80558 MULTIRATE SIGNAL PROCESSING

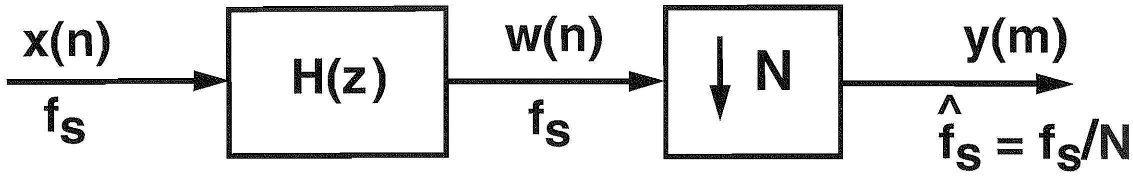
Part II: Design and Implementation of Efficient Decimators and Interpolators

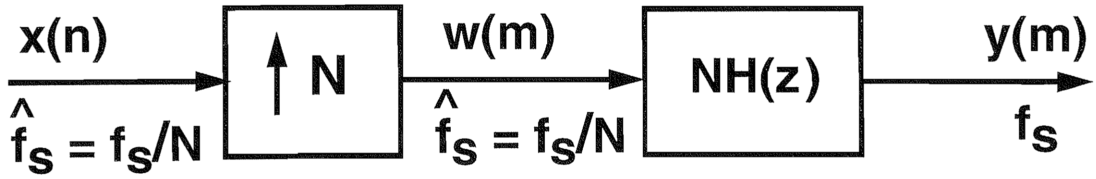
- The purpose of this part is to review a number different techniques for constructing efficient filters for decimation and interpolation purposes.
- Both efficient implementation forms as well as various methods for designing decimation and interpolation filters for these implementation forms are considered.
- This part has been divided into the following subparts:
- II.A: One-Stage Desimation and Interpolation Structures
- II.B: Conventional Multistage Implementations
- II.C: Special Filter Structures
- II.D: Nth-Band IIR Filters
- II.E: Nth-Band FIR Filters
- II.F: Half-Band FIR Filters
- II.G: Half-Band IIR Filters
- II.H: Use of Conventional and Modified Comb (Running Sum) Structures as a First Stage for Multistage Decimator Implementaions

Part II.A: One-Stage Decimation and Interpolation Structures

- To emphasize the duality between the decimators and interpolators we first consider the implementation and design of decimators and interpolators using the one-stage structures as shown on the next page.
- In the case of the decimator, the input sampling rate is f_s and the output sampling rate after decimation by an integer factor N is f_s/N.
- The role of the filter to preserve the signal components for $|f| \leq (f_s/2)/N$ and attenuate the signal components aliasing from the region $(f_s/2)/N \leq |f| \leq f_s/2$ into $|f| \leq (f_s)/N$ using Case A, Case B, or Case C specifications considered in Part I.
- In the case of the interpolator, the input sampling rate is f_s/N and the output sampling rate after interpolation by an integer factor N is f_s.
- The role of the filter to preserve the original baseband components for $|f| \leq (f_s/2)/N$ and attenuate the images in the the region $(f_s/2)/N \leq |f| \leq f_s/2$ using the Case A, Case B, or Case C specifications.
- Recall that the amplitude response of $H(z)$ in the interpolation case approximates N in the passband.

One-Stage (Single-Stage) Decimator and Interpolator Structures to be Considered





• In the sequel, it will be shown to be benefical to carry out the sampling rate alteration in several stages.

• As will be seen later, also in these cases is it very useful to express the overall system by means of certain identities in the one-stage (single-stage) equivalent forms as shown on the previous page.

• This makes the both the synthesis and analysis of the overall system more straightforward.

• We recall that given N and α in Cases A, B, and C the passband region is given in terms of the 'real' frequency and the angular frequency $\omega = 2\pi f/f_s$ as $[0,\ \alpha(f_s/2)/N]$ and $[0,\ \alpha\pi/N]$, respectively.

• The stopband regions are in terms of the 'real' frequency

$$X_s = \begin{cases} [(f_s/2)/N,\ f_s/2] & \text{for Case A} \\ \bigcup_{k=1}^{\lfloor N/2 \rfloor} [\frac{(2k-\alpha)f_s/2}{N},\ \min(\frac{(2k+\alpha)f_s/2}{N},\ f_s/2)] & \\ & \text{for Case B} \\ [(2-\alpha)(f_s/2)/N,\ f_s/2] & \text{for Case C.} \end{cases} \tag{1}$$

• In terms of the angular frequency, the stopband regions are

$$\Omega_s = \begin{cases} [\pi/N,\ \pi] & \text{for Case A} \\ \bigcup_{k=1}^{\lfloor N/2 \rfloor} [\frac{(2k-\alpha)\pi}{N},\ \min(\frac{(2k+\alpha)\pi}{N},\ \pi)] & \\ & \text{for Case B} \\ [(2-\alpha)\pi/N,\ \pi] & \text{for Case C.} \end{cases} \tag{2}$$

Direct-Form FIR Filter Realizations for Decimation

• Consider decimation by an integer factor N using an FIR filter with the transfer function

$$H(z) = \sum_{n=0}^{M} h(n) z^{-n}. \tag{3}$$

• The input-output relation for the filter in the time domain can be expressed as

$$w(n) = \sum_{k=0}^{M} h(k) x(n-k). \tag{4}$$

• The output after decimation by a factor of N is then

$$y(m) = w(Nm). \tag{5}$$

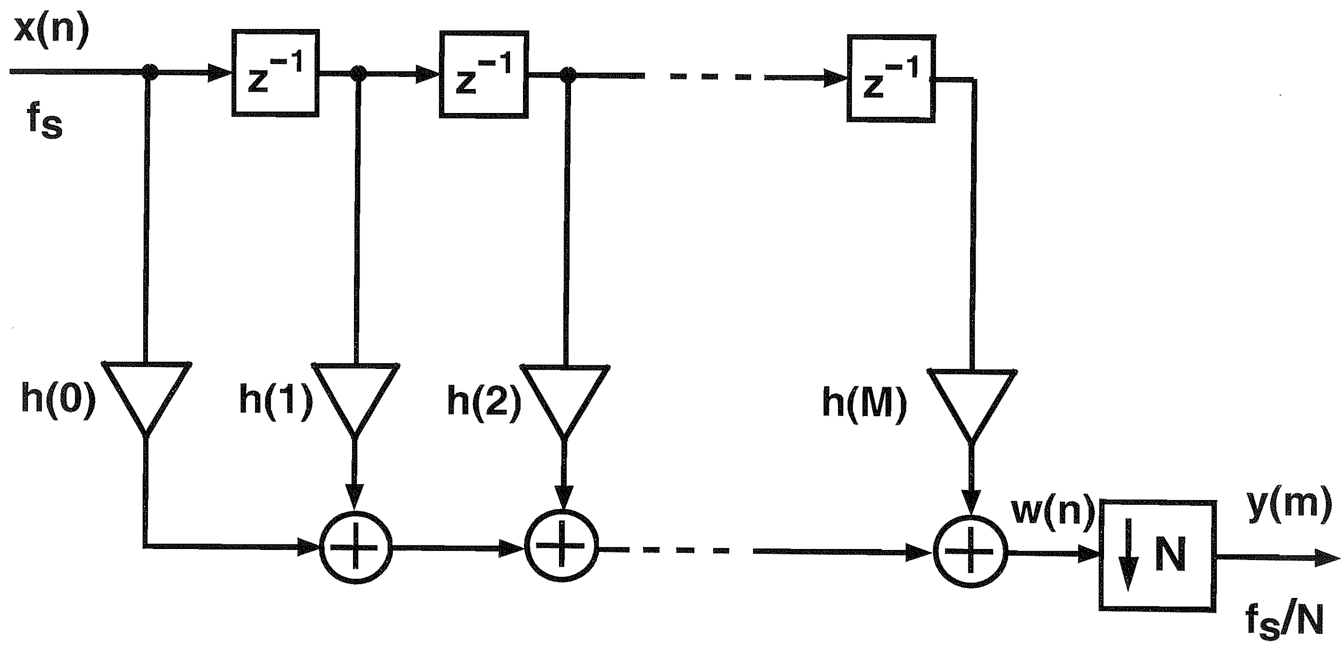
• The first structure of the next page shows an implementation where all the outputs are evaluated out of which only every Nth output sample is picked up. Therefore, for this structure, a plenty of vain computations are performed.

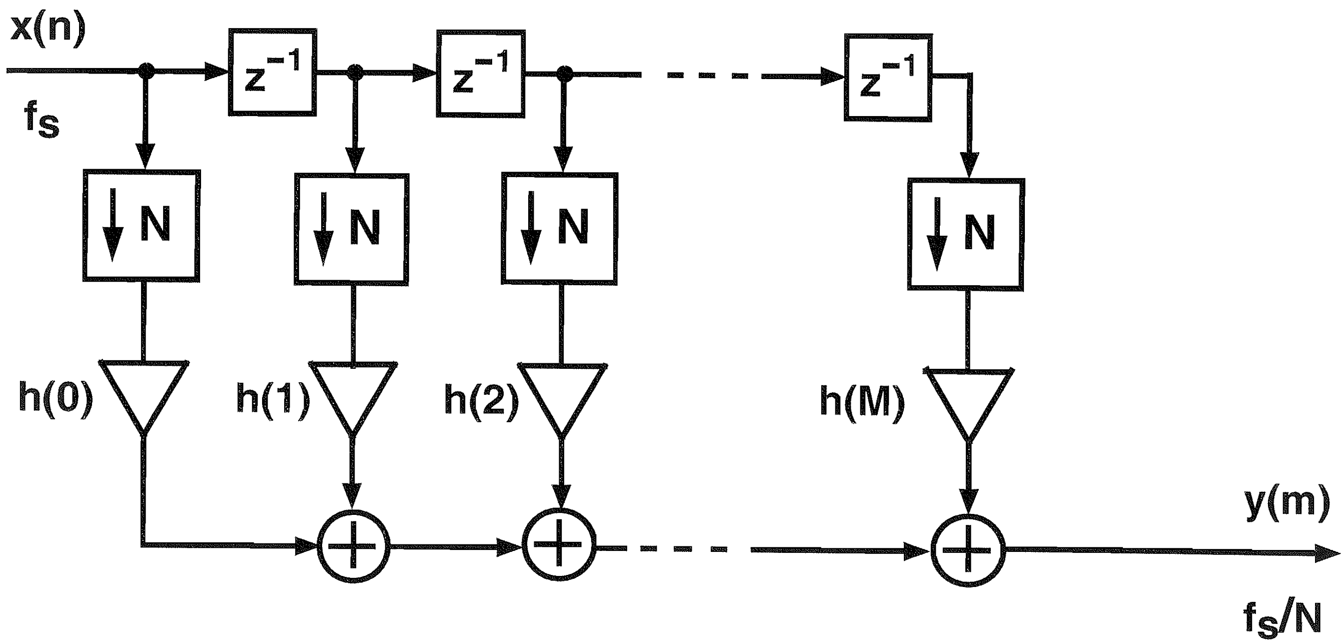
• The number of multiplications per input sample is in this case $(M+1)$.

