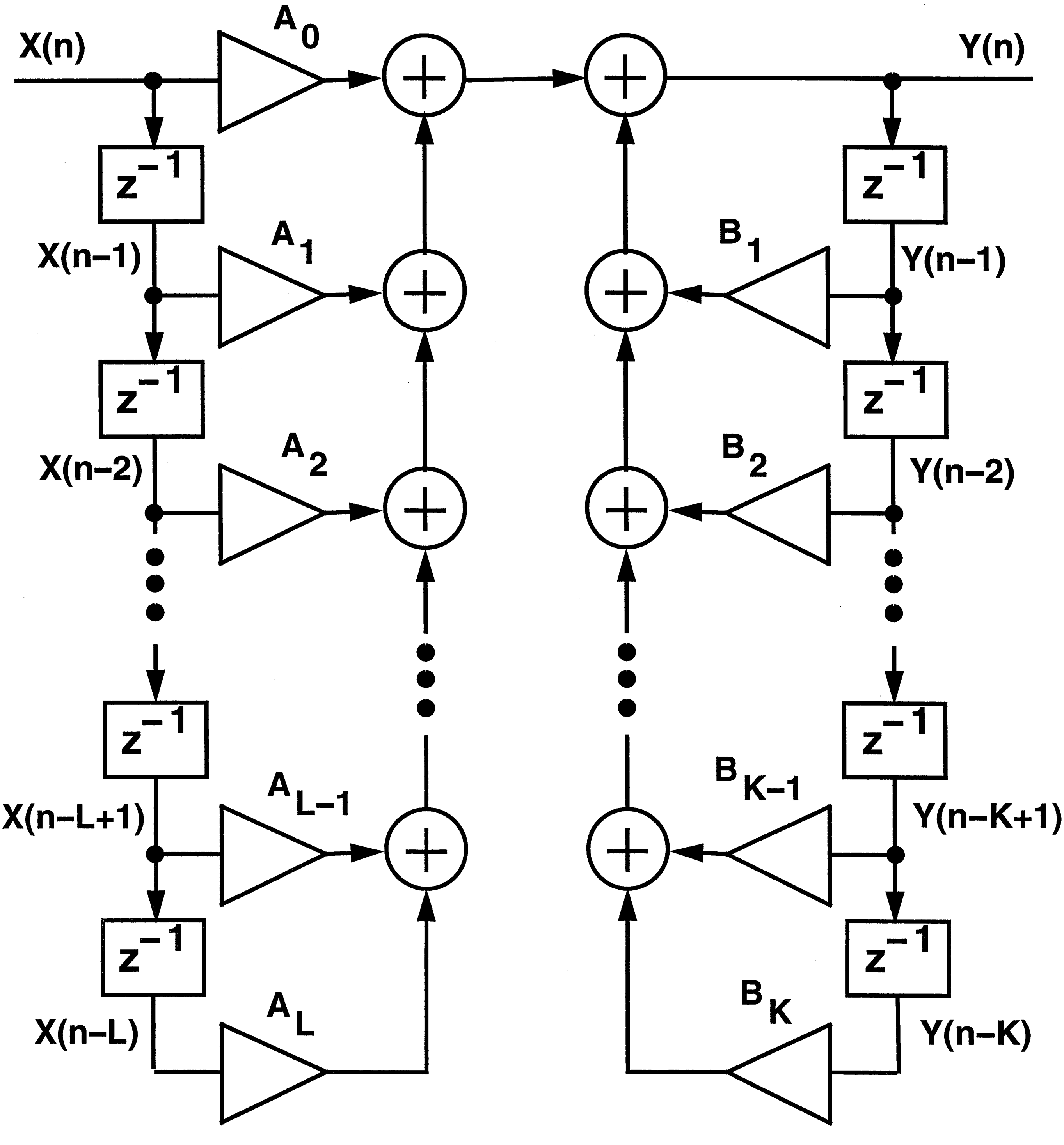
Step II: Digital Filtering



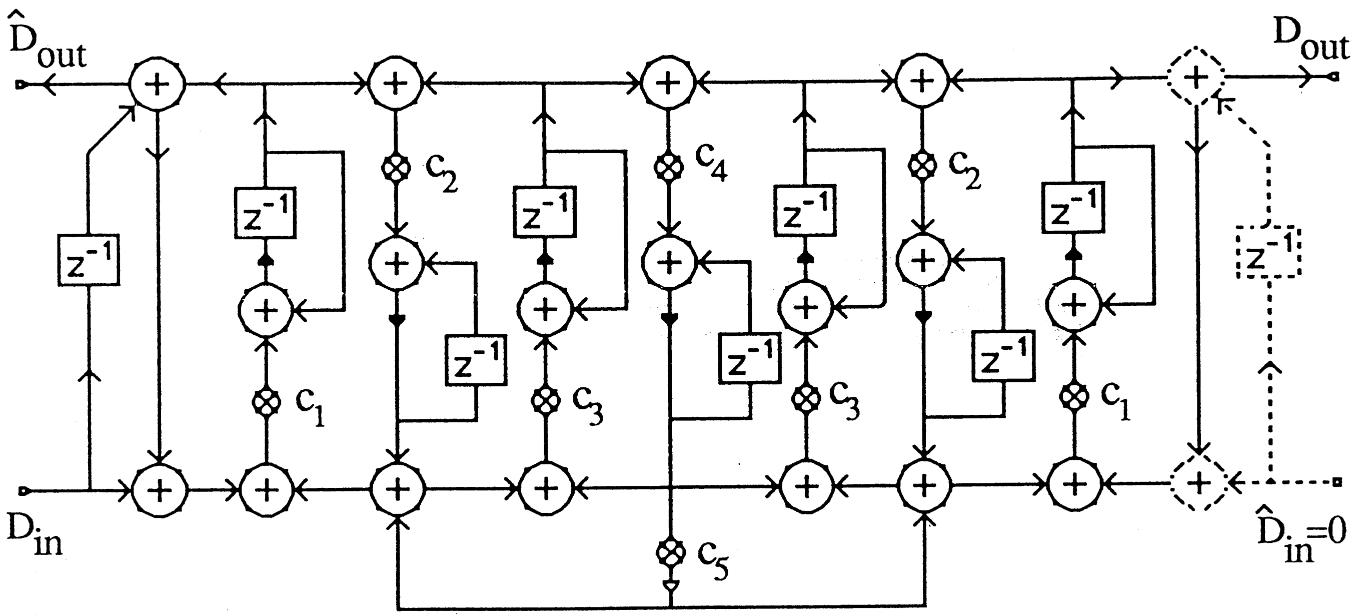
What is a digital filter?

A general digital filter having the following input-output relation (difference equation):

$$Y(n) = \sum_{l=0}^{L} A_l \, X(n-l) + \sum_{k=1}^{K} B_k Y(n-k)$$



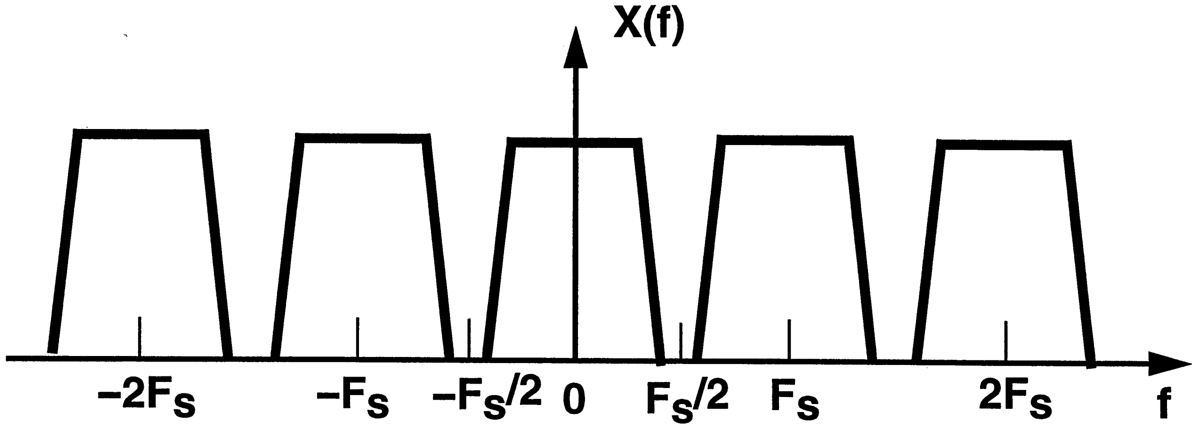
The same difference equation can be implemented in several ways. An exotic structure:

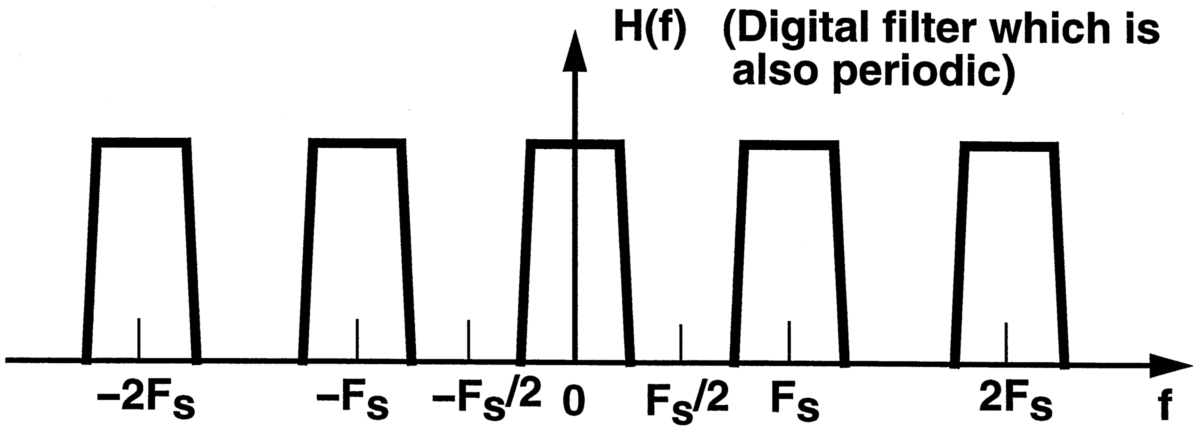


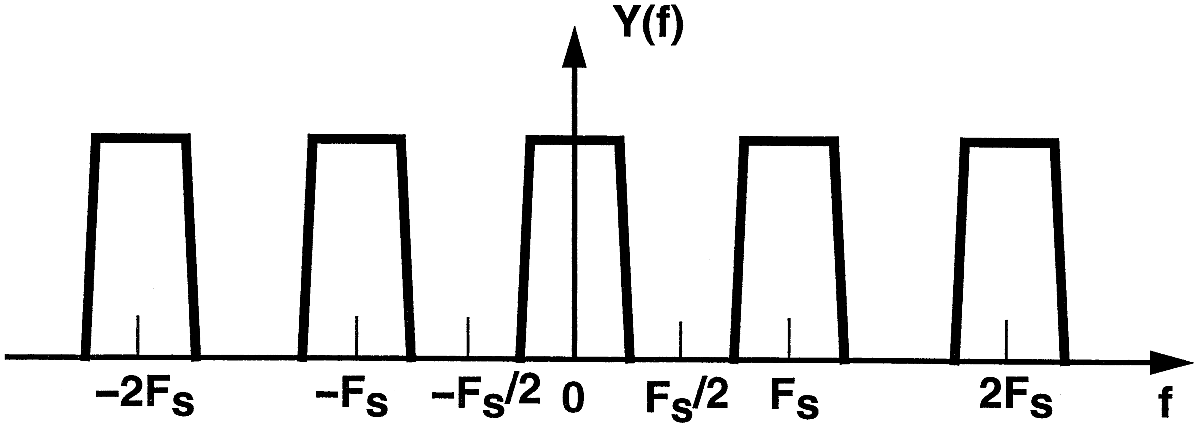
Factors affecting on selecting a proper structure:

- **The effects of finite wordlength (noise, oscillations, coefficient sensitivity)**
- **Easy realizability (signal processors, VLSI circuits)**

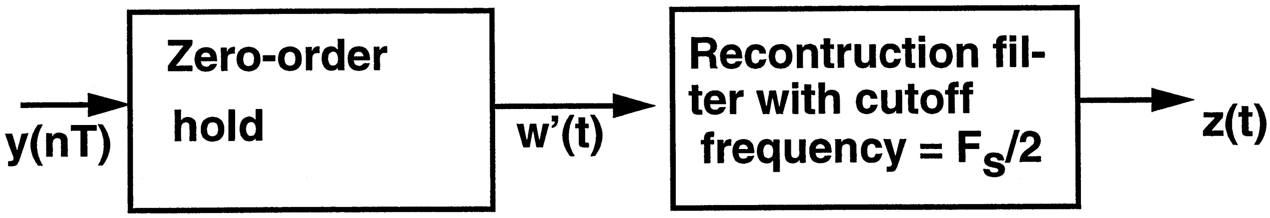
FREQUENCY DOMAIN



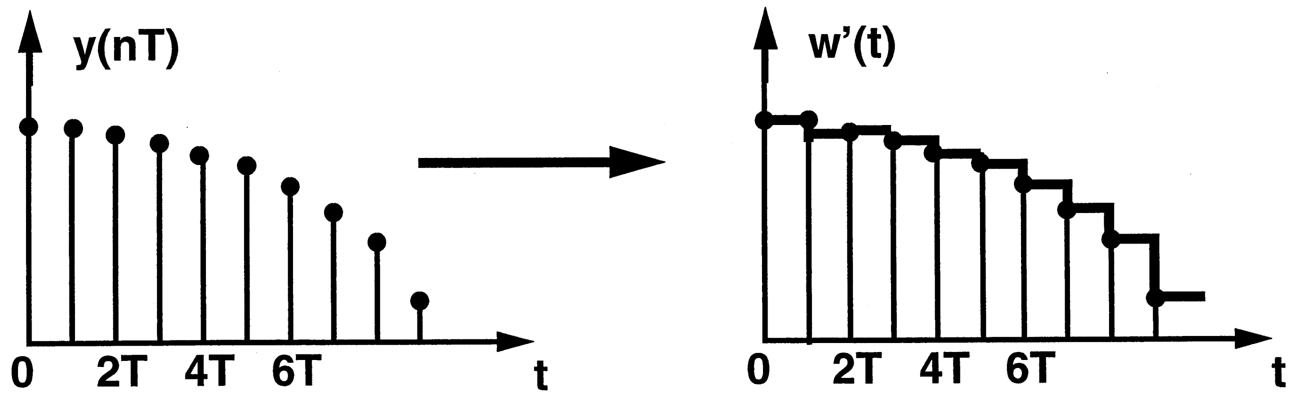


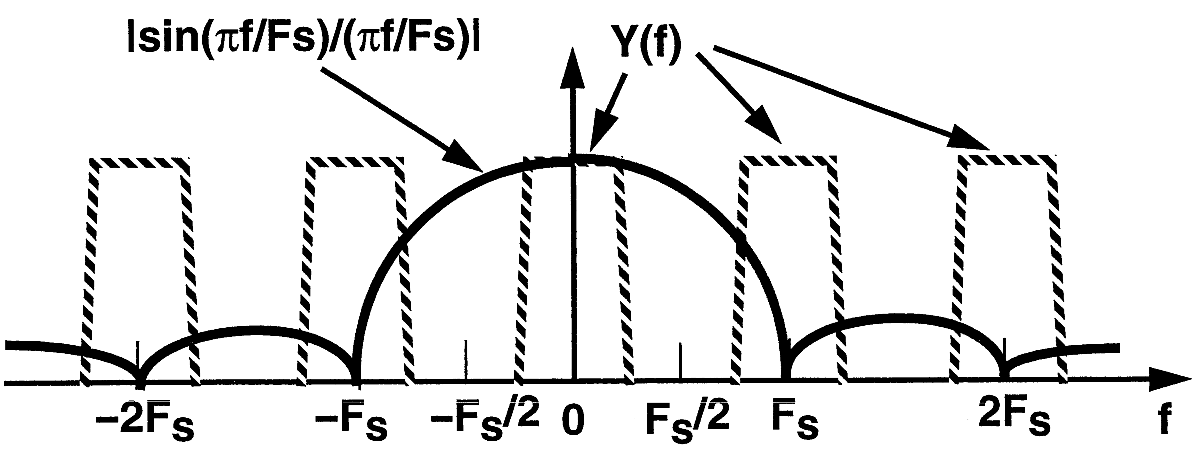


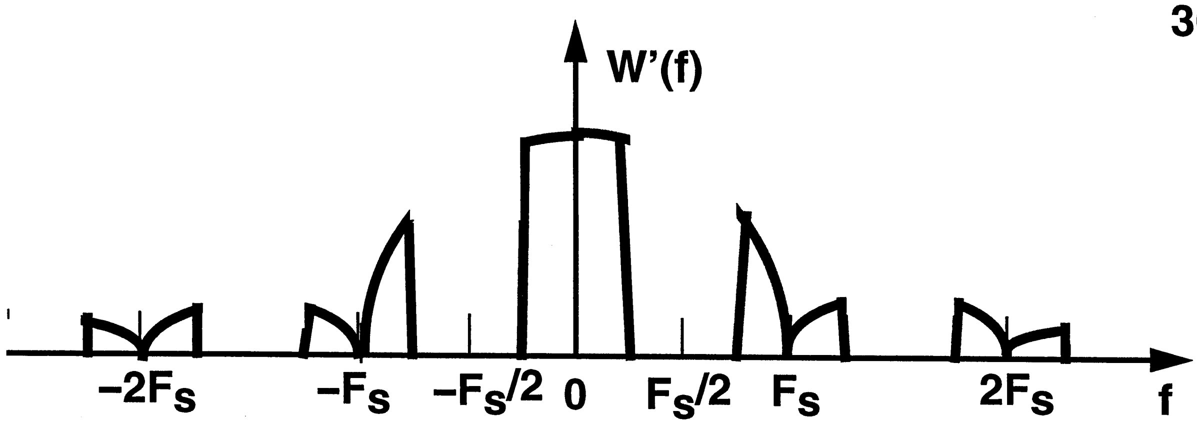
Step III: Converting the discrete-time signal into the analog form:

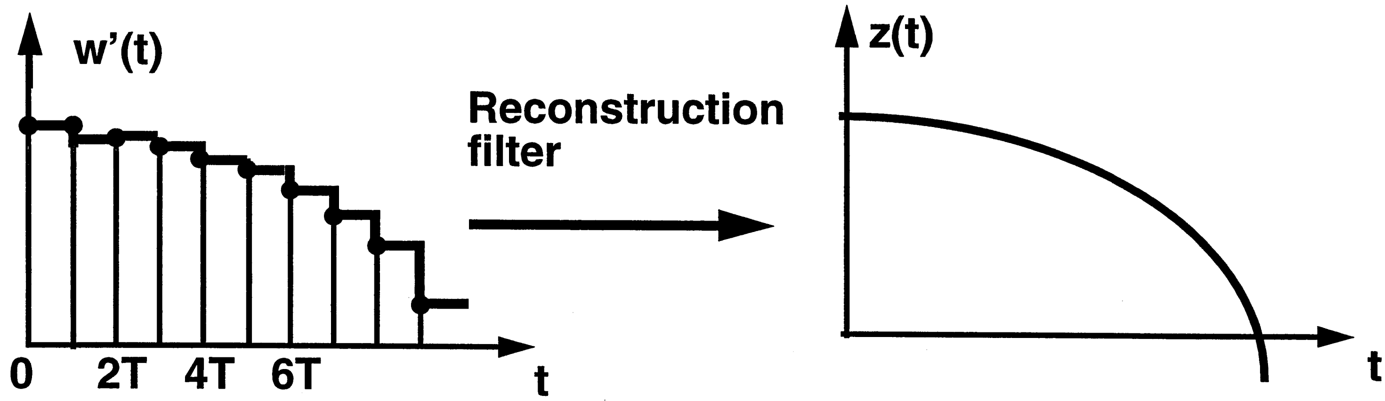


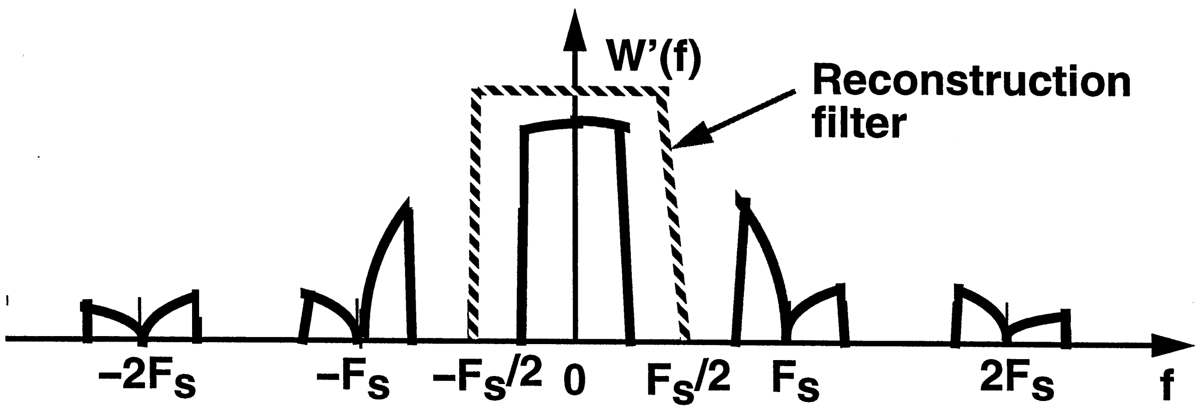
Zero-order hold

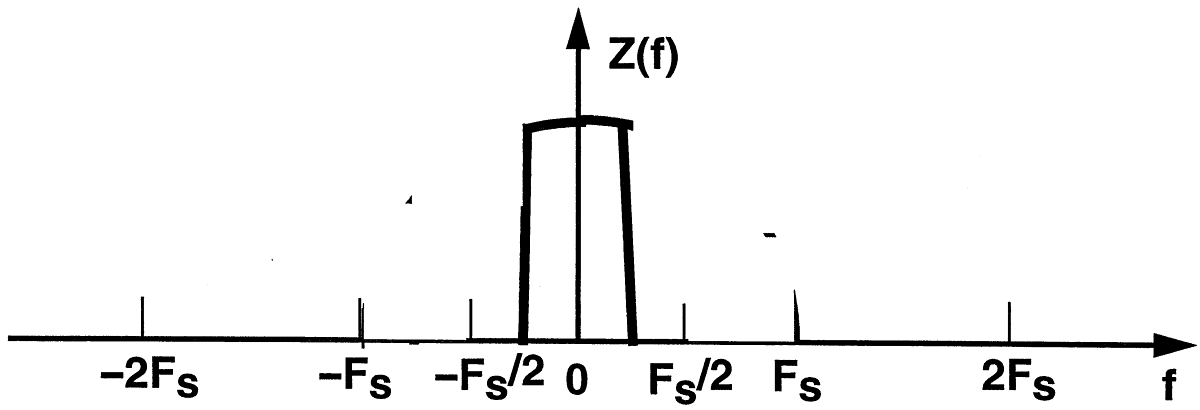





W'(f)
−2Fs
−Fs
−Fs/2
0
Fs/2
Fs
2Fs
f


w'(t)
0
2T
4T
6T
t
Reconstruction filter
z(t)
t


W'(f)
Reconstruction filter
−2Fs
−Fs
−Fs/2
0
Fs/2
Fs
2Fs
f


Z(f)
−2Fs
−Fs
−Fs/2
0
Fs/2
Fs
2Fs
f

Nowadays, high accuracy A/D and D/A conveters are implemented using oversampling (sampling frequency is much higher than F_S) and sigma-delta modulation.

In these cases, the requirements for the analog antialiasing and reconstruction filters become much milder.

Appendix C gives a more thorough discussion on what is happening when processing a continuous-time signal with the aid of a discrete-time system.

SOME ADVANTAGES OF DIGITAL SIGNAL PROCESSING

- Signals are easy to store, transmit, and compress in digital form.
- The performance of a digital signal processing algorithms remain the same.
- The accuracy of digital filters is much higher than that of their analog counterparts.
- Very complicated algorithms are easily implementable using signal processors and VLSI circuits.
- Using programmable processors or VLSI circuits it is fast to test the performances of various algorithms.
- Algorithms can be implemented using milions of similar very small silicon area VLSI circuits.
- There exist several algorithms which cannot be realized using analog filters such as fast Fourier transform, filters for image processing, adaptive filters, and filters having exactly linear phase (there is no phase distortion).

- Using multiplexing, the same filter can be used for processing several signals at the same time.

BASIC DISADVANTAGE

- There is the upper limit for the achievable sampling frequency, nowadays around 100 MHz. The high frequency parts of the TV and radio sets must still be implemented using analog signal processing

DIRCRETE-TIME SIGNALS: SEQUENCES

- The signals processed by digital filters or more generally by digital signal processing algorithms are sequence of numbers. A sequence of numbers x, in which the nth number of the sequence is denoted by $x[n]$, is formally written as

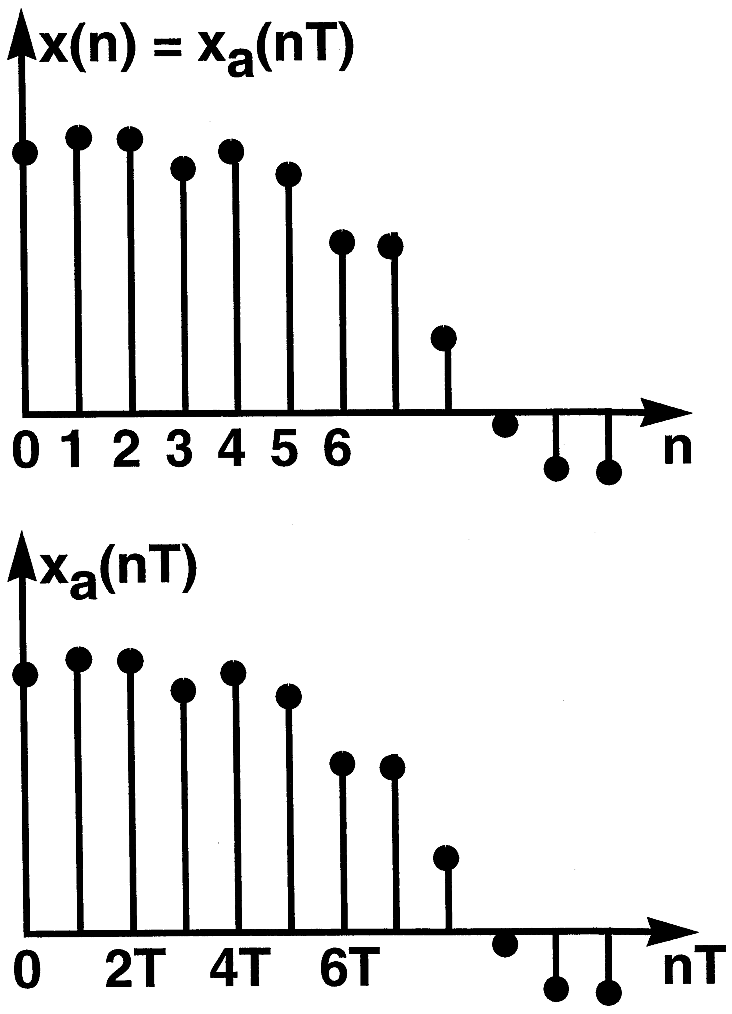
$$x = \{x[n]\}, \qquad -\infty < n < \infty.$$

- For simplicity, we will denote usually the entire sequence by $x[n]$, instead of $\{x[n]\}$.

