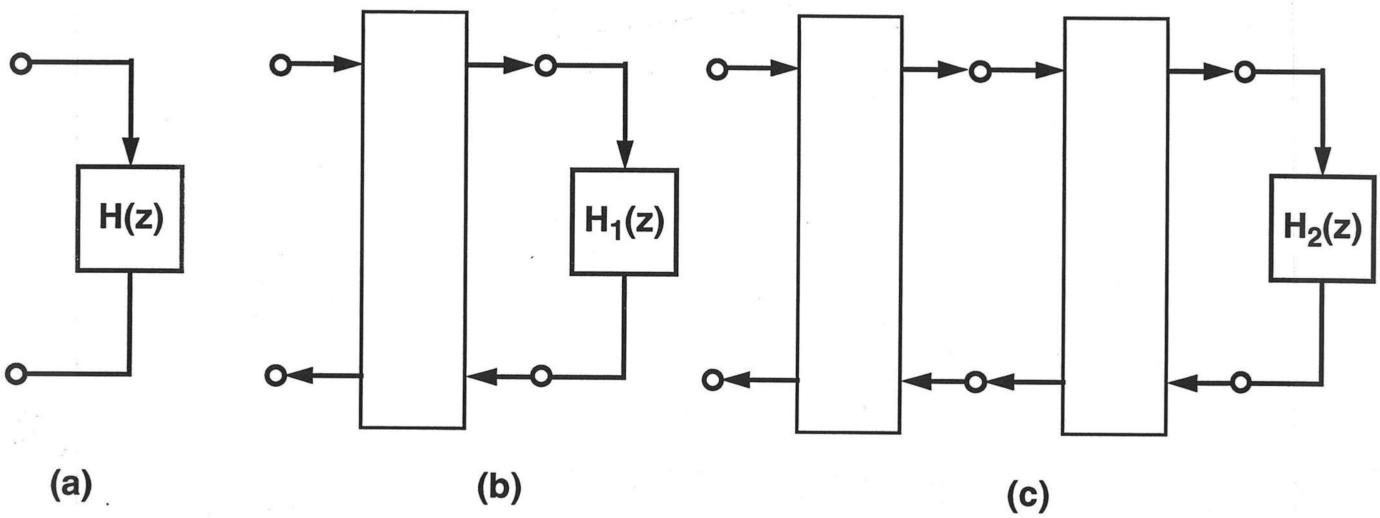
# 80558 MULTIRATE SIGNAL PROCESSING

## Part IV: Design of FIR filters using multirate and complementary filtering

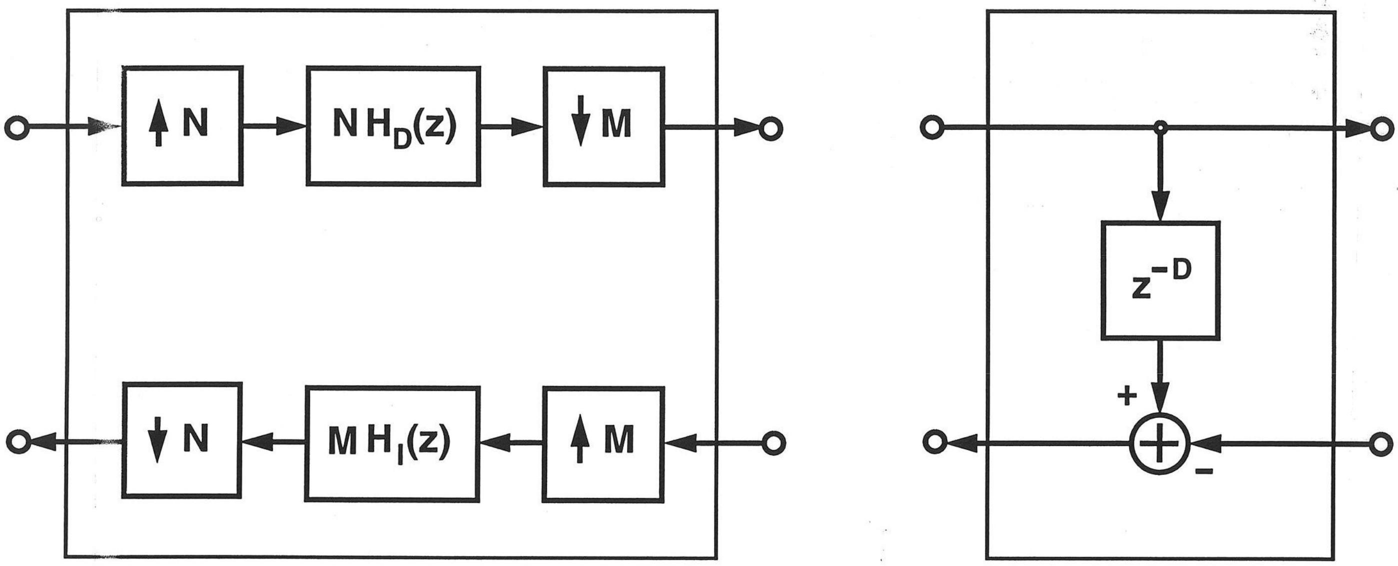
- This part concentrates on designing FIR filters in such a manner that the number of multiplications per input sample becomes very small also for filters having a very narrow transition bandwidths.
- This is achieved by using multirate and complementary filtering in a proper manner.
- This pile contains a pile of lecture notes, two conference articles, and one short (unpublished) article.

# MULTIRATE DESIGN OF FIR FILTERS

- A general approach is obtained by combining multirate filtering and complementary filtering (more details can be found in the enclosed articles).
- Basic principle: Synthesize $H(z)$ as a cascade of a two-port and a termination filter such that the addition and multiplication rates are deduced. Do the same for $H_1(z)$ and continue.



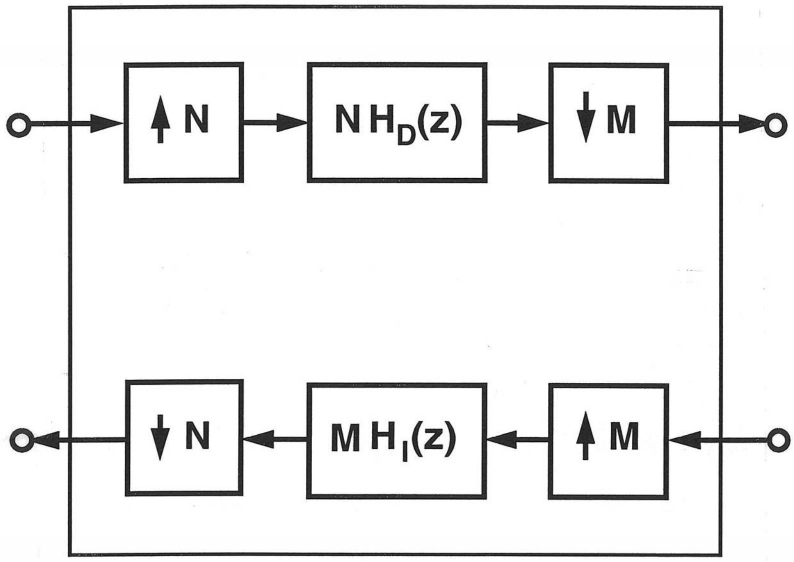
# Basic blocks




(a) (b)

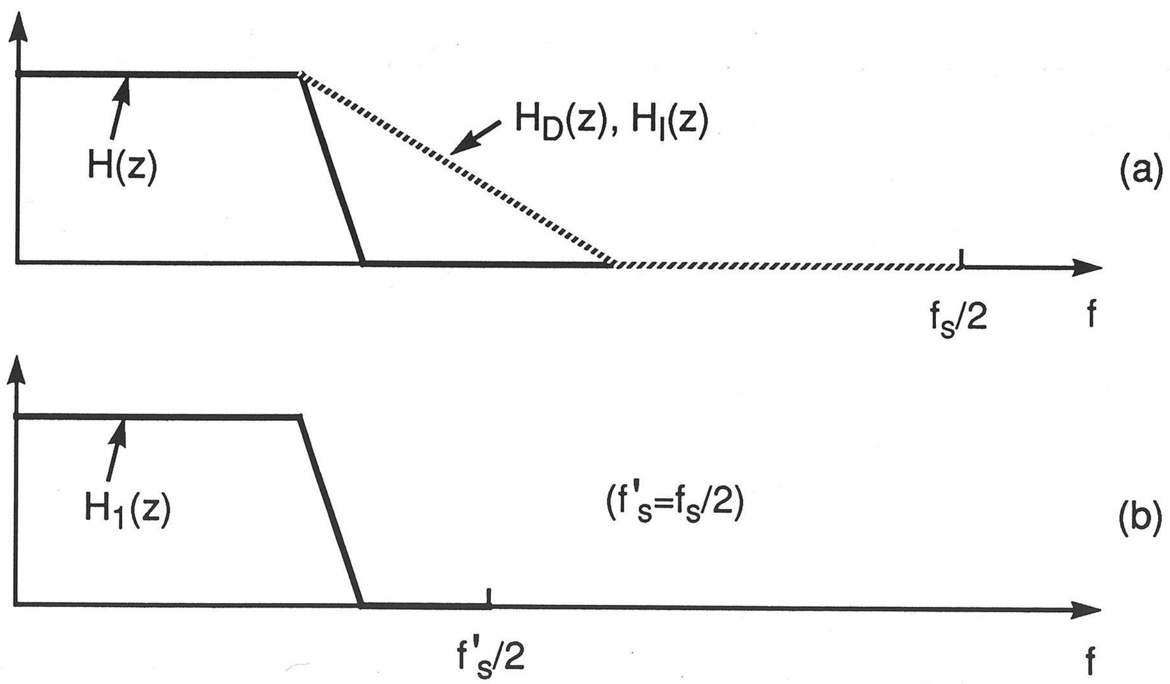
# Narrowband filter

Building block:




(a)

- Both the sampling rate and the order of the termination are reduced by $r = M/N$. The saving in the multiplication rate is thus $r^2$.



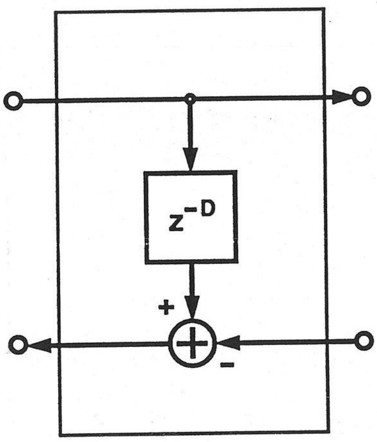
## Wideband filter

- Block (b) maps the design of a wideband filter into the design of a narrowband filter and the process can be continued.