• In order to derive a more efficient implementation, Eqs. (4) and (5) are combined to give

$$y(m) = \sum_{k=0}^{M} h(k) x(Nm-k). \tag{6}$$

Implementations of an FIR filter of order M for Decimation by an Integer Factor N

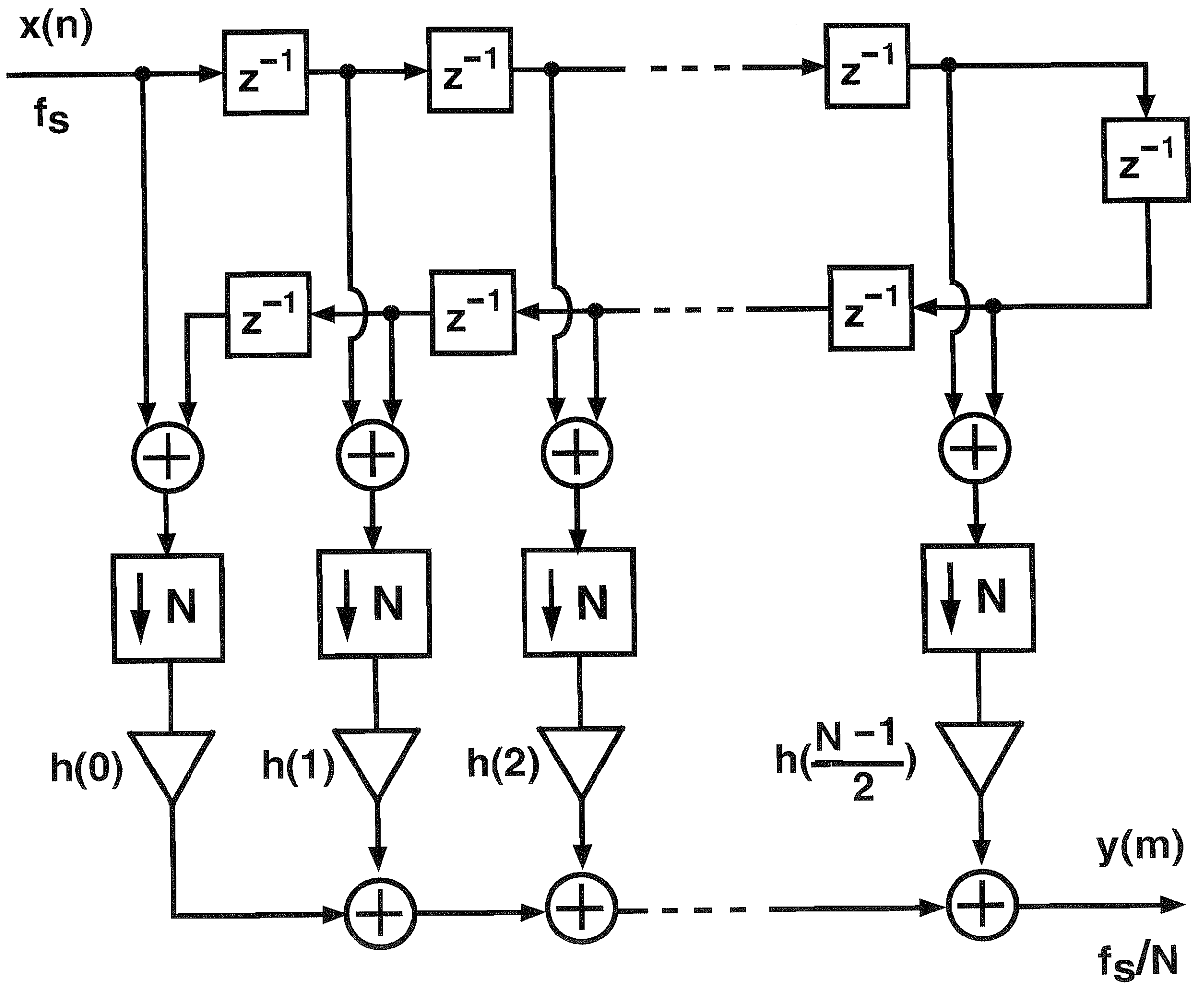




- Since there are no feedback loops, only every Nth output of the filter with transfer function $H(z)$ can be computed.
- The second structure of the previous page shows the corresponding efficient implementation where the multiplications are performed only every Nth time instant compared to the input sampling rate.
- In this structure, the data goes through the delays like for the conventional FIR filter.
- The key idea is to pick up the data from the delays for evaluating the output only every Nth time instant compared to the input sampling rate.
- This reduces the number of multiplications per input sample to $(M+1)/N$, as is desired.

Efficient Implementation of a Linear-Phase FIR Filter of Order M for Decimation by an Integer Factor N

- The multiplication rate can be further reduced using a direct-form structure exploiting the coefficient symmetry, as shown below for M even and $h(M-n) = h(n)$ for $n = 0, 1, \cdots, M/2 - 1$. The number of multiplications per input sample reduces to $(M/2 + 1)/N$.
- For M odd and $h(M-n) = h(n)$ for $n = 0, 1, \cdots, (M-1)/2$, a similar structure exists, requiring $(M+1)/(2N)$ multiplications per input sample.



Transposed Direct-Form FIR Filter Realizations for Interpolation

- Consider interpolation by an integer factor N using an FIR filter with the transfer function

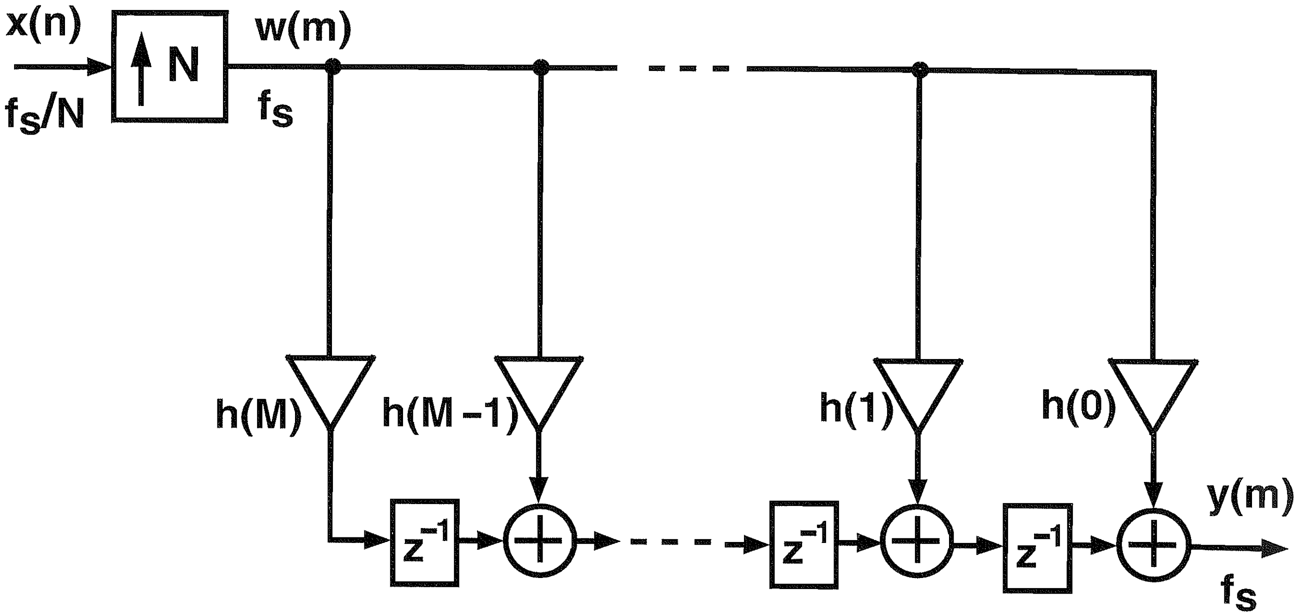
$$H(z) = \sum_{n=0}^{M} h(n) z^{-n}. \tag{7}$$