- In some cases, these sequences are directly in the digital form, whereas in many cases, the sequnces are generated by sampling a continuous-time signal $x_a(t)$ with sampling interval of T, yielding

$$x[n] = x_a(nT).$$

- The two basic representation forms for a sequence obtained by sampling a continuous-time signal are shown below:



Sampling interval T is included

A TYPICAL EXAMPLE: SAMPLING A SPEECH SIGNAL

- The following figure shows a continuous-time speech signal of length 32 ms [figure (a)] and the corresponding discrete-time signal obtained when using the sampling interval of $T = 125\ \mu$s [figure (b)].

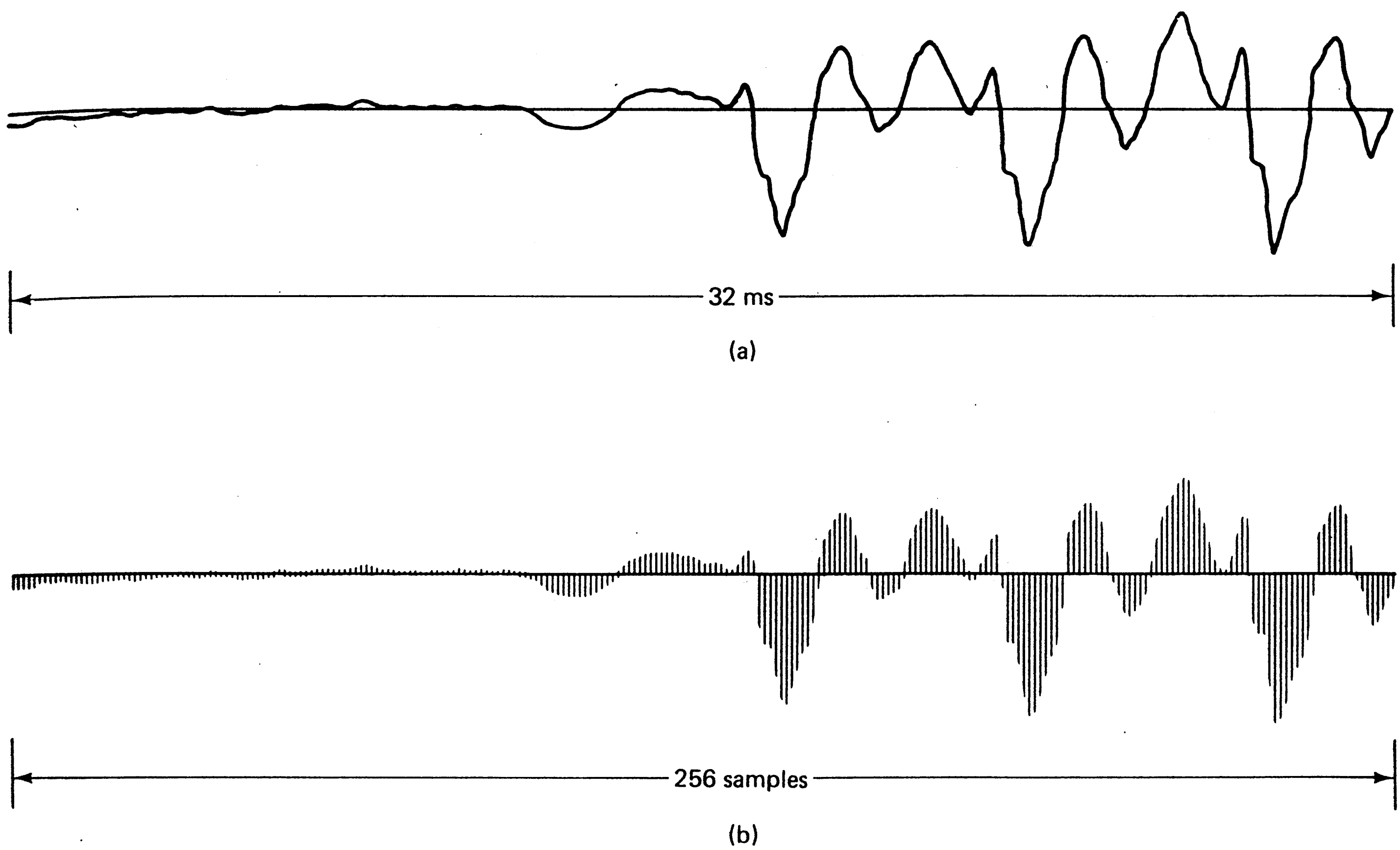


Figure 2.2 (a) Segment of a continuous-time speech signal. (b) Sequence of samples obtained from part (a) with $T = 125\ \mu$s.

CLASSIFICATION OF DICSRETE-TIME SIGNALS

- There exist several ways of classifying discrete-time signals. One alternative is to categorize them into:

- **Random signals:** In this case, the signal samples have no correlation with each other.

- **Transient signals:** In this case, the signal has some pulse shape and is usually of finite duration.

- **Periodic signals:** In this case, the signal satisfies $x[n \pm N] = x[n] \; \forall n$, where N is an integer and is called a period.

- The following transparency shows typical waveforms for the above three signal types.

CLASSIFICATION OF DICSRETE-TIME SIGNALS INTO RANDOM, TRANSIENT, AND PERIODIC SIGNALS

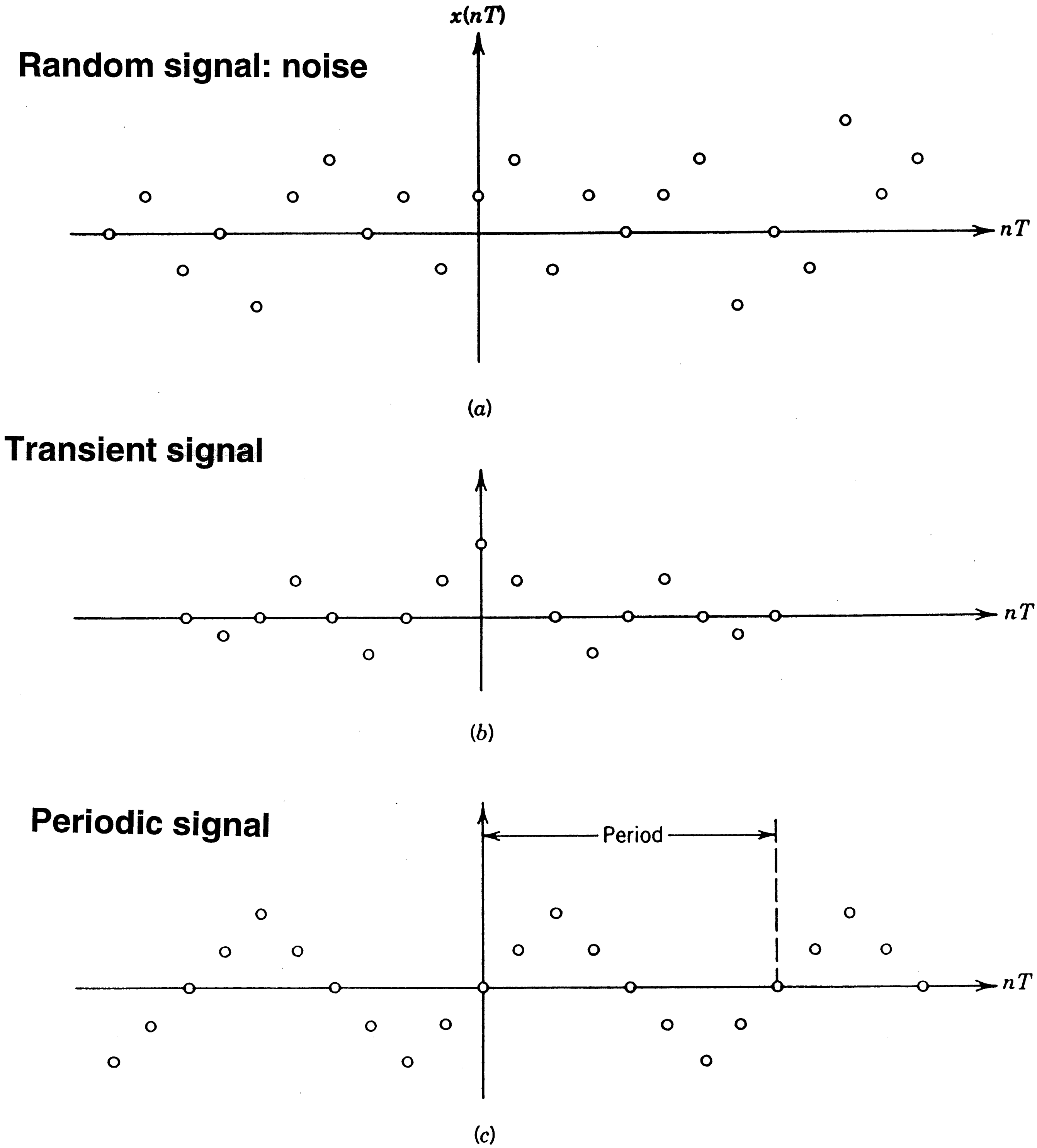


Figure 2.3 Typical signal representations: (*a*) random signal; (*b*) transient signal; (*c*) periodic signal.

DISCRETE-TIME SIGNALS OF PARTICULAR INTEREST

- Like for continuous-time signals, there are some basic discrete-time signals of particular interest.

- We start by giving the mathematical definitions. The actual waveforms are then given on the later transparencies.

- Unit sample or impulse:

$$x[n] = \delta[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0. \end{cases}$$

- Unit step:

$$x[n] = u[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise}. \end{cases}$$

- Unit ramp:

$$x[n] = nu[n].$$

  Here, the use of the unit step indicates that $x[n] = 0$ for $n < 0$.

- Exponential signal:

$$x[n] = A\alpha^n.$$

- Sinusoidal signal:

$$x[n] = r\sin(n\omega_0 + \phi).$$

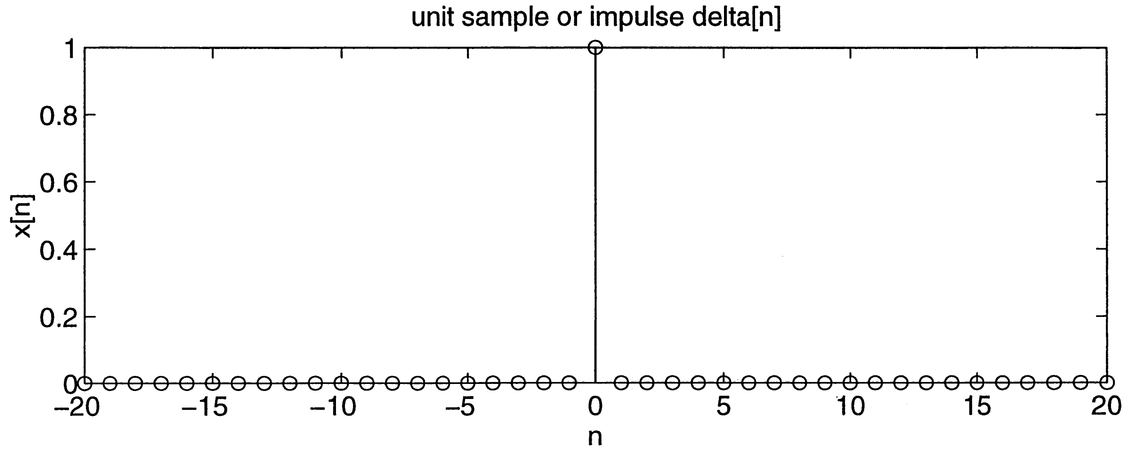
- Damped sinusoidal signal:

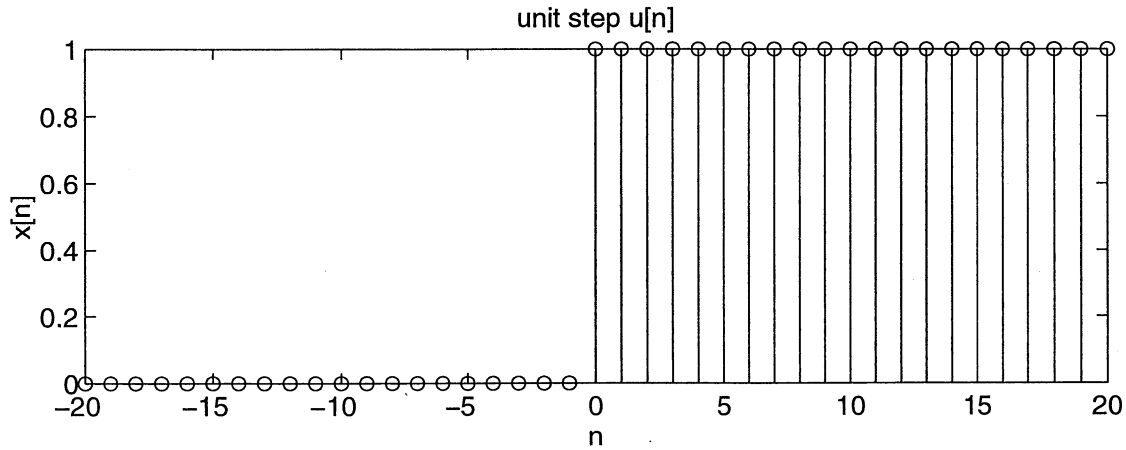
$$x[n] = r^n \sin(n\omega_0 + \phi)u[n], \quad r < 1.$$

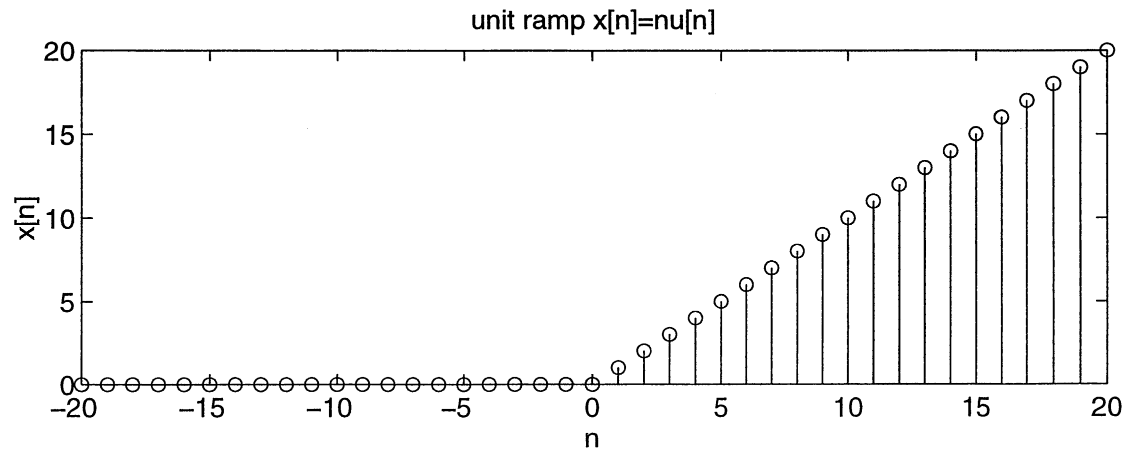
- Expanding sinusoidal signal:

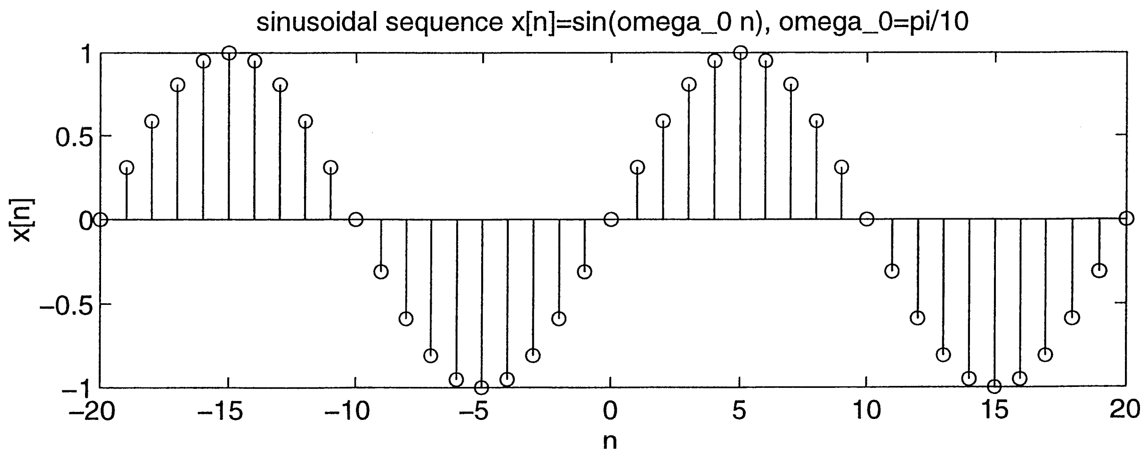
$$x[n] = r^n \sin(n\omega_0 + \phi)u[n], \quad r > 1.$$

WAVEFORMS FOR SOME ELEMENTARY DISCRETE-TIME SIGNALS

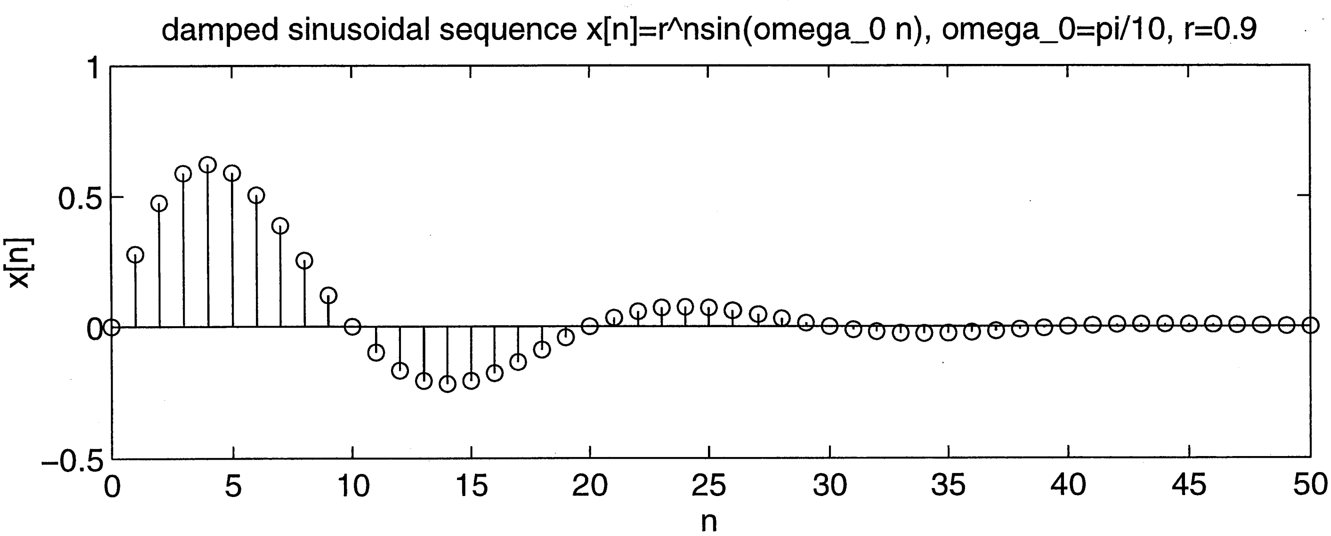


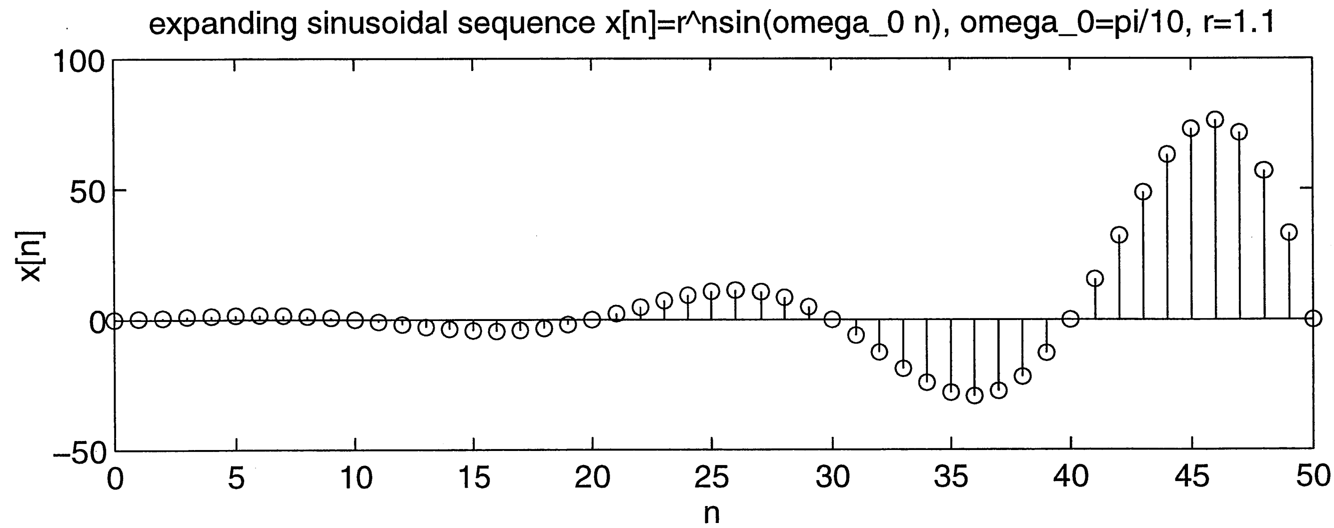




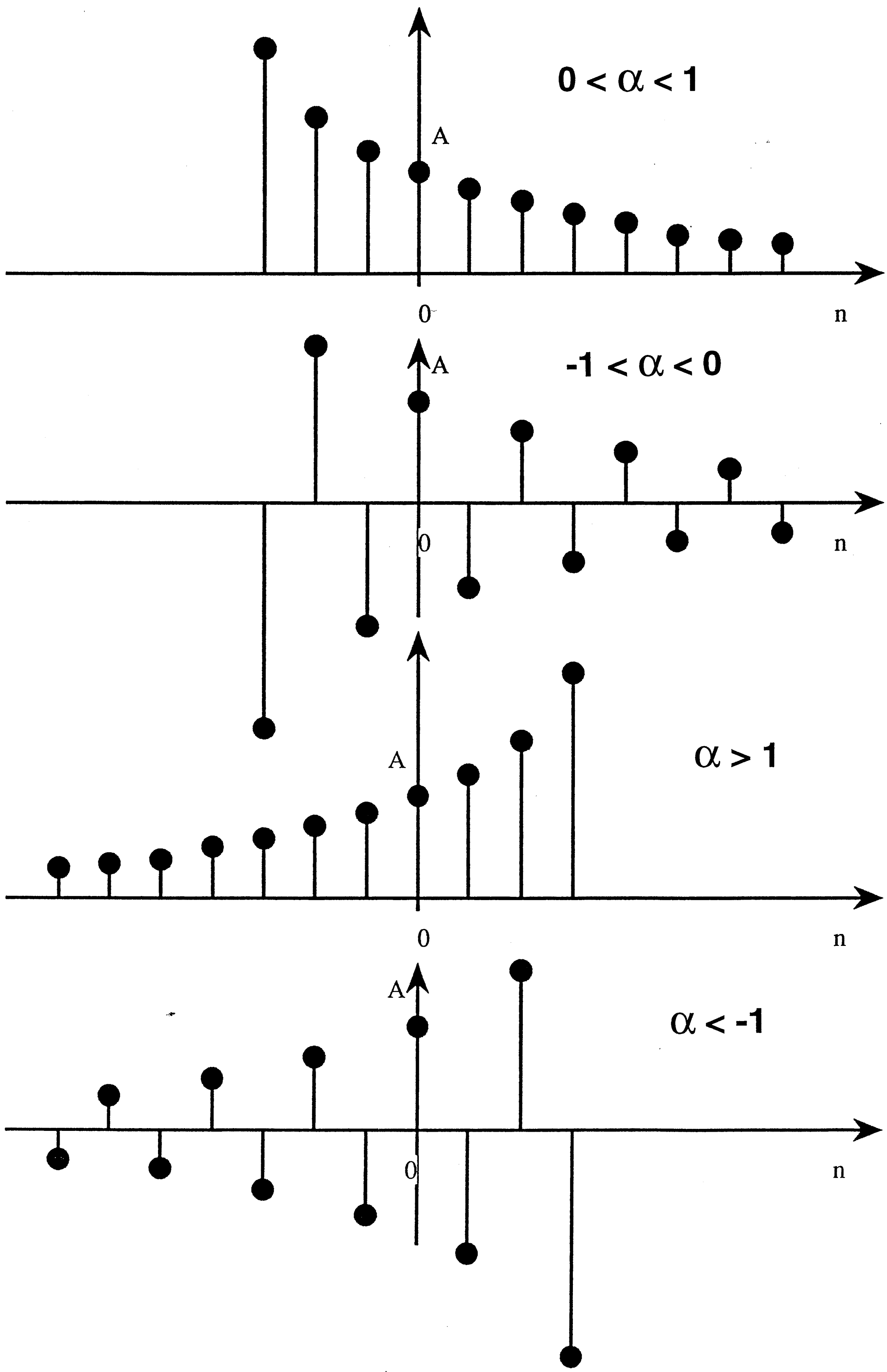


WAVEFORMS FOR SOME ELEMENTARY DISCRETE-TIME SIGNALS

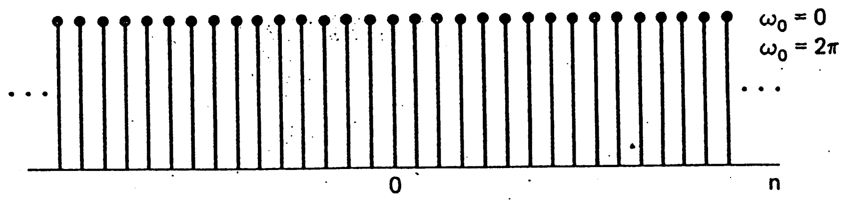




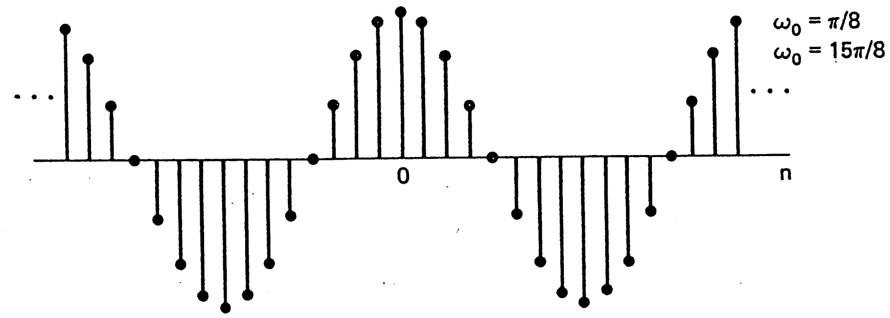
WAVEFORMS FOR EXPONENTIAL DISCRETE-TIME SIGNALS $x[n] = A\alpha^n$.