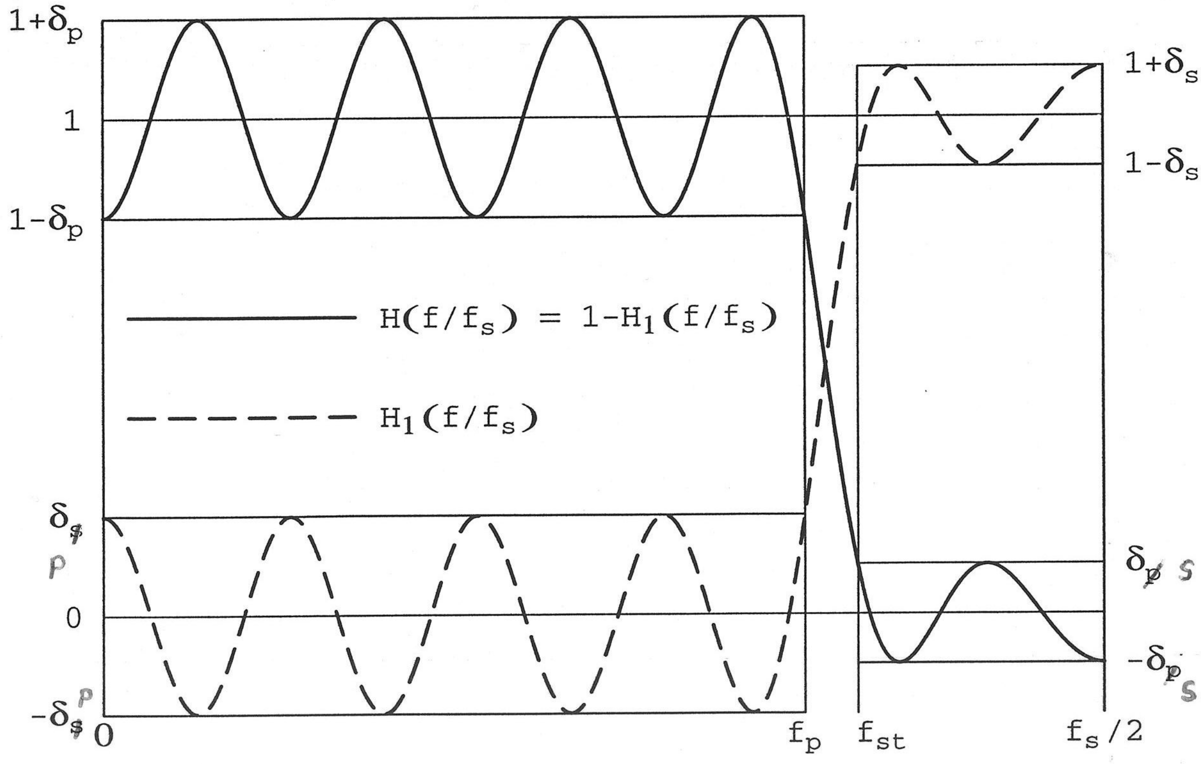
$$H(z) = z^{-D} - H_1(z)$$

$$H_1(f/f_s) = h_1[D] + \sum_{n=1}^{D} 2h_1[D-n]\cos(2\pi n f/f_s)$$

$$H(f/f_s) = 1 - H_1(f/f_s)$$




(b)



## Narrowband Case: Edges of the termination less than one fourth of the sampling rate

- Use block (a) with $r = 2$ ($N = 1,\ M = 2$).

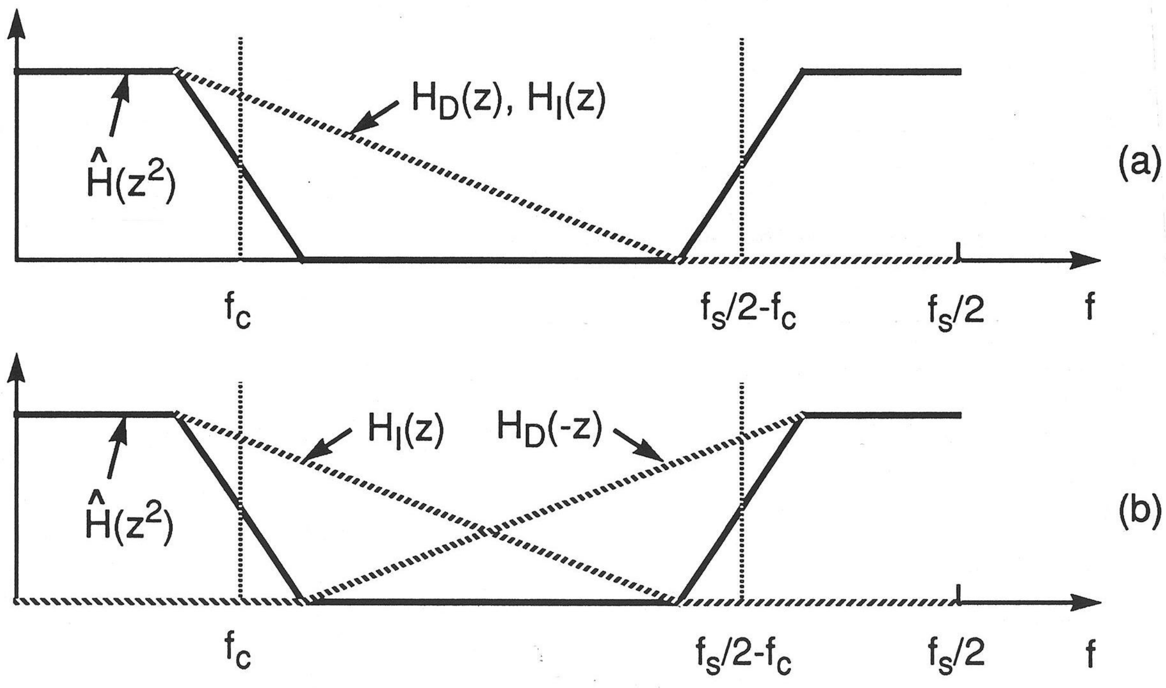
$$Y(z) = F_1(z)X(z) + F_2(z)X(-z),$$

where

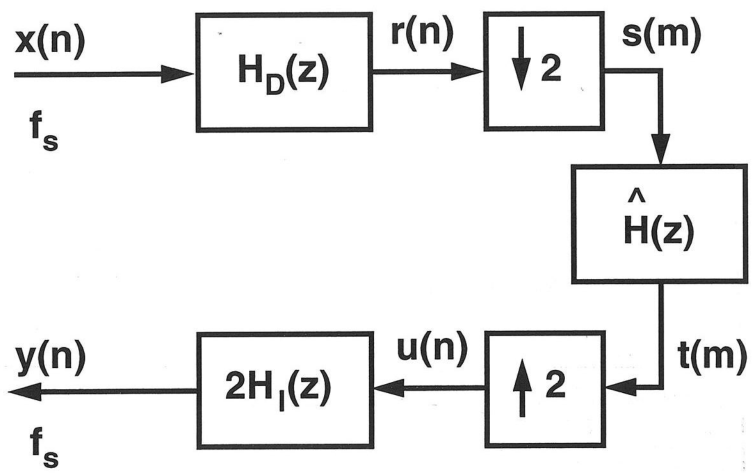
$$F_1(z) = \widehat{H}(z^2)H_D(z)H_I(z)$$

and

$$F_2(z) = \widehat{H}(z^2)H_D(-z)H_I(z).$$



## Derivation




- The $z$-transforms of $x(n)$, $r(n)$, $s(m)$, $t(m)$, $u(n)$, and $y(n)$ are related through

$$R(z) = H_D(z)X(z), \;\; S(z) = (1/2)[R(z^{1/2}) + R(-z^{1/2})],$$

$$T(z) = \widehat{H}(z)S(z), \;\; U(z) = T(z^2), Y(z) = 2H_I(z)U(z).$$

- By combining these results, we arrive at the equations of the previous transparency.
- Using the substitution $z = e^{j\frac{2\pi f}{f_s}}$ with $f_s$ being the input and output sampling rate in the equations of the previous transparency gives

$$Y(e^{j\frac{2\pi f}{f_s}}) = F_1(e^{j\frac{2\pi f}{f_s}})X(e^{j\frac{2\pi f}{f_s}}) + F_2(e^{j\frac{2\pi f}{f_s}})X(e^{j\frac{2\pi(f+f_s/2)}{f_s}}),$$

where

$$F_1(e^{j\frac{2\pi f}{f_s}}) = \widehat{H}(e^{j\frac{4\pi f}{f_s}})H_D(e^{j\frac{2\pi f}{f_s}})H_I(e^{j\frac{2\pi f}{f_s}})$$

and

$$F_2(e^{j\frac{2\pi f}{f_s}}) = \widehat{H}(e^{j\frac{4\pi f}{f_s}})H_D(e^{j\frac{2\pi(f+f_s/2)}{f_s}})H_I(e^{j\frac{2\pi f}{f_s}}).$$