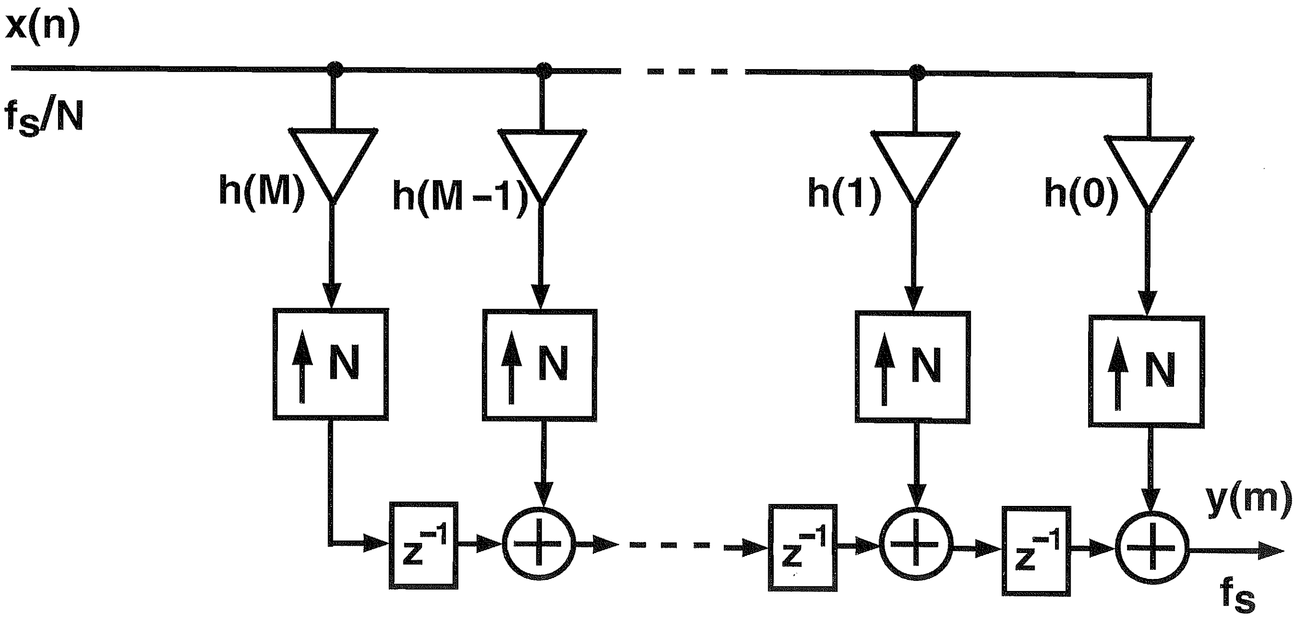
- In order to arrive at an efficient implementation, a transposed direct-form structure is used after interpolation. This gives the first implementation of the next page.
- The input to the FIR filter is given by

$$w(m) = \begin{cases} x(m/N), & m = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

- Hence, only every Nth input sample to the FIR filter is non-zero.
- Therefore, in this structure, several zero-valued input samples are multiplied by the filter coefficients without any contribution to the overall output.
- This problem can be overcome by using the second structure of the next page.
- In this structure, the data goes through the delays in the output part like for the conventional FIR filter at the higher output sampling rate.

Implementations of an FIR Filter of order M for Interpolation by an Integer Factor N



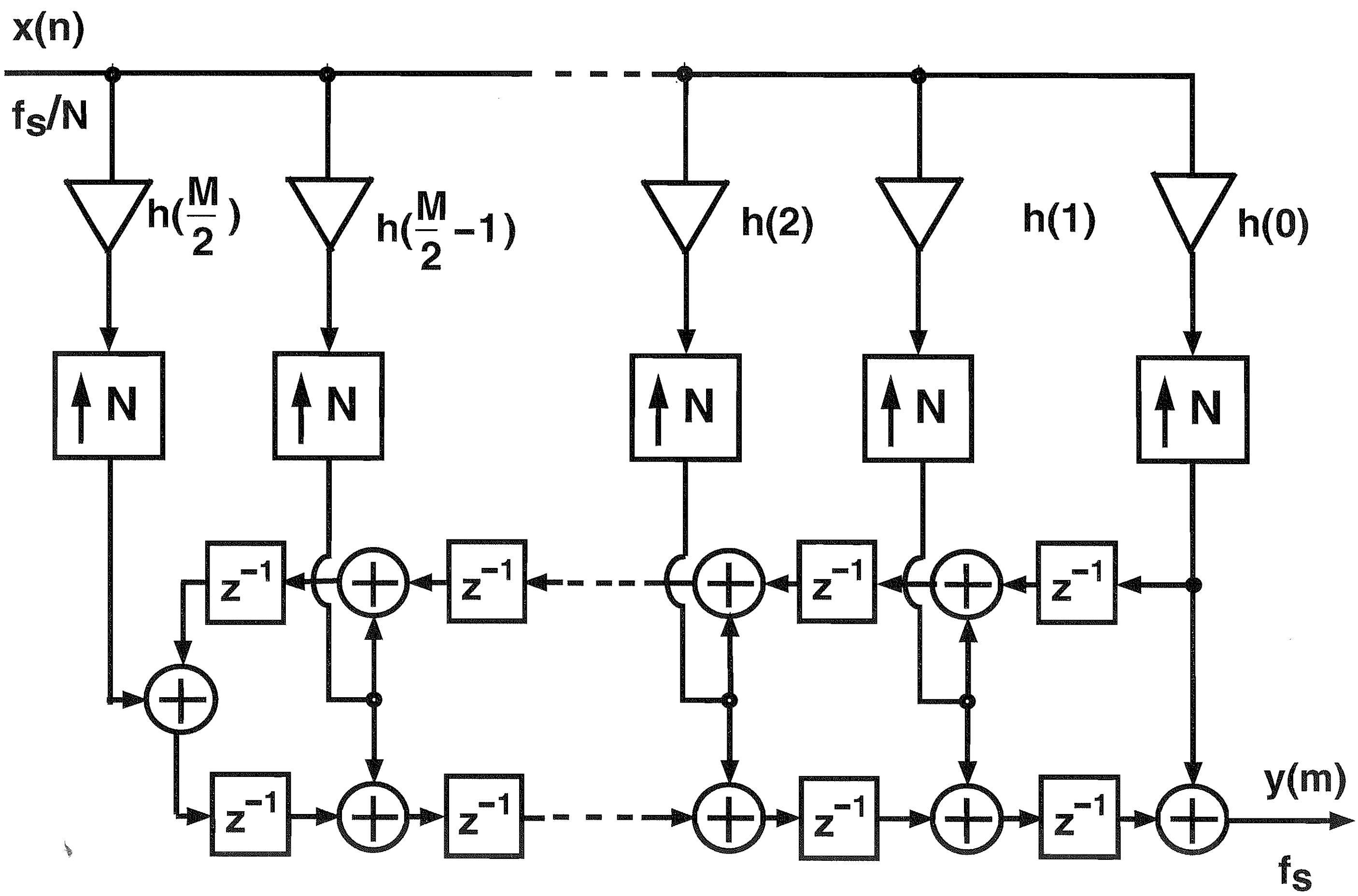


- The key idea is to process only every Nth time instant compared to the output sampling rate of f_s the existing 'real' non-zero input data samples.
- These samples are multiplied by the coefficients and the results are feeded to the adders between the delays (and to the input of the first delay).
- This corresponds to the case where the existing input samples are multiplied at the lower input sampling rate of f_s/N.
- This reduces the number of multiplications per output sample from M to M/N.

Efficient Implementation of a Linear-Phase FIR Filter of Order M for Interpolation by an Integer Factor N

• The multiplication rate can be further decreased using a transposed direct-form structure exploiting the coefficient symmetry, as shown in the following figure for M even and $h(M-n) = h(n)$ for $n = 0, 1, \cdots, M/2 - 1$. The number of multiplications per output sample is now $(M/2+1)/N$.

• Like in the decimation case, there exists a similar stucture for M odd and $h(M-n) = h(n)$ for $n = 0, 1, \cdots, (M-1)/2$, requiring $(M+1)/(2N)$ multiplications per output sample.



Conventional IIR Filters for Decimation and Interpolation

• Consider decimation by an integer factor N using an IIR filter with the following transfer function:

$$H(z) = \frac{\sum_{n=0}^{M} a(n) z^{-n}}{1 - \sum_{n=1}^{M} b(n) z^{-n}}. \tag{9}$$

• This filter suffers from the drawback that the fact that only every Nth output is needed cannot be expoloited due to the feedback loop.

• The diagram of the next page shows a direct-form II structure, where the fact that only every Nth output is needed has been exploited in the feedforward part.

