DESIGN OF FIR FILTERS USING PERIODIC SUBFILTERS AS BASIC BUILDING BLOCKS

- One approach to reduce the cost of implementation of an FIR filter is to construct the overall filter using subfilters whose transfer function is of the form $F(z^L)$.
- The transfer function $F(z^L)$ is obtained by replacing in a conventional transfer function $F(z)$ each unit delay by L unit delays.
- There exist several design techniques some of which are now reviewed.
- We concentrate on the frequency-response masking approach of Lim and the Jing-Fam approach.

Frequency-Response Masking Approach

- In this approach, the overall transfer function is constructed as

$$H(z) = F(z^L)G_1(z) + [z^{-LN_F/2} - F(z^L)]G_2(z),$$

where

$$F(z^L) = \sum_{n=0}^{N_F} f(n)z^{-nL}, \quad f(N_F - n) = f(n),$$

$$G_1(z) = z^{-M_1}\sum_{n=0}^{N_1} g_1(n)z^{-n}, \quad g_1(N_1 - n) = g_1(n),$$

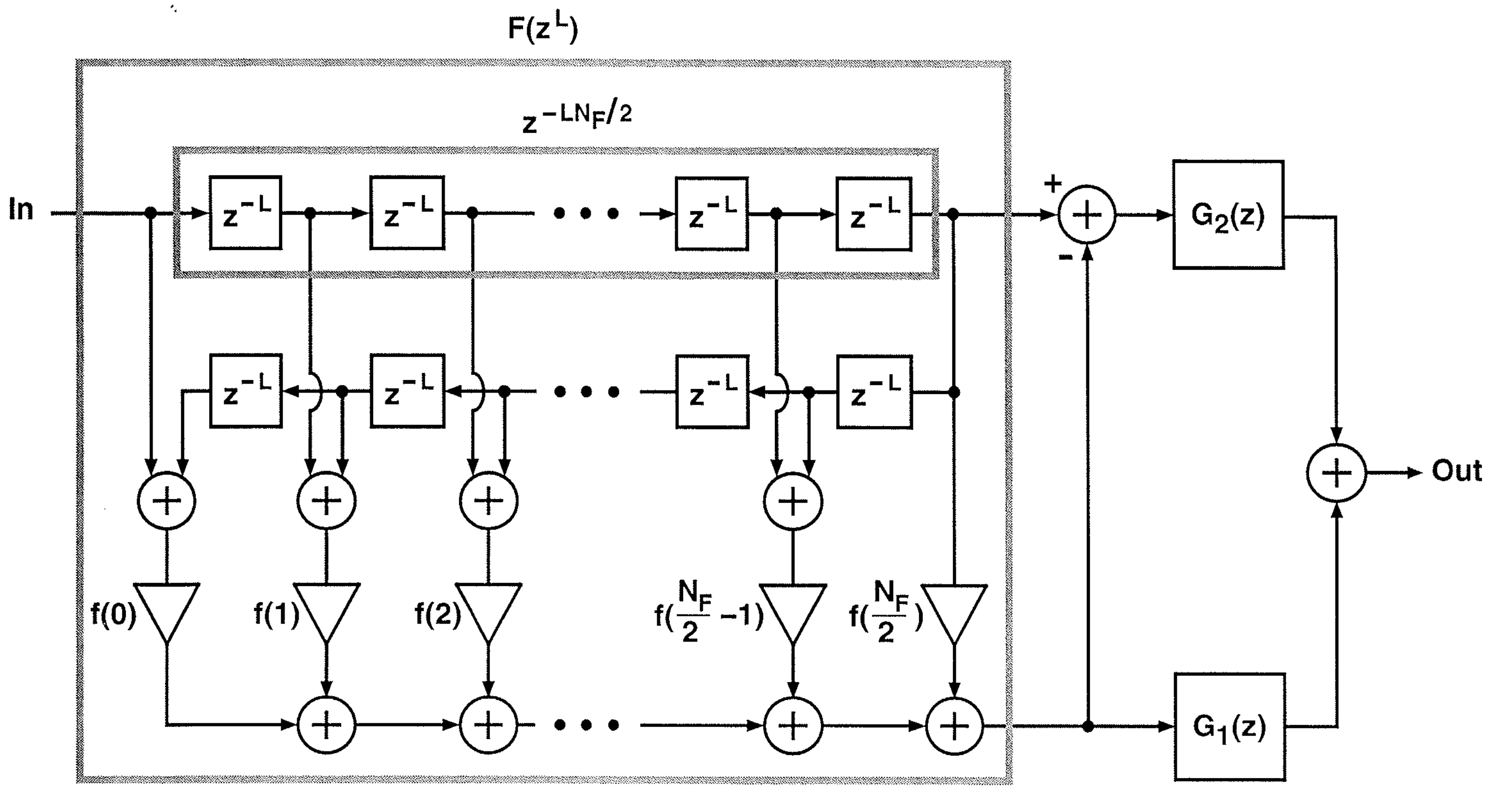
and

$$G_2(z) = z^{-M_2}\sum_{n=0}^{N_2} g_2(n)z^{-n}, \quad g_2(N_2 - n) = g_2(n).$$

- Here, N_F is even, whereas both N_1 and N_2 are either even or odd. For $N_1 \geq N_2$, $M_1 = 0$ and $M_2 = (N_1 - N_2)/2$, whereas for $N_1 < N_2$, $M_1 = (N_2 - N_1)/2$ and $M_2 = 0$.
- These selections guarantee that the delays of both of the terms of $H(z)$ are equal.

Efficient Implementation for the Overall Filter

- The delay term $z^{-LN_F/2}$ is shared with $F(z^L)$.
- Also, $G_1(z)$ and $G_2(z)$ can share their delays if a transposed direct-form implementation (exploiting the coefficient symmetry) is used.



Frequency Response of the Overall Filter

- The frequency response of the overall filter can be written as

$$H(e^{j\omega}) = H(\omega)e^{-j(LN_F+\max\{N_1,N_2\})\omega/2},$$

where $H(\omega)$ denotes the *zero-phase frequency response* of $H(z)$ and can be expressed as

$$H(\omega) = H_1(\omega) + H_2(\omega),$$

where

$$H_1(\omega) = F(L\omega)G_1(\omega), \;\; H_2(\omega) = [1 - F(L\omega)]G_2(\omega) \qquad (A)$$

with

$$F(\omega) = f(N_F/2) + 2\sum_{n=1}^{N_F/2} f(N_F/2 - n)\cos n\omega \qquad (B)$$

and

$$G_k(\omega) = \begin{cases} g_k(N_k/2) + 2\sum_{n=1}^{N_k/2} g_k(N_k/2 - n)\cos n\omega, & N_k \text{ even} \\ 2\sum_{n=0}^{(N_k-1)/2} g_k(\frac{N_k-1}{2} - n)\cos[(n+1/2)\omega], & N_k \text{ odd} \end{cases}$$

for $k = 1, 2$.

Filter Efficiency

- The efficiency as well as the synthesis of $H(z)$ are based on the properties of the pair of transfer functions $F(z^L)$ and $z^{-LN_F/2} - F(z^L)$, which can be generated from the pair of *prototype* transfer functions

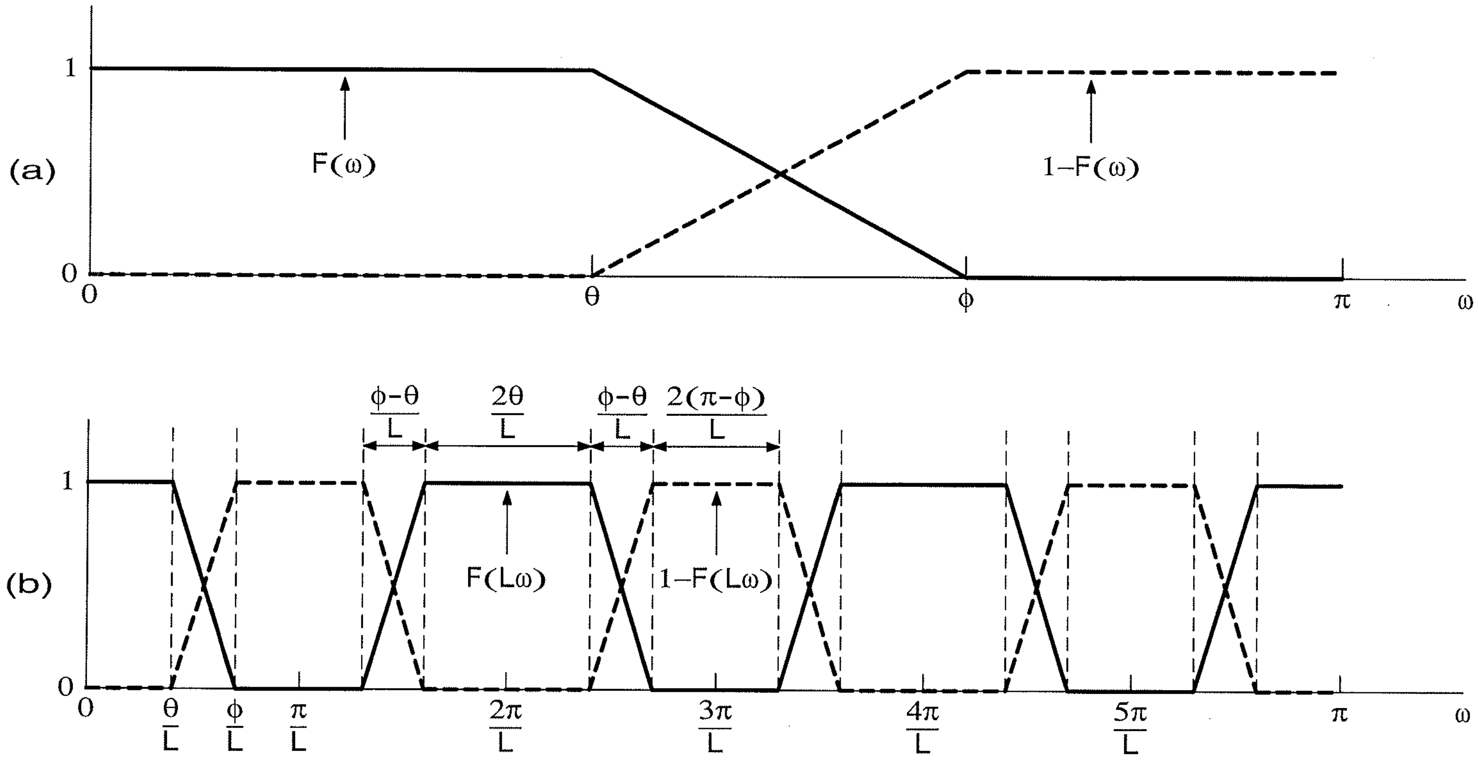
$$F(z) = \sum_{n=0}^{N_F} f(n) z^{-n}$$

and $z^{-N_F/2} - F(z)$ by replacing z^{-1} by z^{-L}.

- The order of the resulting filters is increased to LN_F, but since only every Lth impulse response value is nonzero, the filter complexity (number of adders and multipliers) remains the same.

- The above prototype pair forms a *complementary* filter pair since their zero-phase frequency responses, $F(\omega)$ and $1 - F(\omega)$ with $F(\omega)$ given by Eq. (B), add up to unity.

Generation of a complementary periodic filter pair by starting with a lowpass-highpass complementary pair for $L = 6$.



- Figure (a) illustrates the relations between the responses $F(\omega)$ and $1-F(\omega)$ in the case where $F(z)$ and $z^{-N_F/2} - F(z)$ is a lowpass-highpass filter pair with edges at θ and ϕ.
- The substitution $z^{-L} \mapsto z^{-1}$ preserves the complementary property resulting in the *periodic* responses $F(L\omega)$ and $1-F(L\omega)$, which are frequency-axis compressed versions of the prototype responses such that the interval $[0,\ L\pi]$ is shrunk onto $[0,\ \pi]$ [see Figure (b)].

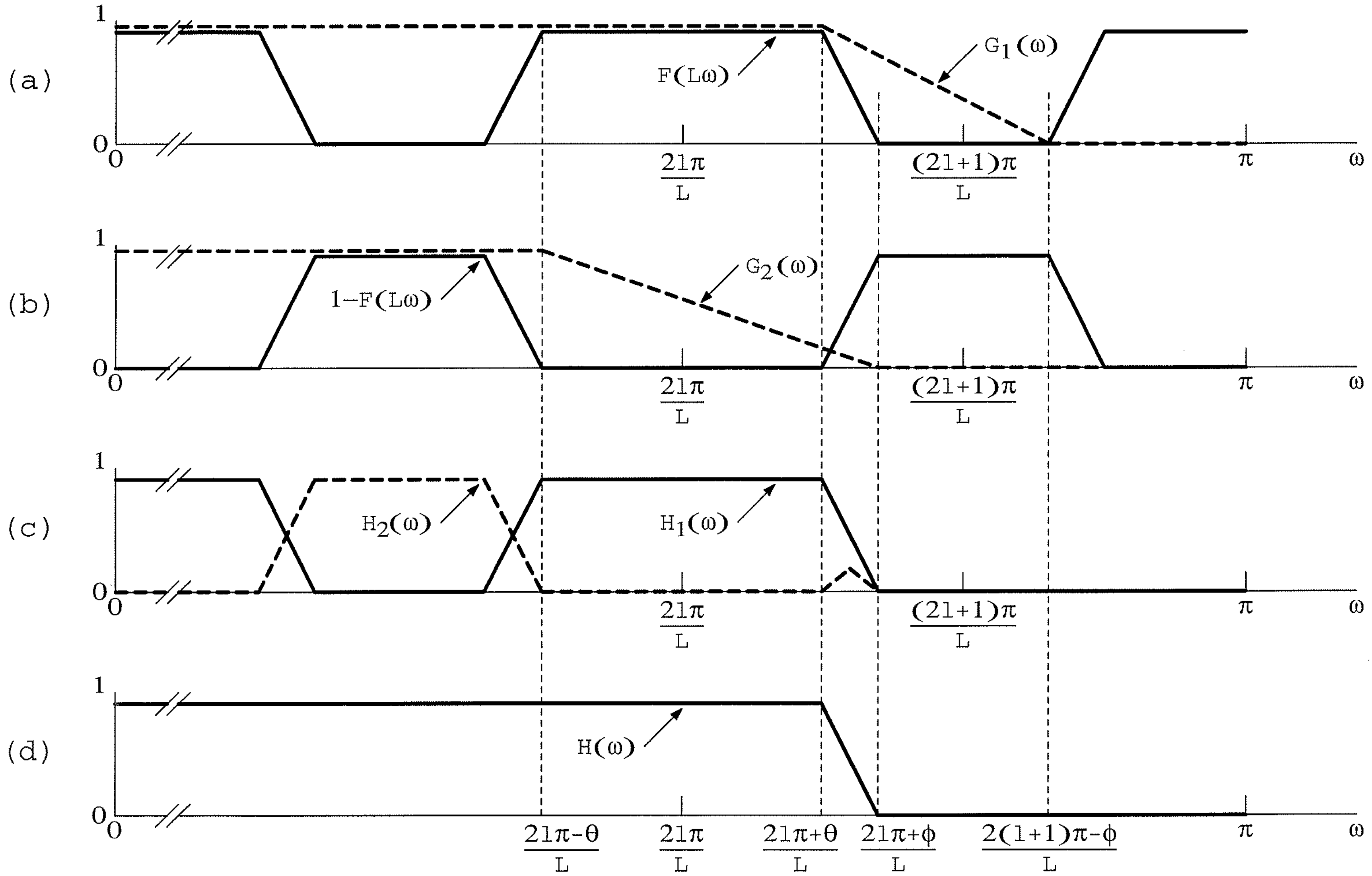
- Since the periodicity of the prototype responses is 2π, the periodicity of the resulting responses is $2\pi/L$ and they contain several passband and stopband regions in the interval $[0, \pi]$.
- For a lowpass filter $H(z)$, one of the transition bands provided by $F(z^L)$ or $z^{-LN_F/2} - F(z^L)$ is used as that of the overall filter.