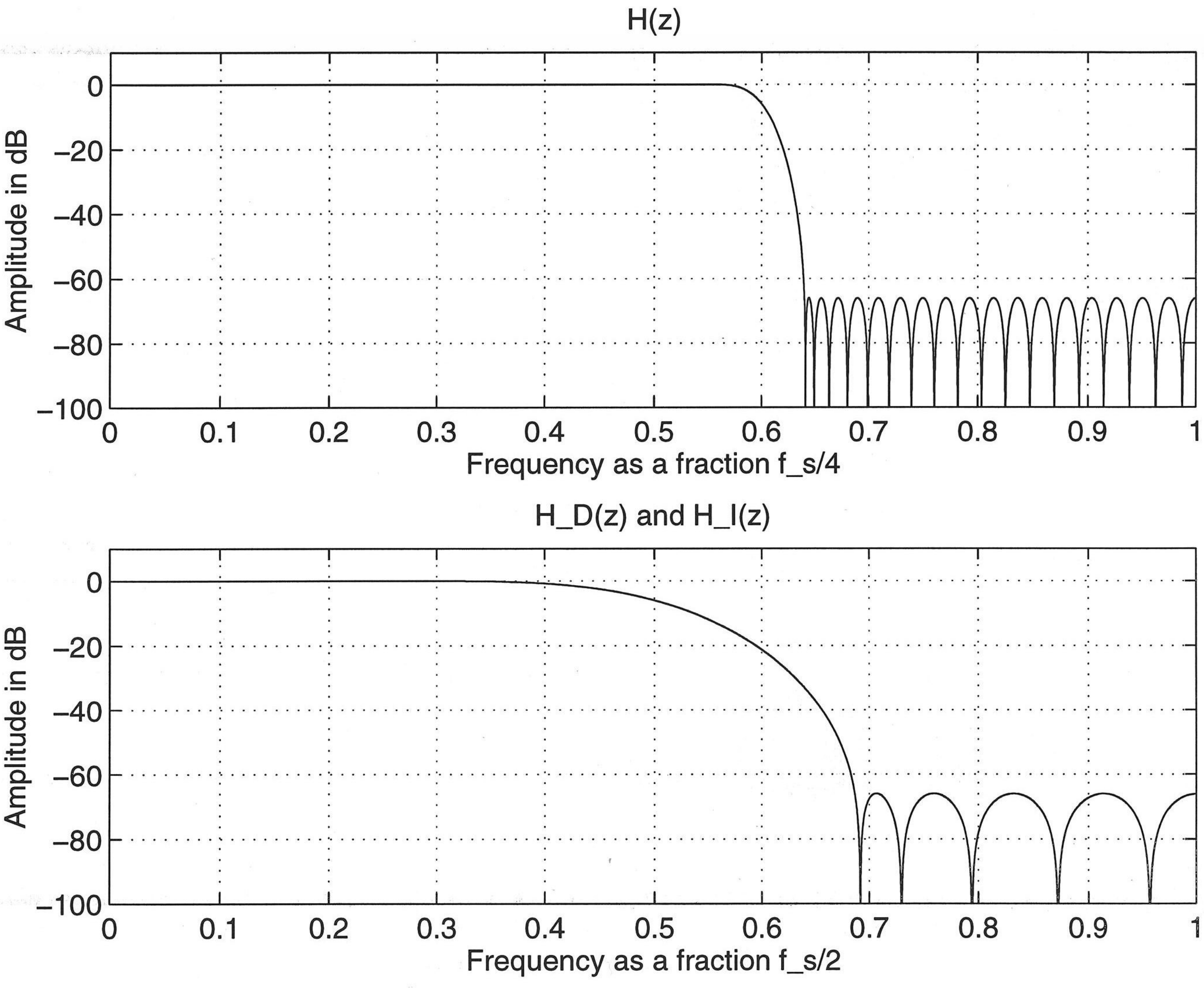
- Here, $X(e^{j2\pi\frac{(f+f_s/2)}{f_s}})$ is an aliased term, which is desired to be attenuted.
- If the desired passband and stopband edges are $f_p$ and $f_{\rm st}$, then $\widehat{H}(z)$ is designed to have edges these edges for the sampling rate of $f_s/2$. The resulting $\widehat{H}(z^2)$ for the sampling rate of $f_s$ has then the desired edge values as well as an extra passband and transition band around $f = f_s/2$ (see the figures on transparency 5).
- The stopband and passband edges of the extra transition band are at $f_s/2 - f_{\rm st}$ and $f_s/2 - f_p$.
- $H_D(z)$ and $H_I(z)$ are designed to have the passband edge at $f_p$ and stopband edge at $f_s/2 - f_{\rm st}$.
- As seen from the figures of transparency 5, these $H_D(z)$ and $H_I(z)$ preserve the passband of $H(z^2)$ and attenuate the extra passband and transition band, giving the desired $F_1(z)$.
- In the case of $F_2(z)$, $H_D(-z)$ and $H_I(z)$ attenuate the passbands and transition bands of $H(z^2)$ around $f = f_s/2$ and $f = 0$, resulting in a small amplitude response of $F_2(z)$, as is desired. This means that the aliased term is attenuated.

**Example 1:** $f_p = 0.28(f_s/2)$, $f_{\rm st} = 0.32(f_s/2)$, $\delta_p = 0.0015$, $\delta_s = 0.0005$.

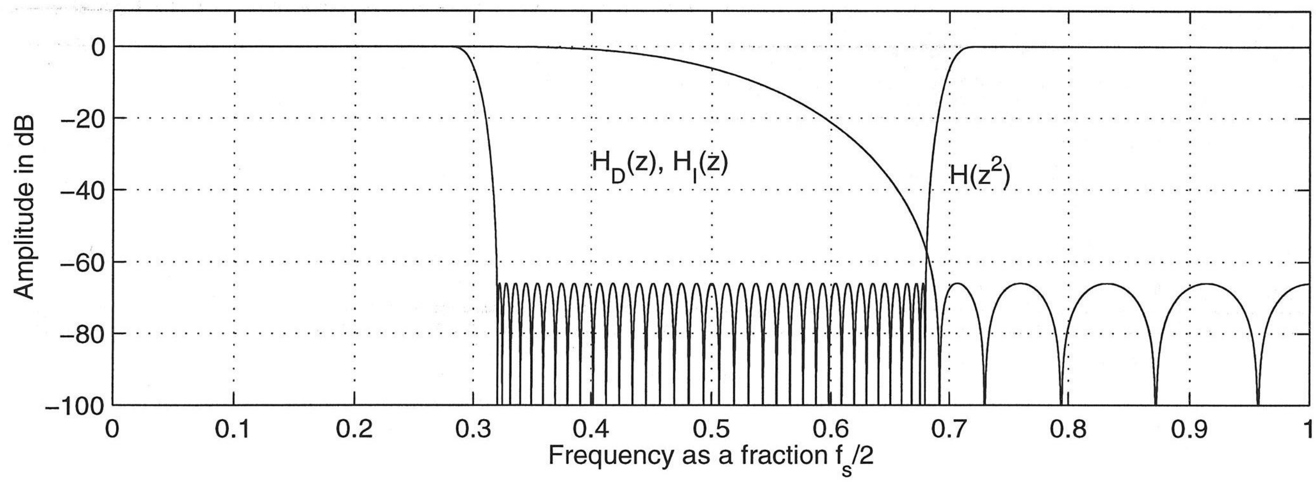
- $\widehat{H}(z)$ is designed to have passband and stopband edges at $f_p$ and $f_{\rm st}$ for the sampling rate of $f_s/2$ and the passband and stopband ripples of 0.0005.
- $H_D(z)$ and $H_I(z)$ are desired to be designed as lowpass halfband filters with the stopband edge at $f_s/2-(2f_{\rm st}+f_p)/3$ [the corresponding passband edge is $(2f_{\rm st}+f_p)/3$] and the passband and stopband ripples of 0.0005.
- It has been observed experimentally that the stopband edge can start at $f_s/2-(2f_{\rm st}+f_p)/3$ instead of $f_s/2-f_{\rm st}$.
- We use half-band filters because of their efficient implementations (this will be considered later).
- The order of $\widehat{H}(z)$ is 92, requiring 47 multipliers, whereas the order of both $H_D(z)$ and $H_I(z)$ is 18 (by assuming that the easily implementable center tap of value 1/2 requires no multipliers, both filters require only 5 multipliers).
- The following four transparencies illustrate the characteristics of the overall design.
- It is seen that the passband ripple for $F_1(z)$ (trans-

fer function for the unaliased term) is the sum of the passband ripples of $\widehat{H}(z)$, $H_D(z)$, and $H_I(z)$, that is, it is equal to $3 \cdot 0.0005 = 0.0015$.

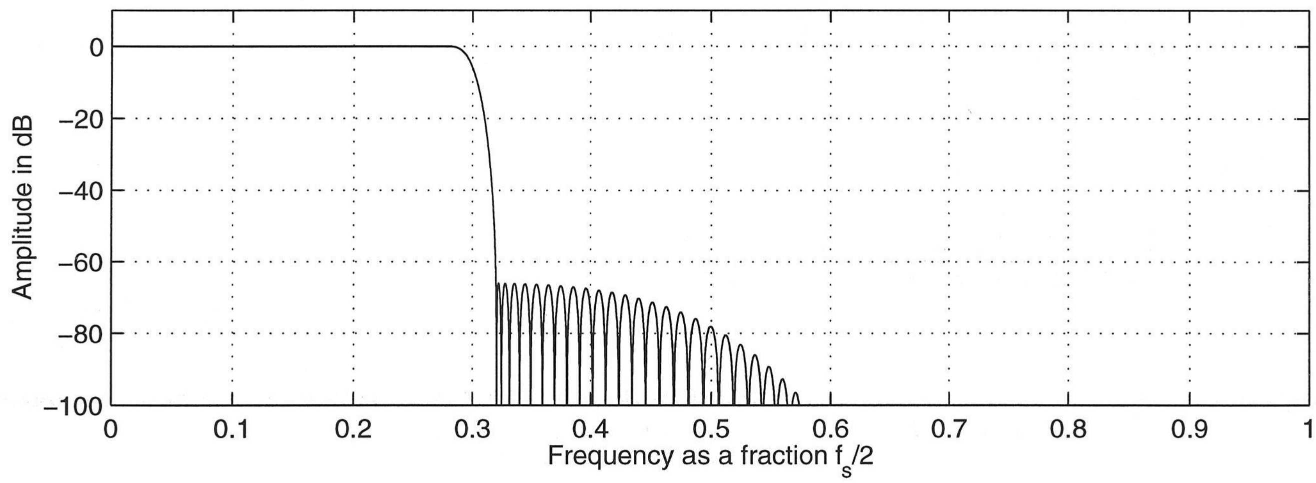
- The maximum stopband ripple is that of $\widehat{H}(z)$, that is, 0.0005.
- The minimum order of a direct FIR filter design to meet the same criteria is 169, requiring 85 multiplications per input sample. The proposed design requires only 28.5 multiplications per input sample.
- It is also seen from the following transparencies that the maximum amplitude value of $F_2(z)$ (transfer function for the aliased term) is equal to the stopband ripples of the subfilters, that is, 0.0005. The aliased components are thus attenuated by at least 66 dB.