• The time-domain equations are given by

$$w(n) = x(n) + \sum_{k=1}^{M} b(k) w(n-k) \tag{10}$$

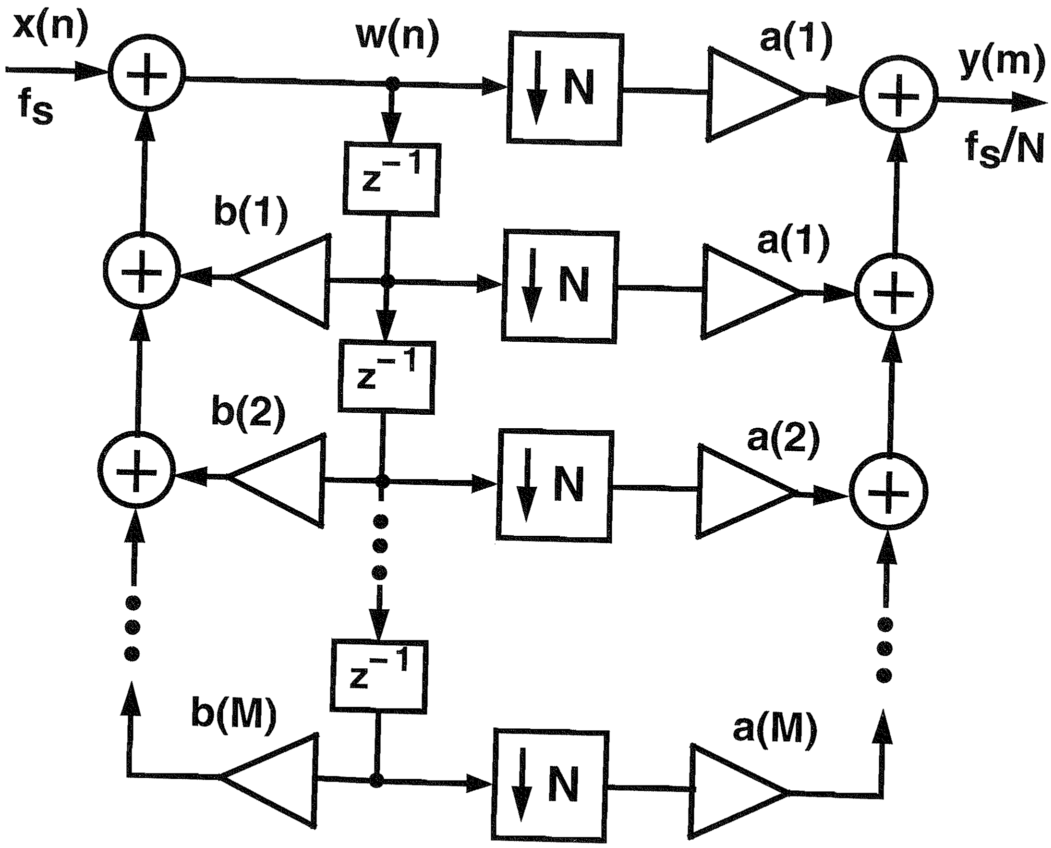
and

$$y(m) = \sum_{k=0}^{M} a(k) w(Mm - k). \tag{11}$$

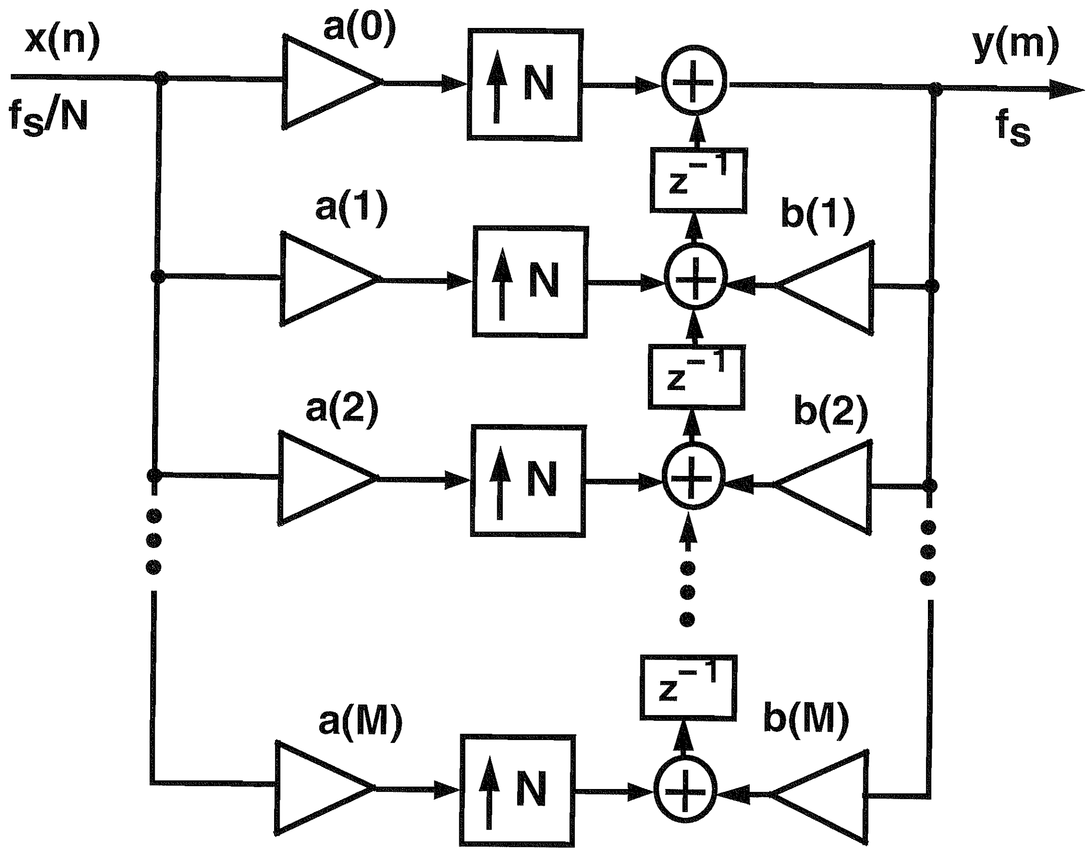
• Note that every sample value $w(n)$ is necessary needed due to the feedback loop.

• Page 16 shows the corresponding transposed direct-form II structure for the interpolator.

Implementation of a Conventional IIR Filter of order M for Decimation by an Integer Factor N



Implementation of a Conventional IIR Filter of order M for Interpolation by an Integer Factor N



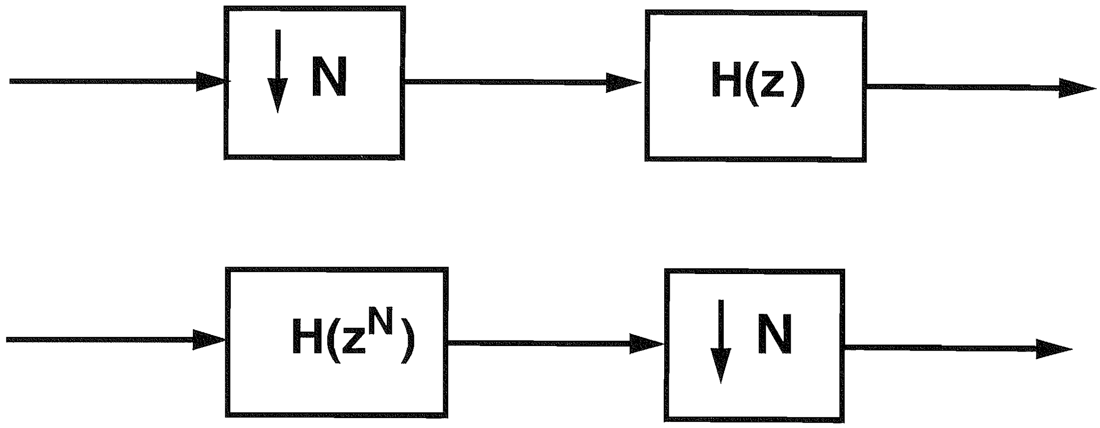
Identities for Decimators and Interpolators

• Two important identities that are exploited in the sequel in building efficient decimator and interpolator structures as well as in analysing their performances are shown on the next page.

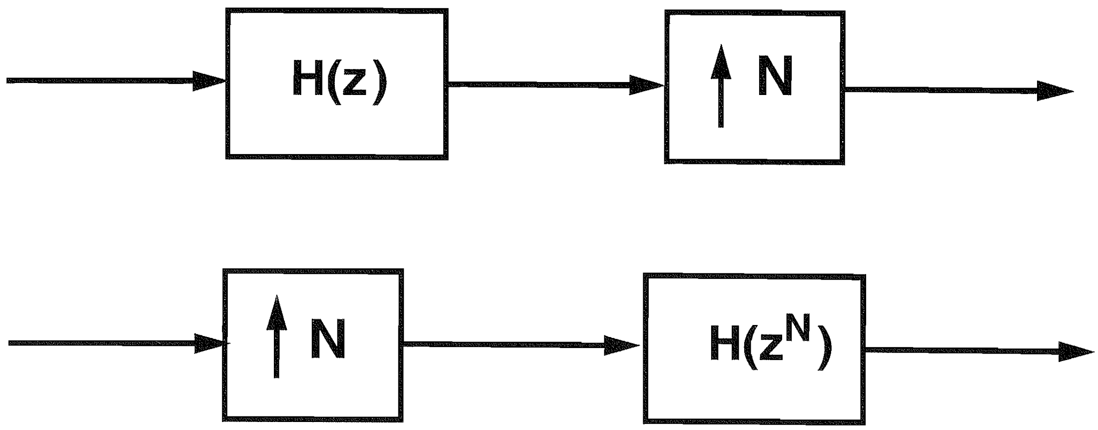
• If there is a transfer function $H(z)$ after decimation by N [before interpolation by N], this transfer can be moved before the decimation block [after the interpolation block] by changing it to be $H(z^N)$ and vice versa.

• Note that $H(z^N)$ is obtained from $H(z)$ by replacing each unit delay z^{-1} by z^{-N}, that is, a block of N delays.

Identity for the decimator



Identity for the interpolator



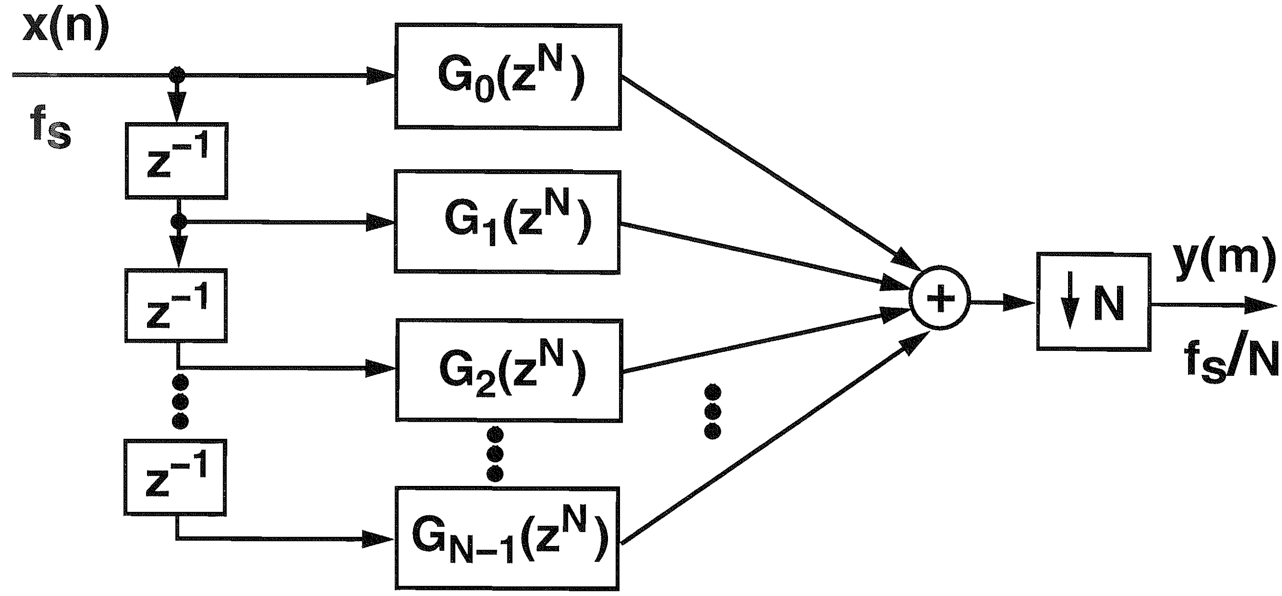
Polyphase Structures for Decimation

- Very important implementation forms for both decimators and interpolators are the so-called polyphase structures.
- When decimating by a factor of N, the first step is to express the overall transfer function as