Case A design of a lowpass filter using the frequency-response masking technique

- In the first case, denoted by Case A, the edges are given by (see the figure shown below)

$$\omega_p = (2l\pi + \theta)/L, \quad \omega_s = (2l\pi + \phi)/L,$$

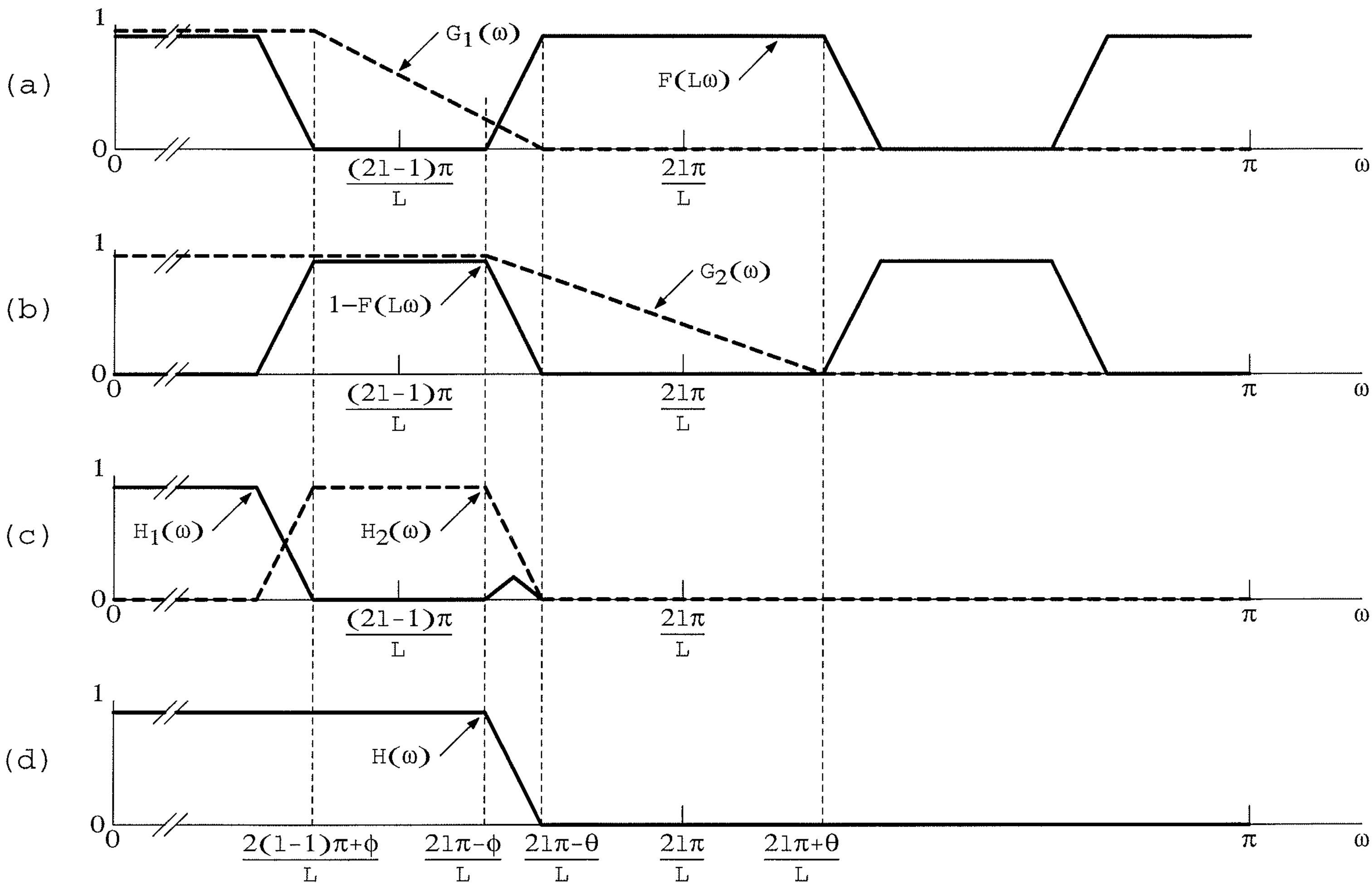
where l is a fixed integer.



Case B design of a lowpass filter using the frequency-response masking technique

- In the second case, referred to as Case B, the edges are given by (see the figure shown below)

$$\omega_p = (2l\pi - \phi)/L, \quad \omega_s = (2l\pi - \theta)/L.$$



Overall Efficiency

- In both cases, the widths of the transition bands are $(\phi - \theta)/L$, which is only $1/L$-th of that of the prototype filters.
- Since the filter order is roughly inversely proportional to the transition band width, this means that the arithmetic complexity of the periodic transfer functions to provide one of the transition bands is only $1/L$-th of that of a conventional nonperiodic filter.
- Note that the orders of both the periodic filters and the corresponding nonperiodic filters are approximately the same, but the conventional filter does not contain zero-valued impulse response samples.

How to exploit the attractive properties of the periodic filters ?

- In order to exploit the attractive properties of the periodic transfer functions, the two low-order masking filters $G_1(z)$ and $G_2(z)$ are designed such that the subresponses $H_1(\omega)$ and $H_2(\omega)$ as given by Eq. (A) approximate in the passband $F(L\omega)$ and $1 - F(L\omega)$, respectively, so that their sum approximates unity, as is desired.
- In the filter stopband, the role of the masking filters is to attenuate the extra unwanted passbands and transition bands of the periodic responses.

Conditions in Case A

- In Case A, this is achieved by selecting the edges of $G_1(z)$ and $G_2(z)$ as (see transparency 8)

$$\omega_p^{(G_1)} = \omega_p = [2l\pi + \theta]/L, \quad \omega_s^{(G_1)} = [2(l+1)\pi - \phi]/L$$
$$\omega_p^{(G_2)} = [2l\pi - \theta]/L, \quad \omega_s^{(G_2)} = \omega_s = [2l\pi + \phi]/L.$$

- Since $F(L\omega) \approx 0$ on $[\omega_s, \ \omega_s^{(G_1)}]$, the stopband region of $G_1(z)$ can start at $\omega = \omega_s^{(G_1)}$, instead of $\omega = \omega_s$.

- Similarly, since $H_1(\omega) \approx F(L\omega) \approx 1$ and $[1 - F(L\omega)] \approx 0$ on $[\omega_p^{(G_2)}, \ \omega_p]$, the passband region of $G_2(\omega)$ can start at $\omega = \omega_p^{(G_2)}$, instead of $\omega = \omega_p$.

Actual Synthesis

- The design of $F(L\omega)$ can be performed conveniently using linear programming as proposed by Lim.
- Another, computationally more efficient, alternative is to use the Remez algorithm.
- Its use is based on the fact that

$$|E_H(\omega)| \leq 1 \quad \text{for} \quad \omega \in \Omega_p^{(F)} \cup \Omega_s^{(F)},$$

where

$$E_H(\omega) = W_H(\omega)[H(\omega) - D(\omega)]$$

is satisfied when $F(\omega)$ is designed such that the maximum absolute value of the error function given in the table of the next transparency becomes less than or equal to unity on $[0,\ \theta] \cup [\phi,\ \pi]$.

Conditions in Case B

- For Case B designs, the required edges of the two masking filters $G_1(z)$ and $G_2(z)$ are (see transparency 9)

$$\omega_p^{(G_1)} = [2(l-1)\pi + \phi]/L, \quad \omega_s^{(G_1)} = \omega_s = [2l\pi - \theta]/L$$

$$\omega_p^{(G_2)} = \omega_p = [2l\pi - \phi]/L, \quad \omega_s^{(G_2)} = [2l\pi + \theta]/L.$$

Filter Design

- Based on the observations of Lim, the design of $H(z)$ with passband and stopband ripples of δ_p and δ_s can be accomplished for both Case A and Case B in the following two steps:

1. Design $G_k(z)$ for $k = 1, 2$ using either the Remez algorithm or linear programming such that $G_k(\omega)$ approximates unity on $[0, \ \omega_p^{(G_k)}]$ with tolerance $0.85\delta_p \cdots 0.9\delta_p$ and zero on $[\omega_s^{(G_k)}, \ \pi]$ with tolerance $0.85\delta_s \cdots 0.9\delta_s$.

2. Design $F(L\omega)$ such that the overall response $H(\omega)$ approximates unity on

$$\Omega_p^{(F)} = \begin{cases} [[2l\pi - \theta]/L, \ [2l\pi + \theta]/L] & \text{for Case A} \\ [[2(l-1)\pi + \phi]/L, \ [2l\pi - \phi]/L] & \text{for Case B} \end{cases}$$

with tolerance δ_p and approximates zero on

$$\Omega_s^{(F)} = \begin{cases} [[2l\pi + \phi]/L, \ [2(l+1)\pi - \phi]/L] & \text{for Case A} \\ [[2l\pi - \theta]/L, \ [2l\pi + \theta]/L] & \text{for Case B} \end{cases}$$

with tolerance δ_s.

Error Function for Designing $F(\omega)$

$$E_F(\omega) = W_F(\omega)[F(\omega) - D_F(\omega)],$$

where

$$D_F(\omega) = [u(\omega) + l(\omega)]/2$$

$$W_F(\omega) = 2/[u(\omega) - l(\omega)]$$

with

$$u(\omega) = \min\big(\Psi_1(\omega) + \psi_1(\omega), \Psi_2(\omega) + \psi_2(\omega)\big)$$

$$l(\omega) = \max\big(\Psi_1(\omega) - \psi_1(\omega), \Psi_2(\omega) - \psi_2(\omega)\big)$$

$$\Psi_k(\omega) = \frac{D_H[h_k(\omega)] - G_2[h_k(\omega)]}{G_1[h_k(\omega)] - G_2[h_k(\omega)]}, \quad k = 1, 2$$

$$\psi_k(\omega) = \frac{1/W_H[h_k(\omega)]}{|G_1[h_k(\omega)] - G_2[h_k(\omega)]|}, \quad k = 1, 2$$

and

$$h_1(\omega) = (2l\pi + \omega)/L$$

$$h_2(\omega) = \begin{cases} (2l\pi - \omega)/L & \text{for } \omega \in [0, \ \theta] \\ [2(l+1)\pi - \omega]/L & \text{for } \omega \in [\phi, \ \pi] \end{cases}$$

for Case A and

$$h_1(\omega) = (2l\pi - \omega)/L$$

$$h_2(\omega) = \begin{cases} (2l\pi + \omega)/L & \text{for } \omega \in [0, \ \theta] \\ [2(l-1)\pi + \omega]/L & \text{for } \omega \in [\phi, \ \pi] \end{cases}$$

for Case B.

- For Step 2 of the above algorithm, $D_H(\omega) = 1$ and $W_H(\omega) = 1/\delta_p$ on $\Omega_p^{(F)}$, whereas $D_H(\omega) = 0$ and $W_H(\omega) = 1/\delta_s$ on $\Omega_s^{(F)}$, giving for $k = 1, 2$

$$D_H[h_k(\omega)] = \begin{cases} 1 & \text{for } \omega \in [0,\ \theta] \\ 0 & \text{for } \omega \in [\phi,\ \pi], \end{cases}$$

$$W_H[h_k(\omega)] = \begin{cases} 1/\delta_p & \text{for } \omega \in [0,\ \theta] \\ 1/\delta_s & \text{for } \omega \in [\phi,\ \pi] \end{cases}$$

for Case A and

$$D_H[h_k(\omega)] = \begin{cases} 1 & \text{for } \omega \in [0,\ \theta] \\ 0 & \text{for } \omega \in [\phi,\ \pi], \end{cases}$$

$$W_H[h_k(\omega)] = \begin{cases} 1/\delta_s & \text{for } \omega \in [0,\ \theta] \\ 1/\delta_p & \text{for } \omega \in [\phi,\ \pi] \end{cases}$$

for Case B.

- Even though the resulting error function looks very complicated, it is straightforward to use the subroutines EFF and WATE in the Remez algorithm described by McClellan, Parks, and Rabiner for optimally designing $F(z)$.

Reductions of the Orders of $G_1(z)$ and $G_2(z)$

- The order of $G_1(z)$ can be considerably reduced by allowing larger ripples on the those regions of $G_1(z)$ where $F(L\omega)$ has one of its stopbands.
- As a rule of thumb, the ripples on these regions can be selected to be ten times larger.
- Similarly, the order of $G_2(z)$ can be decreased by allowing (ten times) larger ripples on those regions where $F(L\omega)$ has one of its passbands.

Practical Filter Design

- In practice, ω_p and ω_s are given and l, L, θ, and ϕ must be determined.
- To ensure that Case A yields a desired solution with $0 \leq \theta < \phi \leq \pi$, it is required that (see transparency 8)

$$\frac{2l\pi}{L} \leq \omega_p, \quad \omega_s \leq \frac{(2l+1)\pi}{L}$$

for some positive integer l.

- In this case,

$$l = \lfloor L\omega_p/(2\pi) \rfloor, \;\; \theta = L\omega_p - 2l\pi, \;\; \phi = L\omega_s - 2l\pi.$$

- Similarly, to ensure that Case B yields a desired solution with $0 \leq \theta < \phi \leq \pi$, it is required that (see transparency 9)

$$\frac{(2l-1)\pi}{L} \leq \omega_p, \quad \omega_s \leq \frac{2l\pi}{L}$$

for some positive integer l.