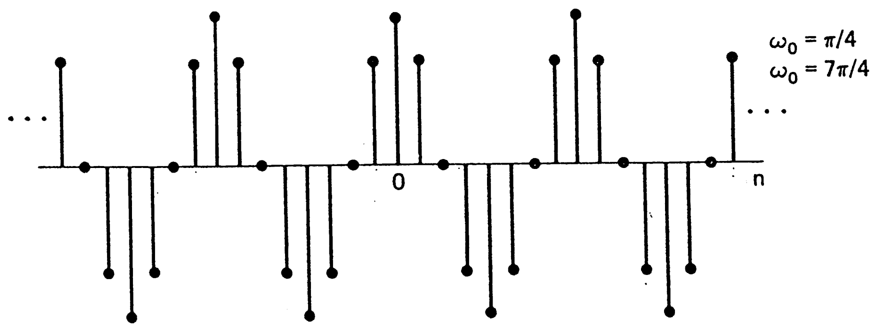
WAVEFORMS FOR SIGNALS $x[n] = \cos(n\omega_0)$ FOR VARIOUS VALUES OF ω_0



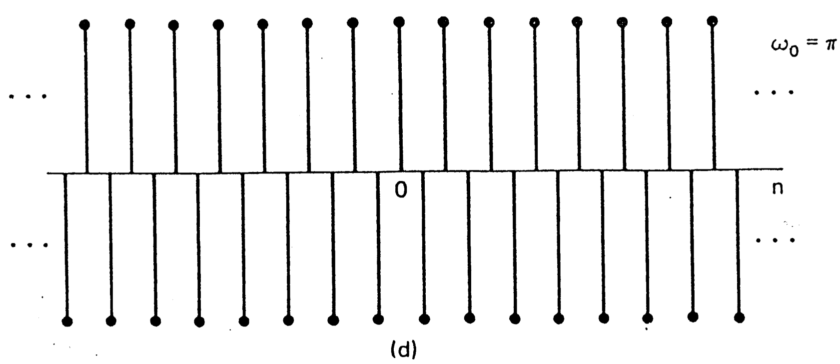
(a)



(b)



(c)

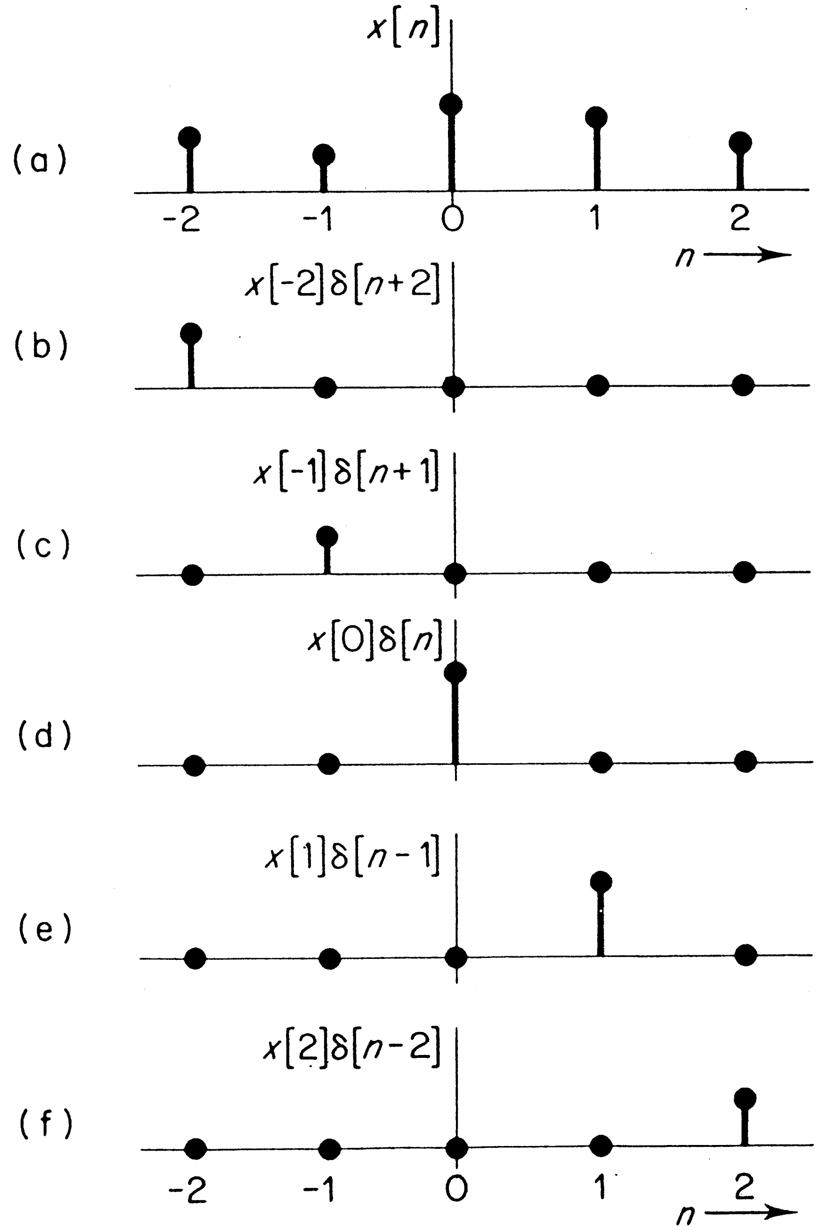


(d)

USEFUL REPRESENTATION FORM FOR AN ARBITRARY SIGNAL

- Later on we will utilize the fact that any sequence is expressible as a sum a weighted shifted unit samples as follows (see the figure shown below)

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$



DIGITAL SIGNALS AND DISCRETE-TIME SIGNALS

- In the above, we considered signals which where discrete in time, that is, discrete-time signals.
- However, in practice, when processed by digital signal processing algorithms, the signals must be quantized to binary numbers.
- These signals with sample values also discrete are called **digital** signals.

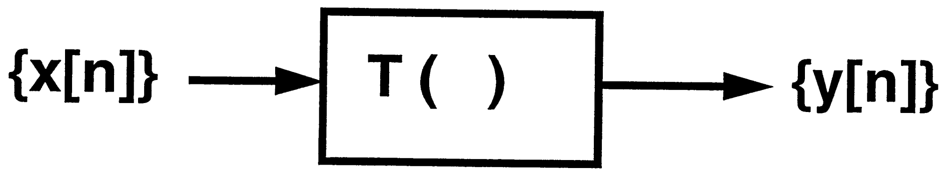
DISCRETE-TIME SYSTEMS

- A discrete-time system is defined mathematically as a transformation or an operator $T(\ )$ mapping uniquely an input sequence (exitation) $\{x[n]\}$ into an output sequence (response) $\{y[n]\}$.

- This can be denoted by

$$\{y[n]\} = T(\{x[n]\})$$

and is indicated pictorically in the figure shown below:



In the following, we use the notation $x[n]$ for the entire sequence even though, strightly speaking, $x[n]$ is the nth sample of the sequence.

Examples:

- Delay: $y[n] = x[n-10]$
- Moving averages:

$$y[n] = \frac{1}{2K+1} \sum_{k=n}^{2K} x[n-k]$$

$$y[n] = \frac{1}{2K+1} \sum_{k=n-K}^{K} x[n-k]$$

VARIOUS KINDS OF DISCRETE-TIME SYSTEMS

MEMORYLESS SYSTEMS

- The output $y[n]$ at every value of n depends only on the input $x[n]$ at the same value of n.
- Systems with $y[n] = e^{x[n]}$ and $y[n] = x[n]^2$ are memoryless, whereas a system with $y[n] = x[n-10]$ is not.

LINEAR SYSTEMS

- A system is defined to be **linear** if and only if

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= aT(x_1[n]) + bT(x_2[n]) \\ &= ay_1[n] + by_2[n], \end{aligned}$$

where $y_1[n]$ and $y_2[n]$ are the responses of the system to the exitations $x_1[n]$ and $x_2[n]$, respectively.

- In other words, the principle of superposition is valid for linear systems.

- A linear system is thus characterized by the following properties:

- **Additivity property:**

$$T(x_1[n] + x_2[n]) = T(x_1[n]) + T(x_2[n]) = y_1[n] + y_2[n].$$

- **Scaling property:**

$$T(ax[n]) = aT(x[n]) = ay[n].$$

Example 1: $y[n] = T(x[n]) = \frac{1}{2K+1} \sum_{k=n-K}^{n+K} x[k]$

$$T(ax_1[n] + bx_2[n]) = \frac{1}{2K+1} \sum_{k=n-K}^{n+K} (ax_1[k] + bx_2[k])$$

$$= a\frac{1}{2K+1} \sum_{k=n-m}^{n+m} x_1[k] + b\frac{1}{2K+1} \sum_{k=n-m}^{n+m} x_2[k]$$
$$= aT(x_1[n]) + bT(x_2[n])$$

$\Rightarrow$ The system is linear.

Example 2: $T(x[n]) = e^{x[n]}$

$$T(ax_1[n] + bx_2[n]) = e^{(ax_1[n]+bx_2[n])} = e^{ax_1[n]} \cdot e^{bx_2[n]}$$
$$= \left(e^{x_1[n]}\right)^a \cdot \left(e^{x_2[n]}\right)^b = (T(x_1[n]))^a \cdot (T(x_2[n]))^b$$
$$\neq aT(x_1[n]) + bT(x_2[n])$$

$\Rightarrow$ The system is not linear.

TIME-INVARIANT (SHIFT-INVARIANT) SYSTEMS

- A **time-invariant** system is characterized by the property that if $y[n]$ is the response to $x[n]$, then $y[n-k]$ is the response to $x[n-k]$, where k is a positive or negative integer. In other words, $T(x[n]) = y[n] \Rightarrow T(x[n-k]) = y[n-k]$.

- For this system, a time shift or a delay of the exitation causes a corresponding shift in the response.

Example 1: $y[n] = T(x[n]) = \frac{1}{2K+1} \sum_{k=n-K}^{n+K} x[k]$

$$T(x[n-r]) = \frac{1}{2K+1} \sum_{k=n-m}^{n+m} x[k-r]$$

$$\overset{l=k-r}{=} \frac{1}{2K+1} \sum_{l=(n-r)-m}^{(n-r)+m} x(l) = y[n-r]$$

$\Rightarrow$ The system is time-invariant.

Example 2: $y[n] = T(x[n]) = nx[n]$

$$T(x[n-k]) = nx[n-k] \neq y[n-k] = [n-k]x[n-k]$$

$\Rightarrow$ The system is not time-invariant.

CAUSALITY

- A system is **causal** if the output $y[n]$ at any $n = n_0$ depends on the input $x[n]$ only for $n \leq n_0$. This implies that if $x_1[n] = x_2[n]$ for $n \leq n_0$, then $y_1[n] = y_2[n]$ for $n \leq n_0$. For a causal system, only the input samples occuring at $n = n_0$ or before this time instant have an effect on the response. This system is nonanticipative.

Example 1: $y[n] = T(x[n]) = \frac{1}{2K+1} \sum_{k=n-K}^{n+K} x[k]$

- $y[n_0]$ depends also on the input samples occuring at $n = n_0 + k$ for $k = 1, 2, \cdots, K$, that is, samples occuring after the instant $n = n_0$.

$\Rightarrow$ The system is anticipative and not causal.

Example 2: $y[n] = T(x[n]) = \frac{1}{2K+1} \sum_{k=n-2K}^{n} x[k]$

- $y[n_0]$ depends on $x[n_0 - 2K]$, $x[n_0 - 2K + 1]$, $\cdots$, $x[n_0]$, that is, on input samples occuring at $n = n_0$ and before this instant.

$\Rightarrow$ The system is causal.

STABILITY

- A system is **stable**, if every bounded input sequence ($|x[n]| \leq B_x < \infty \; \forall n \in \mathbf{Z}$) produces a bounded output ($|y[n]| \leq B_y < \infty \; \forall n \in \mathbf{Z}$). Here, B_x ja B_y are fixed positive finite values.

Example 1: $y[n] = T(x[n]) = \frac{1}{2K+1} \sum_{k=n-K}^{n+K} x[k]$

- Since the output at any time instant is the average of $2K+1$ bounded input samples, it is bounded, that is, the system is obviously stable.

Example 2: $y[n] = T(x[n]) = nx[n]$

- As a bounded input sequence with $B_x = 1$, we use the unit step as given by

$$x[n] = u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0. \end{cases}$$

- The output sequence is then

$$y[n] = \begin{cases} 0, & n < 0 \\ n, & n \geq 0. \end{cases}$$

- $y[n] = n \to \infty$ when $n \to \infty$.

$\Rightarrow$ The system is not stable.

LINEAR TIME-INVARIANT (LTI) SYSTEMS

- This course concentrates on systems which are both linear and time-invariant (later on often abbreviated as LTI).

- According to the discussion on transparency 45, any exitation can be written as a sum of of a weighted shifted unit samples as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k],$$

where

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases}$$

- If the system is linear, then the response to this exitation is expressible as

$$y[n] = T(\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]) = \sum_{k=-\infty}^{\infty} x[k]T(\delta[n-k]).$$

- If the system is both linear and time-invariant, then

$$T(\delta[n-k]) = h[n-k],$$

where $h[n]$ is the response to $\delta[n]$.

- In this case, the response to the exitation given by

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

can be written in the following simple form:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

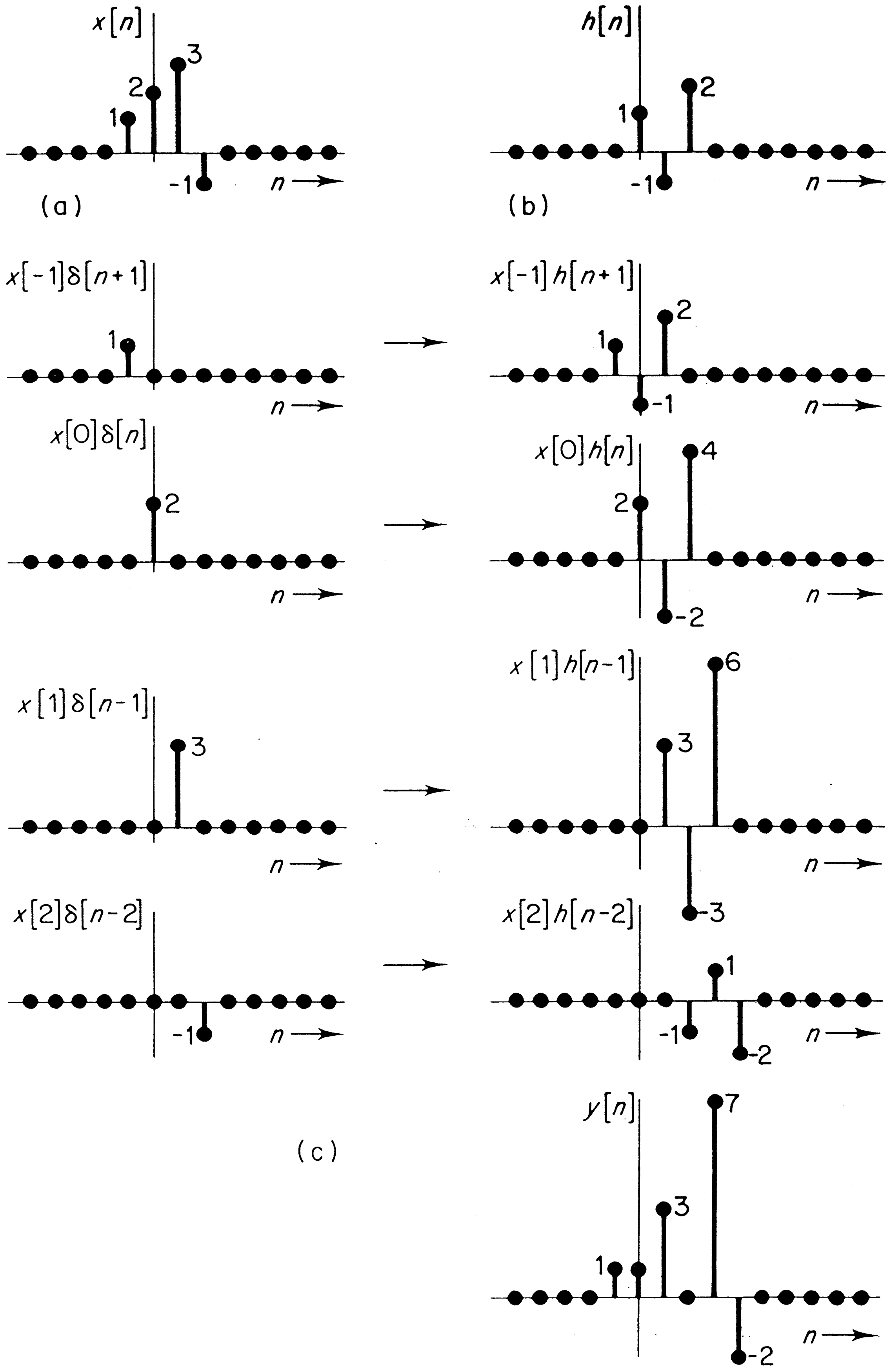
- The above equation is commonly called the **convolution sum** and written as

$$y[n] = x[n] * y[n].$$

- Our equation shows that the response for an arbitrary input sequence can be expressed as a weighted shifted sum of the impulse responses with the weights being the samples of the input sequence.

- This gives us a rather simple technique for evaluating the output sequence for any input sequence. This is illustrated on the following transparency.

EVALUATION OF THE OUTPUT USING THE CONVOLUTION SUM



PROPERTIES OF LTI SYSTEMS

1. $x[n] * h[n] = h[n] * x[n]$

2. $x[n] * (h_1[n] + h_2[n]) = h_1[n] * x[n] + h_2[n] * x[n]$

3. If we cascade two systems with impulse responses $h_1[n]$ ja $h_2[n]$, then the impulse response of the resulting system is given by $h_1[n] * h_2[n]$.

4. For the corresponding parallel connection, the impulse response is given by $h_1[n] + h_2[n]$.

5. The system is causal $\iff h[n] = 0, n < 0$.

6. The system is stable $\iff S = \sum_{k=-\infty}^{\infty} |h[k]| < \infty$.

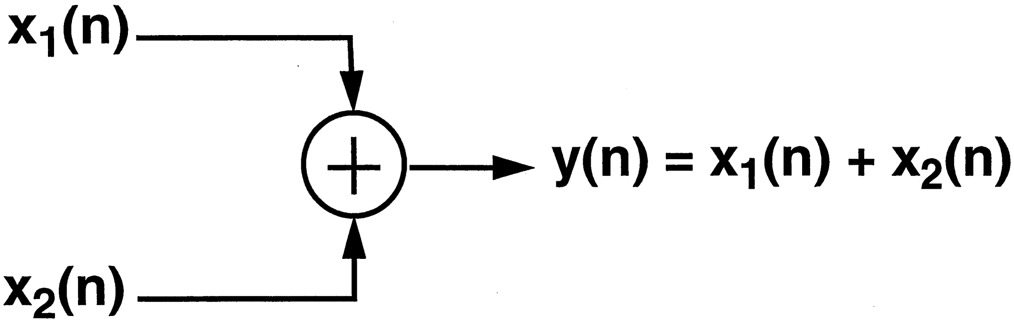
Building blocks for causal LTI sytems , that is for digital filters

— Adder

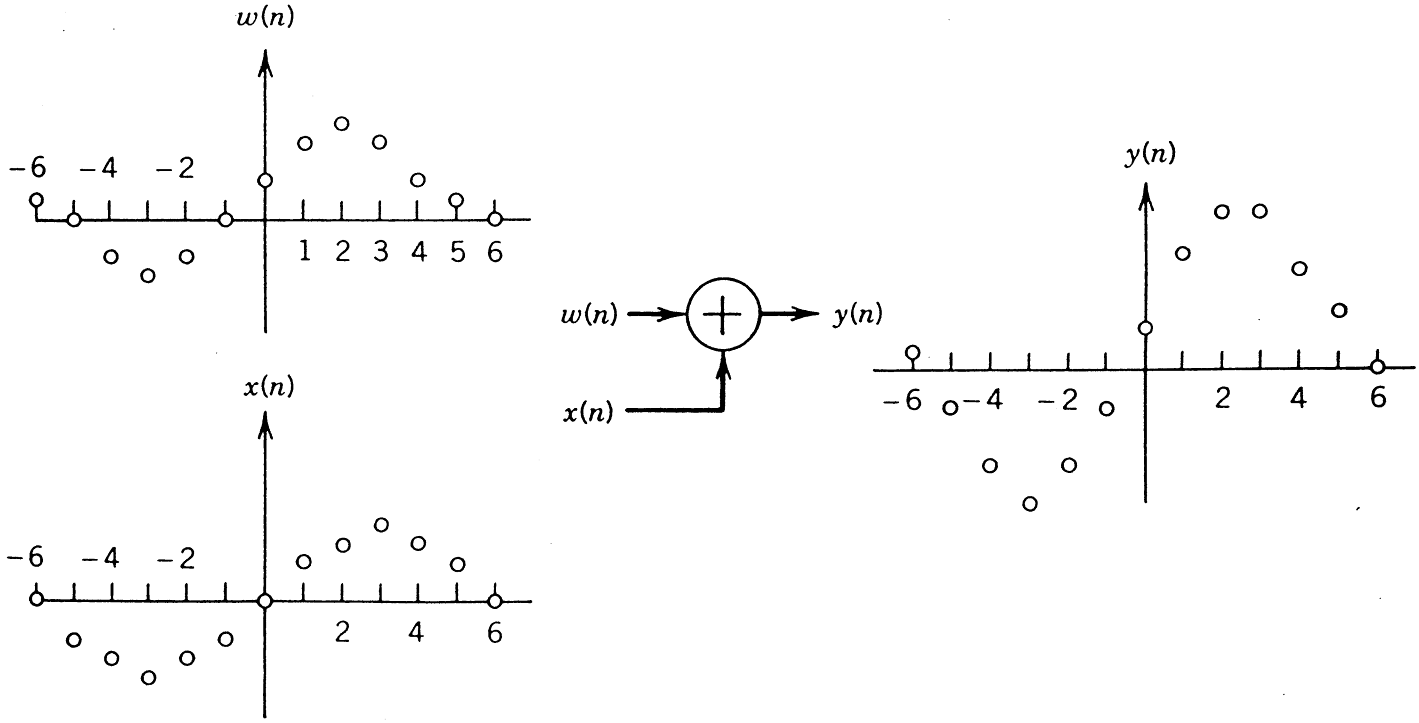
— Delay

— Multiplier

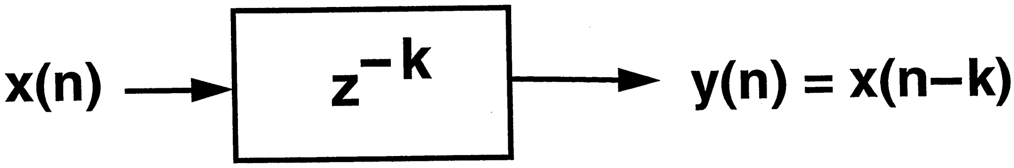
Adder



Examples:



Delay:



Examples:

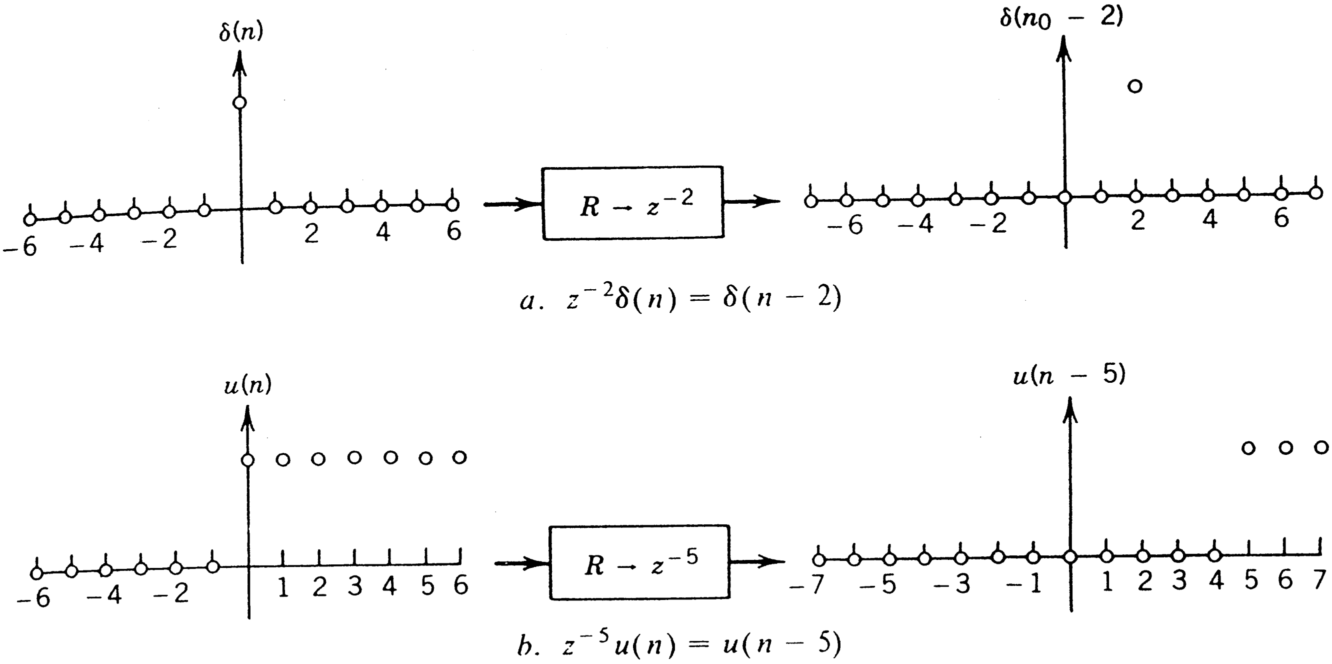
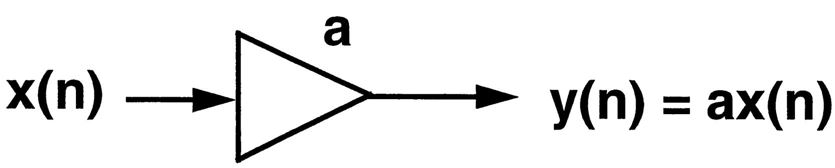
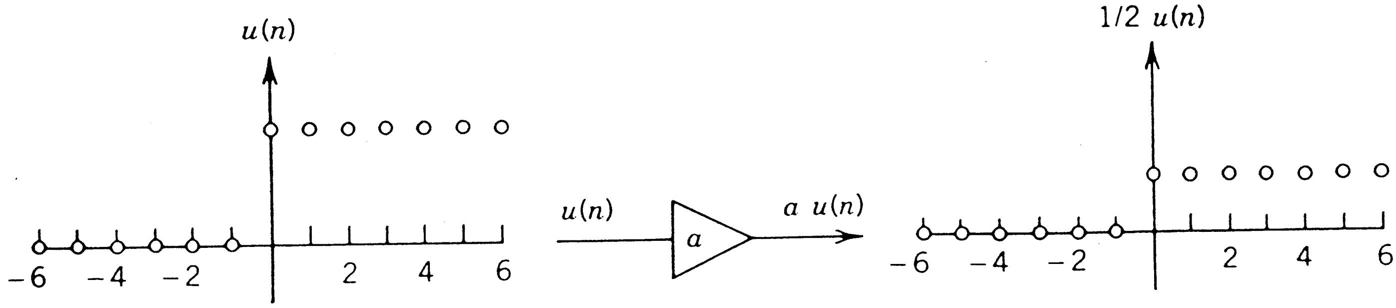


Figure 2.6 Shift operation.