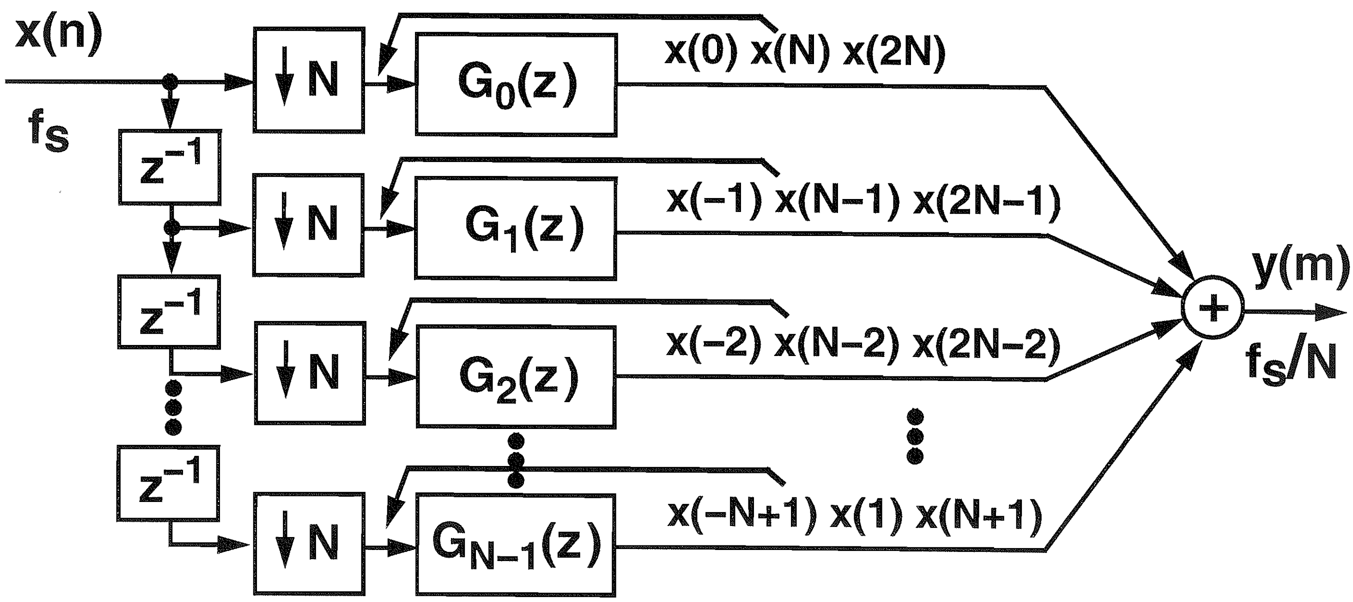
$$H(z) = \sum_{k=0}^{N-1} z^{-k} G_k(z^N). \tag{12}$$

- This transfer function is a sum of N branch filters with transfer functions $z^{-l}G_l(z^N)$ for $l = 0, 1, \cdots, N-1$.
- Here, $G_l(z^N)$ is obtained from a 'conventional' transfer function $G_l(z)$ by replacing z^{-1} by z^{-N}.
- Note that for the lth branch there are l delay terms (for $l = 0$, there are no delays).
- The first diagram of the next page gives a very inefficient implementation form. Only the number of the additional delay terms has been minimized.
- A significant simplification can be achieved by transferring the decimation block before each $G_l(z^N)$. According to the identities considered on Page 18, the $G_l(z^N)$'s for $l = 0, 1, \cdots, N-1$ are replaced in this case by the $G_l(z)$'s, as shown by the second diagram of the next page.

Polyphase Structures for Decimation: Intermediate Implementation Forms



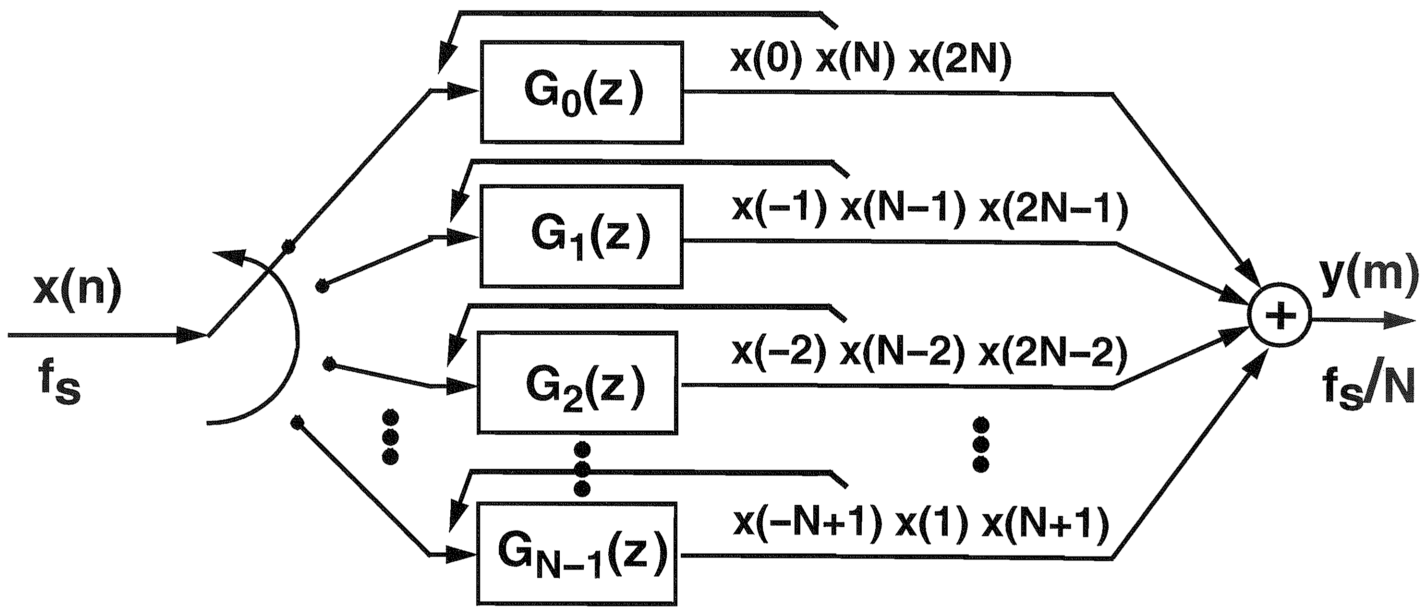
Identical Structure



How to Perform the Decimation Effectively?

- If $x(n)$ in the second diagram of the previous page is the input sample for $G_0(z)$, then the input sample for the $G_l(z)$ for $l = 1, 2, \cdots, N-1$ is, due to the delays, $x(n-l)$'s for $l = 1, 2, \cdots, N-1$.
- Therefore, if the decimation operations occur at the same time and the starting time for the first branch is $n = 0$, then the first samples at the inputs of the $G_l(z)$'s for $l = 0, 1, \cdots, N-1$ are $x(0-l)$ for $l = 0, 1, \cdots, N-1$.
- When decimating for the second time, the input samples for the branches are $x(N-l)$ for $l = 0, 1, \cdots, N-1$ and for the third time $x(2N-l)$ for $l = 0, 1, \cdots, N-1$ and so on.
- Based on this fact, the overall system can be implemented using the commutative structure shown on the next page.
- This system works in such a way that the input data is first divided into the following block of N samples: $\{x(mN-(N-1)), x(mN-(N-2)), \cdots, x(mN)\}$ for $m = 0, 1 \cdots$.

Commutative Decimator Structure



• For the mth block of N samples, the 'rotator' gives the first sample to $G_{N-1}(z)$, the second one to $G_{N-2}(z)$, and, finally, the last one to $G_0(z)$.

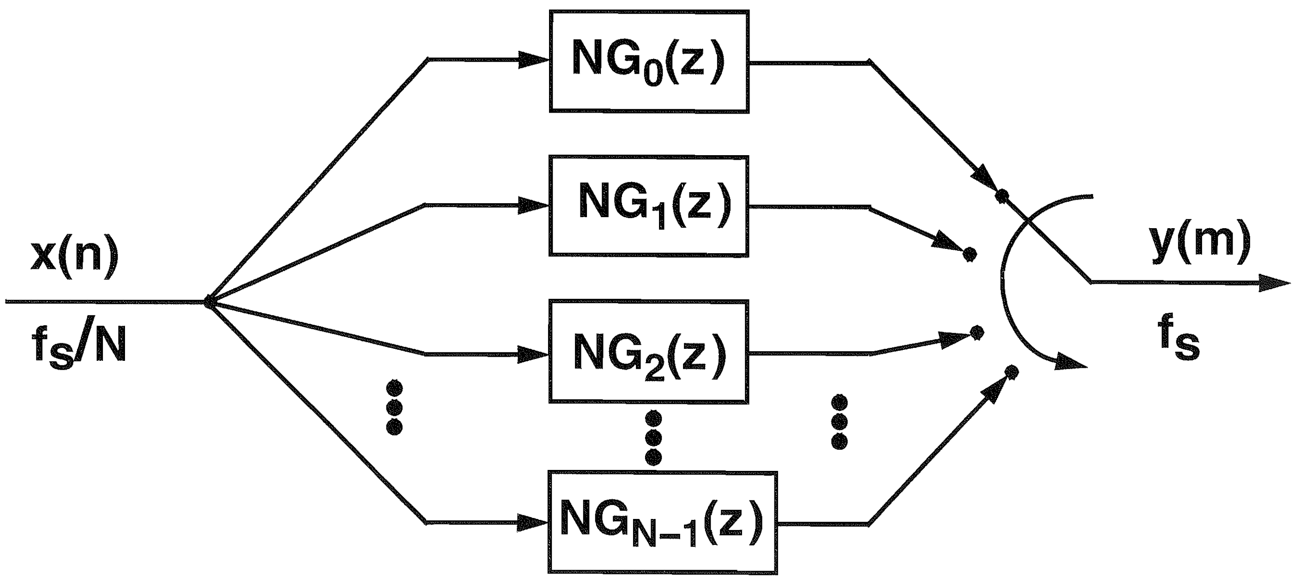
• After sharing the set of N samples, each filter performs only one operation, thereby working at the lower output sampling rate of f_s/N.

• Finally, the outputs of the the branch filters are added to give one new output sample $y(m)$.

• Then, a new set of N input samples for $m+1$ are processed in a similar manner to generate one more output sample $y(m+1)$.