- In this case

$$l = \lceil L\omega_s/(2\pi) \rceil, \;\; \theta = 2l\pi - L\omega_s, \;\; \phi = 2l\pi - L\omega_p.$$

- For any set of ω_p, ω_s, and L, either Case A or Case B (not both) will yield the desired θ and ϕ, provided that L is not too large.
- If $\theta = 0$ or $\phi = \pi$, then the resulting specifications for $F(\omega)$ are meaningless and the corresponding value of L cannot be used.

Optimizing L

- The remaining problem is to determine L to minimize the number of multipliers, which is $N_F/2 + 1 + \lfloor (N_1+2)/2 \rfloor + \lfloor (N_2+2)/2 \rfloor$ or $N_F + N_1 + N_2 + 3$ depending on whether the symmetries in the filter coefficients are exploited or not.
- Hence, in both cases, a good measure of the filter complexity is the sum of the orders of the subfilters.
- Instead of determining the actual minimum filter orders for various values of L, the computational workload can be significantly reduced based on the use of the formula

$$N \approx \Phi(\delta_p, \delta_s)/(\omega_s - \omega_p),$$

where

$$\begin{aligned}\Phi(\delta_p, \delta_s) =& 2\pi[0.005309(\log_{10}\delta_p)^2 + 0.07114\log_{10}\delta_p \\ & - 0.4761]\log_{10}\delta_s \\ & - 2\pi[0.00266(\log_{10}\delta_p)^2 + 0.5941\log_{10}\delta_p \\ & + 0.4278].\end{aligned}$$

- This formula gives the estimated minimum order for a conventional direct-form FIR filter with passband and stopband edges at ω_p and ω_s, respectively, and passband and stopband ripples of δ_p and δ_s.
- Since the widths of transition bands of $F(z)$, $G_1(z)$, and $G_2(z)$ are $\phi - \theta$, $(2\pi - \phi - \theta)/L$, and $(\phi + \theta)/L$, respectively, good estimates for the corresponding filter orders are

$$N_F \approx \frac{\Phi(\delta_p, \delta_s)}{\phi - \theta}, \quad N_1 \approx \frac{L\Phi(\delta_p, \delta_s)}{2\pi - \phi - \theta}, \quad N_2 \approx \frac{L\Phi(\delta_p, \delta_s)}{\phi + \theta}.$$

- For the optimum nonperiodic direct-form design, the transition bandwidth is $\omega_s - \omega_p = (\phi - \theta)/L$, giving

$$N_{\mathrm{opt}} \approx \frac{L\Phi(\delta_p, \delta_s)}{\phi - \theta}.$$

- The sum of the subfilter orders can be expressed in terms of N_{opt} as follows

$$N_{\mathrm{ove}} = N_{\mathrm{opt}}\Big[\frac{1}{L} + \frac{\phi - \theta}{2\pi - \phi - \theta} + \frac{\phi - \theta}{\phi + \theta}\Big]. \qquad (A)$$

- The smallest values of $N_{\rm ove}$ are typically obtained at those values of L for which $\theta + \phi \approx \pi$ and, correspondingly, $2\pi - \theta - \phi \approx \pi$.
- In this case, $N_1 \approx N_2$ and Eq. (A) reduces, after substituting $\phi - \theta = L(\omega_s - \omega_p)$, to

$$N_{\rm ove} = N_{\rm opt}\Big[\frac{1}{L} + 2L(\omega_s - \omega_p)/\pi\Big].$$

- At these values of L, N_F decreases and $N_1 \approx N_2$ increases inversely proportionally to L with the minimum of $N_{\rm ove}$,

$$N_{\rm ove} = 2N_{\rm opt}\sqrt{\frac{2(\omega_s - \omega_p)}{\pi}},$$

taking place at

$$L_{\rm opt} = 1\Big/\sqrt{\frac{2(\omega_s - \omega_p)}{\pi}}\,. \qquad (B)$$

- If for $L = L_{\rm opt}$, $\theta + \phi$ is not approximately equal to π, then L minimizing the filter complexity can be found in the near vicinity of $L_{\rm opt}$.
- The following example illustrates the use of the above estimation formulas.

Example: $\omega_p = 0.4\pi$, $\omega_s = 0.402\pi$, $\delta_p = 0.01$, **and** $\delta_s = 0.001$

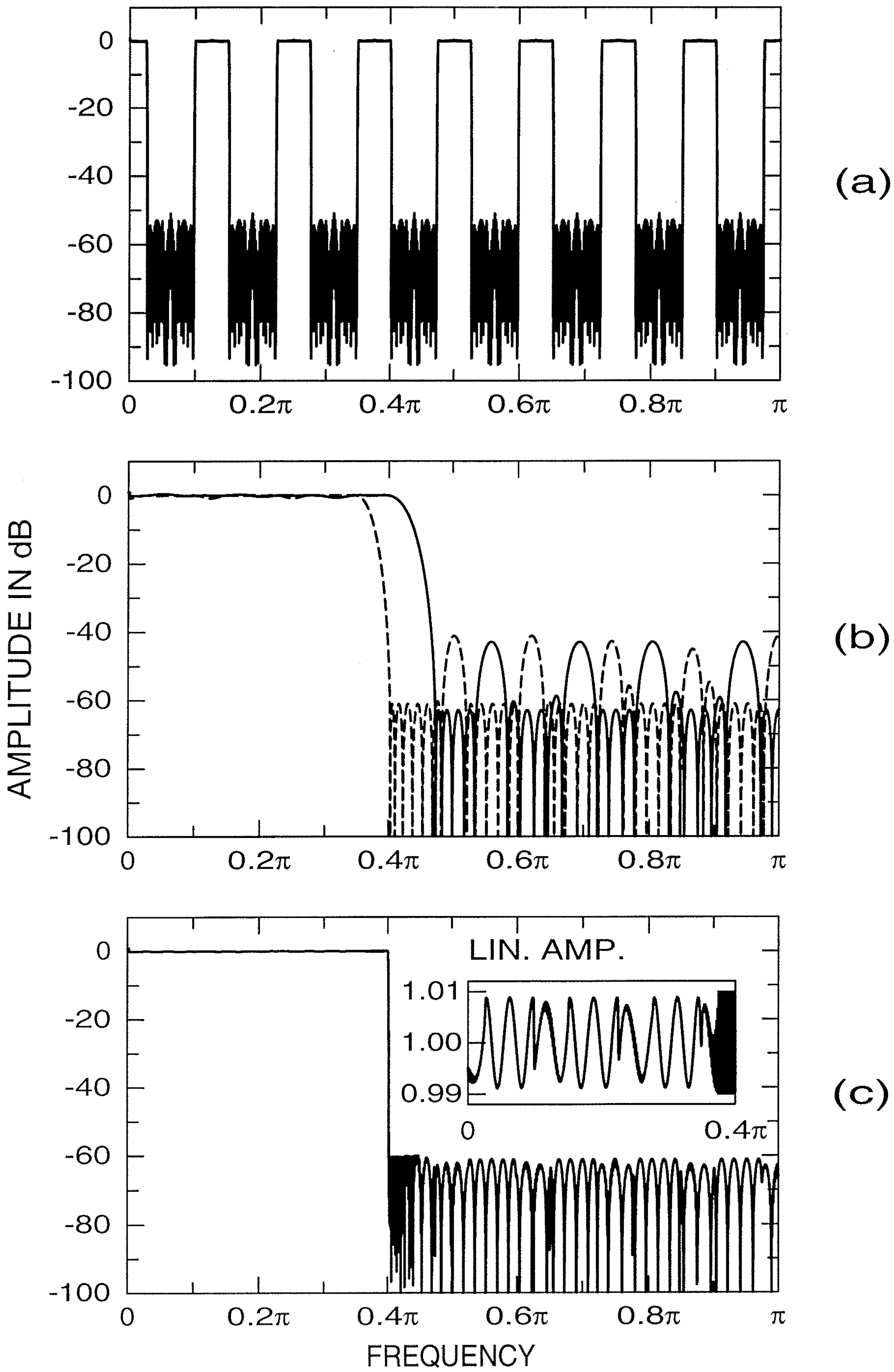
- For the optimum conventional direct-form design, $N_{\rm opt} = 2541$, requiring 1271 multipliers when the coefficient symmetry is exploited. Eq. (B) gives $L_{\rm opt} = 16$.
- The table of transparency 26 shows, for the admissible values of L in the vicinity of this value, l, θ, ϕ, the estimated orders for the subfilter, and the sum of the subfilter orders as well as whether the overall filter is a Case A or Case B design.
- Also with the estimated filter orders of the table, $L = 16$ gives the best result.
- The actual filter orders are $N_F = 162$, $N_1 = 70$, and $N_2 = 98$.
- The responses of the subfilters as well as that of the overall design are depicted in transparency 27.

- The overall number of multipliers and adders for this design are 168 and 330, respectively, which are 13 % of those required by an equivalent conventional direct-form design (1271 and 2541).
- The overall filter order is 2690, which is only 6 % higher than that of the direct-form design (2541).

Estimated Filter Orders for the Admissible Values of L in the Vicinity of $L_{\text{opt}} = 16$

L	Case	l	θ	ϕ	N_F	N_1	N_2	$N_F + N_1 + N_2$
8	B	2	0.784π	0.8π	318	98	26	442
9	B	2	0.382π	0.4π	282	38	58	378
11	A	2	0.4π	0.422π	232	47	69	348
12	A	2	0.8π	0.824π	212	162	38	412
13	B	3	0.774π	0.8π	196	155	43	394
14	B	3	0.372π	0.4π	182	58	92	332
16	A	3	0.4π	0.432π	160	70	98	328
17	A	3	0.8π	0.834π	150	236	54	440
18	B	4	0.764π	0.8π	142	210	58	410
19	B	4	0.362π	0.4π	134	78	128	340
21	A	4	0.4π	0.442π	122	92	128	342
22	A	4	0.8π	0.844π	116	314	68	498

Amplitude responses for a filter synthesized using the frequency-response masking approach



Multistage Frequency-Response Masking Approach

- If the order of $F(z)$ is too high, its complexity can be reduced by implementing it using the frequency-response masking technique.
- Extending this to an arbitrary number of stages results in the multistage frequency-response masking approach, where $H(z)$ is generated iteratively as

$$\begin{aligned}
H(z) \equiv F^{(0)}(z) =& F^{(1)}(z^{L_1})G_1^{(1)}(z) \\
&+ [z^{-L_1 N_F^{(1)}/2} - F^{(1)}(z^{L_1})]G_2^{(1)}(z) \\
F^{(1)}(z) =& F^{(2)}(z^{L_2})G_1^{(2)}(z) \\
&+ [z^{-L_2 N_F^{(2)}/2} - F^{(2)}(z^{L_2})]G_2^{(2)}(z) \\
\vdots \quad & \vdots \quad \vdots \\
F^{(R-1)}(z) =& F^{(R)}(z^{L_R})G_1^{(R)}(z) \\
&+ [z^{-L_R N_F^{(R)}/2} - F^{(R)}(z^{L_R})]G_2^{(R)}(z).
\end{aligned}$$

- Here, the $G_1^{(r)}(z)$'s and $G_2^{(r)}(z)$'s for $r = 1, 2, \cdots, R$ as well as $F^{(R)}(z)$ are the filters to be designed.

- For implementation purposes, $H(z)$ can be expressed in the form shown in the table of the next transparency.
- Transparency 31 shows an efficient implementation for a three-stage filter, where the delay terms z^{-M_3}, z^{-m_2}, and z^{-m_1} can be shared with $F^{(3)}(z^{\widehat{L}_3})$.
- In order to obtain a desired overall solution, the orders of the $G_1^{(r)}(z)$'s and $G_2^{(r)}(z)$'s for $r = 2, 3, \cdots, R$, denoted by $N_1^{(r)}$ and $N_2^{(r)}$ in table of the next transparency, have to be even.

Implementation Form for the Transfer Function in the Multistage Frequency-Response Approach

$$H(z) \equiv F^{(0)}(z^{\widehat{L}_0}) = F^{(1)}(z^{\widehat{L}_1})G_1^{(1)}(z^{\widehat{L}_0}) + [z^{-M_1} - F^{(1)}(z^{\widehat{L}_1})]G_2^{(1)}(z^{\widehat{L}_0})$$

$$F^{(1)}(z^{\widehat{L}_1}) = F^{(2)}(z^{\widehat{L}_2})G_1^{(2)}(z^{\widehat{L}_1}) + [z^{-M_2} - F^{(2)}(z^{\widehat{L}_2})]G_2^{(2)}(z^{\widehat{L}_1})$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$F^{(R-1)}(z^{\widehat{L}_{R-1}}) = F^{(R)}(z^{\widehat{L}_R})G_1^{(R)}(z^{\widehat{L}_{R-1}}) + [z^{-M_R} - F^{(R)}(z^{\widehat{L}_R})]G_2^{(R)}(z^{\widehat{L}_{R-1}}),$$

where

$$\widehat{L}_0 = 1, \quad \widehat{L}_r = \prod_{k=1}^{r} L_k, \quad r = 1, 2, \cdots, R$$