Multiplier:



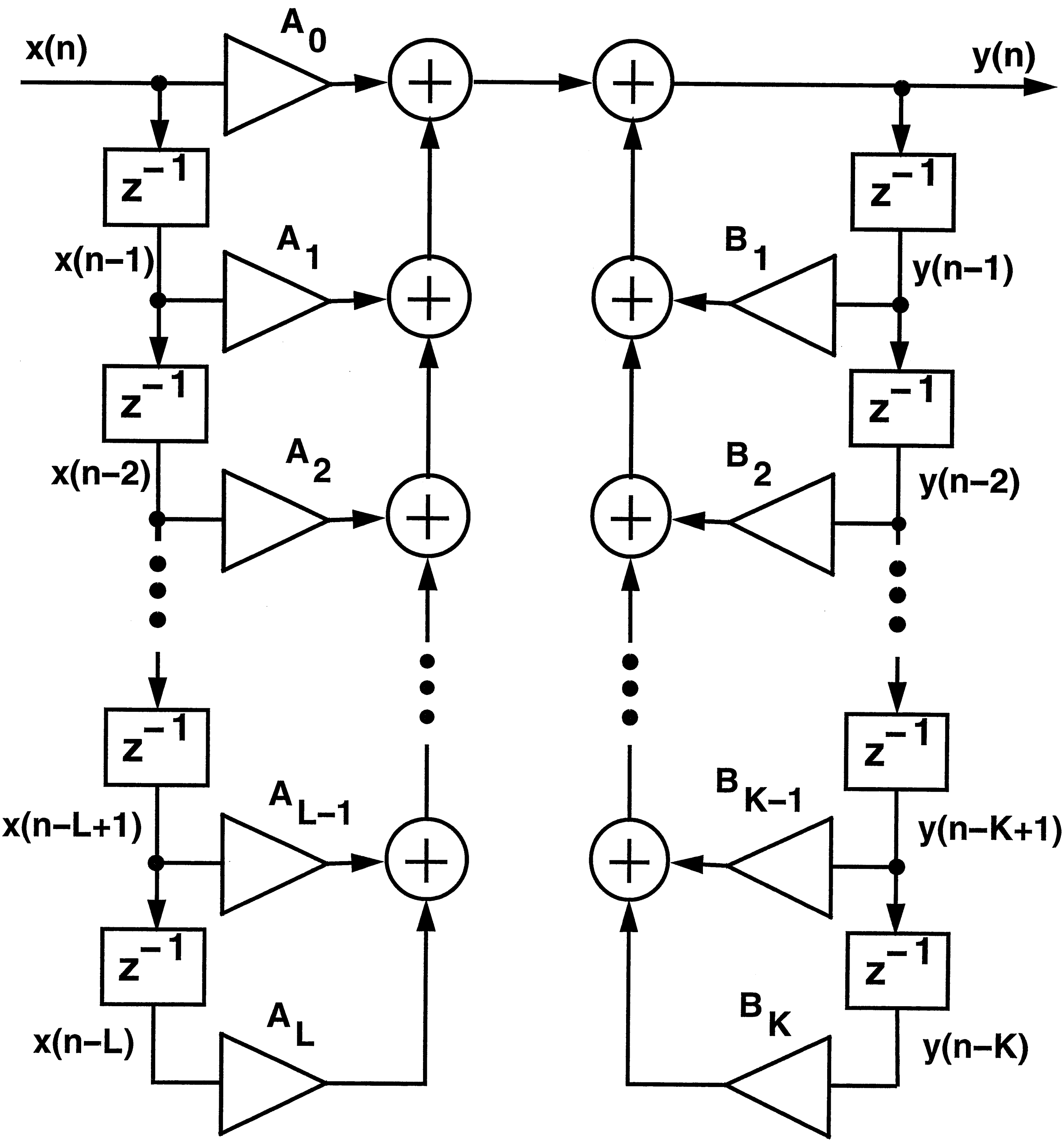
Examples:



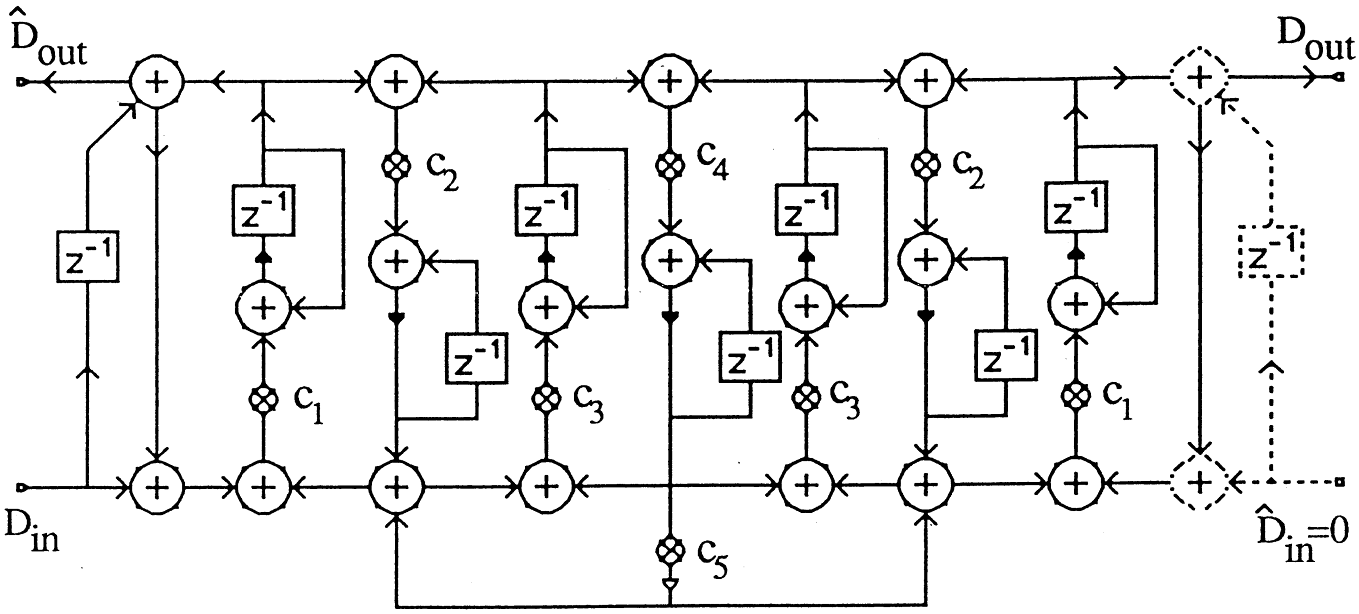
a. Scalar multiplication $a = \frac{1}{2}$

General digital filter having the following input-output relation (difference equation)

$$y(n) = \sum_{l=0}^{L} A_l \, x(n-l) + \sum_{k=1}^{K} B_k \, y(n-k).$$



The same difference equation can be implemented in several ways. An exotic structure:



Factors affecting on selecting a proper structure:

- **The effects of finite wordlength (noise, oscillations, coefficient sensitivity)**
- **Easy realizability (signal processors, VLSI circuits)**

DO WE ALREADY HAVE TOOLS FOR ANALYZING AND SYNTHESIZING LTI SYSTEMS?

- Let us try to find solutions to the following example:
- Consider a causal system characterized by the difference equation $y[n] = ay[n-1] + x[n]$.

(a) What is the unit sample response of our system?

(b) For which values of a the system is stable?

(c) What is the response to the exitation given by

$$x[n] = u[n] - u[n-N] = \begin{cases} 1 & , 0 \leq n \leq N-1 \\ 0 & , \text{otherwise.} \end{cases}$$

(a) We start by determining the response to the unit sample (impulse) exitation given by

$$x[n] = \delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases}$$

- Since our system is causal, the output values are zero until the first nonzero-valued sample enters the system.
- This means that the difference equation must be ex-

pressed in the form $y[n] = ay[n-1] + x[n]$, instead of the form $y[n-1] = (1/a)y[n] - (1/a)x[n]$. In other words, before evaluating the output sample at $n = n_0$, we have to know the input sample at $n = n_0$ as well as the ouput sample at $n = n_0 - 1$.

- For the unit sample exitation, we obtain the following response:

$$x[n] = \delta[n] = 0, n < 0 \;\Rightarrow\; y[n] = h[n] = 0, n < 0$$

$$h[0] = ah[-1] + \delta[0] = 1$$

$$h[1] = ah[0] + \delta[1] = a$$

$$h[2] = ah[1] + \delta[2] = a^2$$

$$\ldots$$

$$h[n] = ah[n-1] = a^2 h[n-2] = \cdots = a^n h[0] = a^n$$

$$\Rightarrow\; h[n] = a^n u[n],$$

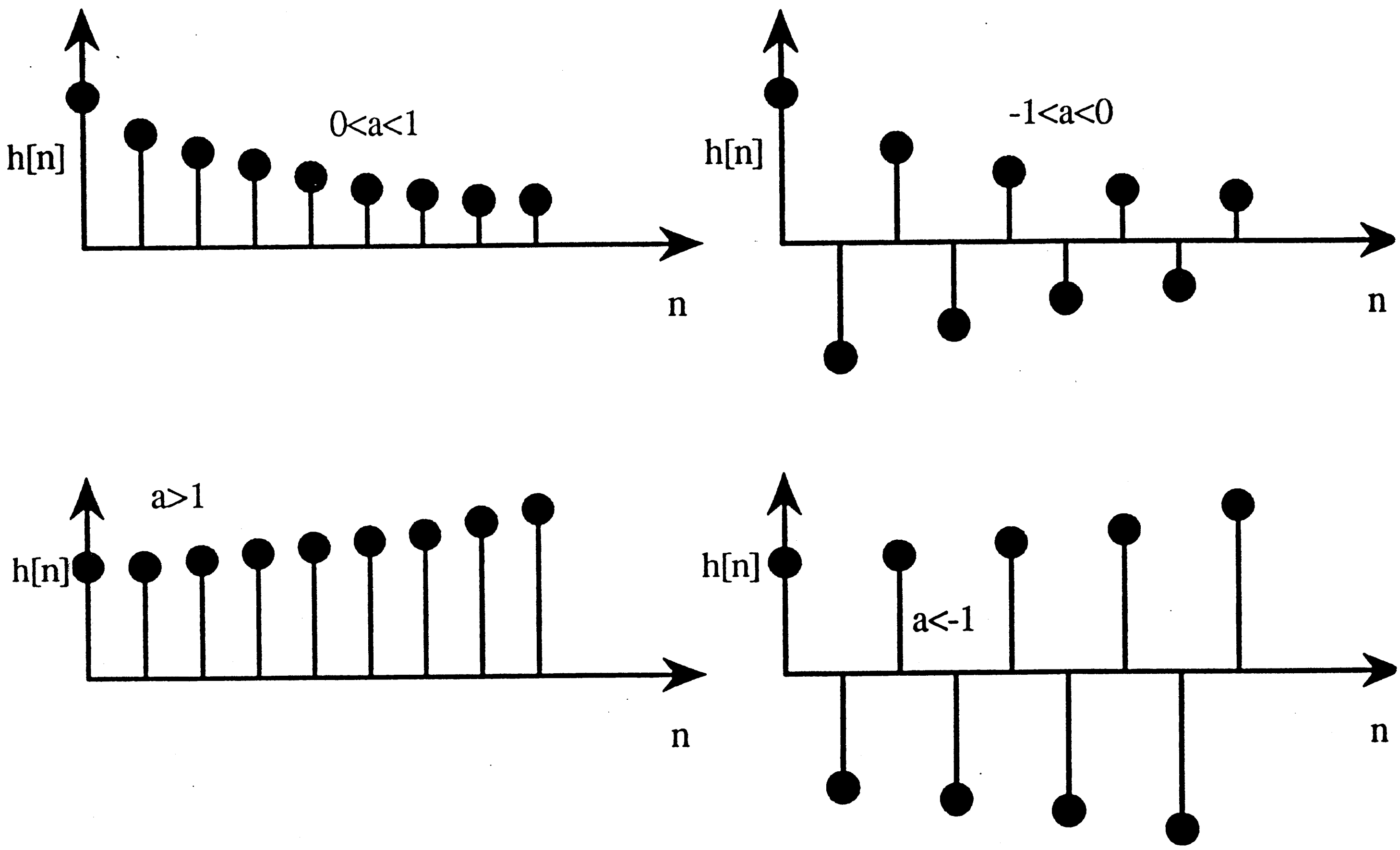
where

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

is the unit step.

WE SUCCEEDED!!

(b) The following figure shows the unit sample responses for various values of a.



- When $0 < a < 1$ or $-1 < a < 0$, the unit sample responses $h[n]$ approaches the value of zero for $n \to \infty$.

- In these cases,

$$S = \sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=0}^{\infty} |a|^k = 1/(1-|a|) < \infty,$$

so that the system is stable (the series $1 + |a|^1 + |a|^2 + |a|^3 + \cdots$ is geometric and converges for $|a| < 1$.)

- When $a > 1$, $h[n]$ approaches ∞ for $n \to \infty$. When $a < -1$, $h[n]$ approaches $+\infty$ and $-\infty$ for even and odd values of n, respectively, for $n \to \infty$.

- In these case, the system is clearly unstable.

- In the $a = 1$ case, $h[n] = 1 \; \forall \; n \geq 0$, whereas in the $a = -1$ case, $h[n] = 1$ and $h[n] = -1$ for even and odd values of n, respectively.

- In these two special cases, the system is on the border-line of being stable or unstable. However, the system must be categorized to be unstable since $h[n]$ does not approach the value of zero for $n \to \infty$.

WE SUCCEEDED AGAIN!!

(c) We evaluate the response $y[n]$ of our system to the exitation $x[n] = u[n] - u[n-N]$ by exploiting the convolution sum:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

- In our case,

$$x[k] = u[n] - u[n-N] = \begin{cases} 1, & 0 \leq k \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$h[n-k] = a^{n-k}u[n-k] = \begin{cases} a^{n-k}, & k \leq n \\ 0, & k > n \end{cases}.$$

$$\Rightarrow \; y[n] = \sum_{k=0}^{N-1} a^{n-k}u[n-k],$$

where

$$u[n-k] = \begin{cases} 1, & k \leq n \\ 0, & k > n. \end{cases}$$

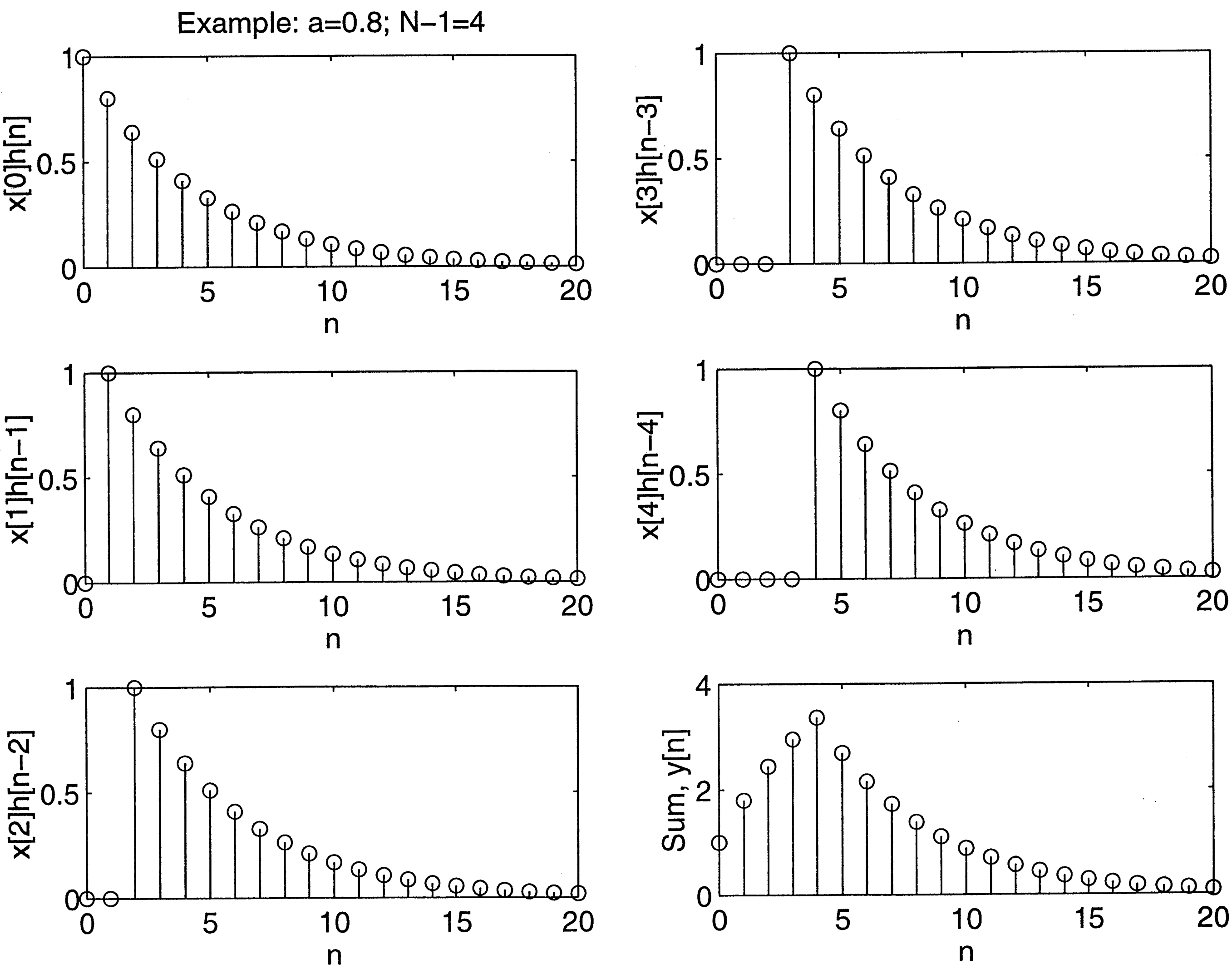
Based on the above equations (see the figure on the next transparency), we obtain

$$y[n] = \begin{cases} 0, & n < 0 \\ \sum_{k=0}^{n} a^{n-k}, & 0 \leq n \leq N-1 \\ \sum_{k=0}^{N-1} a^{n-k}, & n \geq N \end{cases}$$

$$= \begin{cases} 0, & n < 0 \\ a^n \frac{1-a^{-(n+1)}}{1-a^{-1}}, & 0 \leq n \leq N-1 \\ a^n \frac{1-a^{-N}}{1-a^{-1}}, & n \geq N \end{cases}$$

$$= \begin{cases} 0, & n < 0 \\ \frac{1-a^{(n+1)}}{1-a}, & 0 \leq n \leq N-1 \\ a^{n+1} \frac{a^{-N}-1}{1-a}, & n \geq N. \end{cases}$$

WE SUCCEEDED AGAIN!!



COMMENTS

- The above example was very simple and trivial. However, finding the solutions to this example was not very straightforward.

- In addition, it would be benefical to know the frequency-domain behavior of our system as well as to determine easily the system response to any exitation.

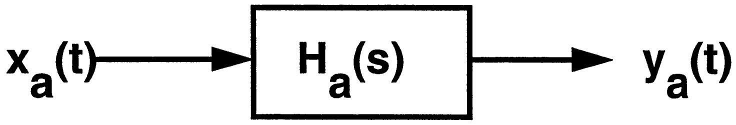
- The stability of the system should be easy to check.

- The solution to the above-mentioned problems is the z-transform which plays the same role for dicrete-time systems as the Laplace-transform does for continuous-time systems.

- Before actually introducing the z-transform, the following two transparencies illustrate the similarities between the z-transform and the Laplace-transform.

Laplace transform

The basic tool for analyaing continuous-time systems is the Laplace transform:



$$Y_a(s)=H_a(s)X_a(s)\,.$$

Frequency response:

$$Y_a(j\Omega) = H_a(j\Omega)X_a(j\Omega)$$

$$Y_a(j2\pi f) = H_a(j2\pi f)X_a(j2\pi f)\,.$$

Response to sinusoidal exitation $x_a(t) = \sin 2\pi f_0 t$:

$$y_a(t) = |H_a(j2\pi f_0 t)| \; \sin\left(2\pi f_0 t + \arg\left(H_a(j2\pi f_0)\right)\right).$$

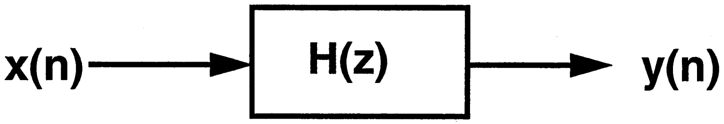
$H_a(s)$ is the transfer function of a continuous-time system telling everything about the performance of the system.

$y_a(t)$ can be determined in the following complicated manner:

$$y_a(t) = \int_{-\infty}^{\infty} x_a(\tau)h_a(t-\tau)d\tau\ .$$

z- transform

The basic tool for analyzing discrete-time systems is the z-transform:



$Y(z)=H(z)X(z)$.

Frequency response: $Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$.