Commutative Interpolator Structure

- The corresponding commutative interpolator structure works in a similar manner as shown below.
- The main difference is that each of the branch filters gets the same input sample and perform one operation at the input sampling rate.
- The sampling rate is increased by N by using the 'rotator' in such a manner that for each input sample, the first filter gives the first output sample, the second filter the second sample, and the, finally, the Nth filter the Nth output sample.
- This increases the sampling rate by a factor of N, as is desired. Note that, because of interpolation, each $G_l(z)$ is multiplied by N.



Polyphase Decomposition for an FIR Filter

• The desired polyphase decomposition for FIR filters is very trivial.

• As an example, we consider the FIR transfer function of page 27:

$$H(z) = \sum_{n=0}^{38} h(n)z^{-n}. \tag{13}$$

• As shown on this page, this transfer function is expressible for $N = 3$ as

$$H(z) = G_0(z^3) + z^{-1}G_1(z^3) + z^{-2}G_2(z^3), \tag{14a}$$

where

$$G_0(z^3) = \sum_{n=0}^{12} h(3n)z^{-3}, \tag{14b}$$

$$G_1(z^3) = \sum_{n=0}^{12} h(3n+1)z^{-3}, \tag{14c}$$

and

$$G_2(z^3) = \sum_{n=0}^{12} h(3n+2)z^{-3}. \tag{14d}$$

• Note that each of these filters contain every Nth (third) sample and the remaining samples are zero-valued.

• The first sample $G_l(z^3)$ for $l = 0, 1, 2$ occurs at $n = 0$, $n = 1$, and $n = 2$, respectively.

• The filters in the commutative structures are for $l = 0, 1, 2$

$$G_l(z) = \sum_{n=0}^{12} h(3n + l)z^{-1} \quad (15)$$

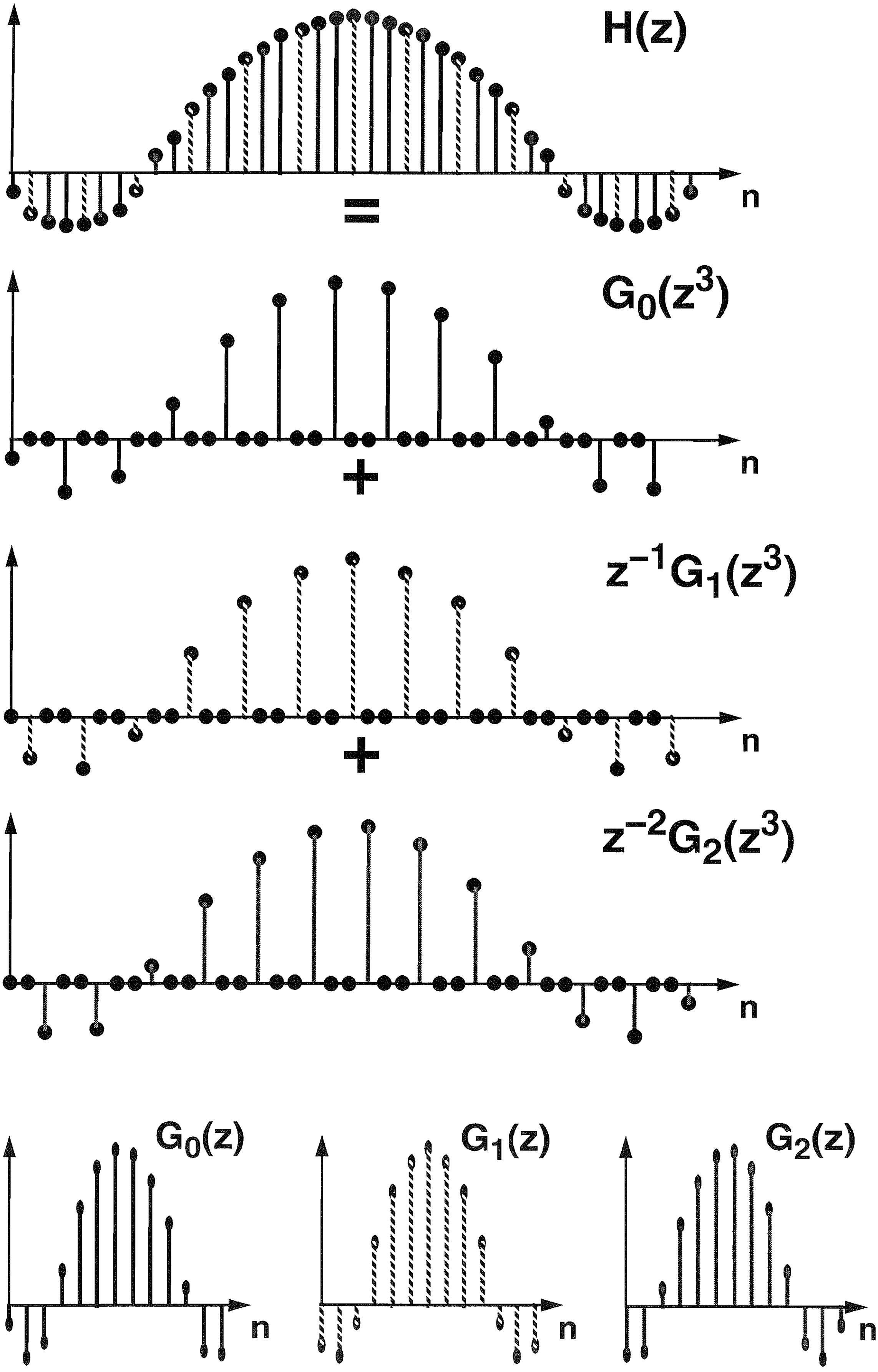
.

• In this example, all the filters are of the same order.

• If for instance, the order of $H(z)$ is 37, then $G_2(z)$ is of order 11 ($h(38) = 0$).

• In the general case, after knowing N and the impulse response of the FIR filter, $G_l(z)$ simply contains the non-zero samples $h(l + Nr)$ for $r = 0, 1, \cdots$. Trivial!

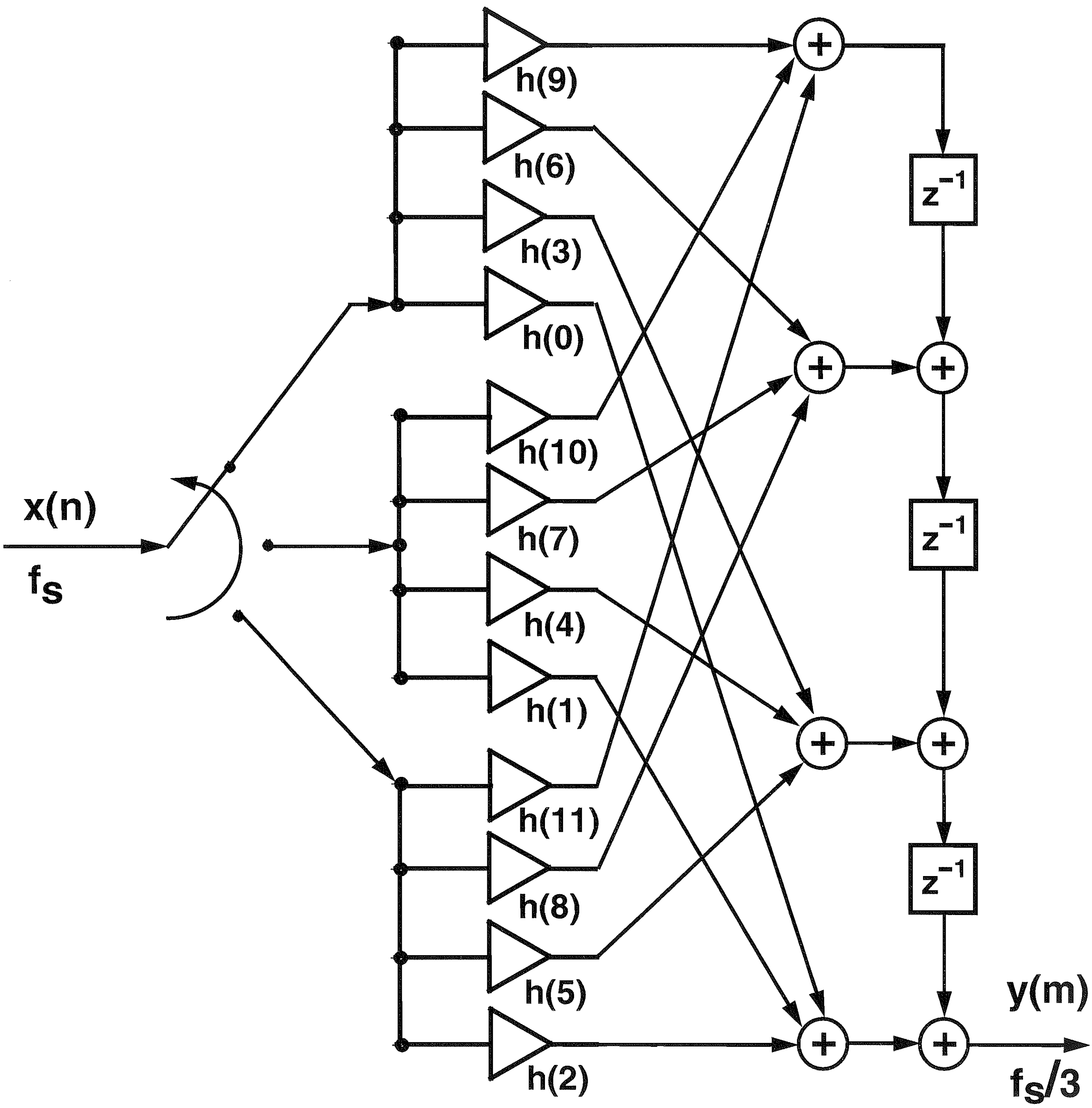
Example Polyphase Decomposition for an FIR Filter in the $N = 3$ Case