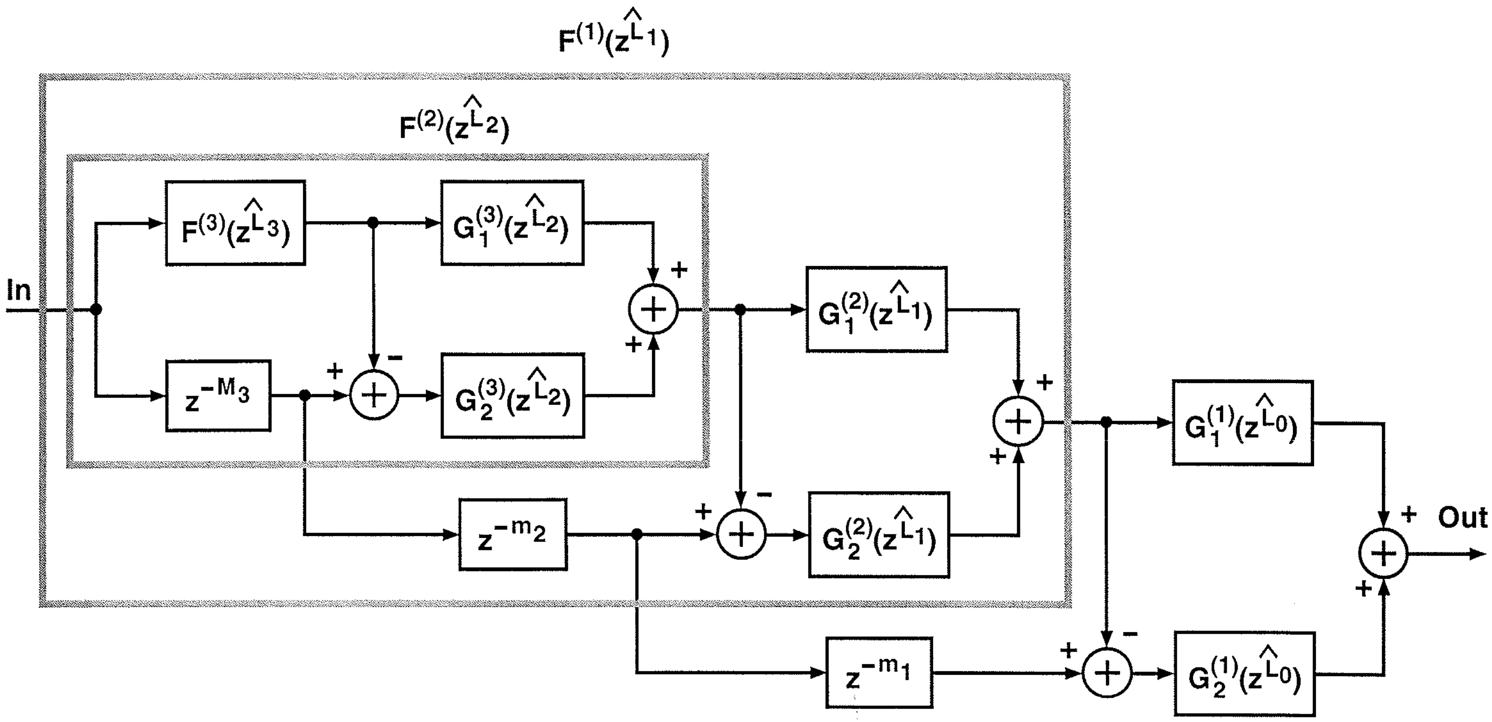
$$M_R = \widehat{L}_R N_F^{(R)}/2, \quad M_{R-r} = M_{R-r+1} + m_{R-r}, \quad r = 1, 2, \cdots, R-1$$

$$m_{R-r} = \widehat{L}_{R-r} \max\{N_1^{(R-r+1)}, N_2^{(R-r+1)}\}/2, \quad r = 1, 2, \cdots, R-1$$

$N_F^{(R)}$ is the order of $F^{(R)}(z)$.

$N_1^{(r)}$ and $N_2^{(r)}$ are the orders of $G_1^{(r)}(z)$ and $G_2^{(r)}(z)$, respectively.

An implementation for a filter synthesized using the three-stage frequency-response masking approach



Multistage Filter Optimization

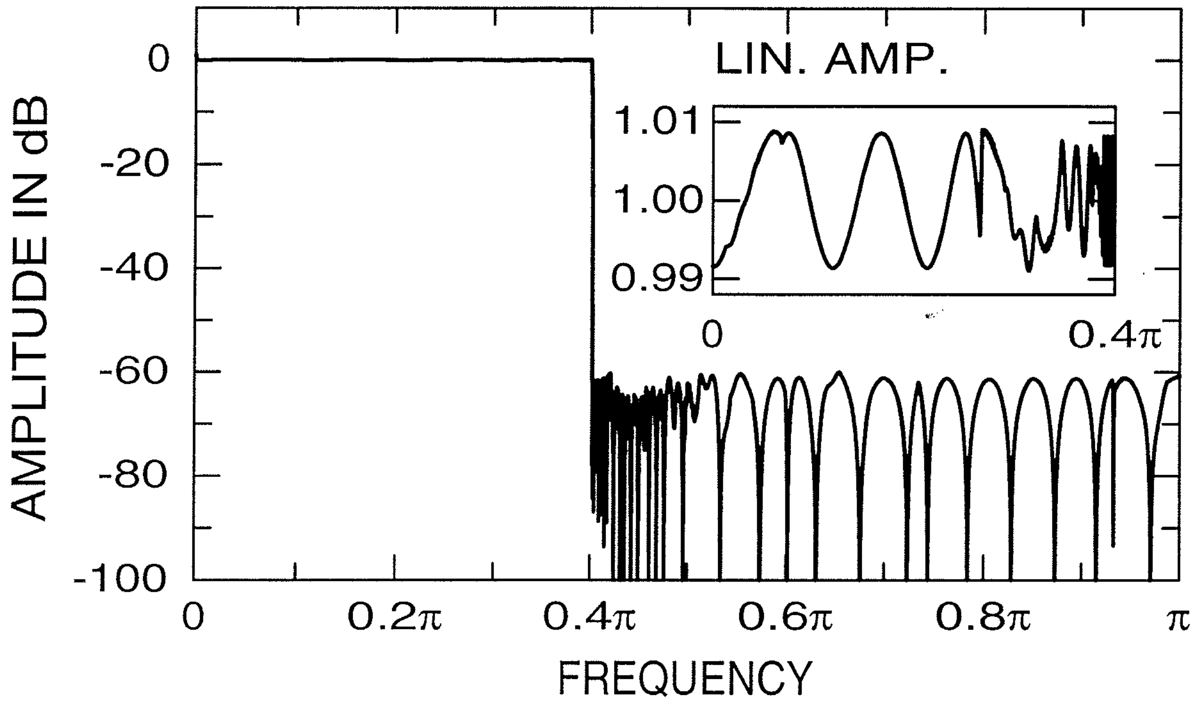
- See the article T. Saramäki, "Design of Computationally Efficient FIR Filters Using Periodic Subfilters as Building Blocks".

Example: $\omega_p = 0.4\pi$, $\omega_s = 0.402\pi$, $\delta_p = 0.01$, **and** $\delta_s = 0.001$

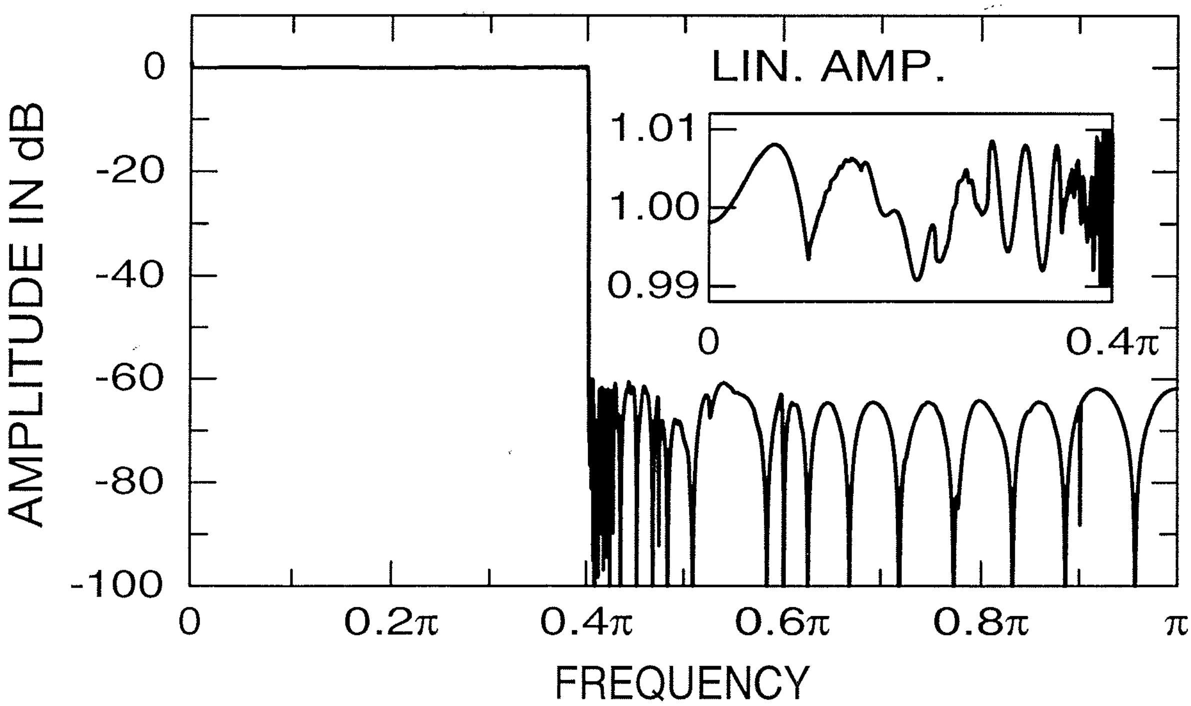
- For a two-stage design, the best result is obtained by $L_1 = L_2 = 6$.
- The minimum orders of $G_1^{(1)}(z)$, $G_2^{(1)}(z)$, $G_1^{(2)}(z)$, $G_2^{(2)}(z)$, and $F^{(2)}(z)$ are 26, 40, 28, 36, and 74, respectively (see Figure (a) in transparency 35).
- Compared with the conventional direct-form FIR filter of order 2541, the number of multipliers and adders required by this design (107 and 204) are only 8 % at the expense of a 15 % increase in the overall filter order (to 2920).
- For a three-stage design, the best result is obtained by $L_1 = L_2 = L_3 = 4$.
- The minimum orders of $G_1^{(1)}(z)$, $G_2^{(1)}(z)$, $G_1^{(2)}(z)$, $G_2^{(2)}(z)$, $G_1^{(3)}(z)$, $G_2^{(3)}(z)$, and $F^{(3)}(z)$ are 16, 28, 18, 24, 16, 32, and 40, respectively (see Figure (b) in transparency 35).

- The number of multipliers and adders (94 and 174) are only 7% of those required by the direct-form equivalent at the expense of a 26 % increase in the overall filter order (to 3196).

Amplitude responses for filters synthesized using the multistage frequency-response masking approach. (a) Two-stage filter. (b) Three-stage filter.



(a)



(b)

Jing-Fam Approach

- Another general approach for designing multiplier-efficient FIR filters has been proposed by Jing and Fam.
- This approach is based on the iterative use of the following two facts:
  - A narrowband filter $(\omega_p, \omega_s < \pi/2)$ can be effectively implemented as

$$H(z) = F(z^L)G(z). \qquad (A)$$

  - A wideband filter $(\omega_p, \omega_s > \pi/2)$ can be implemented as

$$H(z) = z^{-M} - (-1)^M \widehat{H}(-z),$$

where $\widehat{H}(z)$ is a narrowband filter, which can be effectively implemented in the form of Eq. (A).

Design of Narrowband Lowpass Filters

- When ω_s is less than $\pi/2$, then the overall filter can be synthesized in the simplified form

$$H(z) = F(z^L)G(z) = \sum_{n=0}^{N_F} f(n)z^{-nL} \sum_{n=0}^{N_G} g(n)z^{-n},$$

where N_F and N_G can be even or odd.

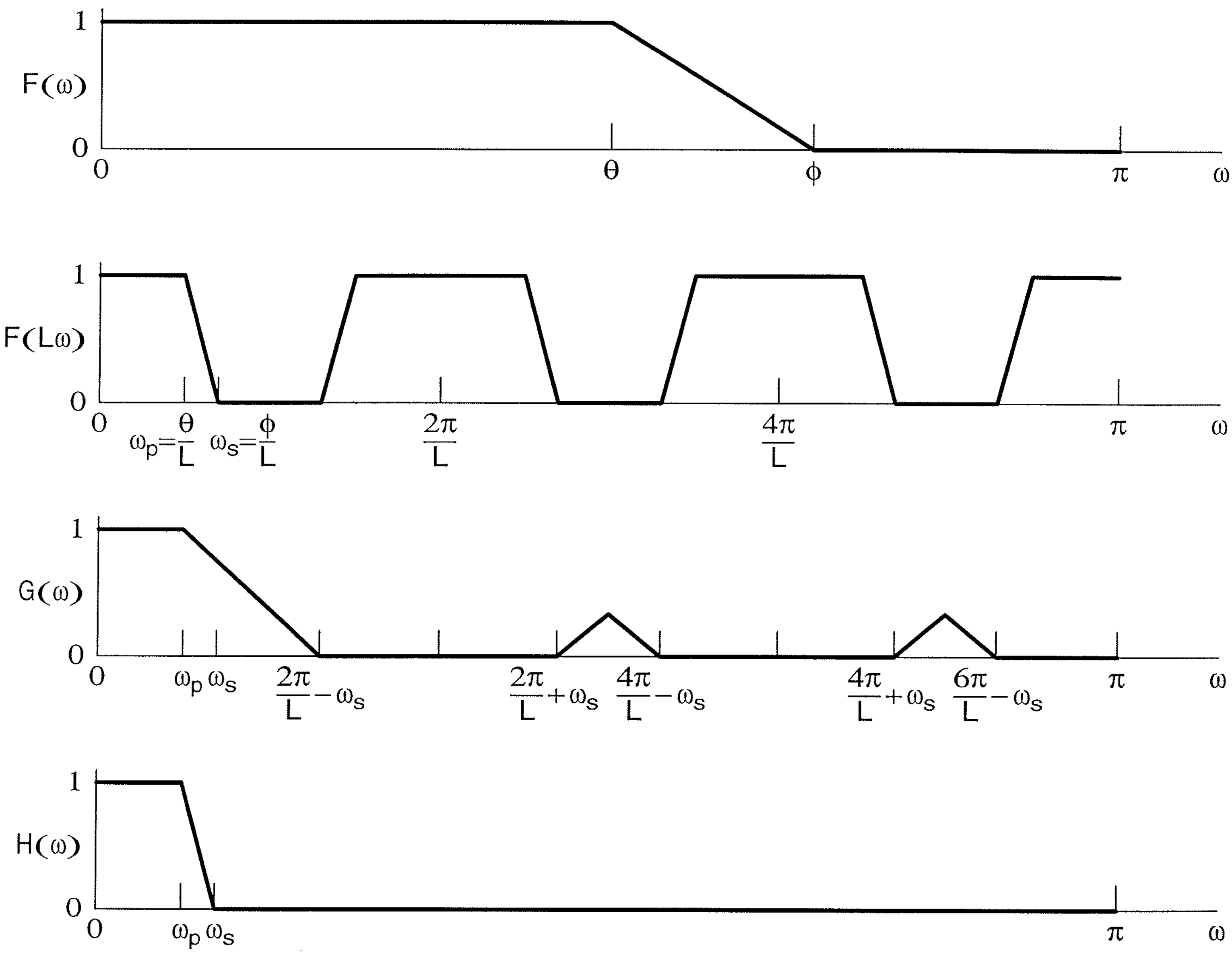
- The zero-phase frequency response of this filter can be written as

$$H(\omega) = F(L\omega)G(\omega).$$

- If the passband and stopband edges of $F(z)$ are θ and ϕ, then the edges of the first transition band of $F(L\omega)$ are (see the figure in the following transparency)

$$\omega_p = \theta/L, \quad \omega_s = \phi/L.$$

Design of a narrowband lowpass filter using a cascade of a periodic and a non-periodic filter



- $F(L\omega)$ does not provide the desired attenuation in the regions where it has extra unwanted passband regions, i.e., in the region

$$\Omega_s(L,\omega_s) = \bigcup_{k=1}^{\lfloor L/2 \rfloor} [k\frac{2\pi}{L} - \omega_s, \min(k\frac{2\pi}{L} + \omega_s, \pi)].$$

- Therefore, the role of the nonperiodic filter $G(z)$ is to provide the desired attenuation in this region.