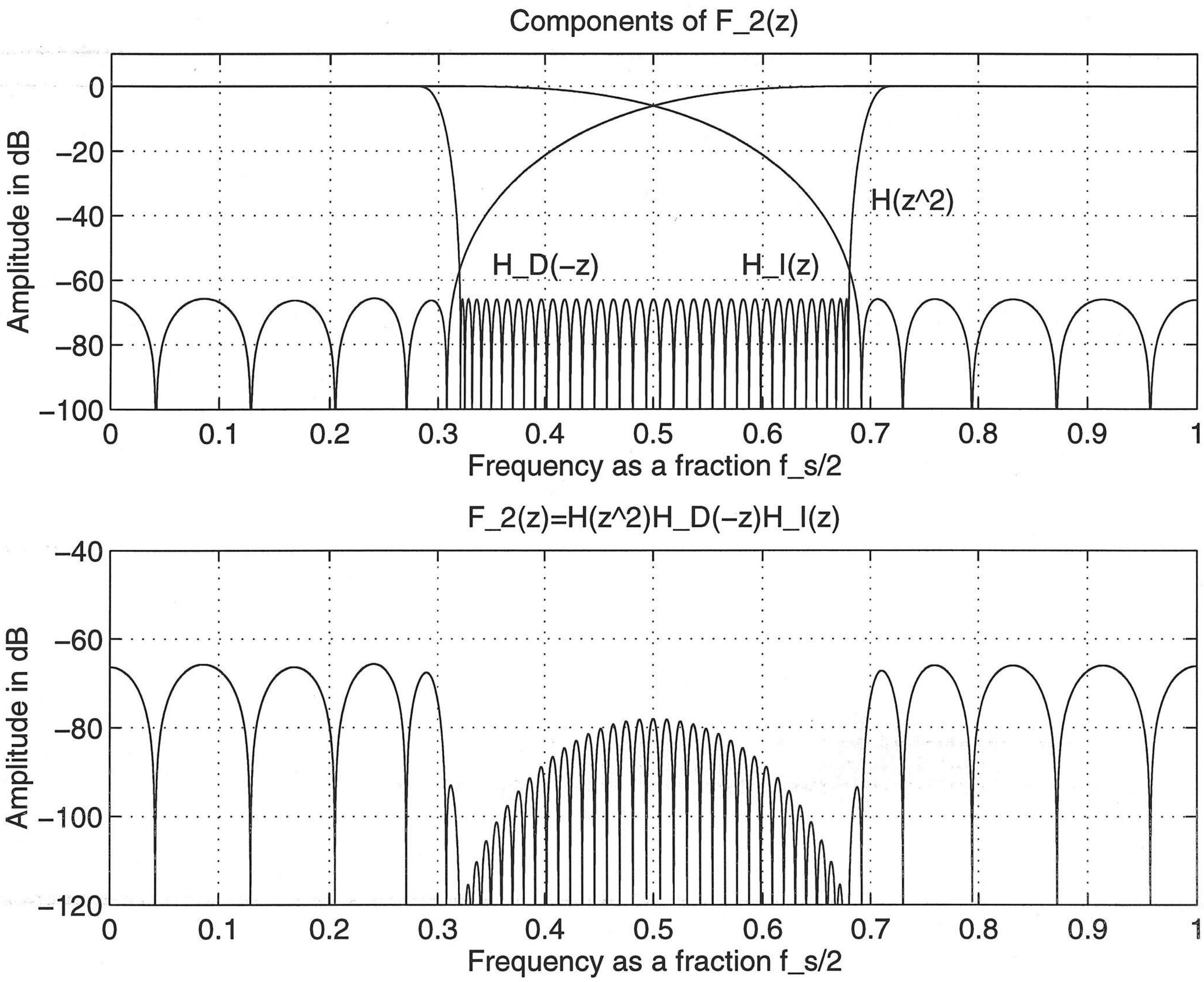

H(z)
Amplitude in dB
0
−20
−40
−60
−80
−100
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Frequency as a fraction f_s/4
H_D(z) and H_I(z)
Amplitude in dB
0
−20
−40
−60
−80
−100
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Frequency as a fraction f_s/2

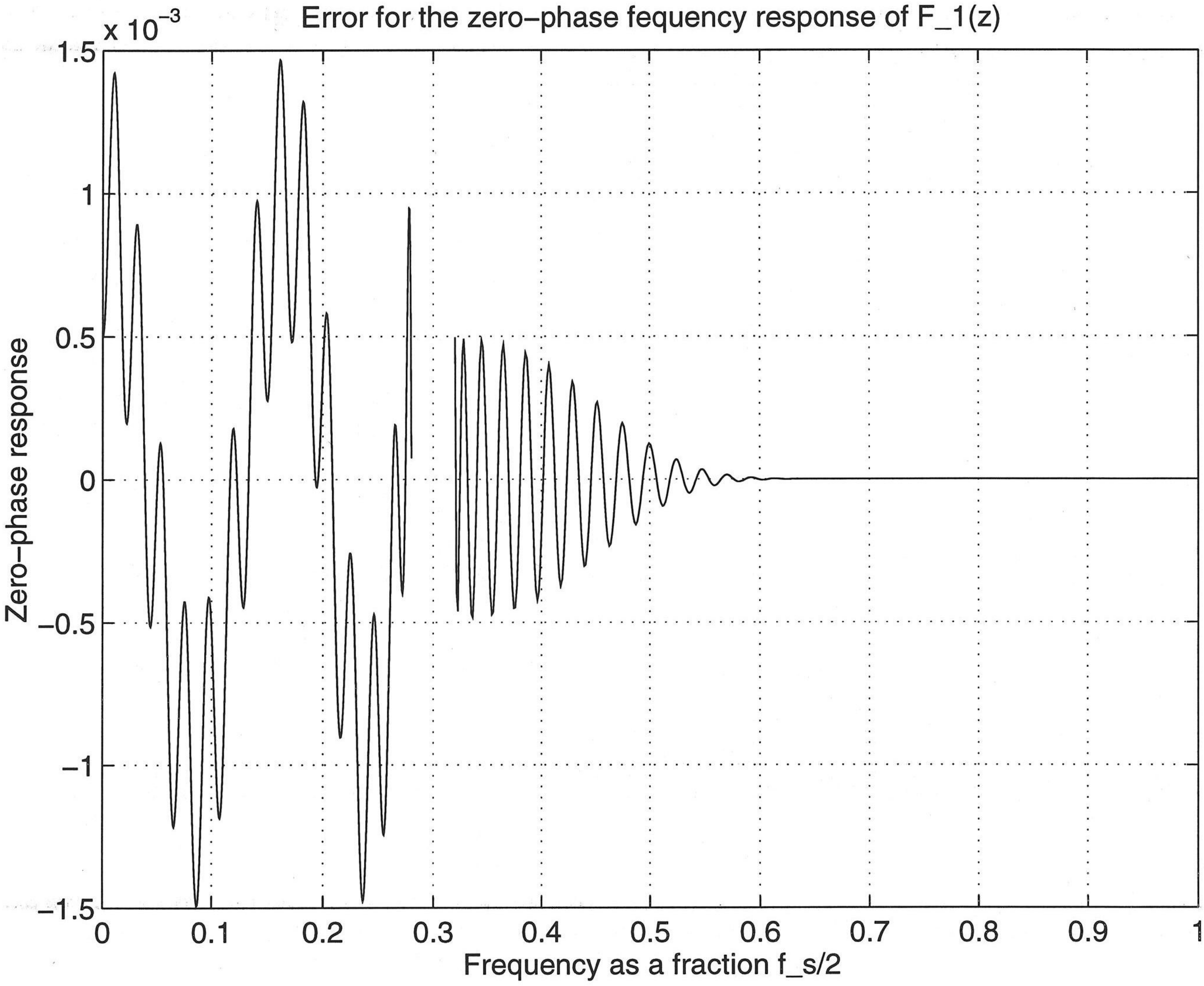
Components of $F_1(z)$




$F_1(z)=H(z^2)H_D(z)H_I(z)$




Components of F_2(z)
Amplitude in dB
0
−20
−40
−60
−80
−100
H(z^2)
H_D(−z)
H_I(z)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Frequency as a fraction f_s/2
F_2(z)=H(z^2)H_D(−z)H_I(z)
Amplitude in dB
−40
−60
−80
−100
−120
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Frequency as a fraction f_s/2


Error for the zero−phase fequency response of F_1(z)
x 10^−3
1.5
1
0.5
0
−0.5
−1
−1.5
Zero−phase response
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Frequency as a fraction f_s/2

## Wideband Case: Edges of the termination larger than one fourth of the sampling rate

- Use block (b) cascaded with block (a) with $r = 2$ $(N = 1,\ M = 2)$.

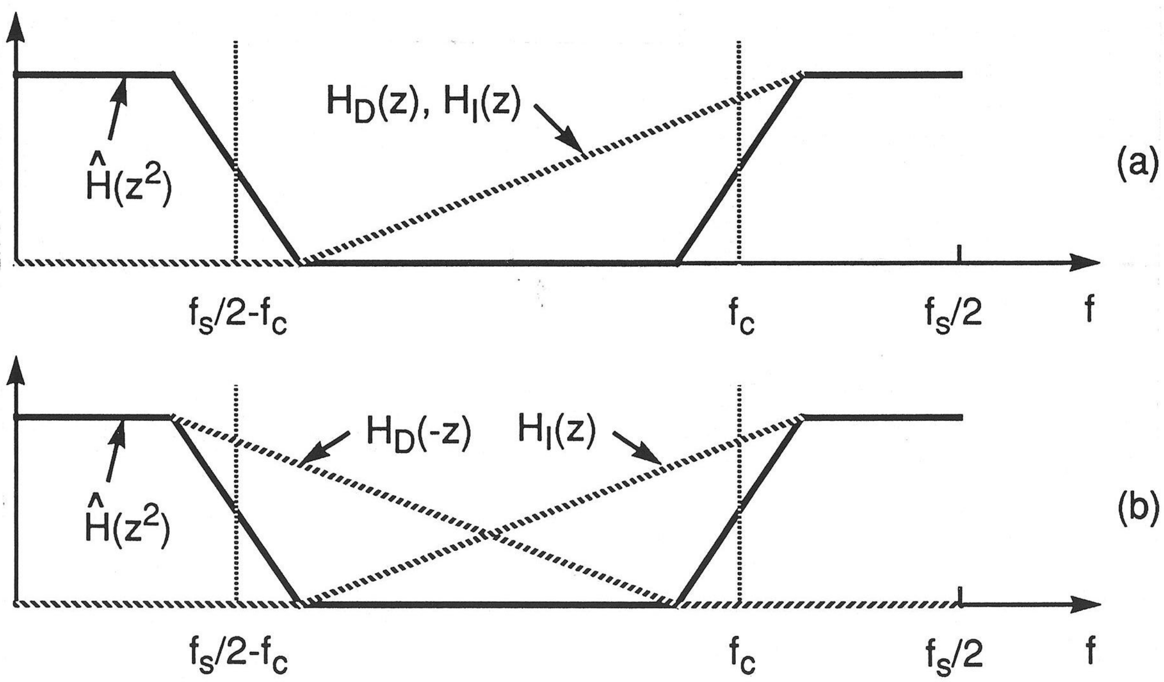
$$Y(z) = [z^{-D} - F_1(z)]X(z) - F_2(z)X(-z),$$

where

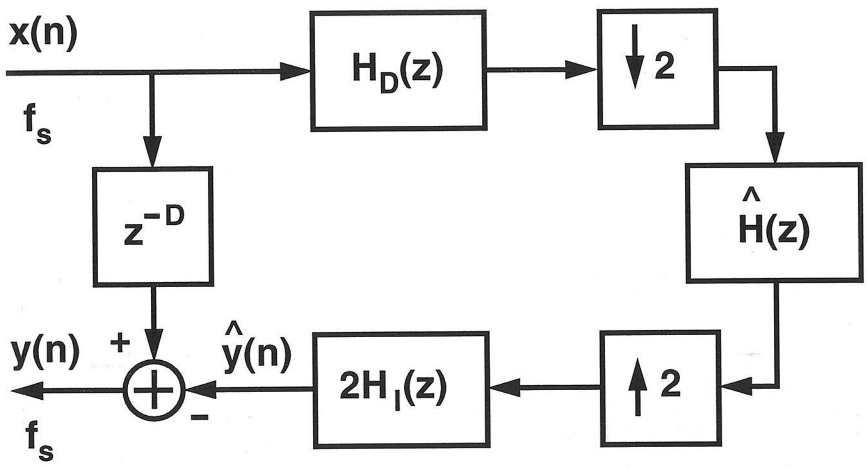
$$F_1(z) = \widehat{H}(z^2)H_D(z)H_I(z)$$

and

$$F_2(z) = \widehat{H}(z^2)H_D(-z)H_I(z).$$



# Derivation




- According to the previous considerations, the $z$-tranforms of $\widehat{y}(n)$ and $x(n)$ are related through

$$\widehat{Y}(z) = F_1(z)X(z) + F_2(z)X(-z),$$

where

$$F_1(z) = \widehat{H}(z^2)H_D(z)H_I(z)$$

and

$$F_2(z) = \widehat{H}(z^2)H_D(-z)H_I(z).$$

- On the other hand,

$$Y(z) = z^{-D}X(z) - \widehat{Y}(z),$$

giving the equations of the previous transparency.