Memory-Saving Implementations for FIR Polyphase Structures

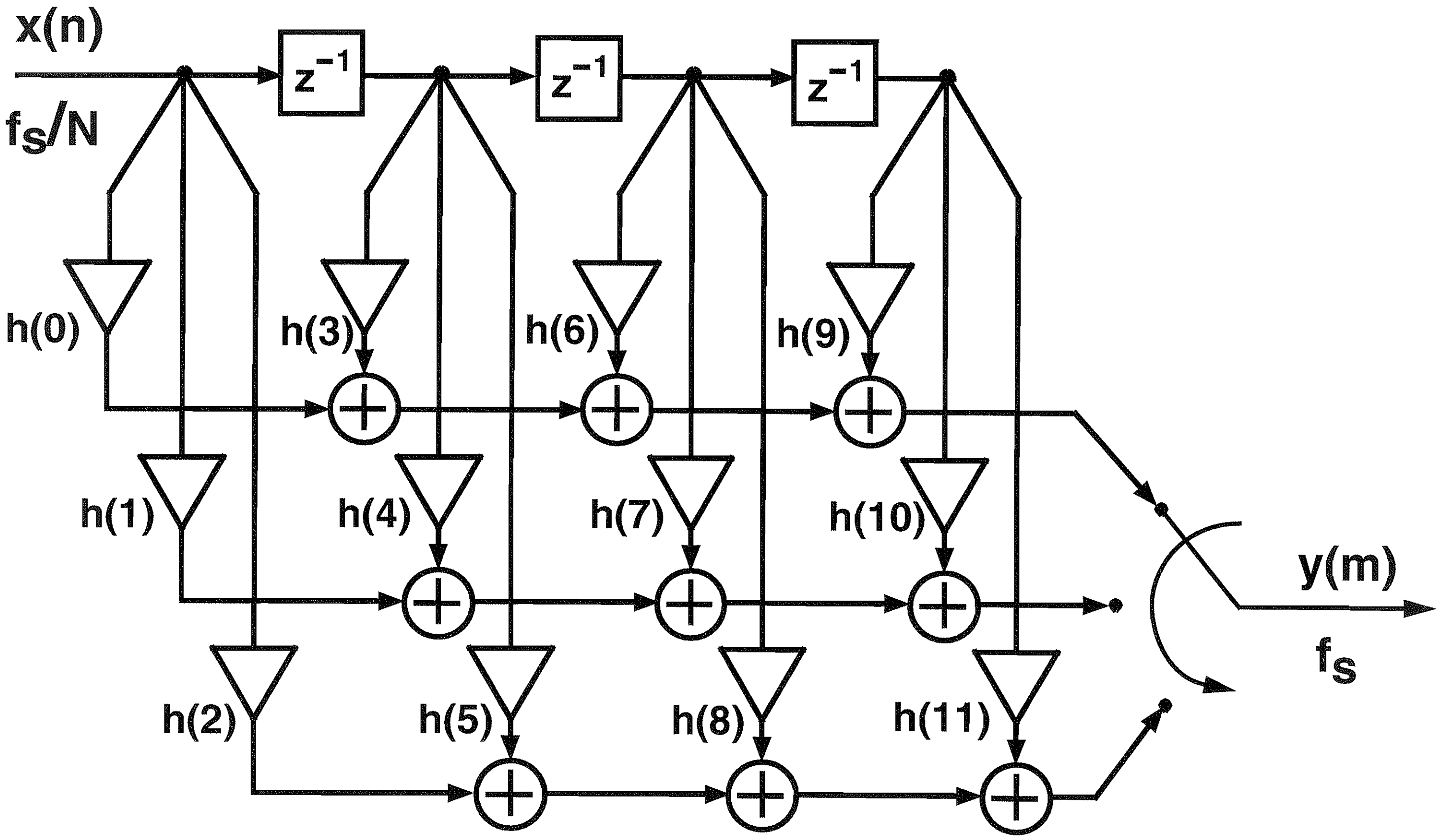
- Compared to the direct-form FIR decimator and interpolator implementations exploiting the coefficient symmetry, the commutative structures suffer from the drawback that the impulse responses of the branch filters (except for a single one for an even overall order) are not symmetric.
- Therefore, the number of multipliers as well as the multiplication rate are higher.
- The advantage is that the number of delays required in the implementations can be reduced to be the highest one among the N branch filters $G_l(z)$.
- For the direct-form implementation, the number of delay or memory elements is equal to the filter order
- The next two pages shows memory-saving structures for both the decimator and interpolator in the case where $N = 3$ and $H(z)$ is of order 11, so that $G_0(z) = h(0)+h(3)z^{-1}+h(6)z^{-2}+h(9)z^{-3}$, $G_1(z) = h(1)+h(4)z^{-1}+h(7)z^{-2}+h(10)z^{-3}$, and $G_0(z) = h(2)+h(5)z^{-1}+h(8)z^{-2}+h(11)z^{-3}$.
- Note that in developing these structures, a transposed direct-form structure (direct-form structure) has been used in the decimation (interpolation) case.

Example Memory-Saving Commutative Decimator Structures for $N = 3$ and an FIR Filter of Order 11



Example Memory-Saving Commutative Interpolator Structures for $N = 3$ and an FIR Filter of Order 11

- Note that in order to preserve the signal energy, the $h(n)$'s have to be multiplied by three.



Useful Polyphase Filters

- As shown earlier, FIR filters can always be expressed in the desired form.
- Nth-band FIR filters (to be considered later) are attractive since one of the branches is a pure delay.
- Half-band FIR filter extremely attractive: one of the two branches is a pure delay term and another branch is a linear-phase FIR filter so that the coefficient symmetry can be exploited.
- Nth-band IIR filters (to be considered later): the branch filters are allpass filters.
  - Best nonlinear-phase filters
  - Also approximately linear-phase filters can be designed by selecting one of the branches to be a pure delay term z^{-K}.

Part II.B: Conventional Multistage Implementations

• If the overall sampling rate alteration ratio N is can be factored into the product

$$N = \prod_{k=1}^{K} N_k, \tag{16}$$

where each N_k is an integer, then the decimators and interpolators can be implemented using K stages as shown by the first two diagrams on the next page.

• The main benefit lies in the fact that for the resulting multistage implementations the number of arithmetic operations reduce due to low-order subfilters.

• Also the number of multiplications and additions per input sample (output sample) in the decimation (interpolation) case becomes significantly lower at the expense of more control.

• For the design and analysis purposes, the first set of two diagrams can be redrawn into the equivalent single-stage forms shown by the second set of two diagrams of the next page.

• For the decimator, the transfer function of the single-stage equivalent is expressible as

$$H(z) = H_1(z)H_2(z^{N_1})H_3(z^{N_1N_2}) \cdots H_K(z^{N_1N_2\cdots N_{K-1}}) \tag{17}$$

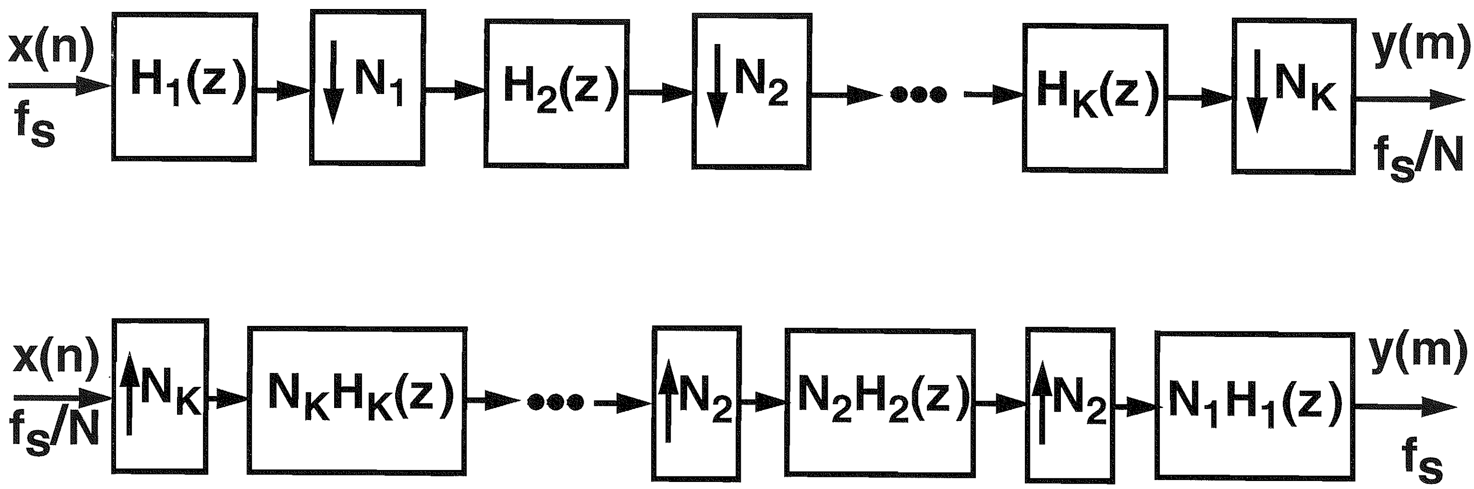
or

$$H(z) = \prod_{k=1}^{K} H_k(z^{\widehat{N}_k}), \tag{18a}$$

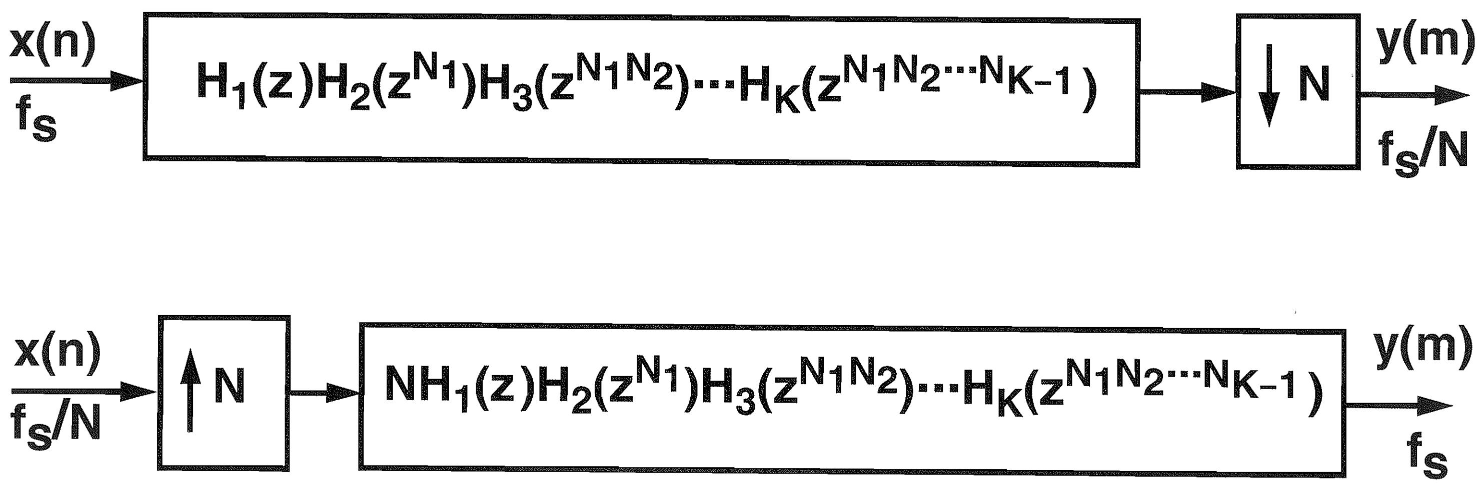
where

$$\widehat{N}_1 = 1, \quad \widehat{N}_k = \prod_{l=1}^{k-1} N_l \text{ for } k = 2, 3, \cdots, K. \tag{18b}$$

- For the interpolator, each $H_k(z)$ is multiplied by N_k.