Response to the sinusoidal exitation $x(n)=\sin\omega_0 n$:

$$y(n) = |H(e^{j\omega_0})| \sin\left(\omega_0 n + \arg(H(e^{j\omega_0}))\right).$$

H(z) is the z-transform of the impulse response h(n), that is, the transfer function of the system

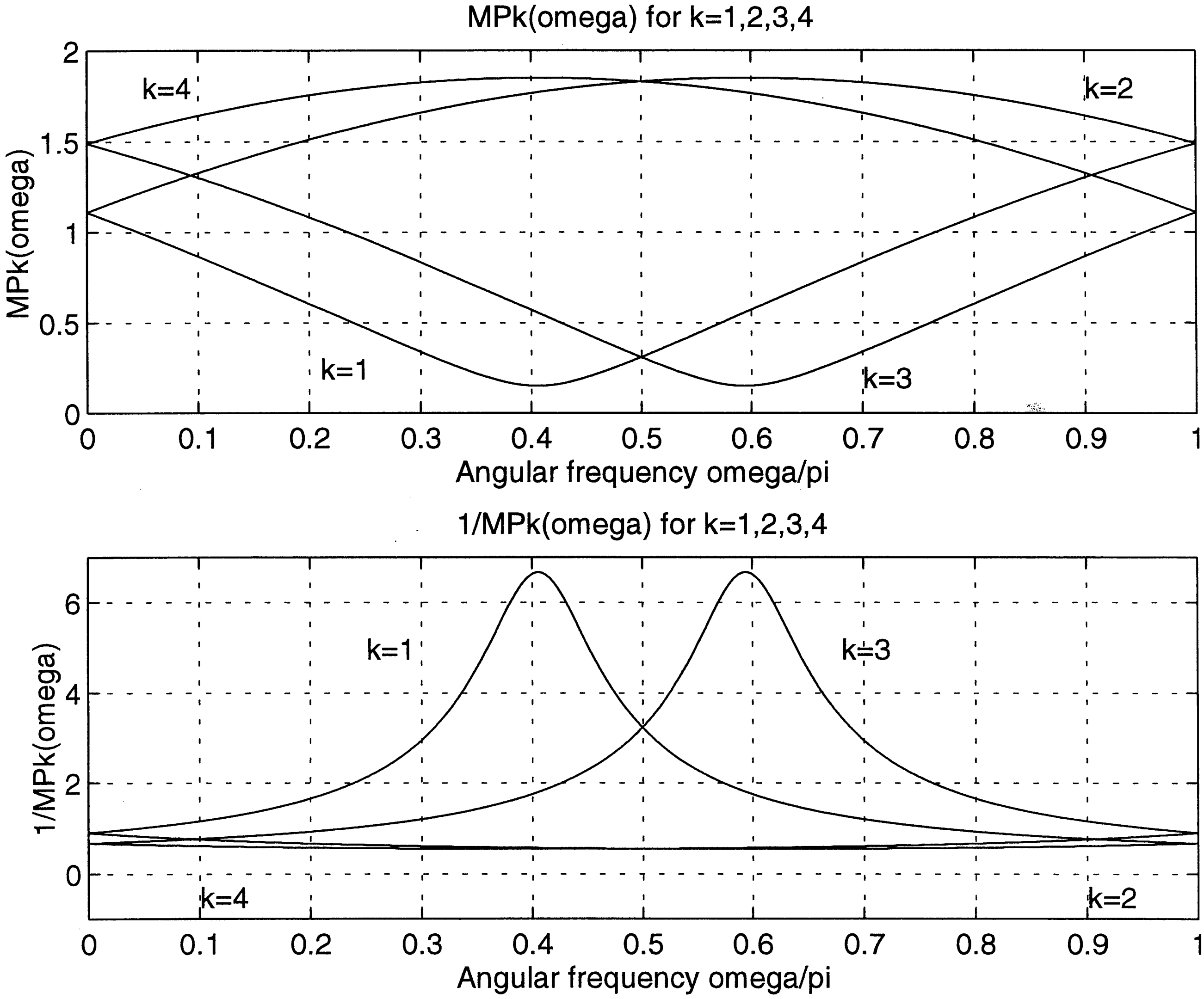
H(z) tells everything about the performance of the system.

y(n) can be determined in the following complicated manner:

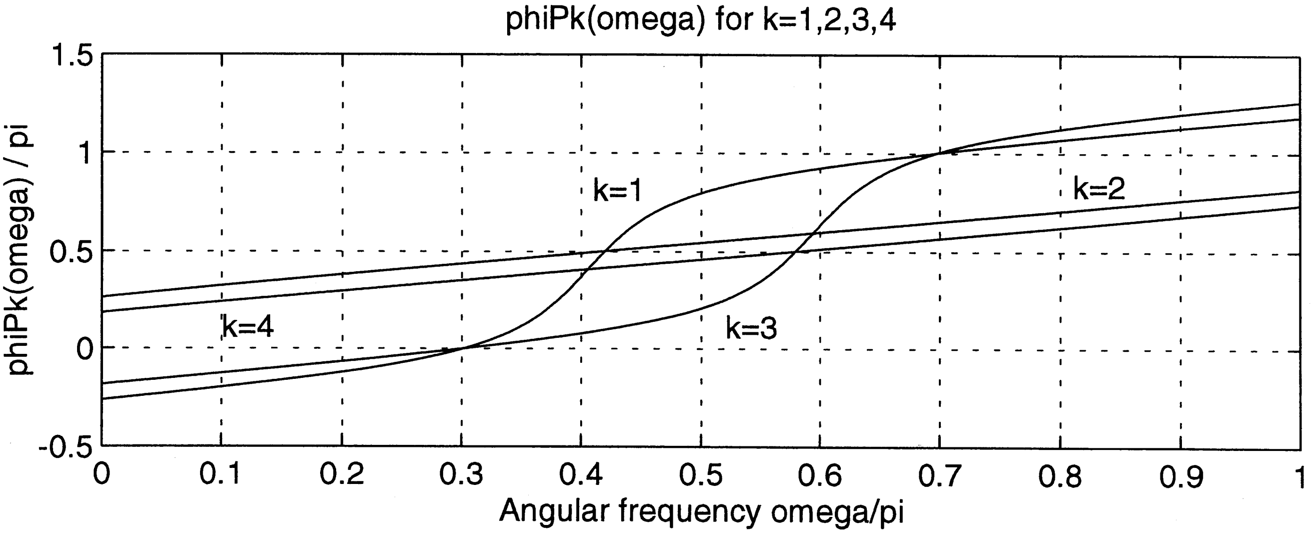
$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) .$$

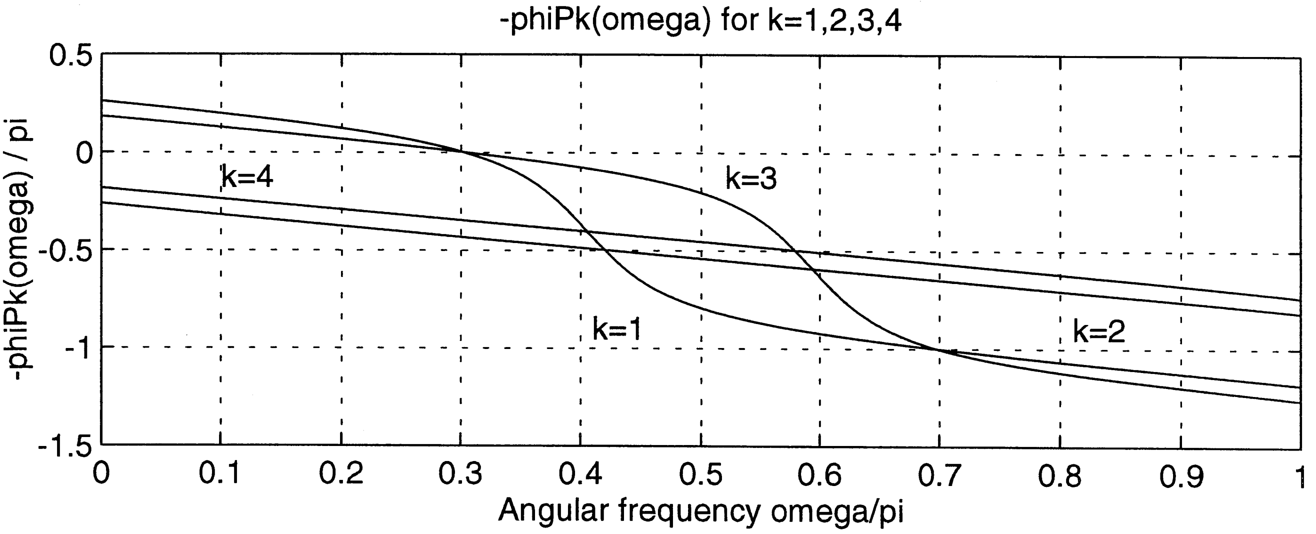
As we shall see later, the relation of ω to the frequency of the analog signal is given by (F_S is the sampling rate)

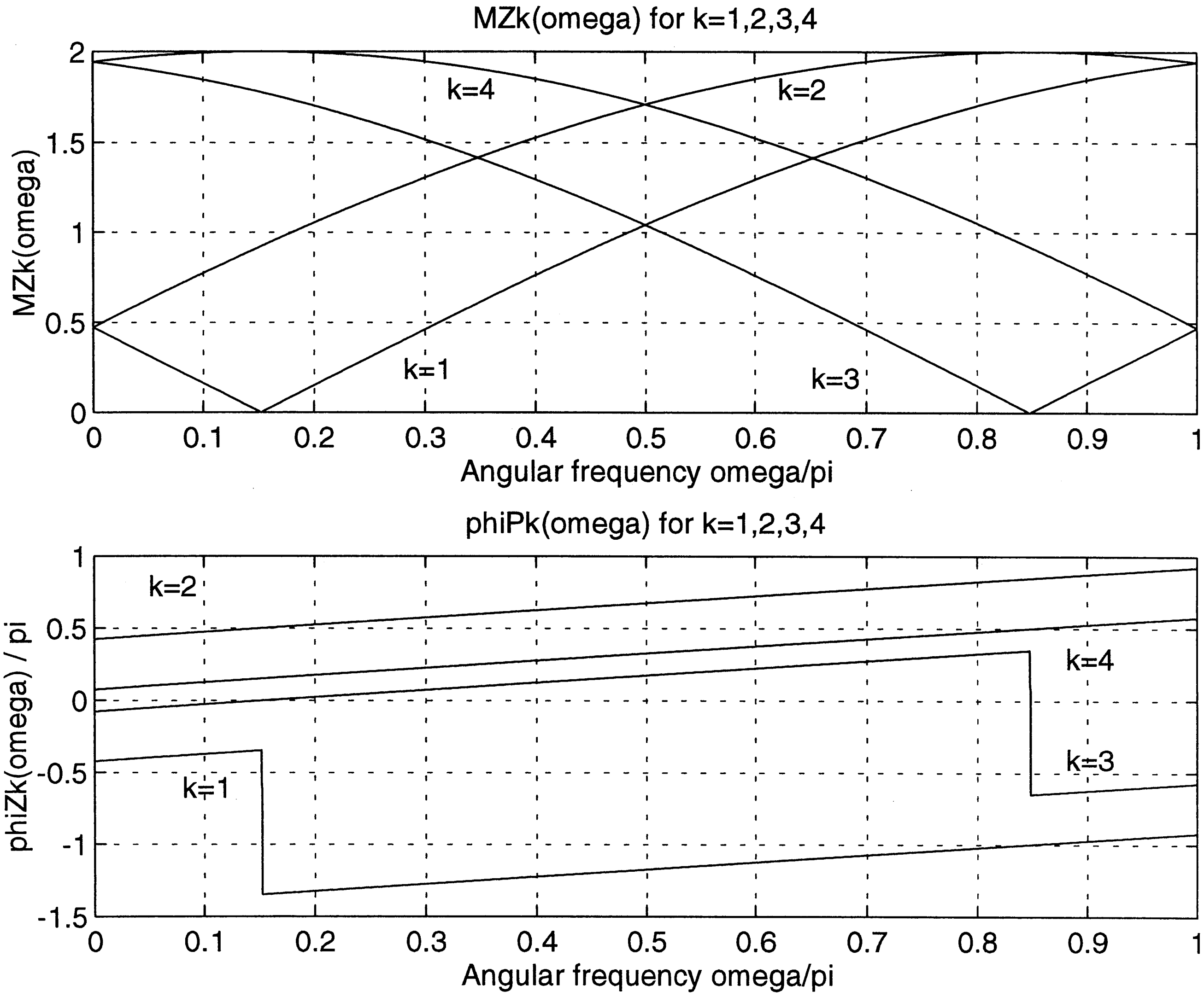
$$\boxed{\omega = 2\pi f/F_S.}$$



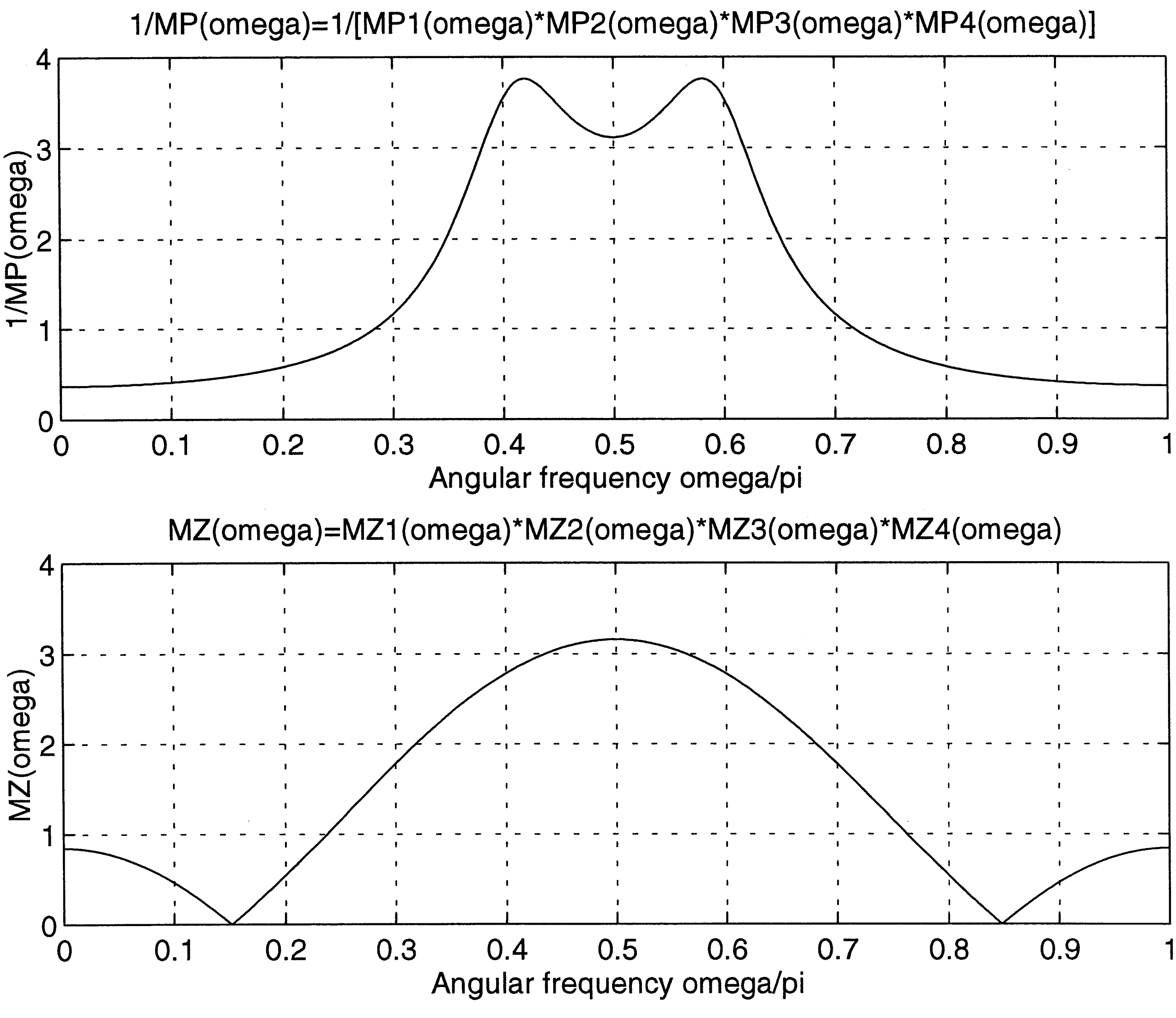
MPk(omega) for k=1,2,3,4
MPk(omega)
k=4
k=2
k=1
k=3
Angular frequency omega/pi
1/MPk(omega) for k=1,2,3,4
1/MPk(omega)
k=1
k=3
k=4
k=2
Angular frequency omega/pi

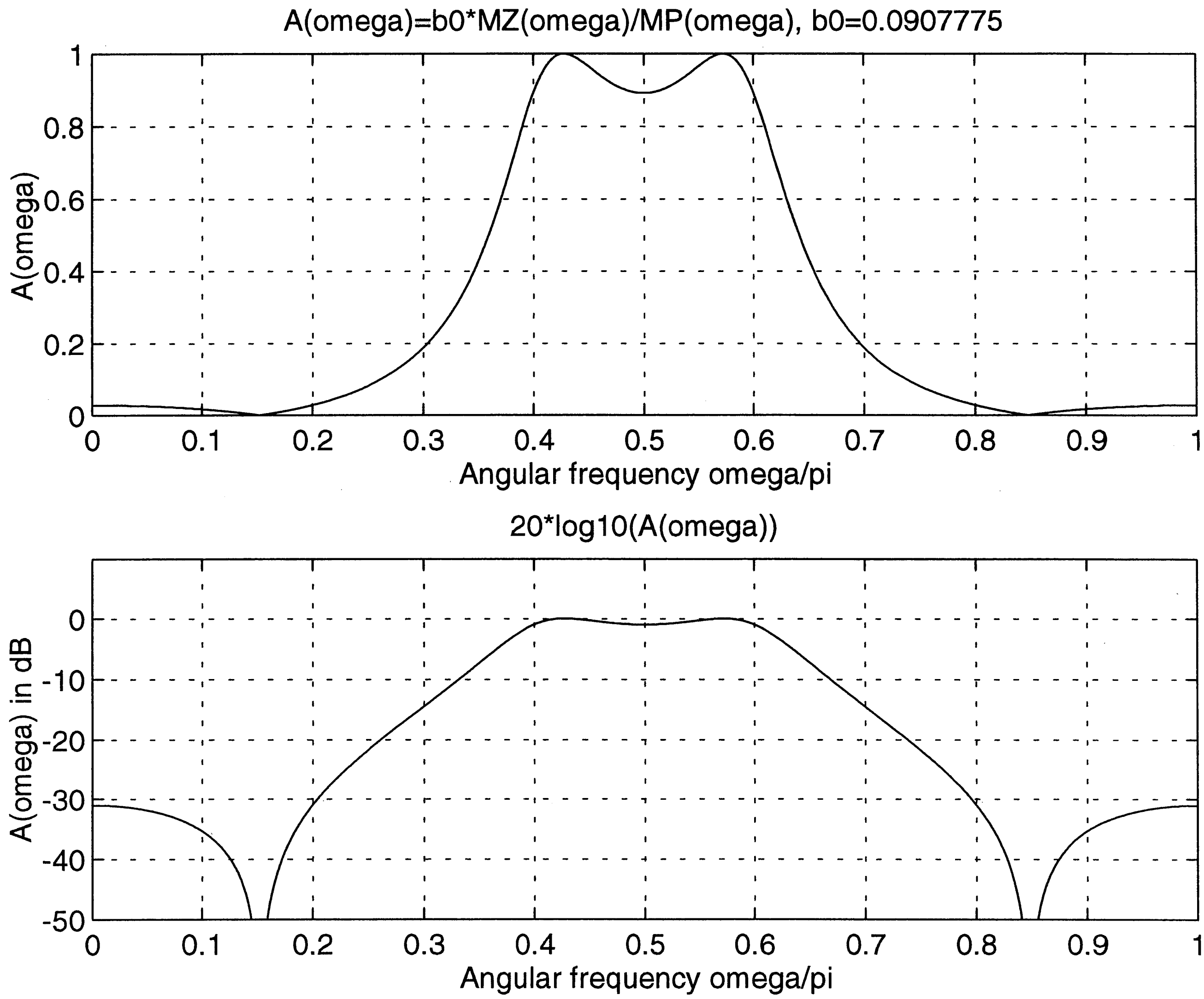

phiPk(omega) for k=1,2,3,4
phiPk(omega) / pi
1.5
1
0.5
0
-0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
k=1
k=2
k=3
k=4
Angular frequency omega/pi

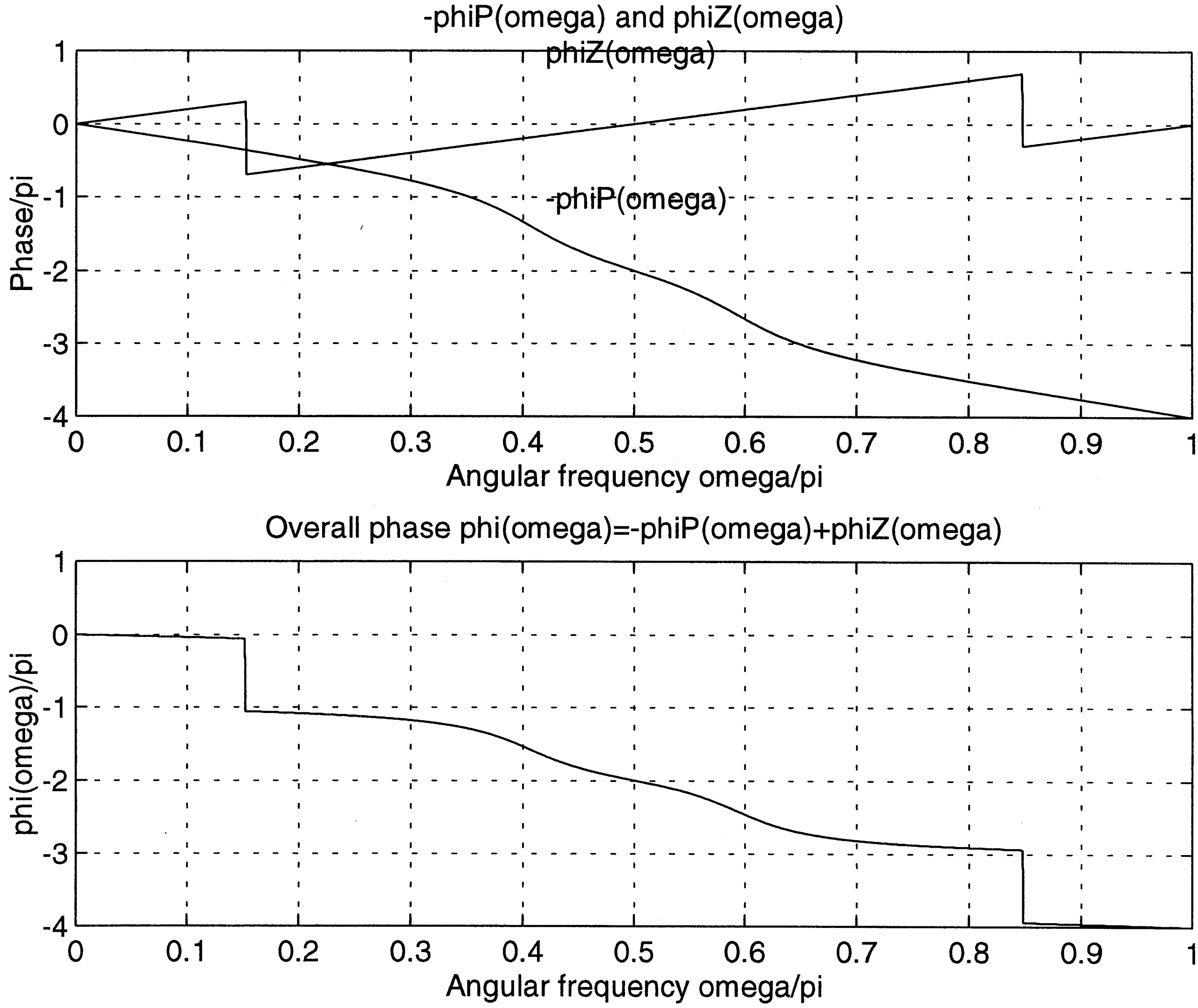

-phiPk(omega) for k=1,2,3,4
-phiPk(omega) / pi
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
k=4
k=3
k=1
k=2
Angular frequency omega/pi



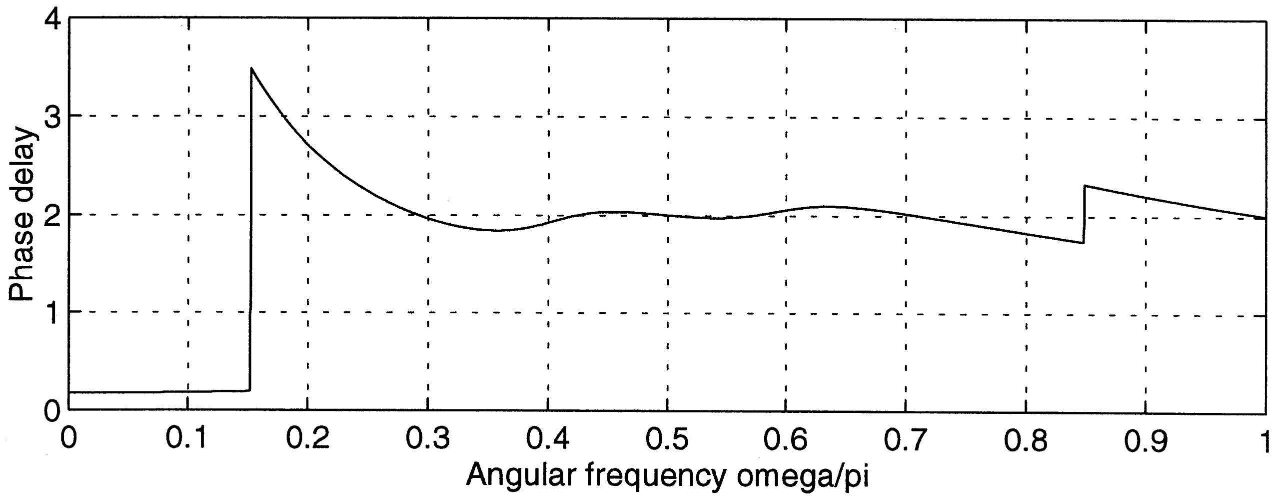
MZk(omega) for k=1,2,3,4
MZk(omega)
Angular frequency omega/pi
k=4
k=2
k=1
k=3
2
1.5
1
0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
phiPk(omega) for k=1,2,3,4
phiZk(omega) / pi
Angular frequency omega/pi
k=2
k=4
k=3
k=1
-0.5
-1
-1.5


1/MP(omega)=1/[MP1(omega)*MP2(omega)*MP3(omega)*MP4(omega)]
1/MP(omega)
0
1
2
3
4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Angular frequency omega/pi
MZ(omega)=MZ1(omega)*MZ2(omega)*MZ3(omega)*MZ4(omega)
MZ(omega)
0
1
2
3
4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Angular frequency omega/pi

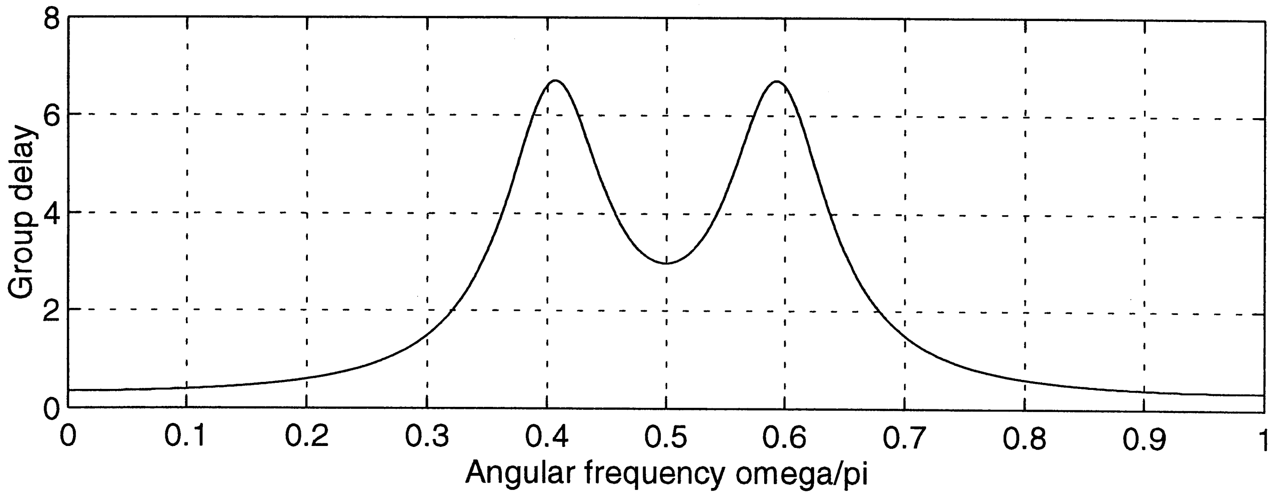

A(omega)=b0*MZ(omega)/MP(omega), b0=0.0907775
A(omega)
0
0.2
0.4
0.6
0.8
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Angular frequency omega/pi
20*log10(A(omega))
A(omega) in dB
-50
-40
-30
-20
-10
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Angular frequency omega/pi



-phiP(omega) and phiZ(omega)
phiZ(omega)
-phiP(omega)
Phase/pi
1
0
-1
-2
-3
-4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Angular frequency omega/pi
Overall phase phi(omega)=-phiP(omega)+phiZ(omega)
phi(omega)/pi
1
0
-1
-2
-3
-4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Angular frequency omega/pi



Phase delay
0
1
2
3
4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Angular frequency omega/pi



Group delay
0
2
4
6
8
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Angular frequency omega/pi

THE z-TRANSFORM

- The z-transform is the most important mathematical tool in analyzing and synthesizing linear time-invariant (LTI) dicrete-time systems.