- As seen from the figures of transparency 10, the basic difference compared to the narrowband case is that now $F_1(z)$ (the complementary filter of the desired one $z^{-D} - F_1(z)$) is formed such that the lower passband and transition band of the periodic filter $\widehat{H}(z^2)$ are now attenuated by $H_D(z)$ and $H_I(z)$.

- Let the passband and stopband edges of $z^{-D} - F_1(z)$ (the stopband and passband edges of $F_1(z)$) be at $f_p > f_s/4$ and $f_{\mathrm{st}} > f_s/4$ ($f_{\mathrm{st}} < f_p$).
- In order for $\widehat{H}(z^2)$ to generate the desired transition band for the sampling frequency $f_s$, $H(z)$ has to be designed to be a lowpass filter with the passband and stopband edges at

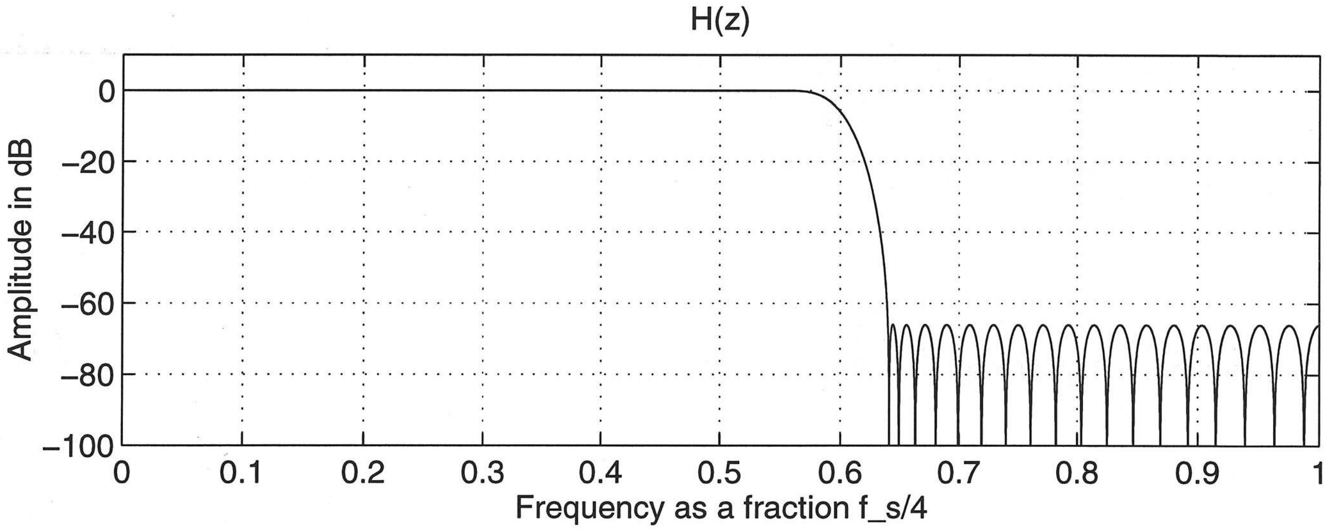
$$\widehat{f_p} = f_s/2 - f_p, \quad \widehat{f_{\mathrm{st}}} = f_s/2 - f_{\mathrm{st}}$$

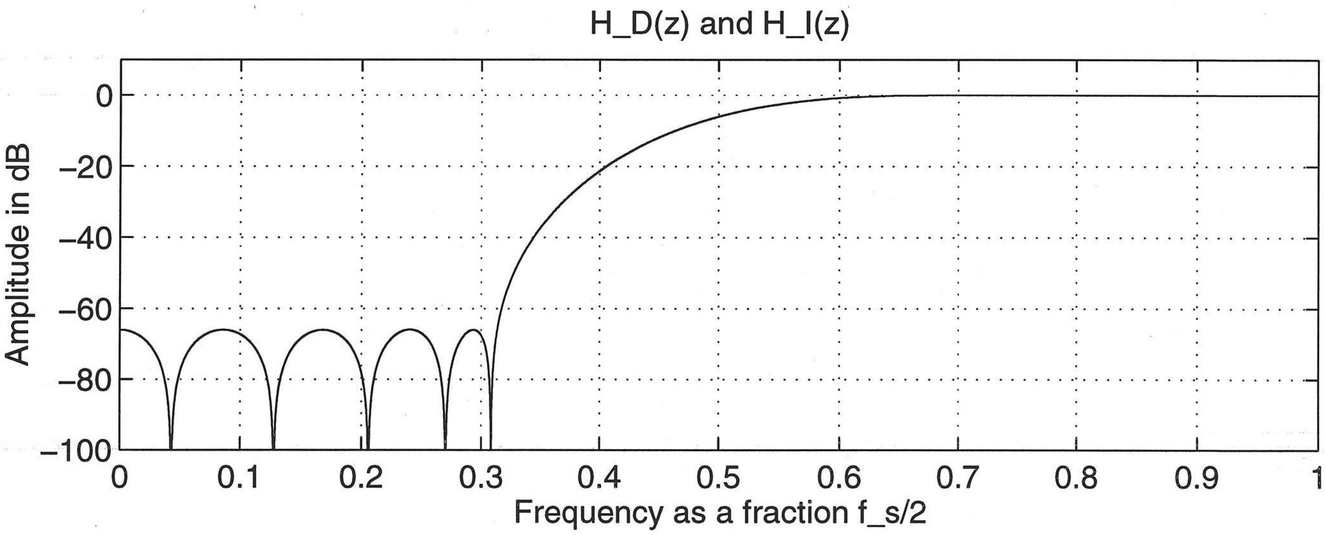
for the sampling frequency of $f_s/2$.

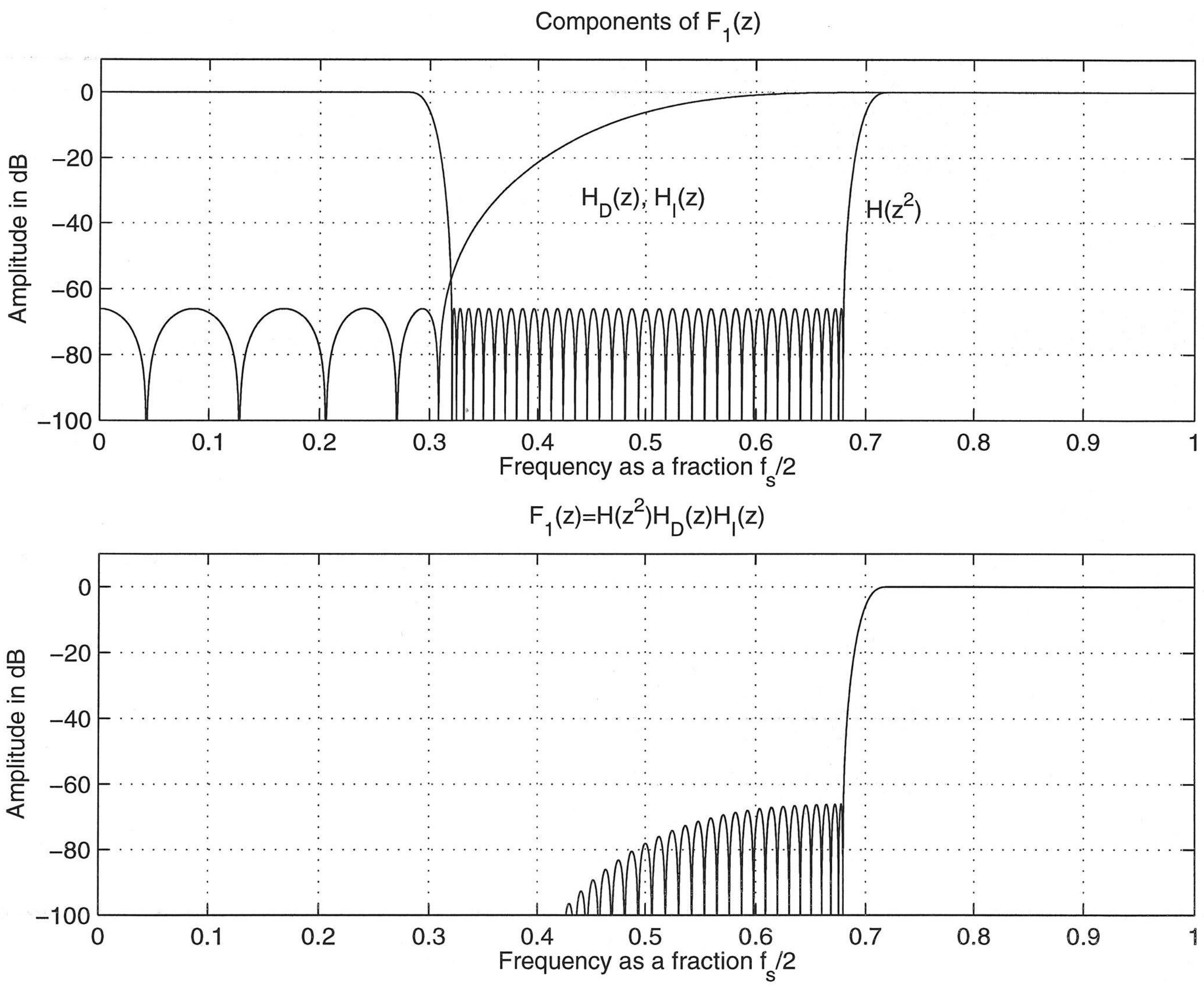
**Example 2: $f_p = 0.72(f_s/2)$, $f_{\rm st} = 0.68(f_s/2)$, $\delta_p = 0.0005$, $\delta_s = 0.0015$.**