Filter Design

- The simplest way to determine $F(z)$ and $G(z)$ such that $H(z)$ is a lowpass design with edges ω_p and ω_s and ripples δ_p and δ_s is to design these subfilters using the MPR algorithm to satisfy

$$\begin{aligned}
1-\delta_p^{(F)} \le F(\omega) \le 1+\delta_p^{(F)} \quad &\text{for} \quad \omega \in [0, L\omega_p] \\
-\delta_s \le F(\omega) \le \delta_s \quad &\text{for} \quad \omega \in [L\omega_s, \pi] \\
1-\delta_p^{(G)} \le G(\omega) \le 1+\delta_p^{(G)} \quad &\text{for} \quad \omega \in [0, \omega_p] \\
-\delta_s \le G(\omega) \le \delta_s \quad &\text{for} \quad \omega \in \Omega_s(L, \omega_s),
\end{aligned}$$

where

$$\delta_p^{(G)} + \delta_p^{(F)} = \delta_p.$$

- The ripples $\delta_p^{(F)}$ and $\delta_p^{(G)}$ can be selected, e.g., to be half the overall ripple δ_p.
- In this case, both $F(z^L)$ and $G(z)$ have $[0, \omega_p]$ as a passband region.

Another Alternative

- The second alternative, resulting in a considerably reduced order of $G(z)$, is to design simultaneously $F(\omega)$ to meet

$$1-\delta_p \le F(\omega)G(\omega/L) \le 1+\delta_p \quad \text{for} \quad \omega \in [0, L\omega_p]$$
$$-\delta_s \le F(\omega)G(\omega/L) \le \delta_s \quad \text{for} \quad \omega \in [L\omega_s, \pi]$$

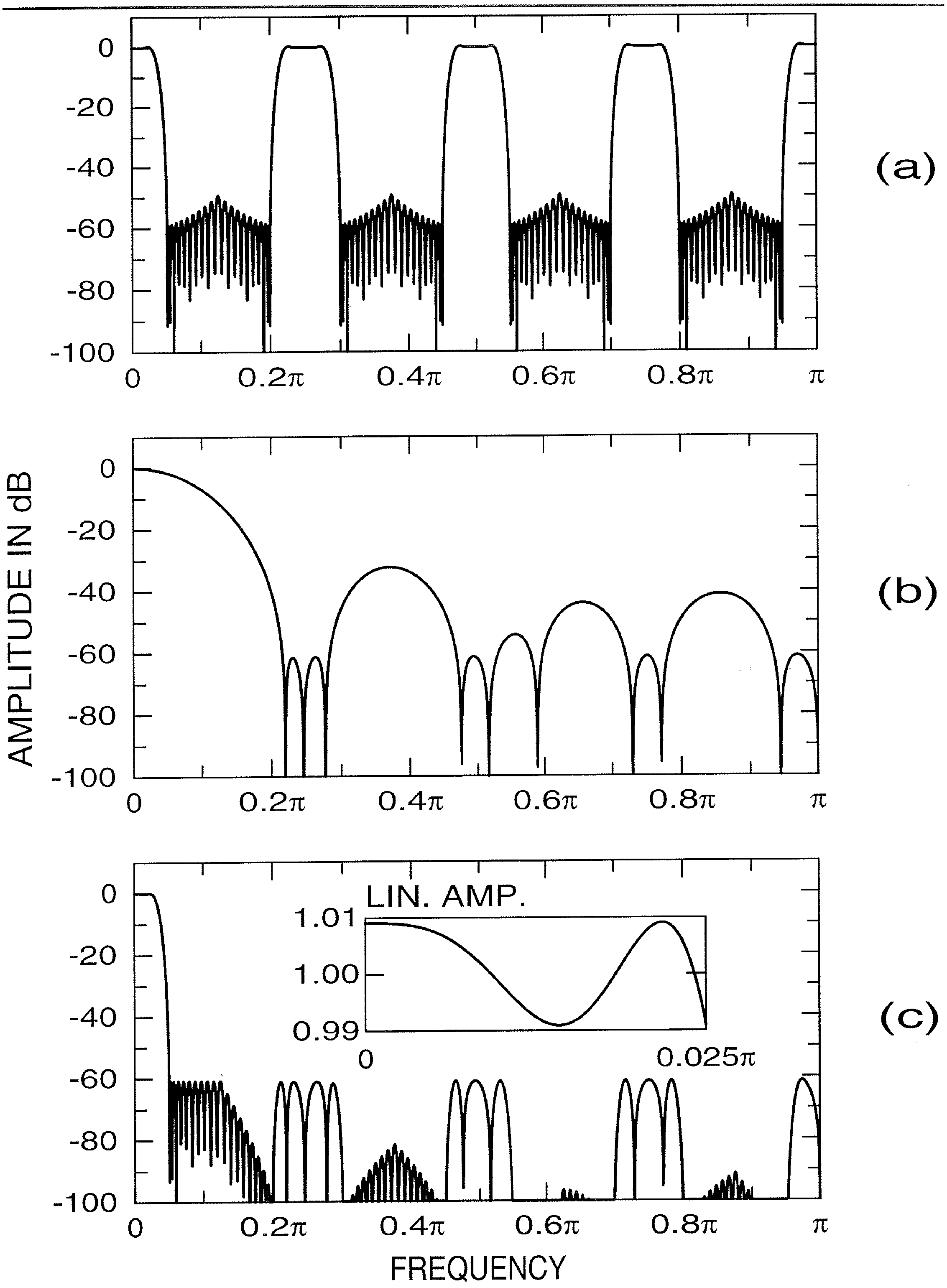
and $G(\omega)$ to meet

$$G(0) = 1$$

$$-\delta_s \le F(L\omega)G(\omega) \le \delta_s \quad \text{for} \quad \omega \in \Omega_s(L, \omega_s).$$

- $G(z)$ has all its zeros on the unit circle and concentrates on providing for the overall filter the desired attenuation on $\Omega_s(L, \omega_s)$ (see the following transparency).
- $F(L\omega)$ equalizes the passband distortion caused by $G(\omega)$ and provides for the overall filter the desired attenuation in its stopband regions.

Optimized Overall Response: $\omega_p = 0.025\pi$, $\omega_s = 0.05\pi$, $\delta_p = 0.01$, **and** $\delta_s = 0.001$. $L = 8$, $N_F = 26$, **and** $N_G = 19$.



Design Algorithm

1. Set $F(\omega) \equiv 1$.

2. Determine $G(\omega)$ using the MPR algorithm to minimize on $[0, \epsilon] \cup \Omega_s(L, \omega_s)$ the peak absolute value of $E_G(\omega) = W_G(\omega)[G(\omega) - D_G(\omega)]$, where

$$D_G(\omega) = \begin{cases} 1 & \text{for } \omega \in [0, \epsilon] \\ 0 & \text{for } \omega \in \Omega_s(L, \omega_s) \end{cases}$$

$$W_G(\omega) = \begin{cases} \alpha & \text{for } \omega \in [0, \epsilon] \\ F(L\omega) & \text{for } \omega \in \Omega_s(L, \omega). \end{cases}$$

3. Determine $F(\omega)$ using the MPR algorithm to minimize on $[0, L\omega_p] \cup [L\omega_s]$ the peak absolute value of $E_F(\omega) = W_F(\omega)[F(\omega) - D_F(\omega)]$, where

$$D_F(\omega) = \begin{cases} 1/G(\omega/L) & \text{for } \omega \in [0, L\omega_p] \\ 0 & \text{for } \omega \in [L\omega_s, \pi] \end{cases}$$

$$W_G(\omega) = \begin{cases} G(\omega/L) & \text{for } \omega \in [0, L\omega_p] \\ \delta_p G(\omega/L)/\delta_s & \text{for } \omega \in [L\omega_s, \pi]. \end{cases}$$

4. Repeat Steps 1 and 2 until the difference between successive solutions is within the given tolerance limits.

Optimization of L

- The remaining problem is to optimize L and the orders of $G(z)$ and $F(z^L)$ to minimize the number of multipliers.
- For the order of $F(z^L)$ in z^L, a good estimate is

$$N_F = N/L,$$

where N is the minimum order of a conventional nonperiodic FIR filter to meet the given overall criteria.
- For the order of $G(z)$, a good estimate has been found to be

$$N_G = \cosh^{-1}(1/\delta_s)\Big[\frac{1}{X(\omega_p, 2\pi/L - (\omega_p + 2\omega_s)/3)} + \frac{L/2}{X(L\omega_p/2, \pi - L(\omega_p + 2\omega_s)/6)}\Big],$$

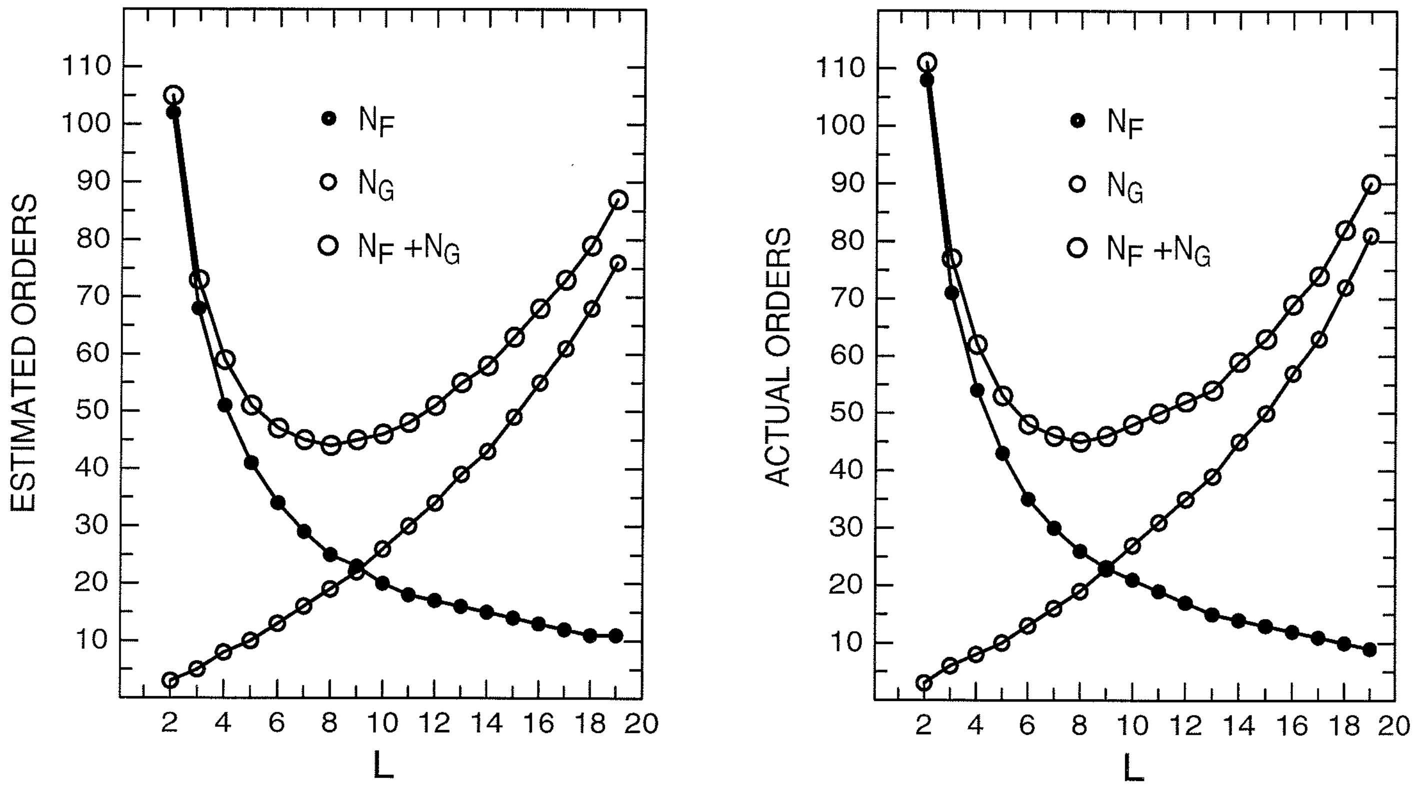
where

$$X(\omega_1, \omega_2) = \cosh^{-1}\Big[\frac{2\cos\omega_1 - \cos\omega_2 + 1}{1 + \cos\omega_2}\Big].$$

- These estimates are rather accurate so that the optimum L can be determined based on these estimates.
- L has to be selected such that the stopband edge of $F(z)$, $\phi = L\omega_s$, is less than π.
- This means that L must be less than π/ω_s.
- After estimating the required orders for $G(z)$ and $F(z)$, the remaining problem is to decrease or increase the orders to find the actual minimum orders.
- Since the frequency-response-shaping responsibilities are very well shared with the subfilters, the minimum orders can be found rather independently.
- First, the minimum order of $F(z)$ can be determined for the estimated order of $G(z)$ and, then, the minimum order of $G(z)$ is determined.