- The z-transform of the sequence $\{x[n]\}$ is defined by

$$X(z) = Z(x[n]) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}.$$

- Several sequences may have the same z-transform.

- In order for the sequence to be unique, the region of convergence of the corresponding z-transform must be known.

- The region of convergence is the region in the z-plane where the series $\sum_{n=-\infty}^{\infty} x[n]z^{-n}$ is absolutely summable, that is, $|X(z)| = \sum_{n=-\infty}^{\infty} |x[n]||z^{-n}| < \infty$.

- This course concentrates mainly on causal sequences $\{x[n]\}$ for which $x[n] = 0$ for $n < 0$.

Example 1:

$$x[n] = \begin{cases} a^n & , n \geq 0 \\ -b^n & , n < 0 \end{cases}$$

$$X(z) = X_1(z) + X_2(z),$$

where

$$X_1(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

and

$$X_2(z) = \sum_{n=-\infty}^{-1} -b^n z^{-n} = -\sum_{n=1}^{\infty} (b^{-1}z)^n.$$

- Both of the above series are geometric and of the form $c + cq + cq^2 + cq^3 + \cdots$. Their sum is $c/(1-q)$ and they converge for $|q| < 1$. Based on this,

$$X_1(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

and the region of convergence is $|az^{-1}| < 1$ or $|z| > a$, whereas

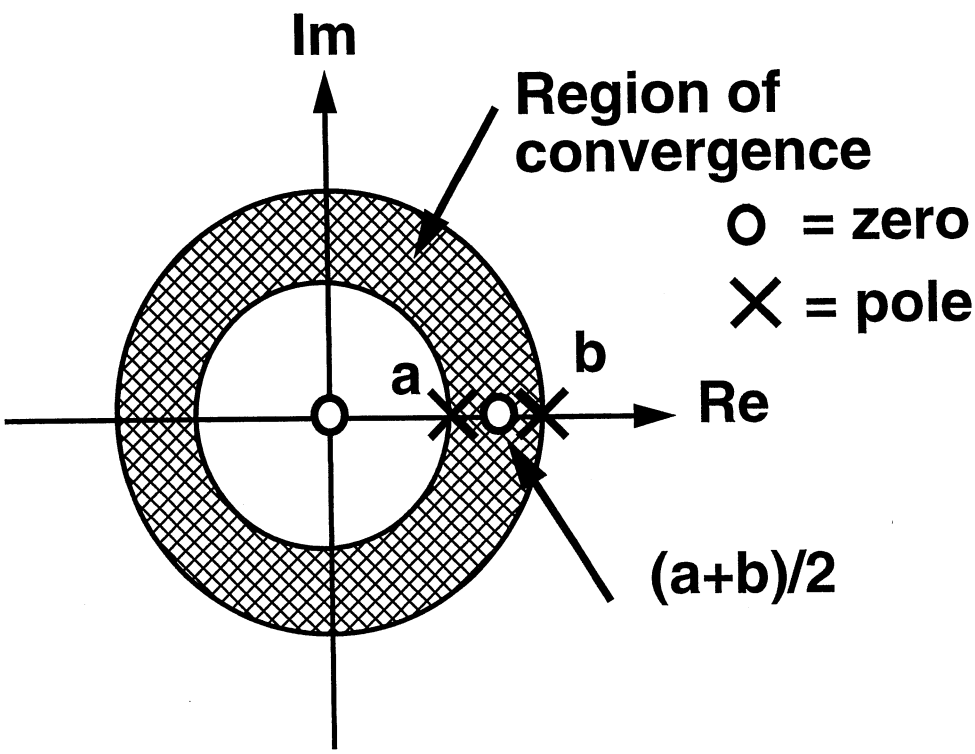
$$X_2(z) = -\frac{b^{-1}z}{1 - b^{-1}z} = \frac{z}{z - b}$$

and the region of convergence is $|b^{-1}z| < 1$ or $|z| < b$.

- For the overall sequence, the region of convergence is thus an annular region of the form $a < |z| < b$ and its z-transform is given by

$$X(z) = X_1(z) + X_2(z) = \frac{z(2z - a - b)}{(z - a)(z - b)}.$$

- The poles of $X(z)$ (the roots of the denominator) are located at the points $z = a$ ja $z = b$ and the zeros (the roots of the numerator) at $z = 0$ ja $z = (a+b)/2$. The poles and zeros as well as the region of the convergence are shown in the figure given below for $a < b$. For $a > b$, $X(z)$ has no region of convergence at all.



Example 2:

$$x[n] = \begin{cases} 0 & , n \geq 0 \\ -a^n - b^n & , n < 0 \end{cases}$$

$$X(z) = X_1(z) + X_2(z),$$

where

$$X_1(z) = \sum_{n=-\infty}^{-1} -a^n z^{-n} = -\sum_{n=1}^{\infty} (a^{-1}z)^n$$

and

$$X_2(z) = \sum_{n=-\infty}^{-1} -b^n z^n = -\sum_{n=1}^{\infty} (b^{-1}z)^n.$$

- In this case, $X_2(z)$ is the same as in Example 1 and has the same region of convergence.

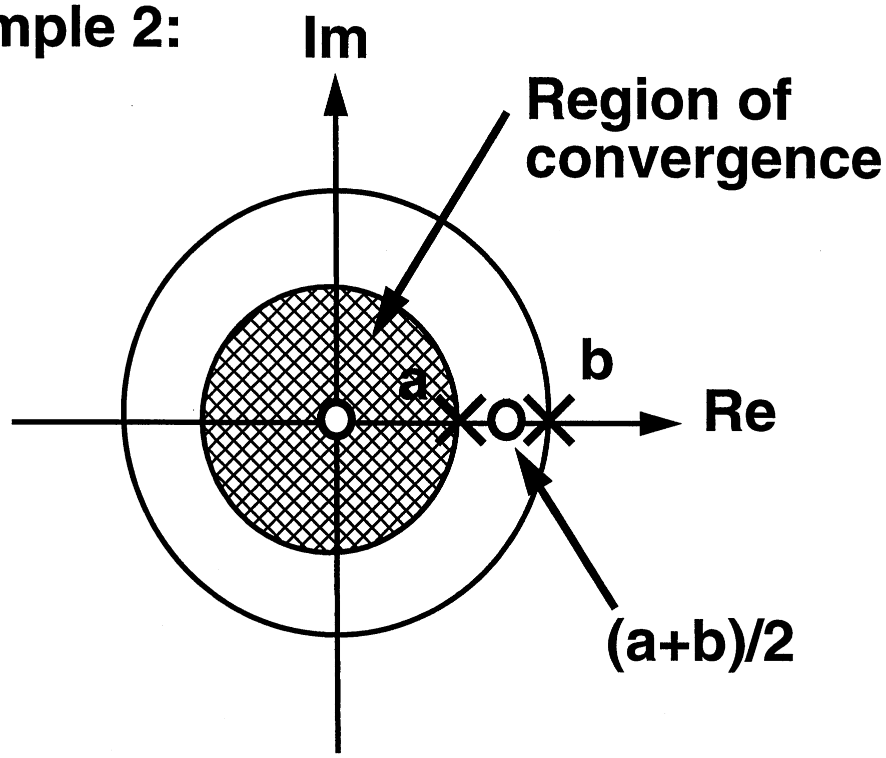
- Also

$$X_1(z) = -\frac{a^{-1}z}{1 - a^{-1}z} = \frac{z}{z - a}$$

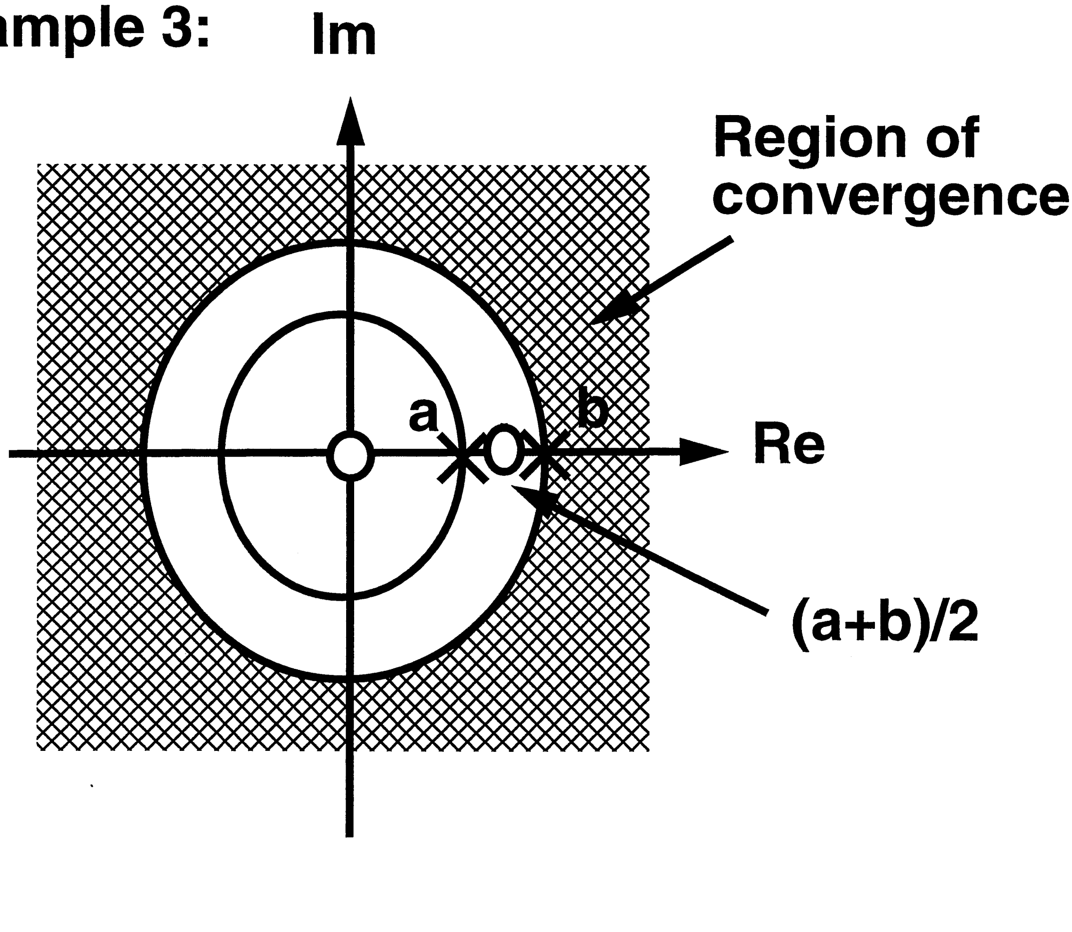
is the same, but the region of convergence is now $|a^{-1}z| < 1$ or $|z| < a$.

- Hence for both Example 1 and this example, $X(z)$ is the same, but the region of convergence is now $|z| < \min\{a, b\}$. See the following transparency.

Example 2:



Example 3:



Example 3:

$$x[n] = \begin{cases} a^n + b^n & , n \geq 0 \\ 0 & , n < 0 \end{cases}$$

$$X(z) = X_1(z) + X_2(z),$$

where

$$X_1(z) = \sum_{n=0}^{\infty} a^{-n} z^n = \sum_{n=1}^{\infty} (a^{-1}z)^n$$

and

$$X_2(z) = \sum_{n=0}^{\infty} b^{-n} z^n = \sum_{n=1}^{\infty} (b^{-1}z)^n.$$

- In this case, $X_1(z)$ is the same as in Example 1 and has the same region of convergence.

- Also

$$X_2(z) = \frac{1}{1 - bz^{-1}z} = \frac{z}{z - b}$$

  is the same, but the region of convergence is now $|bz^{-1}| < 1$ or $|z| > b$.

- $X(z)$ is the same as in Examples 1 and 2, but the region of convergence is now $|z| > \max\{a, b\}$. See the previous transparency.

OBSERVATIONS

- Several sequences may have the same z-transform.
- In order for the sequence to be unique, the region of convergence of the corresponding z-transform must be known.
- If $x[n] = 0$ for $n \geq 0$, then the sequence is called left-sided or anticausal. For an anticausal sequence, the region of convergence of the corresponding z-transform is of the form $|z| < a$. Here, a is the absolute value of the pole of $X(z)$ which is located closest to the origin (Example 2). In general, the poles are complex-valued.
- This course concentrates manly on right-sided or causal sequences $\{x[n]\}$ for which $x[n] = 0$ for $n < 0$. For these sequences, the region of convergence of the corresponding z-transform is of the form $|z| > a$. Here, a is the absolute value of the pole of $X(z)$ which is located fartest away from the origin (Example 3).
- Sequences achieving nonzero values for both $n < 0$

and $n \geq 0$ can be expressed in the form $x(n) = x_1(n) + x_2(n)$, where $x_1(n)$ and $x_2(n)$ are anticausal and causal, respectively (Example 1). For the corresponding z-transforms, denoted by $X_1(z)$ and $X_2(z)$, the regions of convergence are of the forms $|z| < b$ and $|z| > a$, respectively. The overall z-transform is $X(z) = X_1(z) + X_2(z)$ and the region of the convergence is the annular region $a < |z| < b$. For $a > b$, there is no region of convergence.

- For a causal sequence $\{x[n]\}$, the z-transform, also called the one-sided z-transform, is simply given by

$$X(z) = Z(x[n]) = \sum_{n=0}^{\infty} x[n] z^{-n}.$$

Example 4: Determine the z-transforms and the regions of the convergence for the sequences shown below.

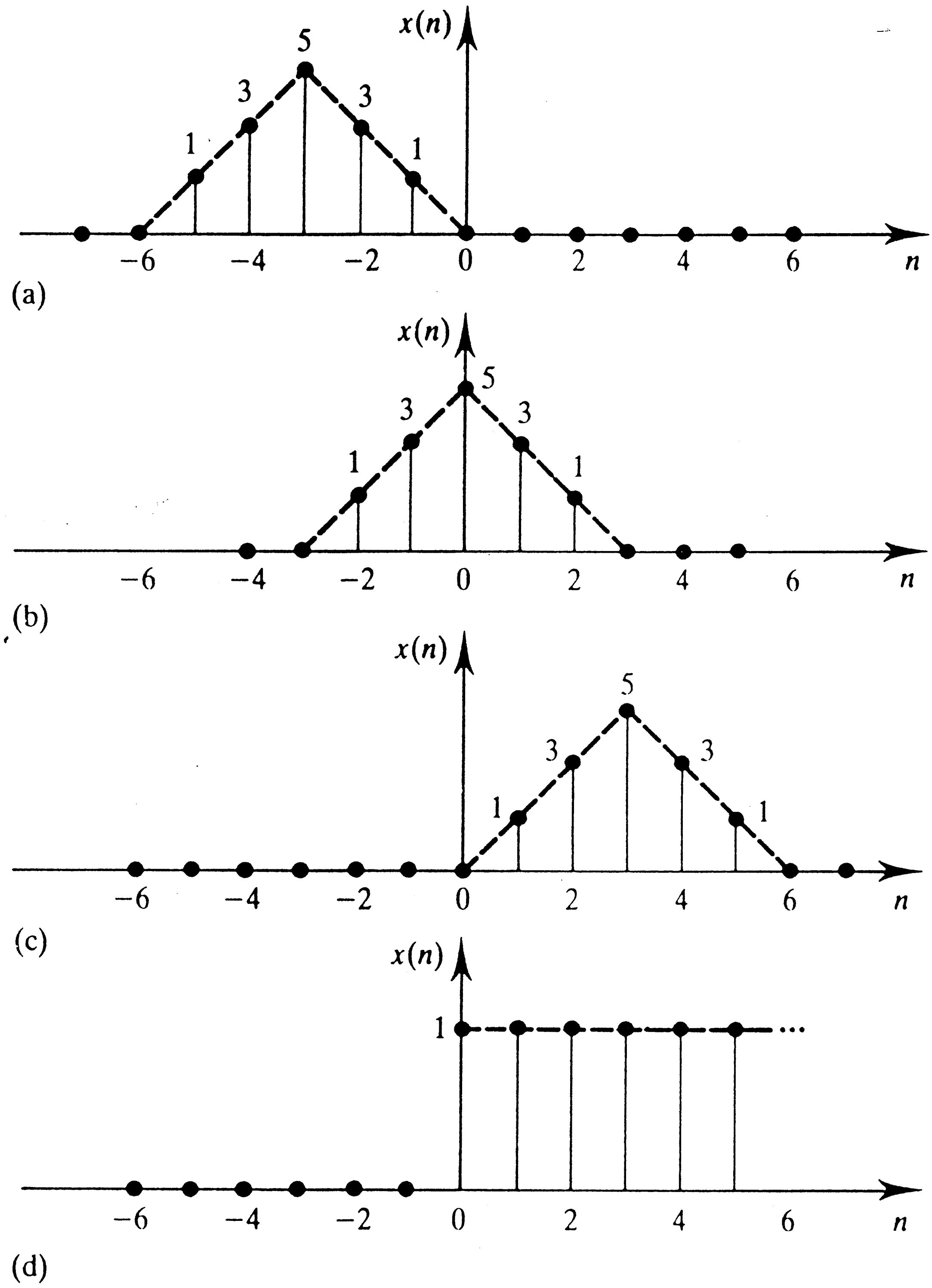


Figure 3.1 Causal and noncausal discrete-time sequences.

- For a shifted unit sample (impulse)

$$x[n] = \delta[n-k] = \begin{cases} 1 & , n = k \\ 0 & \text{otherwise,} \end{cases}$$

the z-transform is obviously given by

$$X(z) = z^{-k}.$$

- The first sequence is expressible as $x[n] = \delta[n+5] + 3\delta[n+4] + 5\delta[n+3] + 3\delta[n+2] + \delta[n+1]$.

- According to the above-mentioned formula, the z-transform is given by $X(z) = z^5 + 3z^4 + 5z^3 + 3z^2 + z^1$ and it is finite everywhere except for the point $z = \infty$. The region of convergence is thus $|z| \neq \infty$.

- The second sequence is expressible as $x[n] = \delta[n+2] + 3\delta[n+1] + 5\delta[n] + 3\delta[n-1] + \delta[n-2]$.

- The corresponding z-transform is given by $X(z) = z^2 + 3z^1 + 5 + 3z^{-1} + z^{-2}$ and it is finite everywhere except for the points $z = 0, \infty$. The region of convergence is thus $|z| \neq 0, \infty$.

- The third sequence is expressible as $x[n] = \delta[n-1] + 3\delta[n-2] + 5\delta[n-3] + 3\delta[n-4] + \delta[n-5]$.

- The corresponding z-transform is given by $X(z) = z^{-1}+3z^{-2}+5z^{-3}+3z^{-4}+z^{-5}$ and it is finite everywhere except for the point $z=0$. The region of convergence is thus $|z| \neq 0$.

- The fourth sequence is the unit step given by

$$x[n] = \begin{cases} 1 & , n \geq 0 \\ 0 & , n < 0. \end{cases}$$

- In the above, we observed that for the sequence described by

$$x[n] = \begin{cases} a^n & , n \geq 0 \\ 0 & , n < 0, \end{cases}$$

the z-transform is given by

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

and the corresponding region of convergence is $|z| > a$.

- Our sequence is the special case for which $a = 1$. Hence, the z-transform is

$$X(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

and the region of convergence is $|z| > 1$.

Example 5: Determine the z-transform and the region of the convergence for the sequence given by

$$x[n] = r^n \cos(n\alpha + \beta)u[n]$$

(r, α ja β are real-valued).

- Using the identity $\cos(n\theta) = (1/2)(e^{jn\theta} + e^{-jn\theta})$, our sequence can be split into two parts as follows:

$$x[n] = x_1[n] + x_2[n],$$

where

$$x_1[n] = (1/2)e^{j\beta}\{re^{j\alpha}\}^n u[n]$$

and

$$x_2[n] = (1/2)e^{-j\beta}\{re^{-j\alpha}\}^n u[n].$$

- The corresponding z-transforms are given by

$$X_1(z) = (1/2)e^{j\beta}\sum_{n=0}^{\infty}(re^{j\alpha}z^{-1})^n$$
$$= (1/2)\frac{e^{j\beta}}{1 - re^{j\alpha}z^{-1}}$$

and

$$X_2(z) = (1/2)e^{-j\beta}\sum_{n=0}^{\infty}(re^{-j\alpha}z^{-1})^n$$
$$= (1/2)\frac{e^{-j\beta}}{1 - re^{-j\alpha}z^{-1}}.$$

- By adding $X_1(z)$ ja $X_2(z)$, we end up with the following z-transform pair:

$$r^n \cos(n\alpha + \beta)u[n] \iff \frac{\cos\beta - r\cos(\alpha+\beta)z^{-1}}{1 - 2r\cos\alpha z^{-1} + r^{-2}}.$$

The region of convergence is $|z| > r$.

- From this transform pair, we obtain directly the following pair:

$$2Rr^n \cos(n\alpha+\beta)u[n] \iff \frac{Re^{j\beta}}{1 - re^{j\alpha}z^{-1}} + \frac{Re^{-j\beta}}{1 - re^{-j\alpha}z^{-1}},$$

which is used later in this course.

- Also many other z-transform pairs can be obtained directly from the above result.

- Some common z-transform pairs are summarized in the following transparency, where ROC stands for the region of convergence.

SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m>0$) or ∞ (if $m<0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^Nz^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

SOME PROPERTIES OF THE z-TRANSFORM

- **THE MOST CRUCIAL PROPERTY: CONVOLUTION:** If the z-transform of the impulse (unit sample) response $h[n]$ of a LTI system is $H(z)$ and the z-transform of the exitation $x[n]$ is $X(z)$, then the z-transform of the response $y[n]$ is simply given by

$$Y(z) = H(z)X(z).$$

  - Earlier we expressed the input-output relation of a LTI system in the time domain in terms of the convolution sum as follows:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k].$$

  - $H(z)$ is called the **transfer function** of the system. As it can be seen later in this course, this function includes all the knowledge about the behavior of our system, such as the stability, the frequency response, the amplitude response, and the phase response.

- **LINEARITY:** If the z-transforms of the sequences $x_1[n]$ ja $x_2[n]$ are $X_1(z)$ ja $X_2(z)$, respectively, then the z-transform of $ax_1[n] + bx_2[n]$ is $aX_1(z) + bX_2(z)$.

- **TIME SHIFTING:** If the z-transform of the sequence $x[n]$ is $X(z)$, then the z-transform of the sequence $x[n-k]$ is $z^{-k}X(z)$.

- **DIFFERENTIATION:** If the z-transform of the sequence $x[n]$ is $X(z)$, then the z-transform of the sequence $nx[n]$ is
$$-z\frac{dX(z)}{dz}.$$

THE z-TRANSFORM OF THE IMPULSE RESPONSE: TRANSFER FUNCTION

- Consider a LTI system (digital filter) obeying the following difference equation:
$$y[n] = \sum_{k=1}^{N} b_k y[n-k] + \sum_{k=0}^{M} a_k x[n-k].$$

- The causality means in this case that the difference equation is determined forward in time. In other words, if $y[n]$ is known at $n = n_0$, then we evaluate $y[n]$ at $n = n_0 + 1$ and so on.

- In many cases, we start determining the values of $y[n]$ at the time instant $n = 0$ and assume that $x[n] = 0$ for $n < 0$ and $y[-1] = y[-2] \cdots = y[-N] = 0$.