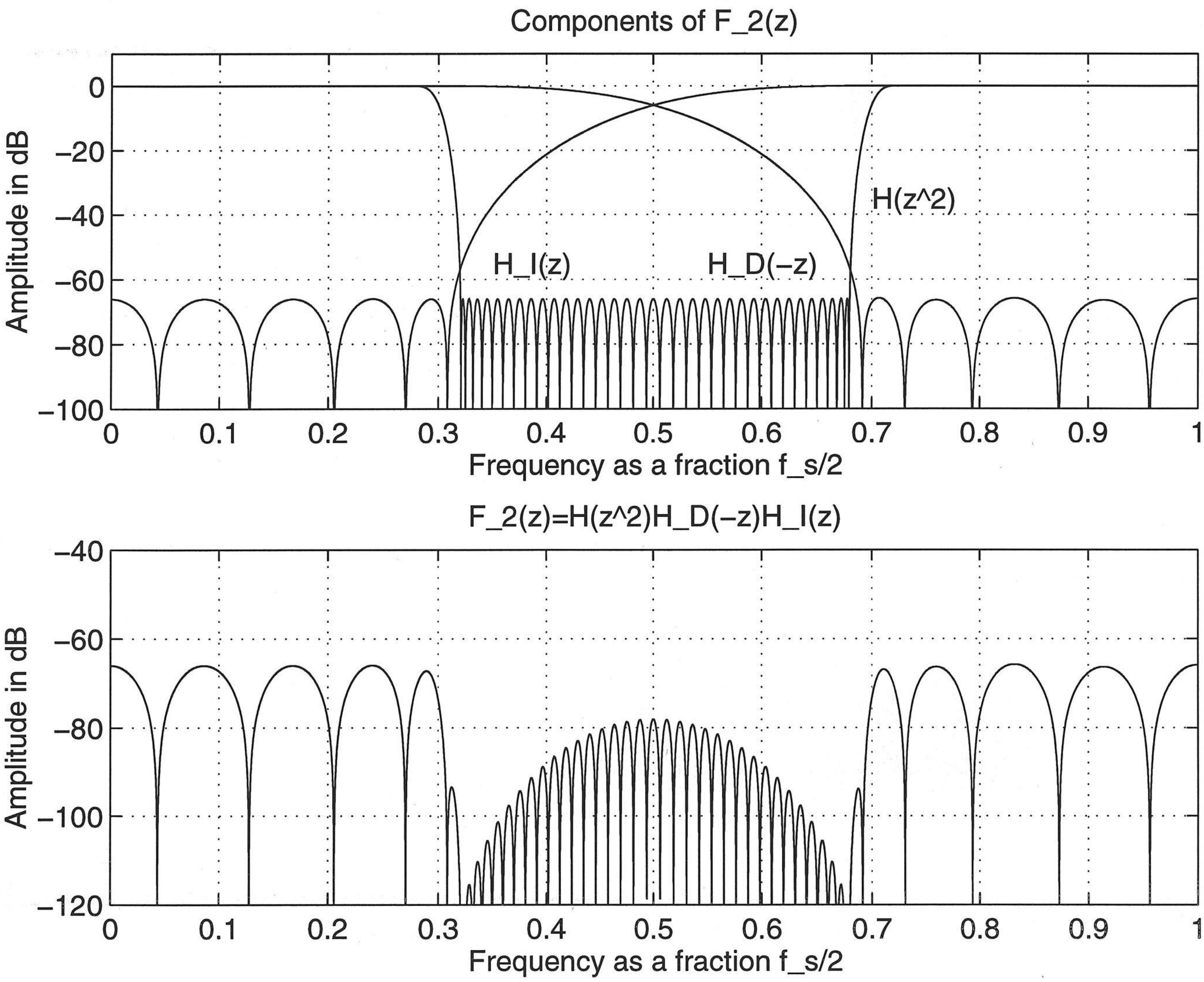
- $\widehat{H}(z)$ is designed to have the passband and stopband edges at $\widehat{f}_p = f_s/2 - f_p = 0.28(f_s/2)$ and $\widehat{f}_{\rm st} = f_s/2 - f_{\rm st} = 0.32(f_s/2)$ for the sampling rate of $f_s/2$. The passband and stopband ripples are again 0.006
- As seen from the following transparencies, the resulting $\widehat{H}(z^2)$ provides the desired transition band for $F_1(z)$.
- $H_D(z)$ and $H_I(z)$ are desired to be designed as highpass halfband filters with the stopband edge at $(2\widehat{f}_{\rm st} + \widehat{f}_p)/3$ [the corresponding passband edge is $f_s/2 - (2f_{\rm st} + f_p)/3$] and the passband and stopband ripples of 0.0005.
- The orders of the subfilters are the same as in Example 1, that is, the order of $\widehat{H}(z)$ is 92 and the orders of both $H_D(z)$ and $H_I(z)$ are 18. In this case, $z^{-D} = z^{-110}$ $(D = 92 + 18)$.
- As seen from the following transparencies, the maximum passband and stopband ripples for $z^{-D} - F_1(z)$ (the stopband and passband ripples for $F_1(z)$) are 0.0005 and 0.0015, respectively.

- The maximum amplitude value of $F_2(z)$ is 0.0005, so that the aliased components are well attenuated.
- Again, the order of the direct FIR filter design to meet the same criteria as $z^{-D} - F_1(z)$ is 169, requiring 85 multiplications per sample. The proposed design requires only 28.5 multiplications per input sample.

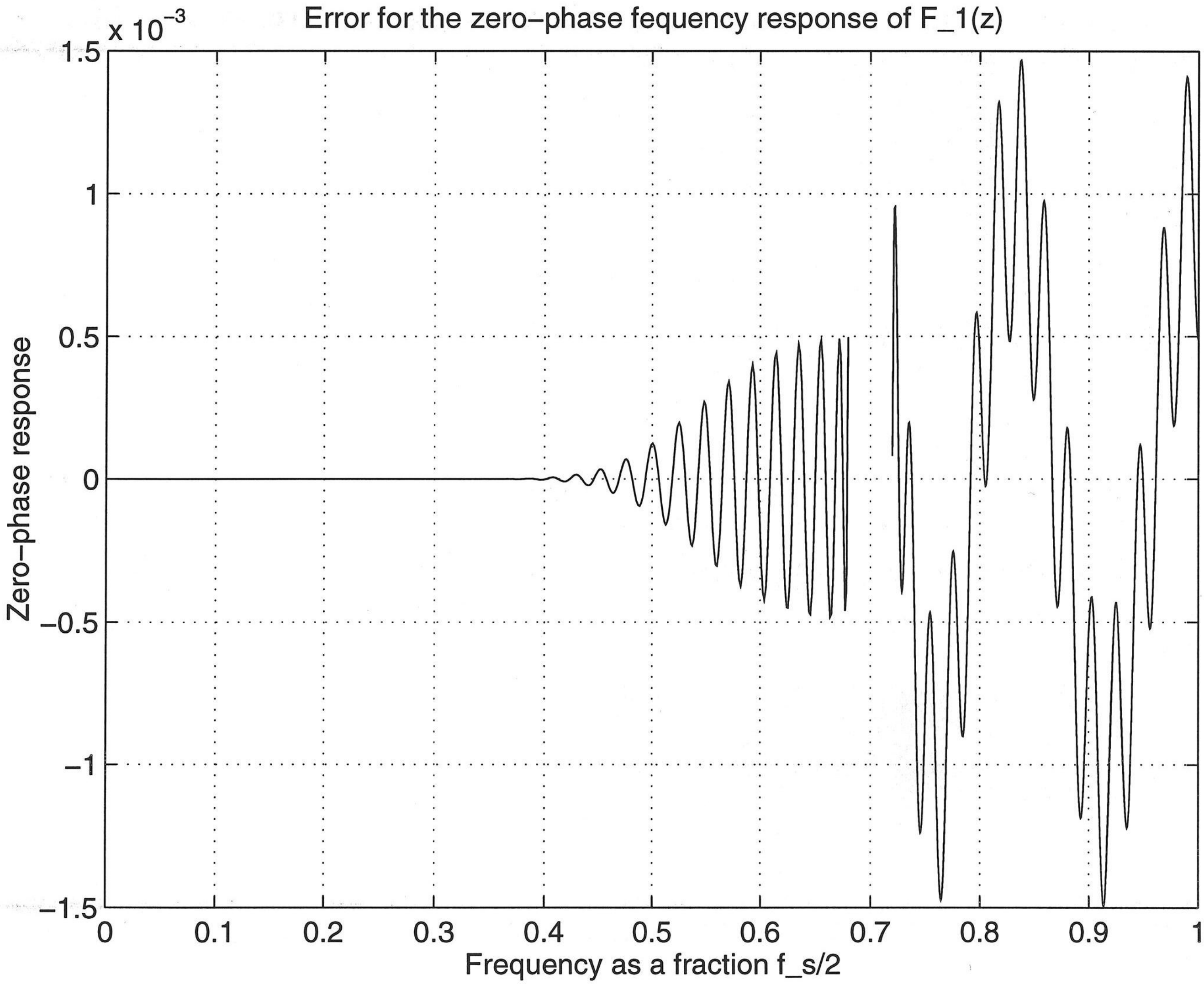

H(z)
Amplitude in dB
0
−20
−40
−60
−80
−100
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Frequency as a fraction f_s/4


H_D(z) and H_I(z)
Amplitude in dB
0
−20
−40
−60
−80
−100
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Frequency as a fraction f_s/2

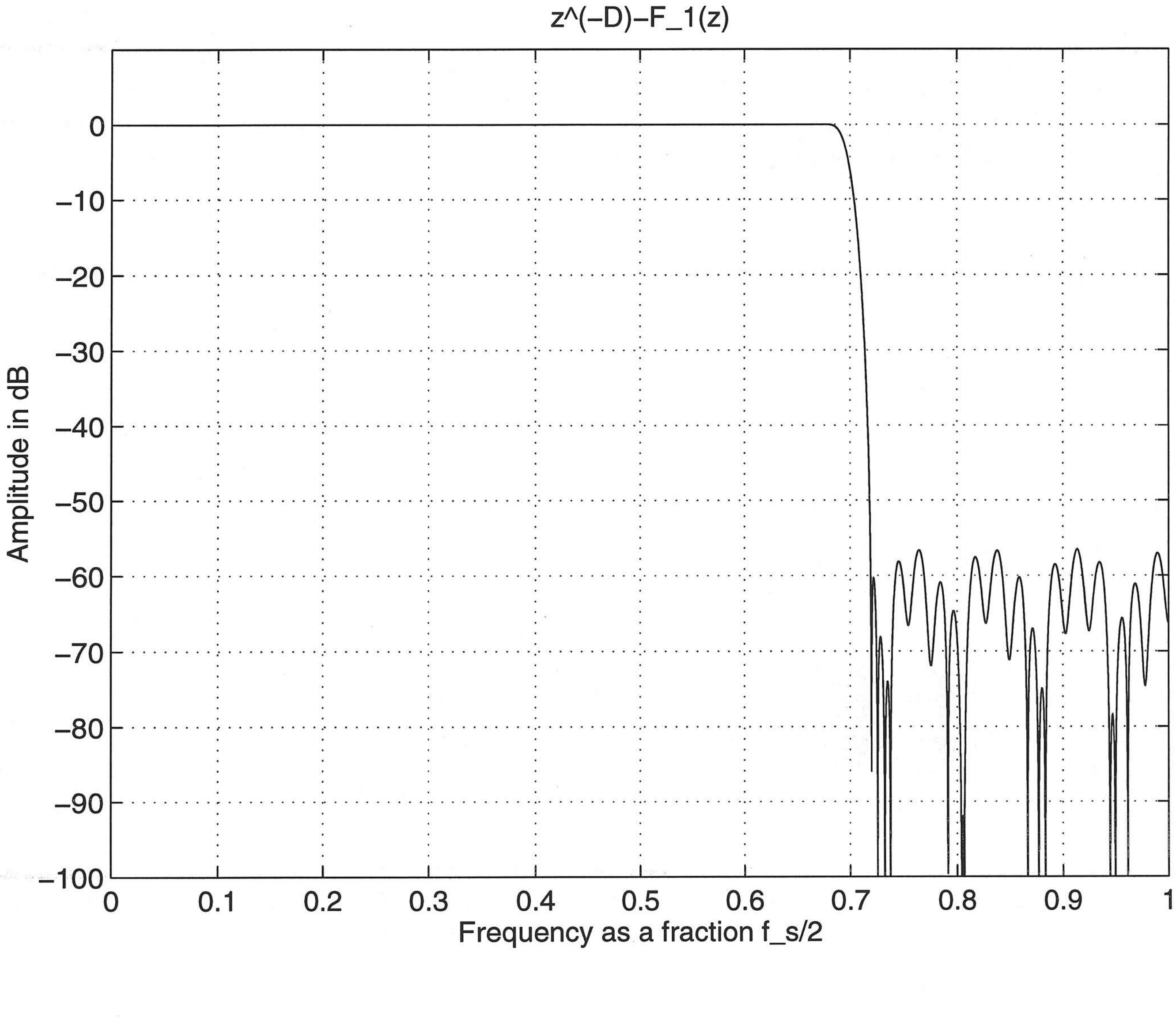

Components of $F_1(z)$
Amplitude in dB
0
−20
−40
−60
−80
−100
$H_D(z)$, $H_I(z)$
$H(z^2)$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Frequency as a fraction $f_s/2$
$F_1(z)=H(z^2)H_D(z)H_I(z)$
Amplitude in dB
0
−20
−40
−60
−80
−100
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Frequency as a fraction $f_s/2$