Example

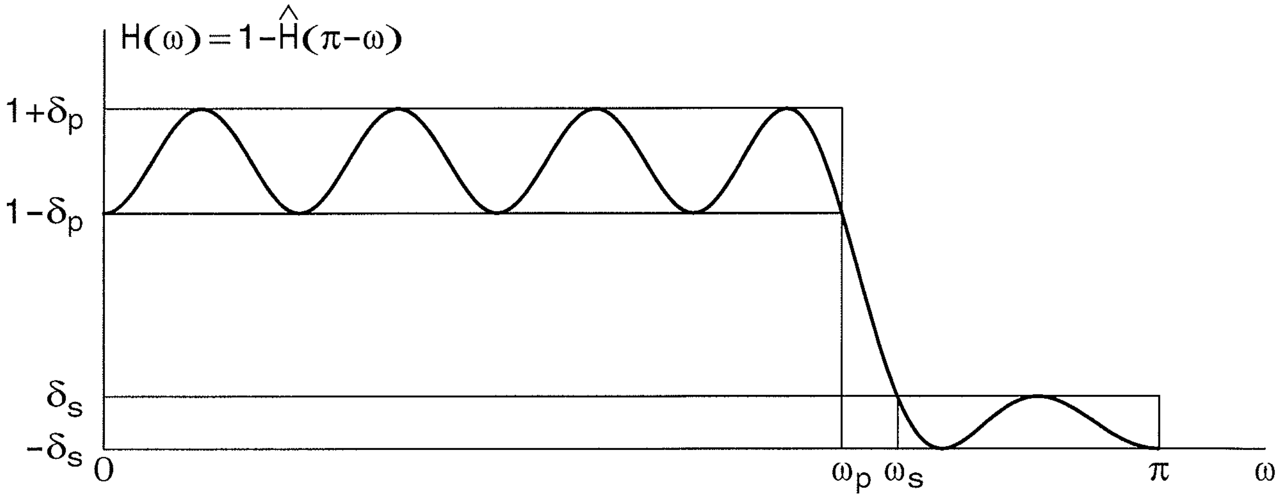
- The specifications are: $\omega_p = 0.025\pi$, $\omega_s = 0.05\pi$, $\delta_p = 0.01$, $\delta_s = 0.001$. The following figure gives for admissible values of L $(2 \leq L \leq 19)$ both the estimated and actual orders for N_F and N_G as well as $N_F + N_G$, which can be used as a measure of the filter complexity.



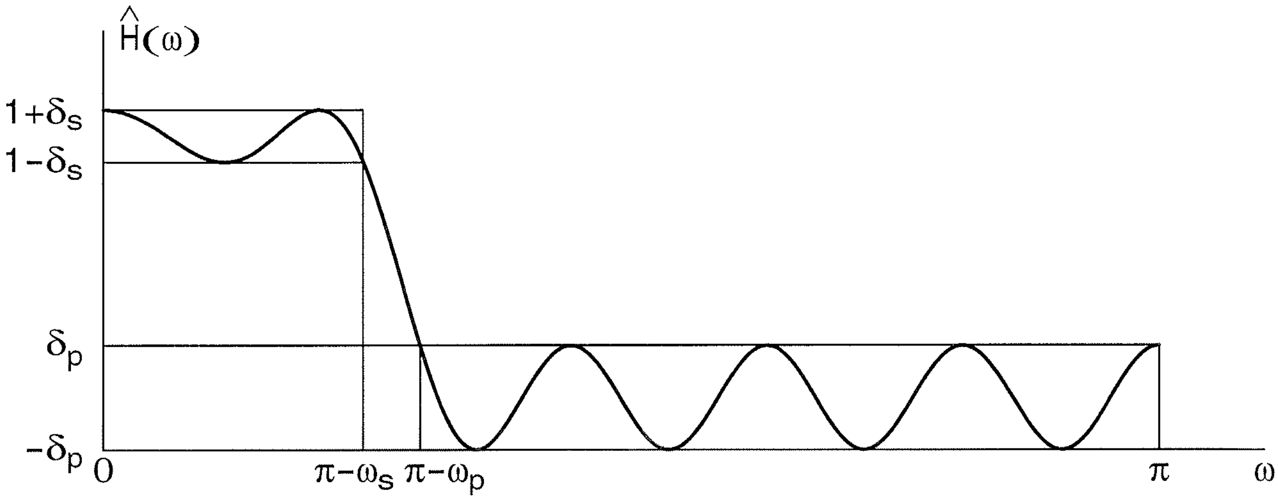
- The estimated orders are very close to the actual ones.
- The overall amplitude response for the optimum value of L, $L = 8$, has been depicted in transparency 42.
- This design ($L = 8$, $N_F = 26$, and $N_G = 19$) requires 24 multipliers and 45 adders.
- The minimum order of a conventional nonperiodic direct-form FIR design is 216, requiring 109 multipliers and 216 adders.
- The price paid for these 80 % reductions in the filter complexity is a 5 % increase in the overall filter order (from 216 to 227).

Design of Wideband Lowpass Filters

- A wideband filter can be implemented as
$$H(z) = z^{-M} - (-1)^M \widehat{H}(-z),$$
where $\widehat{H}(z)$ is a narrowband FIR filter of order $2M$ (see pages 149 and 151 in the section considering the design of FIR filters in the minimax sense).
- The ripples and edges of $H(z)$ and $\widehat{H}(z)$ are related through (see the following figures)
$$\widehat{\omega}_p = \pi - \omega_s, \quad \widehat{\omega}_s = \pi - \omega_p, \quad \widehat{\delta}_p = \delta_s, \quad \widehat{\delta}_s = \delta_p.$$



(a)



(b)

- Hence, if ω_p and ω_s of the desired filter are larger than $\pi/2$, then $\widehat{\omega}_p$ and $\widehat{\omega}_s$ of $\widehat{H}(z)$ are less than $\pi/2$.
- This enables us to design $\widehat{H}(z)$ in the form

$$\widehat{H}(z) = F(z^L)G(z)$$

using the techniques introduced previously.
- The resulting overall transfer function is

$$H(z) = z^{-M} - (-1)^M F((-z)^L)G(-z),$$

where $M = (LN_F + N_G)/2$ is half the order of $F((-z)^L)G(-z)$.
- An implemetation of the resulting filter is shown in the next transparency.
- To avoid half sample delays, the order has to be even.

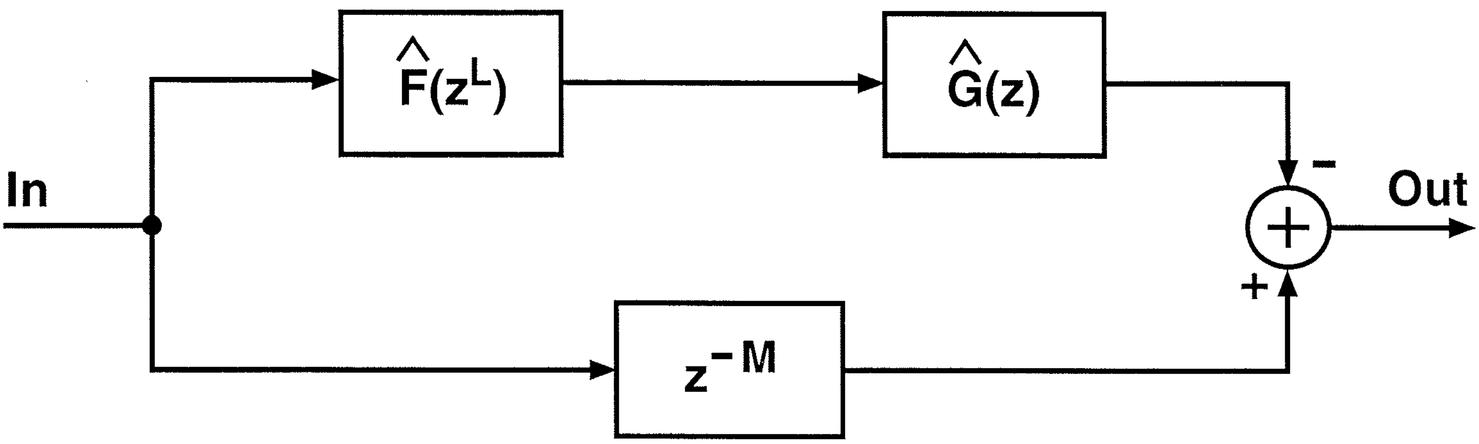
Computationally Efficient Implementation for a Wideband Filter

$$H(z) = z^{-M} - \widehat{F}(z^L)\widehat{G}(z),$$

where

$$\widehat{F}(z^L) = (-1)^M F((-z)^L), \quad \widehat{G}(z) = G(-z).$$

- The delay term z^{-M} can be shared with $\widehat{F}(z^L)$.

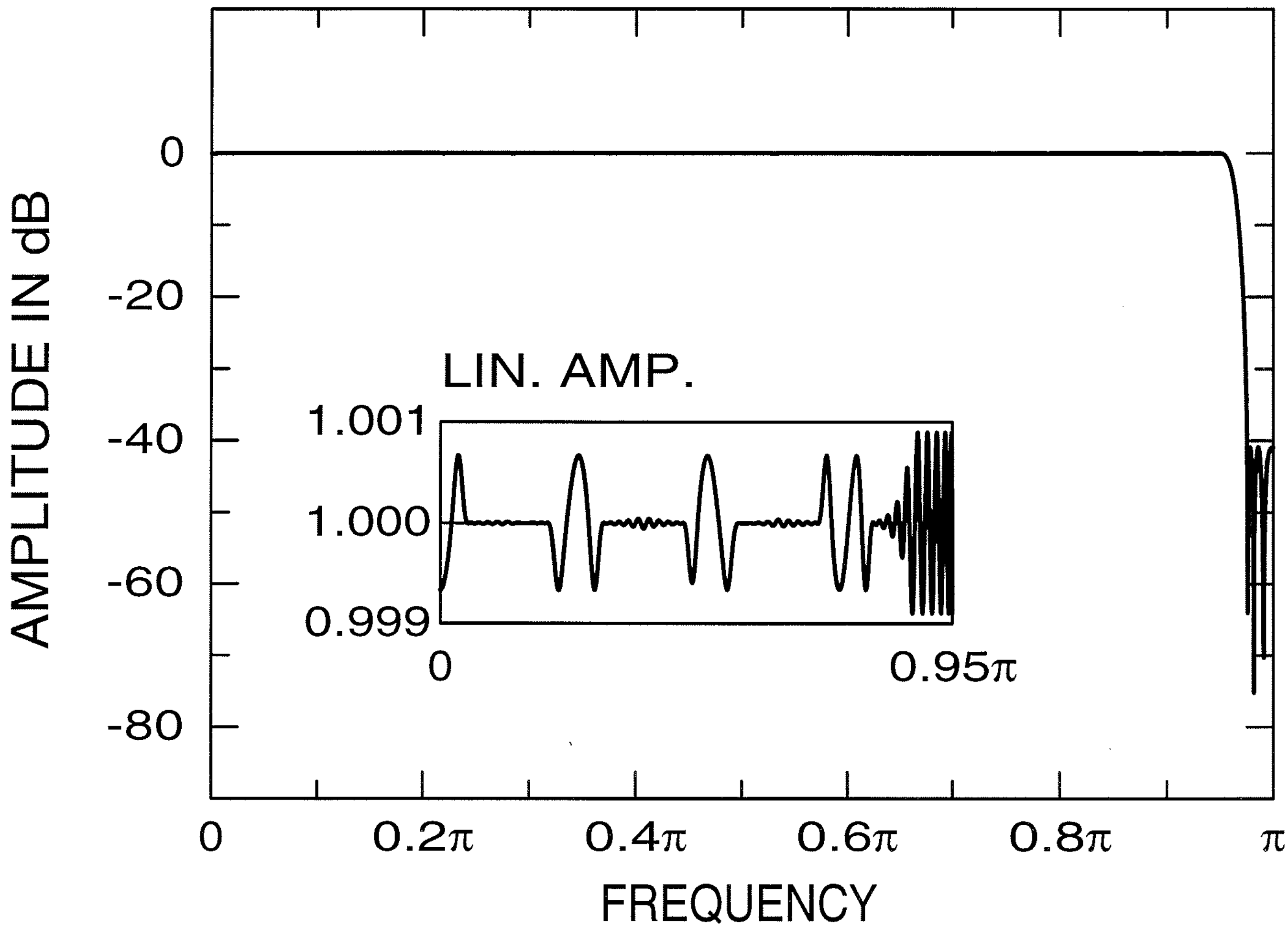


Example: $\omega_p = 0.95\pi$, $\omega_s = 0.975\pi$, $\delta_p = 0.001$, **and** $\delta_s = 0.01$

- The specifications of $\widehat{H}(z)$ become $\widehat{\omega}_p = 0.025\pi$, $\widehat{\omega}_s = 0.05\pi$, $\widehat{\delta}_p = 0.01$, and $\widehat{\delta}_s = 0.001$.
- These are the narrowband specifications of considered in transparencies 42, 46, and 47.
- Therefore, the desired wideband design is obtained by using the earlier subfilters $F(z^L)$ and $G(z)$ ($L = 8$, $N_F = 26$, and $N_G = 19$).
- However, the overall order of the filter is odd (227) and the resulting delay contains a half sample delay.
- Therefore, to achieve the desired solution with even order, the order of $G(z)$ has to be increased by one ($N_G = 20$).
- The amplitude response of the resulting filter is given in the next transparency.

Amplitude Response for a Optimized Wideband Filter

- This design requires 25 multipliers, 46 adders, and 228 delay elements.
- The corresponding numbers for a conventional direct-form equivalent of order 216 are 109, 216, and 216, respectively.



Generalized Designs

- The Jing-Fam approach is based on iteratively using the facts that a narrowband filter can be implemented effectively as $H(z) = F(z^L)G(z)$ and a wideband filter in the form

$$H(z) = z^{-M} - (-1)^M F((-z)^L)G(-z),$$

- If $\omega_s < \pi/2$, then the overall transfer function is generated as

$$H(z) \equiv \widehat{H}_1(z) = G_1(z)F_1(z^{L_1}), \qquad (Aa)$$

where

$$F_1(z) = z^{-M_1} - (-1)^{M_1}\widehat{H}_2(-z), \quad \widehat{H}_2(z) = G_2(z)F_2(z^{L_2}) \qquad (Ab)$$

$$F_2(z) = z^{-M_2} - (-1)^{M_2}\widehat{H}_3(-z), \quad \widehat{H}_3(z) = G_3(z)F_3(z^{L_3}) \qquad (Ac)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$F_{R-2}(z) = z^{-M_{R-2}} - (-1)^{M_{R-2}}\widehat{H}_{R-1}(-z), \quad \widehat{H}_{R-1}(z) = G_{R-1}(z)F_{R-1}(z^{L_{R-1}}) \qquad (Ad)$$

$$F_{R-1}(z) = z^{-M_{R-1}} - (-1)^{M_{R-1}}\widehat{H}_R(-z), \quad \widehat{H}_R(z) = G_R(z). \qquad (Ae)$$

- Here M_r for $r = 1, 2, \ldots, R-1$ is half the order of $\widehat{H}_{r+1}(z)$.