- Strightly speaking, the system characterized by the above difference equation is an LTI system only if the first nonzero input sample occurs at $n = n_0$, then $y[n_0 - 1] = y[n_0 - 2] \cdots = y[n_0 - N] = 0$

- Applying the z-transform to both sides of the above difference equation and using the linear and time-shifting properties (see transparency 89), we obtain

$$Y(z) = \sum_{k=1}^{N} b_k z^{-k} Y(z) + \sum_{k=0}^{M} a_k z^{-k} X(z).$$

- By dividing $Y(z)$ by $X(z)$, we obtain for the z-transform of the impulse response $h[n]$, that is, for the **transfer function**

$$H(z) = Y(z)/X(z) = \frac{\sum_{k=0}^{M} a_k z^{-k}}{1 - \sum_{k=1}^{N} b_k z^{-k}}.$$

- Alternatively, $H(z)$ is expressible in terms of its poles d_k for $k = 1, 2, \cdots, N$ (roots of the denominator) and zeros c_k for $k = 1, 2, \cdots, M$ (roots of the numerator) as follows:

$$H(z) = a_0 \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}.$$

- Since we assume that our system is causal, that is, $h[n] = 0$ for $n < 0$, the region of convergence of $H(z)$ is $|z| > \max\{|d_1|, |d_2|, \ldots, |d_N|\}$. Here, $|d_k|$ is the distance of the pole d_k from the origin.

- On the other hand, for our system to be stable, it is required that

$$\sum_{n=0}^{\infty} |h[n]| < \infty.$$

- Since the above equation is identical to the condition that

$$\sum_{n=0}^{\infty} |h[n]z^{-n}| < \infty$$

for $|z| = 1$, the condition of stability is equivalent to the condition that the region of convergence of $H(z)$ must include the unit circle.

- This implies that for a stable system with transfer function $H(z)$ **all the poles must lie inside the unit circle**.

- The response of our system to the input $x[n] = A\cos(n\omega_0 + \phi)$ is given by

$$y[n] = A|H(e^{j\omega_0})|\cos(n\omega_0 + \phi + \arg\ H(e^{j\omega_0})).$$

- Here,

$$H(e^{j\omega_0}) = H(z)_{|z\,=\,e^{j\omega_0}} = |H(e^{j\omega_0})|e^{j\arg\ H(e^{j\omega_0})}.$$

- That is, $|H(e^{j\omega_0})|$ and arg $H(e^{j\omega_0})$ can be determined by evaluating $H(z)$ at the unit circle point $z = e^{j\omega_0}$ and then expressing the result in polar form.

- $|H(e^{j\omega})|$ giving the change caused by the system to the oscillation amplitude of a sinusoidal signal of frequency ω is called the **amplitude response** of the system.

- arg $H(e^{j\omega})$ giving the change caused by the system to the phase of a sinusoidal signal of frequency ω is called the **phase response** of the system.

- Let us prove that the response really has the above form.

- The input signal is expressible as

$$x[n] = A\cos(n\omega_0 + \phi) = (A/2)(e^{j\phi}e^{jn\omega_0} + e^{-j\phi}e^{-jn\omega_0}).$$

- $y[n]$ is the convolution of $x[n]$ and $h[n]$, that is,

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

$$= (A/2)\sum_{k=0}^{\infty} h[k](e^{j\phi}e^{j(n-k)\omega_0} + e^{-j\phi}e^{-j(n-k)\omega_0})$$

$$= (A/2)e^{j\phi}e^{jn\omega_0}\sum_{k=0}^{\infty} h[k]e^{-jk\omega_0}$$

$$+ (A/2)e^{-j\phi}e^{-jn\omega_0}\sum_{k=0}^{\infty} h[k]e^{jk\omega_0}.$$

- Here,

$$\sum_{k=0}^{\infty} h[k]e^{-jk\omega_0} = \sum_{k=0}^{\infty} h[k]z^{-k}\Big|_{z = e^{j\omega_0}}$$

$$= H(z)\big|_{z = e^{j\omega_0}} = H(e^{j\omega_0}).$$

- Similarly,

$$\sum_{k=0}^{\infty} h[k]e^{jk\omega_0} = H(e^{-j\omega_0}).$$

- By expressing

$$H(e^{j\omega_0}) = |H(e^{j\omega_0})|e^{j\arg H(e^{j\omega_0})}$$

and utilizing the property (to be considered later)

$$H(e^{-j\omega_0}) = |H(e^{j\omega_0})|e^{-j\arg H(e^{j\omega_0})}.$$

gives finally

$$y[n] = (A/2)|H(e^{j\omega_0})|(e^{j[n\omega_0+\phi+\arg\ H(e^{j\omega_0})]})$$
$$+ (A/2)|H(e^{j\omega_0})|(e^{-j[n\omega_0+\phi+\arg\ H(e^{j\omega_0})]})$$
$$= A|H(e^{j\omega_0})|\cos(n\omega_0 + \phi + \arg\ H(e^{j\omega_0})).$$

- We start now considering in more details $H(e^{j\omega})$ which is called the **frequency response** of our system.

FREQUENCY RESPONSE

- If the system transfer function is

$$H(z) = a_0 \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})},$$

then the frequency response is obtained by using the substitution $z = e^{j\omega}$, giving

$$H(e^{j\omega}) = a_0 \frac{\prod_{k=1}^{M}(1 - c_k e^{-j\omega})}{\prod_{k=1}^{N}(1 - d_k e^{-j\omega})}.$$

- By writing

$$1 - c_k e^{-j\omega} = M_{zk}(\omega) e^{j\phi_{zk}(\omega)}$$

and

$$1 - d_k e^{-j\omega} = M_{pk}(\omega) e^{j\phi_{pk}(\omega)},$$

the frequency response is expressible as

$$H(e^{j\omega}) = a_0 \frac{\prod_{k=1}^{M} M_{zk}(\omega) e^{j\phi_{zk}(\omega)}}{\prod_{k=1}^{N} M_{pk}(\omega) e^{j\phi_{pk}\omega)}}.$$

- Based on the above equation, the amplitude and phase responses of our system can be expressed as

$$|H(e^{j\omega})| = a_0 \prod_{k=1}^{M} M_{zk}(\omega) / \prod_{k=1}^{N} M_{pk}(\omega)$$

and

$$\arg\, H(e^{j\omega}) = \sum_{k=1}^{M} \phi_{zk}(\omega) - \sum_{k=0}^{N} \phi_{pk}(\omega).$$

- More detailed forms for the amplitude and phase responses as well as for some other responses decribing the performance of a causal LTI system (digital filter) are given later in the lecture notes in Part 3: Introductory filtering examples.

DETERMINATION OF THE FREQUENCY RESPONSE GRAPHICALLY

- The frequency response of a system with transfer function $H(z)$ can be determined graphically by expressing it in the form
$$H(z) = a_0 z^{N-M} \frac{\prod_{k=1}^{M}(z - c_k)}{\prod_{k=1}^{N}(z - d_k)}.$$
Note that for $N > M$, $H(z)$ has $N - M$ zeros at the origin, whereas for for $N < M$, $H(z)$ has $N - M$ poles at the origin.

- This form is obtained from $H(z)$ considered in the previous transparencies by multiplying both the numerator and denominator by z^N.

- The corresponding frequency response is expressible in the following forms:
$$\begin{aligned} H(e^{j\omega}) &= a_0 e^{j(N-M)\omega} \frac{\prod_{k=1}^{M}(e^{j\omega} - c_k)}{\prod_{k=1}^{N}(e^{j\omega} - d_k)} \\ &= e^{j(N-M)\omega} a_0 \frac{\prod_{k=1}^{M} M_{Zk}(\omega) e^{j\phi_{Zk}(\omega)}}{\prod_{k=1}^{N} M_{Pk}(\omega) e^{j\phi_{Pk}\omega)}} \end{aligned}$$

- Based on this equation, the amplitude and phase responses can be written as

$$|H(e^{j\omega})| = a_0 \prod_{k=1}^{M} M_{Zk}(\omega) / \prod_{k=1}^{N} M_{Pk}(\omega)$$

and

$$\arg H(e^{j\omega}) = (N - M)\omega + \sum_{k=1}^{M} \phi_{Zk}(\omega) - \sum_{k=0}^{N} \phi_{Pk}(\omega).$$

- For the zero at $z = c_k$ [pole at $z = d_k$], $M_{Zk}(\omega)$ and $\phi_{Zk}(\omega)$ [$M_{Pk}(\omega)$ and $\phi_{Pk}(\omega)$] at any value of ω can be determined by drawing a vector from the unit circle point $z = e^{j\omega}$ to the zero [to the pole] (see an example on the following transparency). $M_{Zk}(\omega)$ [$M_{Pk}(\omega)$] is the length of the vector, whereas $\phi_{Zk}(\omega)$ [$\phi_{Pk}(\omega)$] is the corresponding angle.

- As seen from the above equations, the amplitude response at any value of the angular frequency ω, can be obtained by multiplying separately the vector lengths corresponding to the zeros and poles. Finally, the overall amplitude response is obtained by multiplying the result for the zeros by b_0 and then dividing this by the result for the poles.

- The phase response, in turn, is determined by adding to $(N - M)\omega$ the vector angles of the zeros and subtracting the vector angles of the poles.

- The following figure illustrates the above procedure. More details are given in the following pages.

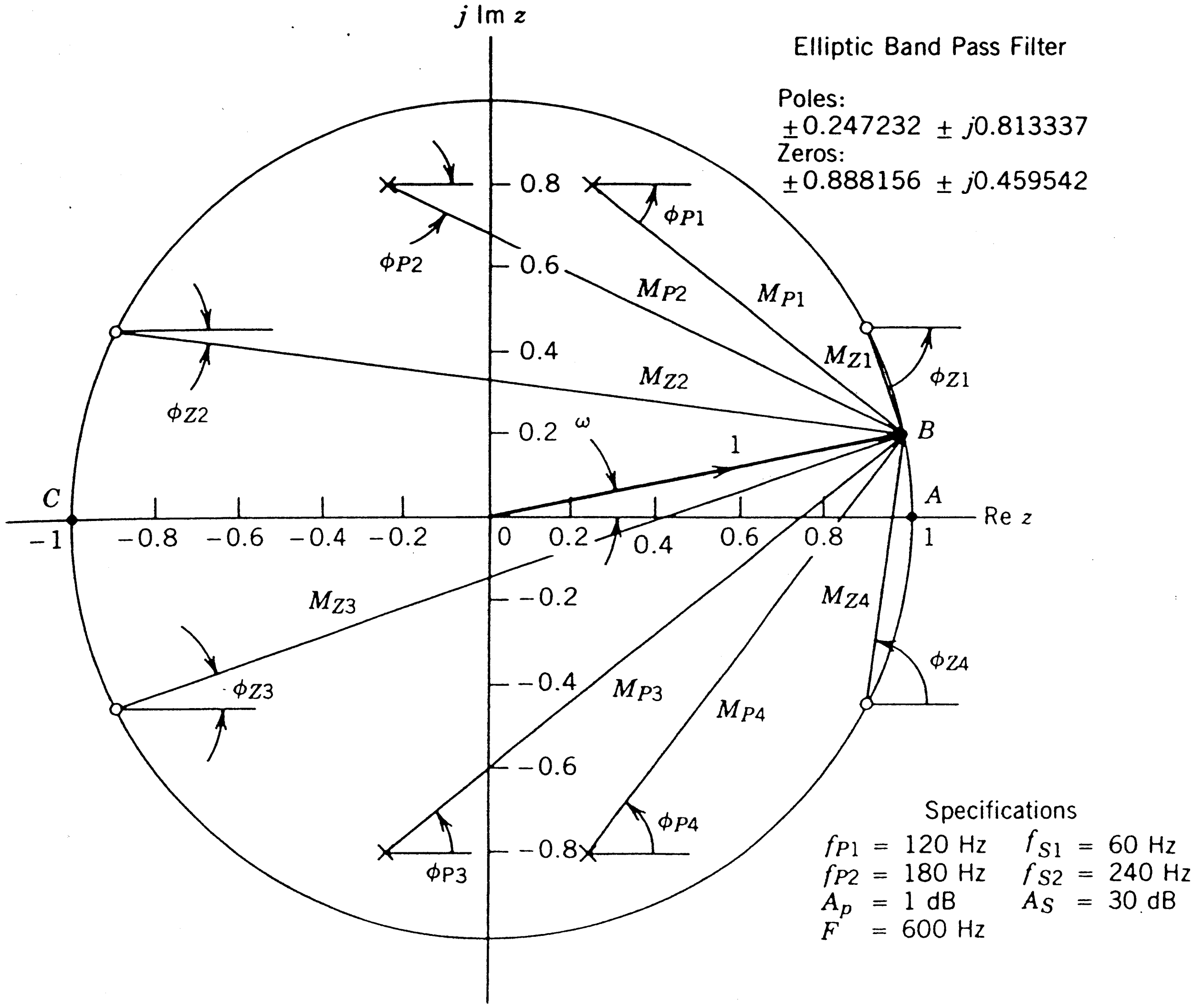


Figure 3.5 Frequency and phase response determination in the *z*-plane.

GROUP DELAY AND PHASE DELAY

- In analyzing and synthesizing digital filters (causal LTI systems) we use often group delay and phase delay responses, instead of the phase response.

- The group delay response of a system with transfer function $H(z)$ is defined by

$$\tau_g(\omega) = -\frac{d}{d\omega}[\arg H(e^{j\omega})].$$

- The phase delay response, in turn, is given by

$$\tau_p(\omega) = -\frac{\arg H(e^{j\omega})}{\omega}.$$

- Here, the phase delay response has a simple physical meaning, as we shall see on the following transparency.

THE MEANING OF THE PHASE DELAY

- Consider a periodic signal given by

$$x[n] = \sum_{k=1}^{K} A_k \cos[n(k\omega_0) + \phi_k].$$

- By filtering this sequence by the transfer function $H(z)$, the output is given by

$$y[n] = \sum_{k=1}^{K} |H(e^{jk\omega_0})| A_k \cos[n(k\omega_0) + \phi + \arg\ H(e^{jk\omega_0})]$$

$$= \sum_{k=1}^{K} |H(e^{jk\omega_0})| A_k \cos[\{n - (-\frac{\arg\ H(e^{jk\omega_0})}{k\omega_0})\}(k\omega_0) + \phi_k].$$

- In the above,

$$-\arg\ H(e^{jk\omega_0})/k\omega_0$$

is the delay casused by the filtering to the kth component. If the phase response is linear, that is,

$$\arg\ H(e^{j\omega}) = -a\omega,$$

then the delay for each component is on a, that is, the phase delay is a constant ($\tau_p(\omega) = -\arg\ H(e^{j\omega})/\omega = a$).

- If also $|H(e^{jk\omega_0})| \approx 1$ kun $k = 1, 2, \cdots, K$, then the waveform of our periodic signal remains practically the same.

- This is very important for instance in processing an ECG-signal, which is practically periodic.

- The following figure shows how the desired waveform can be found from a signal containing both the 50 Hz interference and baseline drift by using a filter with a constant phase delay.

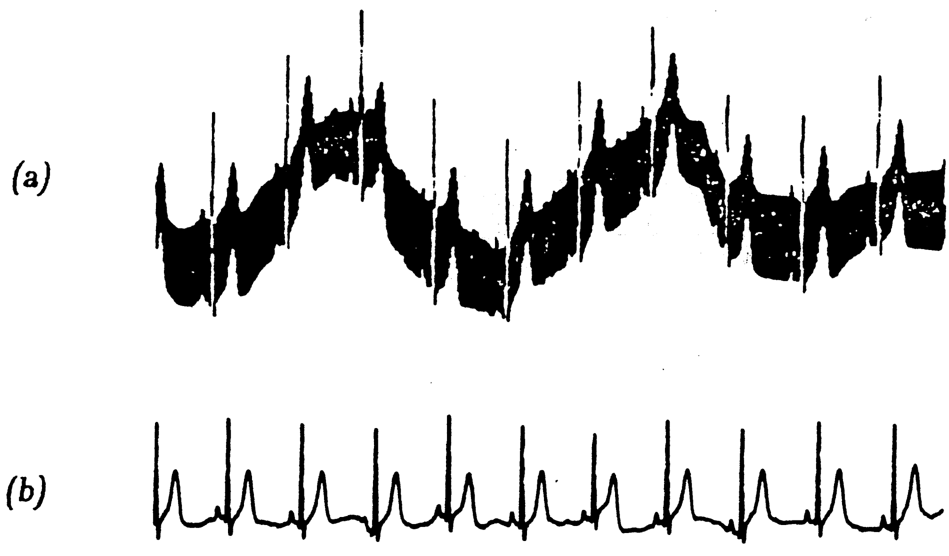


Fig. 10.3. *Elimination of the baseline drift (heart rate 60 beats/min). (a) Original ECG signal. (b) Filtered ECG signal.*

Example 6: The transfer function of a second-order filter is given by

$$H(z) = K\frac{1+z^{-2}}{1+0.81z^{-2}}$$

- Determine K such that the amplitude response achieves the value of unity at the zero frequency.
- Give the pole-zero plot for the filter and determine the amplitude and phase responses.

- The amplitude response at the zero frequency is $H(1)$ ($z = e^{j0} = 1$), giving

$$H(1) = K\frac{1+1}{1+0.81\cdot 1} = 1$$

$$\Rightarrow$$

$$K = 0.905.$$

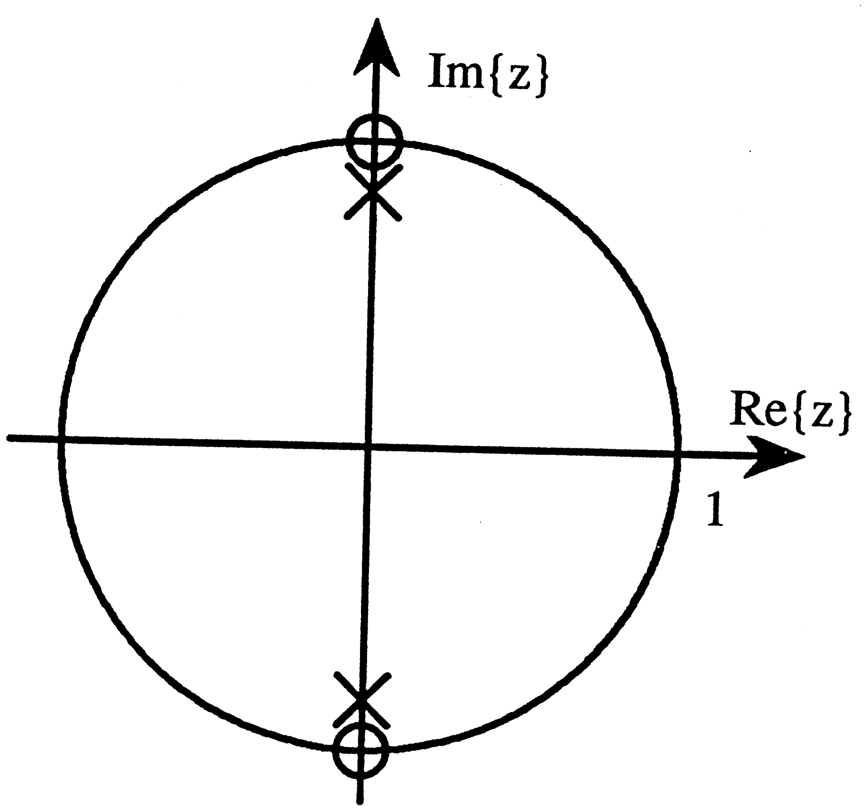
- The zeros of our filter are the roots of the numerator and they are located at the points

$$z_{ze1,\ ze2} = \pm j = e^{\pm j\pi/2}.$$

- The poles of our filter are the roots of the denominator and they are located at the points

$$z_{po1,\ po2} = \pm j0.9 = 0.9e^{\pm j\pi/2}.$$

- The pole-zero plot for our filter is given in the following figure.



- The amplitude response of the filter is expressible as

$$|H(e^{j\omega})| = \frac{Kl_1(\omega)l_2(\omega)}{k_1(\omega)k_2(\omega)},$$

where

$$l_1(\omega) = |e^{j\omega} - z_{ze1}|$$

$$l_2(\omega) = |e^{j\omega} - z_{ze2}|$$

$$k_1(\omega) = |e^{j\omega} - z_{po1}|$$

$$k_2(\omega) = |e^{j\omega} - z_{po2}|.$$

- $l_1(\omega)$ and $l_2(\omega)$ are the distance from the unit circle point $e^{j\omega}$ to the zeros z_{ze1} and z_{ze2}.

- $k_1(\omega)$ ja $k_2(\omega)$ are the corresponding distances from the poles z_{pe1} and z_{pe2}.

- These distances can be expressed as

$$l_1(\omega) = |\cos\omega + j\sin\omega - j| = |\cos\omega + j(\sin\omega - 1)|$$
$$= \sqrt{(\cos\omega))^2 + (\sin\omega - 1)^2}$$

$$l_2(\omega) = \sqrt{(\cos\omega))^2 + (\sin\omega + 1)^2}$$

$$k_1(\omega) = \sqrt{(\cos\omega))^2 + (\sin\omega - 0.9)^2}$$

$$k_2(\omega) = \sqrt{(\cos\omega))^2 + (\sin\omega + 0.9)^2}.$$

- The resulting amplitude response is shown on the next transparency.

- The phase response is expressible as

$$\begin{aligned}\phi(\omega) =& \angle(e^{j\omega} - z_{ze1}) + \angle(e^{j\omega} - z_{ze2}) \\ &- \angle(e^{j\omega} - z_{po1}) - \angle(e^{j\omega} - z_{po2}),\end{aligned}$$

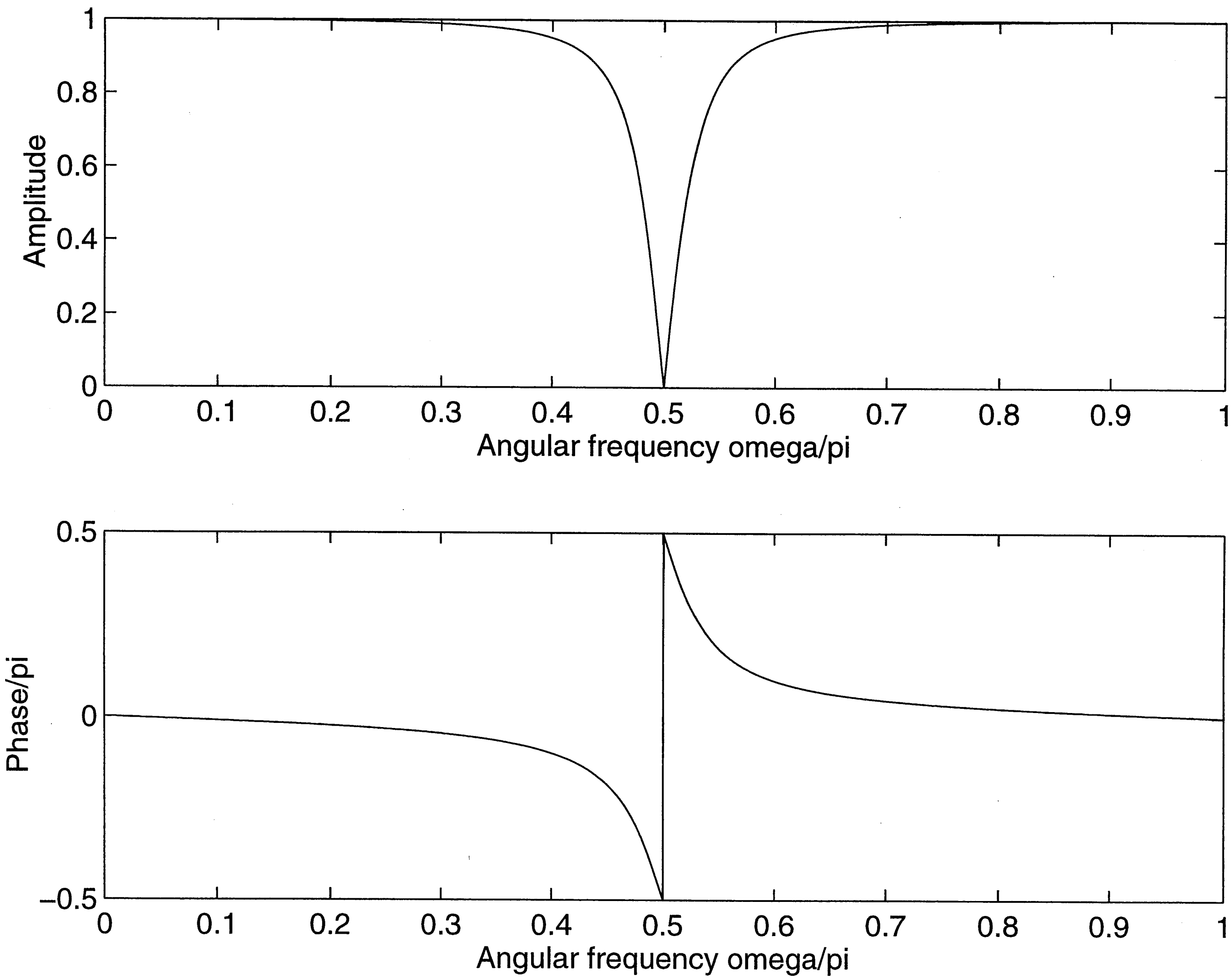
where $\angle(x)$ denotes the angle of x.

- This $\phi(\omega)$ can be rewritten as (see the figure on the next transparency)

$$\begin{aligned}\phi(\omega) =& \arctan\frac{\sin\omega - 1}{\cos\omega} + \arctan\frac{\sin\omega + 1}{\cos\omega} \\ &-\arctan\frac{\sin\omega - 0.9}{\cos\omega} - \arctan\frac{\sin\omega + 0.9}{\cos\omega}.\end{aligned}$$

- Note that there a jump of π in the phase response at $\omega = \pi/2$.

Amplitude and phase responses for the system of Example 6



Properties of the frequency response

- Since $H(e^{j\omega+2k\pi}) = H(e^{j\omega})$ is valid for all integers k, $H(e^{j\omega})$ is periodic with periodicity equal to 2π.

- The next transparency shows the amplitude and phase response for the system of Example 6 in a wider angular frequency range.