Components of F_2(z)
Amplitude in dB
H(z^2)
H_I(z)
H_D(−z)
Frequency as a fraction f_s/2
F_2(z)=H(z^2)H_D(−z)H_I(z)
Amplitude in dB
Frequency as a fraction f_s/2



Error for the zero–phase fequency response of F_1(z)
x 10^-3
1.5
1
0.5
0
−0.5
−1
−1.5
Zero–phase response
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Frequency as a fraction f_s/2


z^(−D)−F_1(z)
Amplitude in dB
0
−10
−20
−30
−40
−50
−60
−70
−80
−90
−100
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Frequency as a fraction f_s/2

## Intermediate Case: Edges of the termination close to one fourth of the sampling rate

- Use block (a) with $r = 3/2$ ($N = 3,\ M = 2$).

$$Y(z) = V(z^{1/3}) + V(z^{1/3}e^{j2\pi/3}) + V(z^{1/3}e^{j4\pi/3}),$$

where

$$V(z) = F_1(z)X(z^3) + F_2(z)X(-z^3)$$

with

$$F_1(z) = \widehat{H}(z^2)H_D(z)H_I(z)$$

and

$$F_2(z) = \widehat{H}(z^2)H_D(-z)H_I(z).$$

- The relation between the input of $H_D(z)$ and the output of $H_I(z)$ is the same as in the narrowband case.

## Differences

- The input sampling rate of $H_D(z)$ is now $f_s' = 3f_s$ and the input of $H_D(z)$ contains one and a half periods of the original input signal spectrum (see Fig. (a) in the next transparency).
- The specifications for the overall filter consisting of the termination filter, the decimator, and the interpolator are the same except that the sampling rate is now $f_s' = 3f_s$ (see Fig. (b) in the next transprency).
- The second basic difference compared to the narrowband case is that there are also aliased terms when finally decimating by three.
- However, because of filtering, the components aliased from the region $[f_s/2, 3f_s/2]$ are very small (see Fig. (c) in the next transparency).

**Filtering in the intermediate case. (a) Periodic input signal spectrum of $H_D(z)$. (b) Transfer function from the input of $H_D(z)$ to the output of $H_I(z)$. (c) Output signal spectrum of $H_I(z)$ before sampling rate reduction by 3.**

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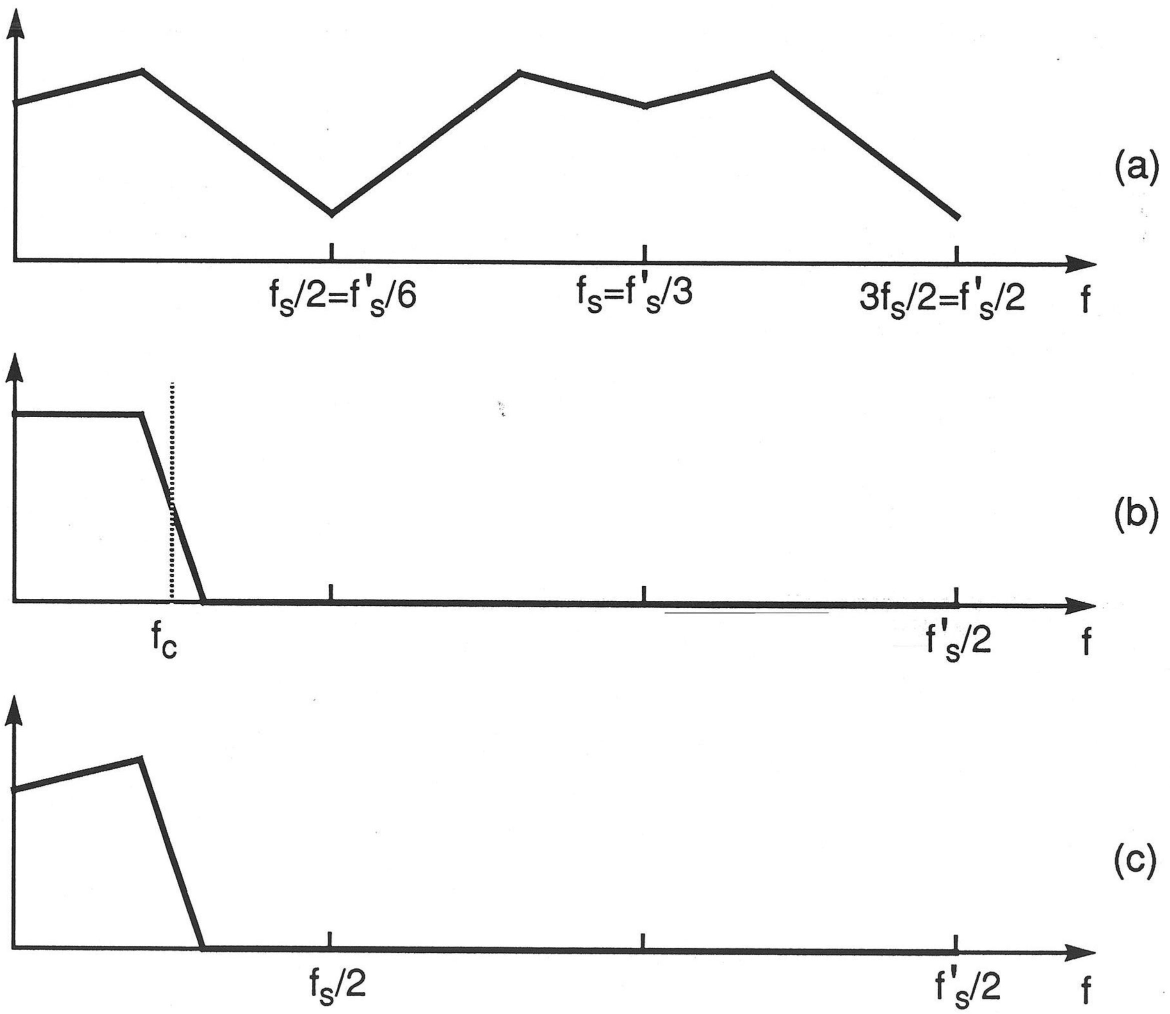
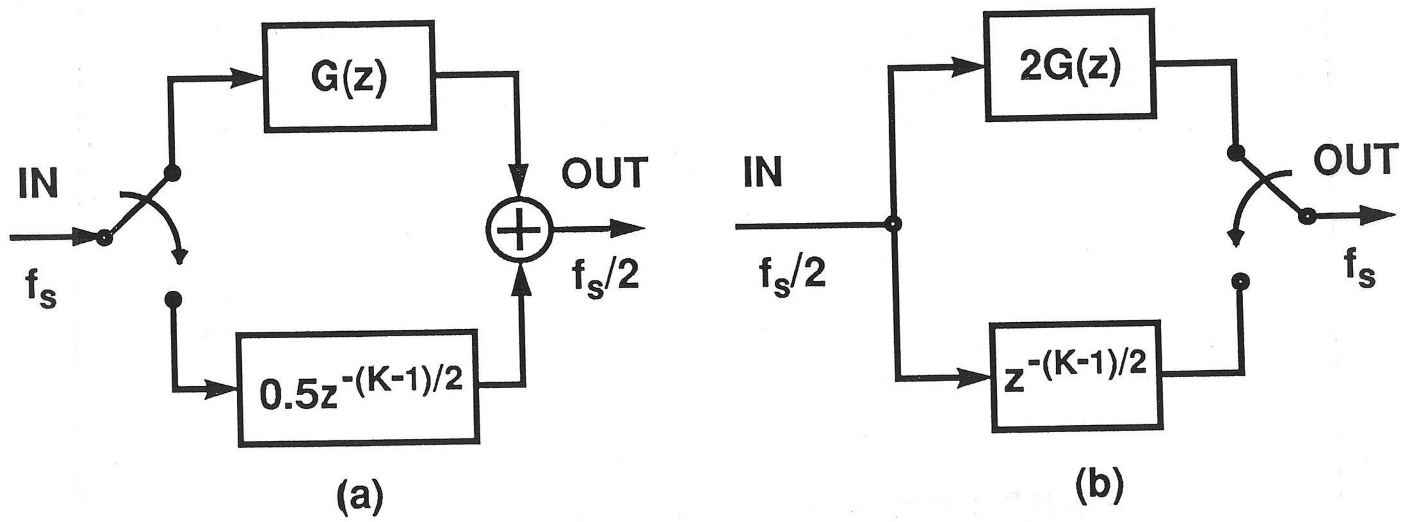


## What happens?

- The sampling frequency of the termination filter $\widehat{H}(z)$ is in this case $\widehat{f_s} = 3/2 f_s$, i.e., 3/2 times that of the overall filter $H(z)$.
- The advantage of using $r = 2/3$ lies, however, in the following facts:
  - The relative transition bandwidths of $H_D(z)$ and $H_I(z)$ are very wide, resulting in very low filter orders.
  - The passband and stopband edges of the termination $\widehat{H}(z)$ are close to $\widehat{f_s}/3$ so that the design procedure can be easily repeated using the building block (a) with $r = 2$ ($N = 1$ and $M = 2$).