Explanation

- The basic idea is to convert iteratively the design of the narrowband overall filter into the designs of narrowband transfer functions $\widehat{H}_r(z)$ for $r = 2, 3, \cdots, R$ until the transition bandwidth of the remaining $\widehat{H}_R(z) = G_R(z)$ becomes large enough and, correspondingly, its complexity (the number of multipliers) is low enough.
- The desired conversion is performed by properly selecting the L_r's and designing the low-order filters $G_r(z)$ for $r = 1, 2, \cdots, R-1$.
- In order to determine the conditions for the L_r's as well as the design criteria for the $G_r(z)$'s, we consider the rth iteration, where

$$\widehat{H}_r(z) = G_r(z)F_r(z^{L_r}) \qquad (Ba)$$

with

$$F_r(z) = z^{-M_r} - (-1)^{M_r}\widehat{H}_{r+1}(-z). \qquad (Bb)$$

- Let the ripples of $\widehat{H}_r(z)$ be $\widehat{\delta}_p^{(r)}$ and $\widehat{\delta}_s^{(r)}$ and the edges be located at $\omega_p^{(r)} < \pi/2$ and $\omega_s^{(r)} < \pi/2$.
- Since $F_r(z)$ is implemented in the form of Eq. (Bb), it cannot alone take care of shaping the passband response of $\widehat{H}_r(z)$.
- Therefore, the simultaneous criteria for $G_r(z)$ and $F_r(z)$ are stated as

$$\begin{aligned} 1-\delta_p^{(F_r)} \le F_r(\omega) \le 1+\delta_p^{(F_r)} \quad & \text{for} \quad \omega \in [0, L_r\omega_p^{(r)}] \\ -\widehat{\delta}_s^{(r)} \le F_r(\omega) \le \widehat{\delta}_s^{(r)} \quad & \text{for} \quad \omega \in [L_r\omega_s^{(r)}, \pi] \\ 1-\delta_p^{(r)} \le G_r(\omega) \le 1+\delta_p^{(r)} \quad & \text{for} \quad \omega \in [0, \omega_p^{(r)}] \\ -\widehat{\delta}_s^{(r)} \le G_r(\omega) \le \widehat{\delta}_s^{(r)} \quad & \text{for} \quad \omega \in \Omega_s(L_r, \omega_s^{(r)}), \end{aligned}$$

where $\delta_p^{(r)}$ is the passband ripple selected for $G_r(z)$,

$$\delta_p^{(F_r)} = \widehat{\delta}_p^{(r)} - \delta_p^{(r)},$$

and

$$\Omega_s(L, \omega_s) = \bigcup_{k=1}^{\lfloor L/2 \rfloor} [k\frac{2\pi}{L} - \omega_s, \min(k\frac{2\pi}{L} + \omega_s, \pi)].$$

- L_r has to be determined such that the edges of $F_r(z)$, $L_r\omega_p^{(r)}$ and $L_r\omega_s^{(r)}$, become larger than $\pi/2$ and, correspondingly, the edges of $\widehat{H}_{r+1}(z)$, $\omega_p^{(r+1)} = \pi - L_r\omega_s^{(r)}$ and $\omega_p^{(r+1)} = \pi - L_r\omega_s^{(r)}$, become less than $\pi/2$.
- Since $F_r(z)$ and $\widehat{H}_{r+1}(z)$ interchange the ripples, the ripple requirements for $\widehat{H}_{r+1}(z)$ are $\widehat{\delta}_p^{(r+1)} = \widehat{\delta}_s^{(r)}$ and $\widehat{\delta}_s^{(r+1)} = \widehat{\delta}_p^{(r)} - \delta_p^{(r)}$.

Design Equations

- The criteria for the $G_r(z)$'s for $r = 1, 2, \cdots, R$, can thus be stated as

$$\begin{aligned} 1 - \delta_p^{(r)} \le G_r(\omega) \le 1 + \delta_p^{(r)} \quad & \text{for} \quad \omega \in [0,\ \omega_p^{(r)}] \\ -\delta_s^{(r)} \le G_r(\omega) \le \delta_s^{(r)} \quad & \text{for} \quad \omega \in \Omega_s^{(r)}, \end{aligned}$$

where

$$\Omega_s^{(r)} = \begin{cases} \bigcup_{k=1}^{\lfloor L_r/2 \rfloor} [k\frac{2\pi}{L_r} - \omega_s^{(r)},\ \min(k\frac{2\pi}{L_r} + \omega_s^{(r)}, \pi)] & ,r < R \\ [\omega_s^{(R)},\ \pi] & ,r = R. \end{cases} \qquad (C)$$

- Here, the $\omega_p^{(r)}$'s and $\omega_s^{(r)}$'s for $r = 2, 3, \cdots, R$ are determined iteratively as

$$\omega_p^{(r)} = \pi - L_{r-1}\omega_s^{(r-1)}, \quad \omega_s^{(r)} = \pi - L_{r-1}\omega_p^{(r-1)},$$

where $\omega_p^{(1)} = \omega_p$ and $\omega_s^{(1)} = \omega_s$ are the edges of the overall design, and the $\delta_s^{(r)}$'s as

$$\delta_s^{(r)} = \begin{cases} \delta_p - \sum_{\substack{k=1 \\ k \text{ odd}}}^{r-1} \delta_p^{(k)} & \text{for } r \text{ even} \\ \delta_s - \sum_{\substack{k=2 \\ k \text{ even}}}^{r-1} \delta_p^{(k)} & \text{for } r \text{ odd,} \end{cases}$$

where δ_p and δ_s are the ripple values of the overall filter and $\delta_p^{(r)}$ is the passband ripple selected for $G_r(z)$.

- In order for the overall filter to meet the given ripple requirements, $\delta_s^{(R)}$ and the $\delta_p^{(r)}$'s have to satisfy for R even

$$\sum_{\substack{k=2 \\ k \text{ even}}}^{R} \delta_p^{(k)} = \delta_s, \quad \delta_s^{(R)} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{R-1} \delta_p^{(k)} = \delta_p \qquad (Da)$$

or for R odd

$$\sum_{\substack{k=1 \\ k \text{ odd}}}^{R} \delta_p^{(k)} = \delta_p, \quad \delta_s^{(R)} + \sum_{\substack{k=2 \\ k \text{ even}}}^{R-1} \delta_p^{(k)} = \delta_s. \qquad (Db)$$

- In the above, the L_r's have to be determined such that the $\omega_s^{(r)}$'s for $r < R$ become smaller than $\pi/2$.

- It is also desired that for the last filter stage $G_R(z)$, $\omega_s^{(R)}$ is smaller than $\pi/2$.
- If $2\pi/L_r - \omega_s^{(r)} < \pi/2$ for $r < R$ or $\omega_s^{(R)} < \pi/2$, then the arithmetic complexity of $G_r(z)$ can be reduced by designing it in the form

$$G_r(z) = G_r^{(1)}(z^{K_r})G_r^{(2)}(z). \tag{E}$$

- It is preferred to design the subfilters of $G_r(z)$ in such a way that the passband shaping is done entirely by $G_r^{(1)}(z^{K_r})$.
- The number of multipliers in the $G_r(z)$'s for $r = 1, 2, \cdots, R-1$ can be reduced by the fact that the overall filter still meets the given criteria when the stopband regions of these filters are decreased by using in Eq. (D) the substitution

$$(2\omega_s^{(r)} + \omega_p^{(r)})/3 \mapsto \omega_s^{(r)}.$$

- After some manipulations, $H(z)$ as given by Eqs. (A) and (E) can be rewritten in the explicit form shown in the table of the next transparency.

Explicit Form for the Transfer function in the Jing-Fam Approach

$$H(z) = H_1(z^{\widehat{L}_1})[I_2 z^{-\widehat{M}_2} + H_2(z^{\widehat{L}_2})[I_3 z^{-\widehat{M}_3} + H_3(z^{\widehat{L}_3})[\cdots$$
$$[I_{R-1} z^{-\widehat{M}_{R-1}} + H_{R-1}(z^{\widehat{L}_{R-1}})[I_R z^{-\widehat{M}_R} + H_R(z^{\widehat{L}_R})]]\cdots]]],$$

where

$$H_r(z^{\widehat{L}_r}) = H_r^{(1)}(z^{K_r \widehat{L}_r}) H_r^{(2)}(z^{\widehat{L}_r})$$
$$H_r^{(1)}(z) = G_r^{(1)}(J_r^{(1)} z), \quad H_r^{(2)}(z) = S_r G_r^{(2)}(J_r^{(2)} z)$$
$$S_1 = 1, \quad S_r = -(-1)^{\widehat{M}_r/\widehat{L}_r}, \quad r = 2, 3, \cdots, R$$
$$J_1^{(2)} = 1, \quad J_2^{(2)} = -1, \quad J_r^{(2)} = -[J_{r-1}^{(2)}]^{L_{r-1}}, \quad r = 3, 4, \cdots, R$$
$$J_r^{(1)} = [J_r^{(2)}]^{K_r}$$
$$\widehat{L}_1 = 1, \quad \widehat{L}_r = \prod_{k=1}^{r-1} L_k, \quad r = 2, 3, \cdots, R$$
$$\widehat{M}_R = \tfrac{1}{2}\widehat{L}_R N_R, \quad \widehat{M}_{R-r} = \widehat{M}_{R-r+1} + \tfrac{1}{2}\widehat{L}_{R-r} N_{R-r}, \quad r = 1, 2, \cdots, R-2$$
$$I_2 = 1, \quad I_r = [J_{r-1}^{(2)}]^{\widehat{M}_r/\widehat{L}_{r-1}}, \quad r = 3, 4, \cdots, R$$
$$N_r = K_r N_r^{(1)} + N_r^{(2)}$$

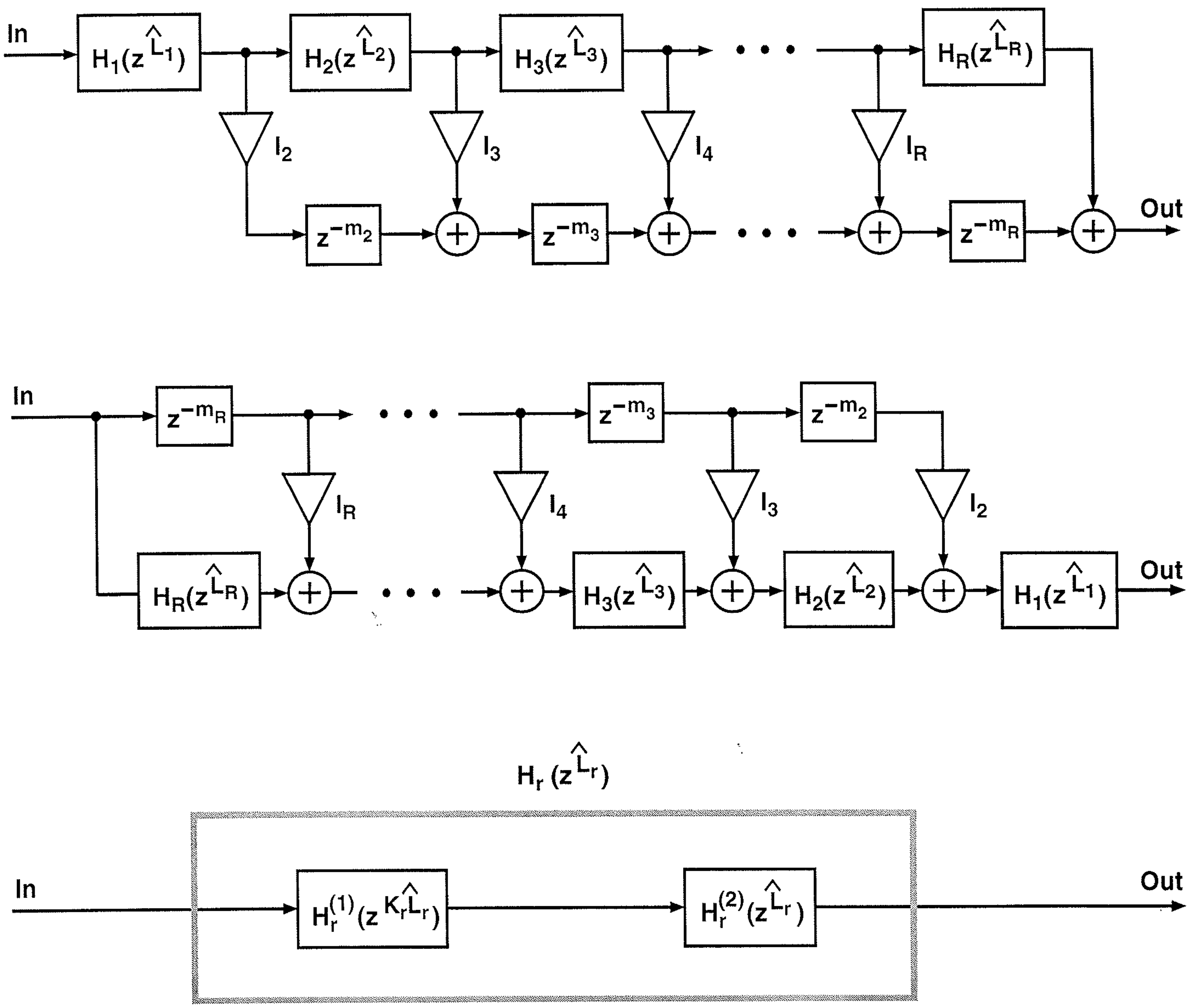
$N_r^{(1)}$ and $N_r^{(2)}$ are the orders of $G_r^{(1)}(z)$ and $G_r^{(2)}(z)$, respectively.

- If $G_r(z)$ is a single-stage design, then $G_r^{(1)}(z^{K_r}) \equiv 1$.
- In order to obtain the desired overall solution, the overall order of $G_r(z)$ for $r \geq 2$, denoted by N_r in the table, has to be even.
- Realizations for the overall transfer function are given in the next transparency, where

$$m_r = \widehat{M}_r - \widehat{M}_{r+1} = \frac{1}{2}\widehat{L}_r N_r,\ r = 2, 3, \cdots, R-1,\ m_R = \widehat{M}_R.$$

- The structure of Fig. (b) is preferred since the delay terms z^{-m_r} can be shared with $H_R^{(1)}(z^{K_R \widehat{L}_R})$ or, if this filter stage is not present, with $H_R^{(2)}(z^{\widehat{L}_R})$.
- If ω_p and ω_s are larger than $\pi/2$, then we set $H(z) \equiv F_1(z)$. In this case, $\delta_p^{(1)} \equiv 0$, $L_1 \equiv 1$, and $G_1(z)$, $\omega_p^{(1)}$, and $\omega_s^{(1)}$ are absent.
- Furthermore, $\omega_p^{(2)} = \pi - \omega_s$ and $\omega_s^{(2)} = \pi - \omega_p$, and $H_1(z)$ is absent in the implementation and in the explicit form of the transfer function.