- The amplitude response as well as the group delay and phase delay responses are even about $\omega = 0$, that is,

$$|H(e^{-j\omega})| = |H(e^{j\omega})|$$

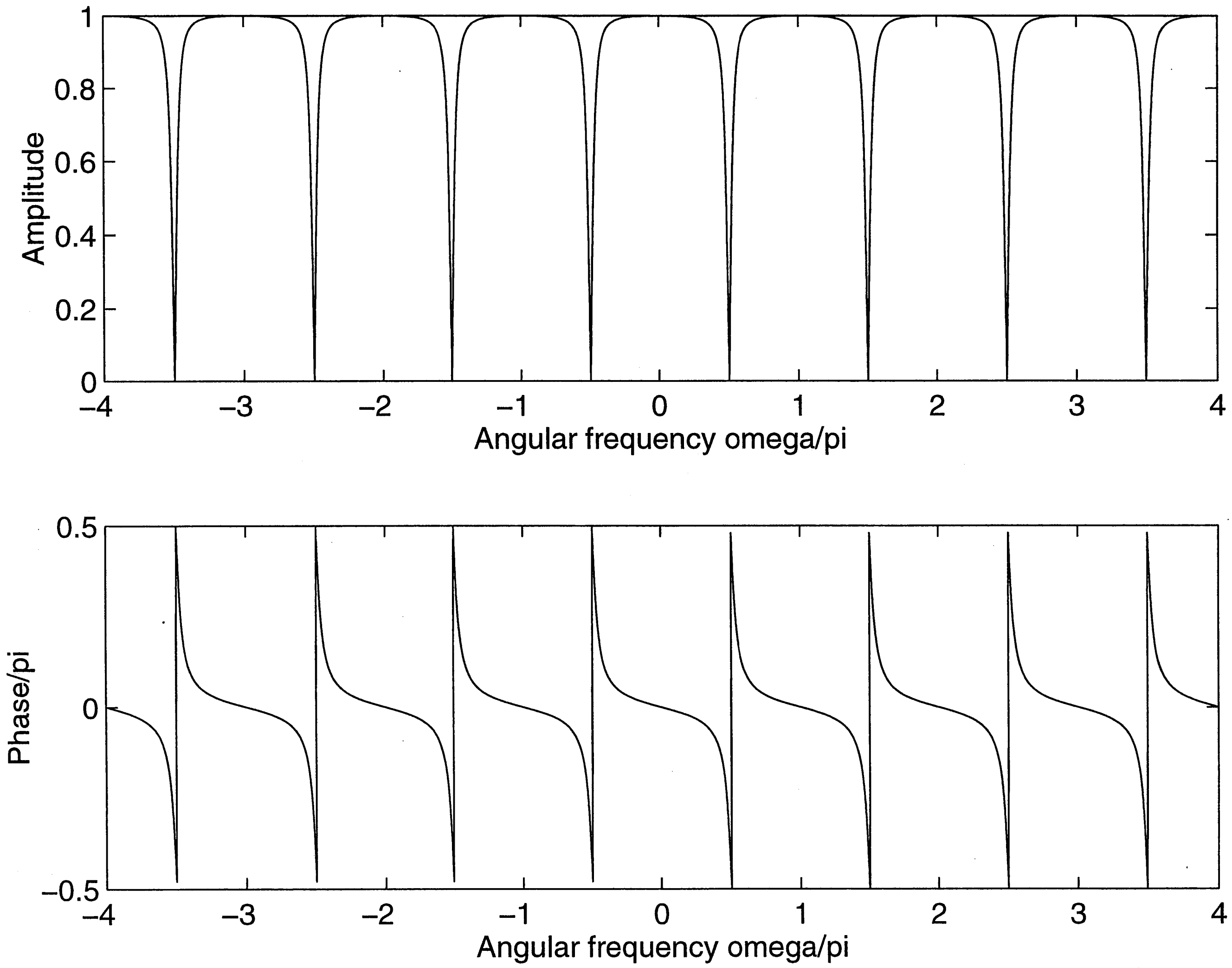
$$\tau_g(-\omega) = \tau_g(\omega)$$

$$\tau_p(-\omega) = \tau_p(\omega).$$

- The phase response is odd about $\omega = 0$, that is,

$$\arg H(e^{-j\omega}) = -\arg H(e^{j\omega}).$$

Amplitude and phase responses for the system of Example 6 in a wider angular frequency range



VARIOUS REPRESENTATION FORMS FOR THE FREQUENCY

- In the above, the frequency response was given in terms of the angular frequency ω.

- If our system with transfer function $H(z)$ is used for processing a continuous-time signal and we assume that both the sampling and resconstruction are perfect, then the relation between the Fourier-transforms of the exitation $x(t)$ and the response $y(t)$ becomes (see Appendix C)

$$Y(2\pi f) = H(e^{j2\pi f/F_s})X(2\pi f),$$

where F_s is the sampling frequency.

- This is valid in the baseband $-F_s/2 \leq f \leq F_s/2$, where f is the 'real frequency'.

- The angular frequency ω is thus related to the real frequency f via

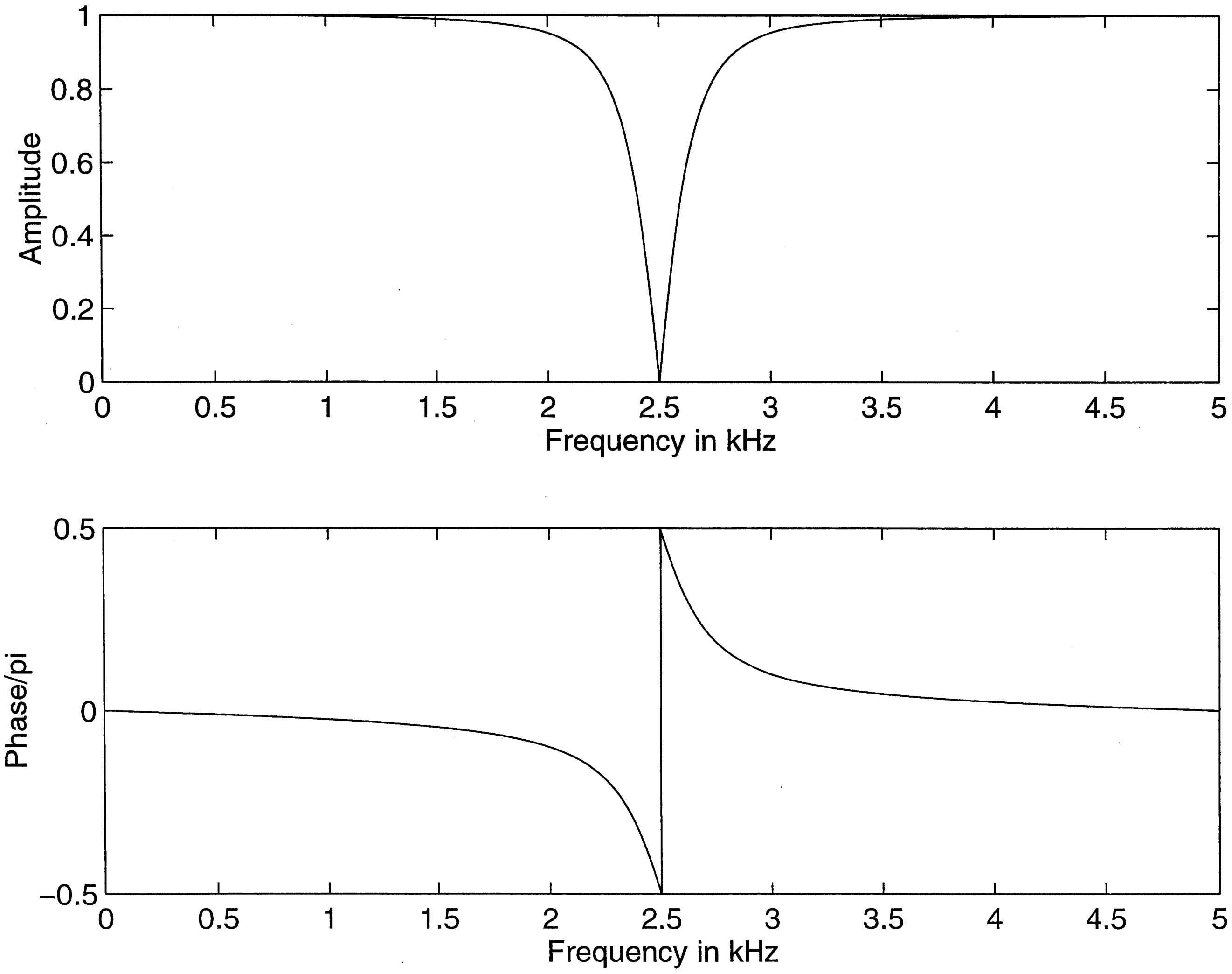
$$\omega = 2\pi f/F_s.$$

- As ω varies from 0 to π, f varies from zero to $F_s/2$. Hence, $\omega = \pi$ corresponds to half the sampling frequency $f = F_s/2$.

- The next transparency shows the amplitude and phase responses for the system of Example 6 for $F_s = 10$ Hz.

- In some cases, the frequency axis is normalized to the sampling rate $F_s = 1$, in which case π corresponds to the frequency $f = 1/2$.

Amplitude and phase responses for the system of Example 6 for $F_s = 10$ Hz.



THE INVERSE z-TRANSFORM

- There exist several techniques to find the sequence $x[n]$ after knowing its z-transform as well as the region of convergence.

- This course concentrates on the technique based on the partial fraction expansion.

- The students are encouraged to read other techniques in textbooks.

- For the sequences considered in this course, the z-transform is expressible as

$$X(z) = \frac{R_M(z)}{P_N(z)} = \frac{\displaystyle\sum_{k=0}^{M} a_k z^{-k}}{1 - \displaystyle\sum_{k=1}^{N} b_k z^{-k}}.$$

- Alternatively, $X(z)$ can be written as

$$X(z) = a_0 \frac{\prod_{k=0}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})},$$

where the d_k's and the c_k's are the poles and zeros of $X(z)$.

- Assuming that there are no repeated poles, $X(z)$ can be expressed as

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}.$$

- Here,

$$A_k = (1 - d_k z^{-1}) X(z)_{|z} = d_k.$$

- For $M < N$, the B_r's are absent.

- Using the notation $b_0 = 1$, B_{M-N}, B_{M-N-1}, $\cdots$, and B_0 can be determined recursively as

$$B_{M-N} = -a_M / b_N$$

and for $k = 1, 2, \cdots, M - N$

$$B_{M-N-k} = \frac{-B_{N-k} - \sum_{l=0}^{k-1} B_{M-N-l} b_{K-r+l}}{b_N}.$$

- If $X(z)$ possesses a sth-order pole at $z = d_i$, then the partial fraction expansion takes the following form:

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1,k\neq i}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - d_i z^{-1})^m}.$$

- The A_k's and B_r's are determined like above.

- The C_m's can be determined as

$$C_m = \frac{1}{(s-m)!(-d_i)^{s-m}} \{ \frac{d^{s-m}}{du^{s-m}} [1 - d_i u]^s X(u^{-1}) \}_{|u} = d_i^{-1}.$$

- The inverse z-transform can now be determined as long as we know the region of convergence, which is in general of the form

$$r_a < |z| < r_b.$$

- For $|d| \leq r_a$, we have the following z-transform pairs:

$$d^n u[n] \iff \frac{1}{1 - dz^{-1}}$$

$$(n+1)d^n u[n] \iff \frac{1}{(1 - dz^{-1})^2}$$

partial fraction expansion takes the following form:

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1,k\neq i}^{N} \frac{A_k}{1-d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1-d_i z^{-1})^m}.$$

- The A_k's and B_r's are determined like above.
- The C_m's can be determined as

$$C_m = \frac{1}{(s-m)!(-d_i)^{s-m}} \{ \frac{d^{s-m}}{du^{s-m}} [1-d_i u]^s X(u^{-1}) \}_{|u\,=\,d_i^{-1}}.$$

- The inverse z-transform can now be determined as long as we know the region of convergence, which is in general of the form

$$r_a < |z| < r_b.$$

- For $|d| \leq r_a$, we have the following z-transform pairs:

$$d^n u[n] \Longleftrightarrow \frac{1}{1-dz^{-1}}$$

$$(n+1)d^n u[n] \Longleftrightarrow \frac{1}{(1-dz^{-1})^2}$$

$$\frac{(n+1)(n+2)}{2!}d^n u[n] \iff \frac{1}{(1-dz^{-1})^3}$$

$$\frac{(n+k-1)!}{n!(k-1)!}d^n u[n] \iff \frac{1}{(1-dz^{-1})^k}$$

- These sequences are nonzero only for $n \leq 0$, that is, they are causal sequences.

- For $|d| \geq r_b$, in turn, we have the following z-transform pairs:

$$-d^n u[-n-1] \iff \frac{1}{1-dz^{-1}}$$

$$-(n+1)d^n u[-n-2] \iff \frac{1}{(1-dz^{-1})^2}$$

$$-\frac{(n+1)(n+2)}{2!}d^n u[-n-3] \iff \frac{1}{(1-dz^{-1})^3}$$

$$-\frac{(-1)^k(-n-1)!}{(-n-k)!(k-1)!}d^n u[-n-k] \iff \frac{1}{(1-dz^{-1})^k},$$

where

$$u[-n-k] = \begin{cases} 1, & n \leq k \\ 0, & \text{otherwise.} \end{cases}$$

- The first sequence is nonzero for $n \leq -1$. The kth sequence is nonzero for $n \leq -k$. These sequences are thus aniticausal.

- For finding the inverse z-transform for $X(z)$ we need in addition the transform pair

$$\delta[n-k] \iff z^{-k}.$$

- It should be pointed out that in the above the d_k's are generally complex-valued.

Example 7: Reconsider the following example which we solved earlier in a rather complicated manner without the use of the z-transform:

- Consider a causal system characterixed by the difference equation $y[n] = ay[n-1] + x[n]$.

(a) What is the unit sample response of our system?

(b) For which values of a the system is stable?

(c) What is the response to the exitation given by

$$x[n] = u[n] - u[n-N] = \begin{cases} 1 & , 0 \leq n \leq N-1 \\ 0 & , \text{otherwise.} \end{cases}$$

(a) Applying the z-transform to both sides of the equation

$$y[n] = ay[n-1] + x[n]$$

and using the linearity and shifting properties of the z-transform (transparency xxx), we obtain

$$Y(z) = az^{-1}Y(z) + X(z) \iff Y(z) = \frac{X(z)}{1 - az^{-1}}.$$

- For the unit sample, $\{\delta[n]\}$,

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1.$$

- The z-transform of the unit sample response, that is, the filter transfer function is thus

$$H(z) = \frac{X(z)}{1 - az^{-1}} = \frac{1}{1 - az^{-1}}.$$

- Since our system is assumed to be causal, the unit sample response satisfies $h[n] = 0$ for $n < 0$ and the region of convergence is $|z| > |a|$.

- Exploting the above-mentioned z-transforms pairs, we obtain

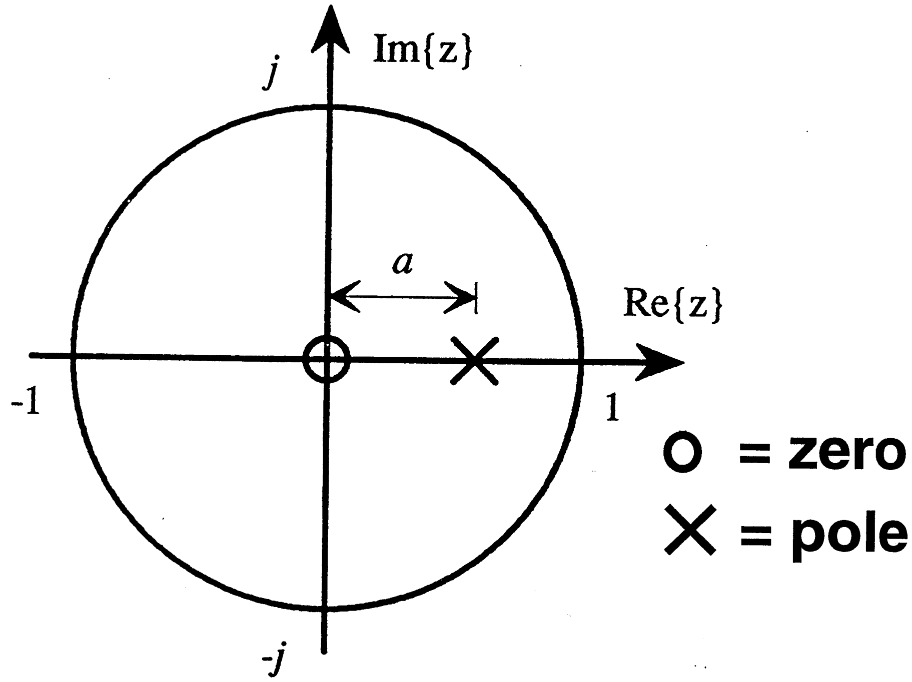
$$H(z) = \frac{1}{1 - az^{-1}} \iff h[n] = a^n u[n].$$

(b) For a stable system, all the poles must be located inside the unit circle, that is, the region of convergence is of the form $|z| > R < 1$ where R is the absolute value of the pole of $H(z)$ which is located fartest away from the origin.

- For our system, the transfer function can be rewritten as

$$H(z) = \frac{z}{z - a},$$

that is, it has a zero $z = 0$ and a pole at $z = a$ (see the figure shown below). Our system is thus stable for $|a| < 1$.



Pole-zero plot for the system of Example 7.

(c) We determine the response of our system $\{y[n]\}$ to the exitation $\{x[n]\} = \{u[n] - u[n-N]\}$ by explot-ing the fact that the z-transform of the output is $Y(z) = H(z)X(z)$, where $H(z)$ is the transfer function determined above, that is, $H(z) = 1/(1-az^{-1})$, and $X(z)$ is the z-transform of of the exitation.

- The z-transform of $\{x[n]\} = \{u[n] - u[n-N]\}$ can be determined by utilizing the facts that the z-transforms of the unit step $u[n]$ and the delayed unit step $u[n-N]$ are $1/(1-z^{-1})$ and $z^{-N}/(1-z^{-1})$, respectively

- We obtain

$$X(z) = \frac{(1-z^{-N})}{1-z^{-1}},$$

- Therefore,

$$Y(z) = H(z)Y(z) = \frac{(1-z^{-N})}{(1-az^{-1})(1-z^{-1})}.$$

- First, we write, using the partial fraction expansion, $1/[(1-az^{-1})(1-z^{-1})]$ as

$$\frac{1}{(1-az^{-1})(1-z^{-1})} = \frac{A_1}{1-az^{-1}} + \frac{A_2}{1-z^{-1}}.$$

- Here,

$$A_1 = \frac{1}{(1-z^{-1})}_{|z=a} = \frac{1}{1-1/a} = \frac{a}{a-1}$$

and

$$A_2 = \frac{1}{(1-az^{-1})}_{|z=1} = \frac{1}{1-a}$$

- Hence,

$$Y(z) = \frac{1}{1-a}\left[\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}}\right] - \frac{z^{-N}}{1-a}\left[\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}}\right]$$

- Since the output is causal, the region of convergence is $|z| > \max\{1, a\}$.

- Utilizing the fact that the z-transform of $x[n-N]$ is related to the z-transform $X(z)$ of $x[n]$ via $z^{-N}X(z)$ and the following z-transform pairs for causal sequences:

$$\frac{1}{1-z^{-1}} \Longleftrightarrow u[n]$$

$$\frac{1}{1-az^{-1}} \Longleftrightarrow a^n u[n],$$

we arrive at

$$y[n] = \frac{1}{1-a}(1-a^{n+1})u[n] - \frac{1}{1-a}(1-a^{n+1-N})u[n-N]$$

$$= \begin{cases} 0, & n < 0 \\ \dfrac{1-a^{(n+1)}}{1-a}, & 0 \le n \le N-1 \\ a^{n+1}\dfrac{a^{-N}-1}{1-a}, & n \ge N. \end{cases}$$

- This is the same result we arrived at earlier without using the z-transform, but with the aid of the z-transform this result was much more easier to obtain.

Example 8: Find the causal sequence $x[n]$ having the following z-transform:

$$X(z) = \frac{z^{-1}}{1 - 0.25z^{-1} - 0.375z^{-2}}.$$

- Locating the poles of $X(z)$ at $z = 0.75$ and $z = -0.5$, $X(z)$ is expressible as

$$X(z) = \frac{z^{-1}}{(1 - 0.75z^{-1})(1 + 0.5z^{-1})}.$$

- Since the order of the numerator is lower than that of the denominator, the corresponding partial fraction expansion is given by

$$X(z) = \frac{A_1}{1 - 0.75z^{-1}} + \frac{A_2}{1 + 0.5z^{-1}}.$$

- Using the formulas described earlier we obtain

$$\begin{aligned} A_1 &= (1 - 0.75z^{-1})X(z)_{|z = 0.75} \\ &= \frac{z^{-1}}{(1 + 0.5z^{-1})}_{|z = 0.75} = \frac{4}{5}. \end{aligned}$$

and

$$\begin{aligned} A_2 &= (1 + 0.5z^{-1})X(z)_{|z = -0.5} \\ &= \frac{z^{-1}}{(1 - 0.75z^{-1})}_{|z = -0.5} = \frac{-4}{5}. \end{aligned}$$

- Therefore,

$$X(z) = \frac{(4/5)}{1 - 0.75z^{-1}} + \frac{-(4/5)}{1 + 0.5z^{-1}}.$$

- Since $x[n]$ is causal, we can exploit the following z-transform pair:

$$\frac{1}{1 - az^{-1}} \Longleftrightarrow a^n u[n],$$

yielding

$$x[n] = \frac{4}{5}[(0.75)^n - (-0.5)^n]u[n].$$

Example 9: Find the causal sequence $x[n]$ having the following z-transform:

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - z^{-1} - 0.3561z^{-2}}.$$

- Locating the poles of $X(z)$ at $z = d, d^* = re^{\pm j\alpha}$, $r = 0.5967$, $\alpha = 0.1838\pi$, $X(z)$ is expressible as

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{(1 - dz^{-1})(1 - d^* z^{-1})}.$$

- Since the orders of the numerator and denominator are the same, the corresponding partial fraction expansion is given by

$$X(z) = B_0 + \frac{A_1}{1 - dz^{-1}} + \frac{A_2}{1 - d^* z^{-1}}$$

.

- Here,

$$B_0 = 1/0.3561 = 2.8082$$

and

$$A_1 = (1 - dz^{-1})X(z)_{|z = d}$$
$$= \frac{1 + 2z^{-1} + z^{-2}}{(1 - d^* z^{-1})}_{|z = d} = Re^{j\beta},$$

where $R = 6.06066$ ja $\beta = -0.5477\pi$.

- A_2 is the complex conjugate of A_1, that is, $A_2 = Re^{-j\beta}$.

- $X(z)$ can thus be rewritten as

$$X(z) = B_0 + \frac{Re^{j\beta}}{1 - re^{j\alpha}z^{-1}} + \frac{Re^{-j\beta}}{1 - re^{-j\alpha}z^{-1}}.$$

- Since $x[n]$ is causal, we can exploit the following z-transform pairs:

$$2Rr^n \cos(n\alpha+\beta)u[n] \iff \frac{Re^{j\beta}}{1 - re^{j\alpha}z^{-1}} + \frac{Re^{-j\beta}}{1 - re^{-j\alpha}z^{-1}}$$

and

$$B_0 \iff B_0\delta[n],$$

giving

$$\begin{aligned} x[n] &= B_0\delta[n] + 2Rr^n \cos(n\alpha + \beta)u[n] \\ &= 2.8082\delta[n] \\ &\quad + 12.1213(0.5967)^n \cos(0.1838\pi n - 0.5477\pi)u[n]. \end{aligned}$$

Example 10: Find the causal sequence $x[n]$ having the following z-transform:

$$X(z) = \frac{z^{-1}}{(1-0.5z^{-1})(1-z^{-1})^2}$$

- $X(z)$ possesses a single pole at $z = 0.5$ and a double pole at $z = 1$.

- Since the numerator order is less than the denominator order, the partial fraction expansion is of the form

$$X(z) = \frac{A_1}{1-0.5z^{-1}} + \frac{C_1}{1-z^{-1}} + \frac{C_2}{(1-z^{-1})^2}.$$

- Here,

$$A_1 = (1-0.5z^{-1})X(z)_{|z=0.5}$$
$$= \frac{z^{-1}}{(1-z^{-1})^2}_{|z=0.5} = 2.$$

- Using the the following formula

$$C_m = \frac{1}{(s-m)!(-d_i)^{s-m}}\{\frac{d^{s-m}}{du^{s-m}}[1-d_i u]^s X(u^{-1})\}_{|u=d_i^{-1}}$$

for a sth-order pole at $z = d_i$, we obtain ($s = 2$ ja

$d_i = 1$)

$$C_1 = \frac{1}{(1)!(-1)^1}\{\frac{d}{du}\frac{[1-u]^2 u}{(1-0.5u)(1-u)^2}\}|u=1$$

$$= -\{\frac{d}{du}\frac{u}{(1-0.5u)}\}|u=1$$

$$= -\{\frac{1-0.5u+0.5u}{(1-0.5u)^2}\}|u=1 = -4$$

and

$$C_2 = \frac{1}{(0)!(-1)^2}\{\frac{[1-u]^2 u}{(1-0.5u)(1-u)^2}\}|u=1$$

$$= \{\frac{u}{(1-0.5u)}\}|u=1 = 2.$$

- Therefore, $X(z)$ takes the following form:

$$X(z) = \frac{2}{1-0.5z^{-1}} + \frac{-4}{1-z^{-1}} + \frac{2}{(1-z^{-1})^2}.$$

- Since our sequence is causal, we can exploit the following z-transform pairs:

$$d^n u[n] \Longleftrightarrow \frac{1}{1-dz^{-1}}$$

and

$$(n+1)d^n u[n] \Longleftrightarrow \frac{1}{(1-dz^{-1})^2},$$

giving

$$x[n] = 2(0.5)^n u[n] - 4(1)^n u[n] + 2(n+1)(1)^n u[n]$$

$$= 2[(n-1) + (0.5)^n]u[n].$$