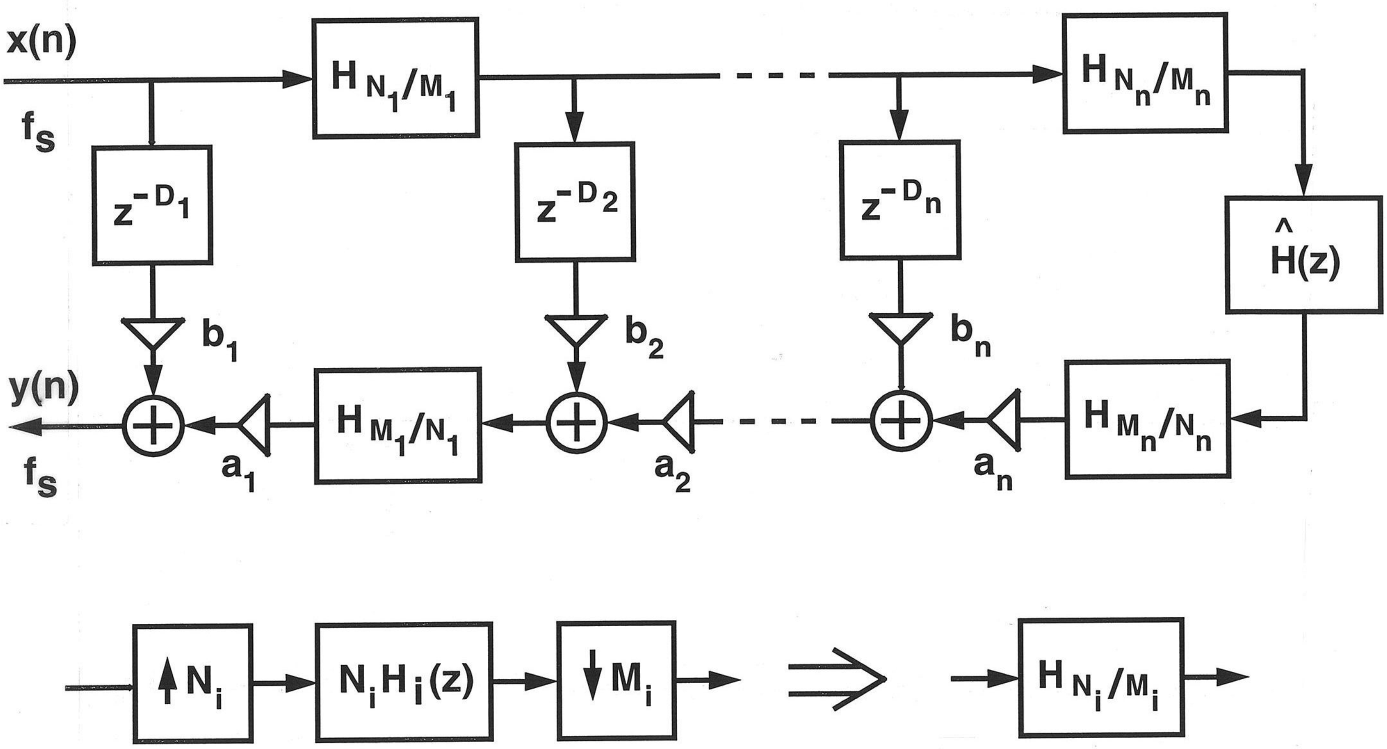
## What do we need?

- Building blocks with $r = 2$ ($N = 1$, $M = 2$) and $r = 2/3$ ($N = 3$, $M = 2$).
- $H_D(z)$ and $H_I(z)$ can be constructed using efficient half-band filters with transfer function of the form $z^{-K} + G(z^2)$ ($K$ is odd and the order of $G(z^2)$ is $K$ in $z^{-2}$.



## Overall structure

- $(a_i, b_i)$ is either $(1, 0)$ or $(-1, 1)$.



**Example: Passband edge = $0.2f_s$, stopband edge = $0.201f_s$, passband and stopband ripples = 0.001.**

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- The given criteria are met by

**1.** Six blocks: (a)-(b)-(a)-(a)-(b)-(a)

  - For all blocks (a) $r = 2$ ($N = 1$ and $M = 2$).

**2.** Nine blocks: (a)-(b)-(a)-(a)-(b)-(a)-(a)-(b)-(a)

  - For all blocks (a) $r = 2$ ($N = 1$ and $M = 2$).

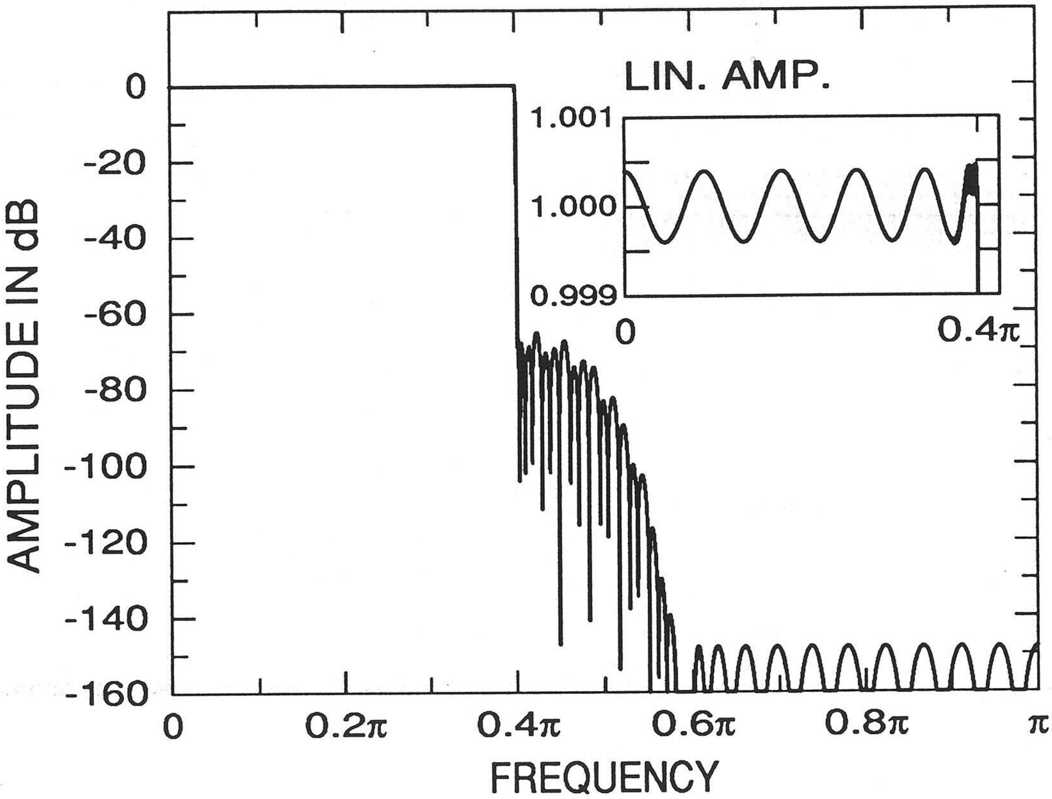
## Six blocks

- The number of multipliers (the orders) for the half-band filters are 11 (42), 4 (14), 11 (42), and 4 (14).
- Extra delay terms are $z^{-1210}$ and $z^{-278}$
- Termination: Sampling rate = $f_s/16$, $\widehat{H}(z) = F(z^2)G(z)$, $F(z)$ of order 136, and $G(z)$ of order 12.
- Overall multiplication rate = 21 multiplications per input sample.
- Direct-form design: Order is 3256, 1629 multipliers.
- Delay: Direct-form: 1628, Multirate design: 2462.
- Delay terms : Direct-form: 3256, Multirate design: 1854.

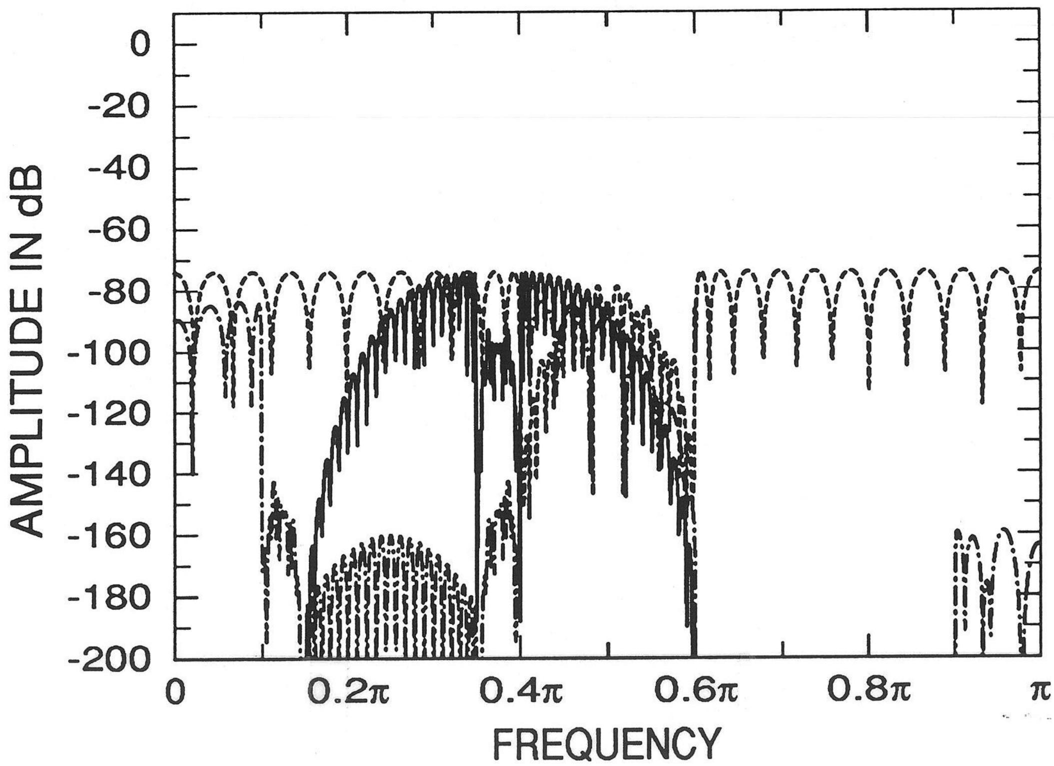
## Overall Response

- For six blocks, the overall output can be written as
$$Y(e^{j\omega}) = \sum_{k=0}^{15} H_k(e^{j\omega}) X(e^{j(\omega - 2k\pi/16)}).$$
- Figure (a) in the next transparency shows $|H_0(e^{j\omega})|$ in dB (amplitude response for the unaliased component).
- Figure (b) shows the three worst amplitude responses $|H_6(e^{j\omega})|$ (solid line), $|H_8(e^{j\omega})|$ (dashed line), and $|H_4(e^{j\omega})|$ (dot-dashed line) for the aliased components.

**Responses for the proposed multirate filter with four decimation and interpolation stages. (a) Response for the unaliased component. (b) Responses for some aliased components.**




(a)




(b)

**Multiplication rate versus the center of the transition band for filters with transition bandwidth of $0.001f_s$ and passband and stopband ripples of 0.001.**

- In contructing this plot, the upper allowable number of stages has been fixed to be 8 and the upper limit for the overall delay has been two and a half times that of the direct-form equivalent (4070).