Implementations for a filter synthesized using the Jing-Fam approach



(a)

(b)

(c)

Filter Optimization

- The remaining problem is to select R, the L_r's, the K_r's, and the ripple values such that the filter complexity is minimized. The following example illustrates this.

Example: $\omega_p = 0.4\pi$, $\omega_s = 0.402\pi$, $\delta_p = 0.01$, $\delta_s = 0.001$

- The only alternative is to select $L_1 = 2$.
- The resulting passband and stopband regions for $G_1(z)$ are (the substitution of Eq. (F) is used)

$$\Omega_p^{(1)} = [0,\ 0.4\pi], \quad \Omega_s^{(1)} = [0.5987\pi,\ \pi].$$

- For $\widehat{H}_2(z)$, $\omega_p^{(2)} = \pi - L_1\omega_s = 0.196\pi$ and $\omega_s^{(2)} = \pi - L_1\omega_p = 0.2\pi$.
- For L_2, there are two alternatives to make the edges of $\widehat{H}_3(z)$, $\omega_p^{(3)} = \pi - L_2\omega_s^{(2)}$ and $\omega_s^{(3)} = \pi - L_2\omega_p^{(2)}$, less than $\pi/2$. These are $L_2 = 3$ and $L_2 = 4$.
- For $R = 5$ stages, there are the following four alternatives to make all the the $\omega_s^{(r)}$'s smaller than $\pi/2$:

$$L_1 = 2, \quad L_2 = 4, \quad L_3 = 3, \quad L_4 = 2$$

$$L_1 = 2, \quad L_2 = 4, \quad L_3 = 4, \quad L_4 = 4$$

$$L_1 = 2, \quad L_2 = 3, \quad L_3 = 2, \quad L_4 = 4$$

$$L_1 = 2, \quad L_2 = 3, \quad L_3 = 2, \quad L_4 = 3.$$

- Among these alternatives, the first one results in an overall filter with minimum complexity.
- In this case, the edges of $\widehat{H}_3(z)$, $\widehat{H}_4(z)$, and $\widehat{H}_5(z) \equiv G_5(z)$ become as shown in the table of the transparency 67.
- The corresponding passband and stopband regions for $G_2(z)$, $G_3(z)$, $G_4(z)$, and $G_5(z)$ are

$$\Omega_p^{(2)} = [0,\ 0.196\pi],\quad \Omega_s^{(2)} = [0.3013\pi,\ 0.6987\pi] \cup [0.8013\pi$$

$$\Omega_p^{(3)} = [0,\ 0.2\pi],\quad \Omega_s^{(3)} = [0.4560\pi,\ 0.8773\pi],$$

$$\Omega_p^{(4)} = [0,\ 0.352\pi],\quad \Omega_s^{(4)} = [0.616\pi,\ \pi],$$

$$\Omega_p^{(5)} = [0,\ 0.2\pi],\quad \Omega_s^{(5)} = [0.296\pi,\ \pi].$$

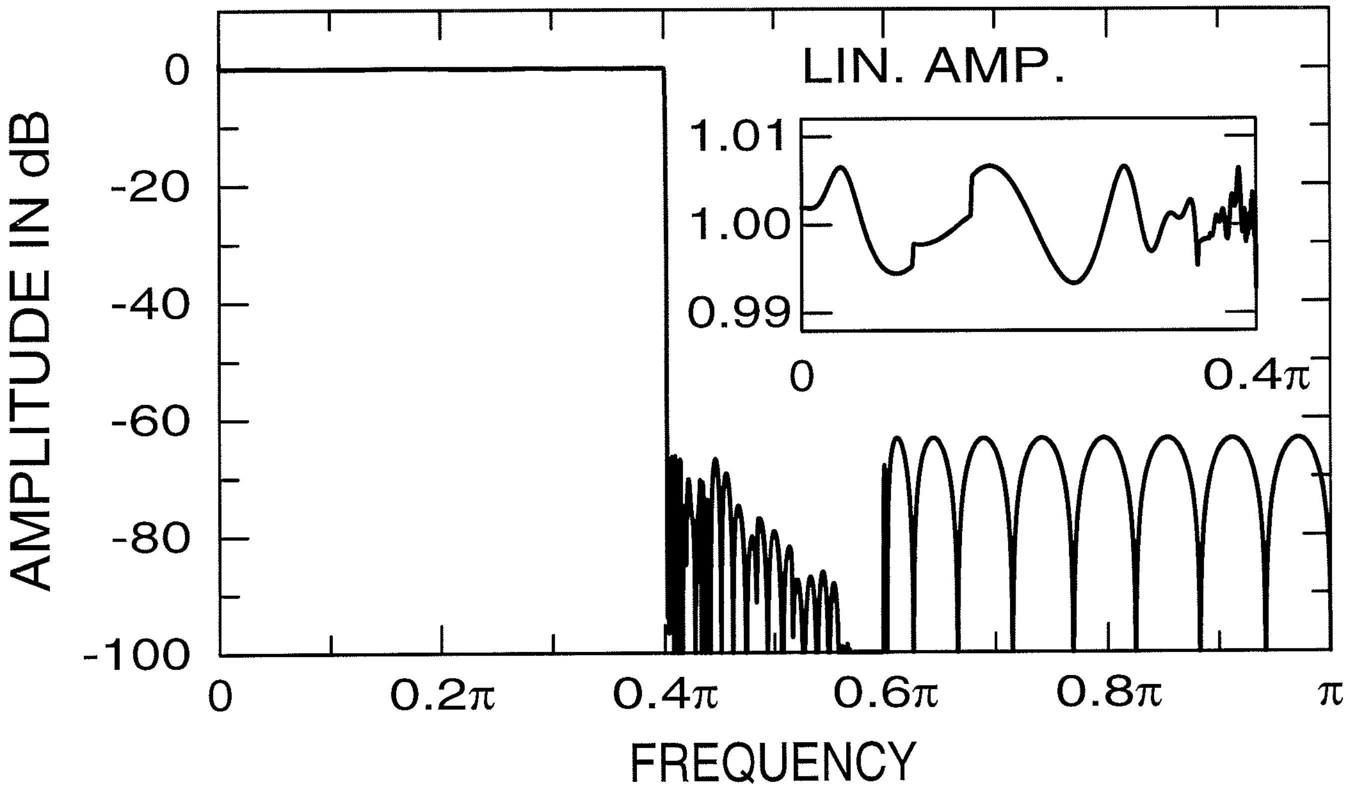
- What remains is to determine the ripple requirements.
- From Eq. (D), it follows for $R = 5$, $\delta_p^{(1)} + \delta_p^{(3)} + \delta_p^{(5)} = \delta_p$ and $\delta_p^{(2)} + \delta_p^{(4)} + \delta_s^{(5)} = \delta_s$.
- By simply selecting the ripple values in these summations to be equal, the required ripples for

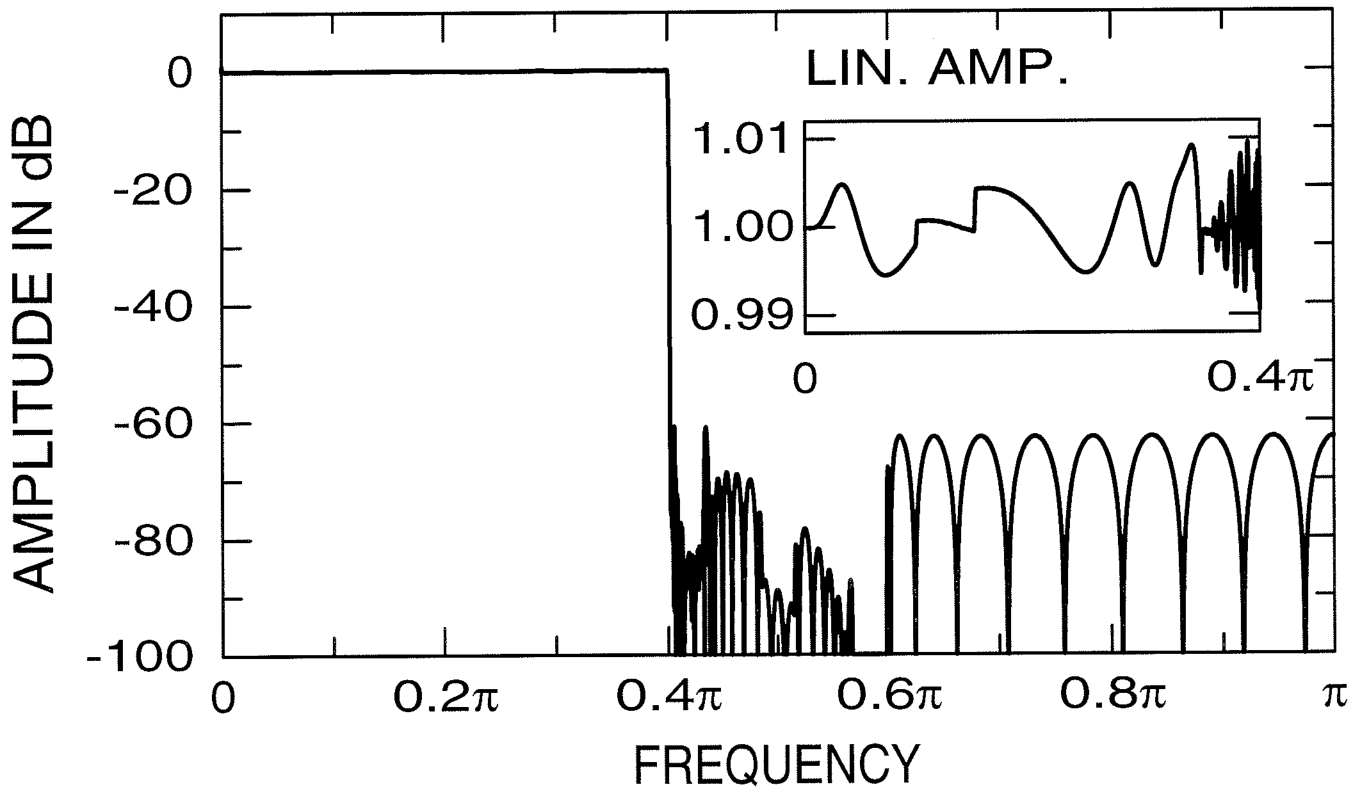
- The first and fourth subfilters are single-stage filters since their stopband edges are larger than $\pi/2$, whereas the remaining three filters are two-stage designs.
- The parameters describing the overall filter are shown in the table, whereas Fig. (a) in transprency 68 depicts the response of this filter.
- The number of multipliers and the order of this design are 78 and 4875, whereas the corresponding numbers for the direct-form equivalent are 1271 and 2541.
- The number of multipliers required by the proposed design is thus only 6 % of that of the direct-form filter.
- Since the complexity of $H_5(z^{\widehat{L}_5})$ is similar to those of the earlier filter stages, $R = 5$ is a good selection in this example.

Data for a Filter Designed Using the Jing-Fam Approach

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
$\omega_p^{(r)}$	0.4π	0.196π	0.2π	0.352π	0.2π
$\omega_s^{(r)}$	0.402π	0.2π	0.216π	0.4π	0.296π
$\delta_p^{(r)}$	$\frac{1}{3}\times 10^{-2}$	$\frac{1}{3}\times 10^{-3}$	$\frac{1}{3}\times 10^{-2}$	$\frac{1}{3}\times 10^{-3}$	$\frac{1}{3}\times 10^{-2}$
$\delta_s^{(r)}$	10^{-3}	$\frac{2}{3}\times 10^{-2}$	$\frac{2}{3}\times 10^{-3}$	$\frac{1}{3}\times 10^{-2}$	$\frac{1}{3}\times 10^{-3}$
L_r	2	4	3	2	—
K_r	—	3	2	—	3
$N_r^{(1)}$	—	20	11	—	22
$N_r^{(2)}$	31	10	8	26	14
N_r	31	70	30	26	80
$\widehat{L}_r$	1	2	8	24	48
$J_r^{(1)}$	—	-1	1	—	-1
$J_r^{(2)}$	1	-1	-1	1	-1
$\widehat{M}_r$	—	2422	2352	2232	1920
I_r	—	1	1	-1	1
S_r	1	1	-1	1	-1
m_r	—	70	120	312	1920

Amplitude responses for filters synthesized using the Jing-Fam approach





Another Design

- The overall filter order as well as the number of multipliers can be decreased by selecting smaller ripple values for the first stages, thereby allowing larger ripples for the last stages.
- Proper selections for the ripple requirements and filter orders are shown in the table of the next transparency.
- The first four filters have been optimized such that their passband variations are minimized.
- The first criteria are met by a half-band filter of order 34, having the passband and stopband edges at 0.4013π and 0.5987π.
- Since every second impulse response coefficient of this filter is zero-valued except for the central coefficient with an easily implementable value of 1/2, this filter requires only 9 multipliers.

Data for Another Filter Designed Using the Jing-Fam Approach

$\delta_p^{(r)}$	7.3×10^{-4}	7.1×10^{-5}	3.5×10^{-4}	12.1×10^{-5}	89.2×10^{-4}
$\delta_s^{(r)}$	10^{-3}	92.7×10^{-4}	92.9×10^{-5}	89.2×10^{-4}	80.8×10^{-5}
K_r	—	3	2	—	2
$N_r^{(1)}$	—	22	13	—	27
$N_r^{(2)}$	34	10	8	24	6

- For the last stage, K_5 is reduced to 2 to decrease the overall filter order. The order of the resulting overall filter [see Fig. (b) in transprency 68] is 3914, which is 54 percent higher than that of the direct-form equivalent. The number of multipliers is reduced to 70.

Comments

- The above Jing-Fam approach cannot be applied directly for synthesizing filters whose edges are very close to $\pi/2$.
- This problem can, however, be overcome by slightly changing the sampling rate or, if this not possible, by shifting the edges by a factor of $\frac{3}{2}$ by using decimation by this factor at the filter input and interpolation by the same factor at the filter output.
- One attractive feature of the Jing-Fam approach is that it can be combined with multirate filtering to reduce the filter complexity even further. This design method will be considered in the course System Level DSP Algorithms.
- When comparing the above designs with the filters synthesized using the multistage frequency-response masking technique, it is observed that

the above designs require slightly fewer multipliers at the expense of an increased overall filter order.

- Both of these general approaches are applicable those specifications that are not very narrowband or very wideband.
- For most very narrowband and wideband cases, filters synthesized in the simplified forms $H(z) = F(z^L)G(z)$ and $H(z) = z^{-M} - F((-z)^L)G(-z)$, respectively, give the best results.
- **There are matlab codes for designing filters using the multiple frequency-response masking approach as well as for designing narrowband filters using periodic filters. These codes will be considered in exercises.**