Single-Stage Equivalents



Design Formulae for Cases A, B, and C

• For the single-stage equivalent, the overall frequency response is expressible as

$$H(e^{j\omega}) = \prod_{k=1}^{K} H_k(e^{j\widehat{N}_k\omega}). \tag{19}$$

• Hence, all the filters execpt for the first one are periodic containing extra passbands and stopbands in the region $[0,\ f_s/2]$ in terms of 'real' frequencies or in $[0,\ \pi]$ in terms of the angular frequencies.

• Based on the periodicities, the passband and stopband regions for the K subfilters can be stated as follows in terms of angular frequencies (this fact becomes clear in connection with examples):

Last stage (Kth stage):

1) The passband region is given in all the cases by

$$\Omega_p^{(K)} = [0, \alpha/N_K]. \tag{20a}$$

2) The stopband region is given by

$$\Omega_s^{(K)} = \begin{cases} [\pi/N_K,\ \pi] & \text{for Case A} \\ \bigcup_{l=1}^{\lfloor N_K/2 \rfloor} [\frac{(2l-\alpha)\pi}{N_K},\ \min(\frac{(2l+\alpha)\pi}{N_K},\ \pi)] & \\ & \text{for Case B} \\ [(2-\alpha)\pi/N_K,\ \pi] & \text{for Case C.} \end{cases} \tag{20b}$$

Stages for $k = 1, 2, \cdots K-1$:

1) The passband region is given in all the cases by

$$\Omega_p^{(k)} = [0, \alpha \widehat{N}_k / N_K], \tag{21}$$

where

$$\widehat{N}_1 = 1, \;\; \widehat{N}_k = \prod_{l=1}^{k-1} N_l \text{ for } k = 2, 3, \cdots, K. \tag{21b}$$

2) The stopband region is given by

$$\Omega_s^{(k)} = \bigcup_{l=1}^{\lfloor N_k/2 \rfloor} [\frac{(2l - \beta_k)\pi}{N_k}, \;\; \min(\frac{(2l + \beta_k)\pi}{N_k}, \;\; \pi)], \tag{22a}$$

where

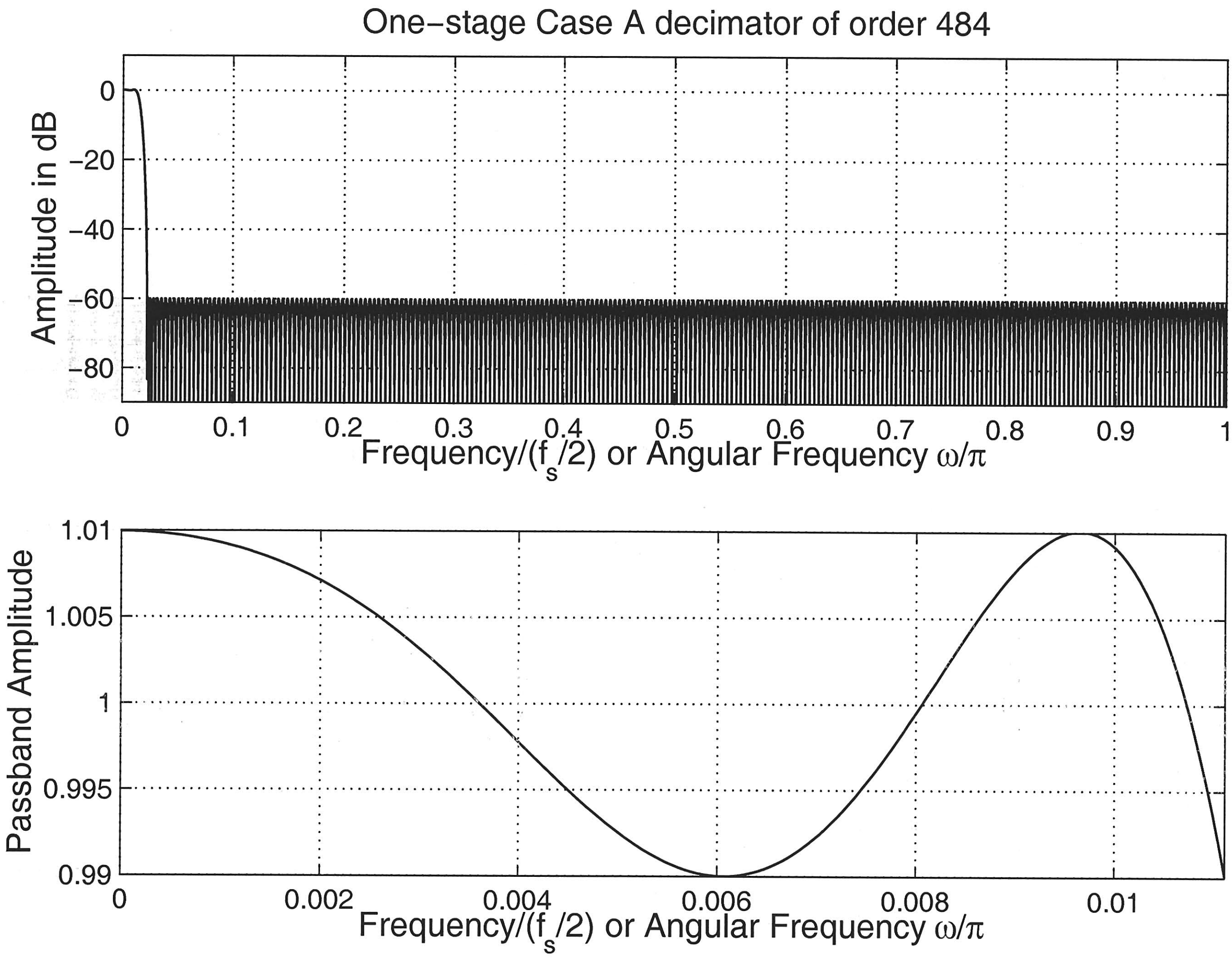
$$\beta_k = \begin{cases} \widehat{N}_{k+1}/N & \text{for Case A} \\ \alpha \widehat{N}_{k+1}/N & \text{for Case B} \\ (2-\alpha)\widehat{N}_{k+1}/N & \text{for Case C.} \end{cases} \tag{22b}$$

- If the passband and stopband ripples for the overall filter are δ_p and δ_s, then the corresponding ripples for the K subfilters are δ_p/K and δ_s.
- Note that if there are K filters in cascade and all of them have the same passband with the maximum deviation from unity equal to δ_p/K, then for the composite filter, $(1 \pm \delta_p/K)^K \approx 1 \pm \delta_p$.
- The examples to be presented illustrate the validity of the above design formulas to achieve the desired performance for the overall multistage implementation.

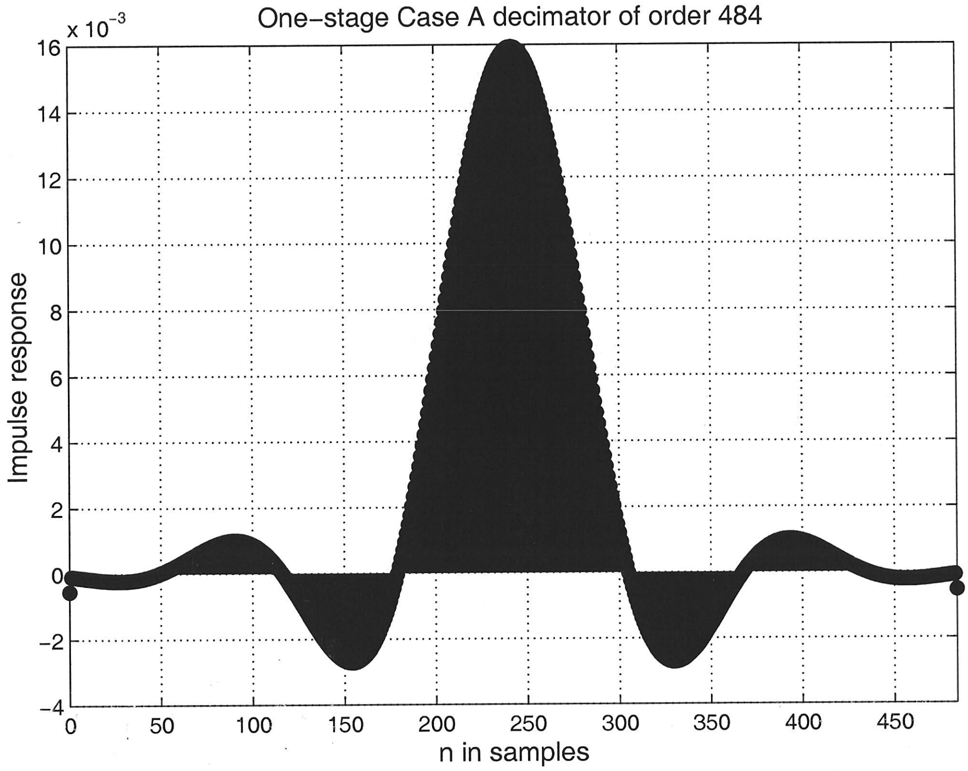
Example Multistage Designs

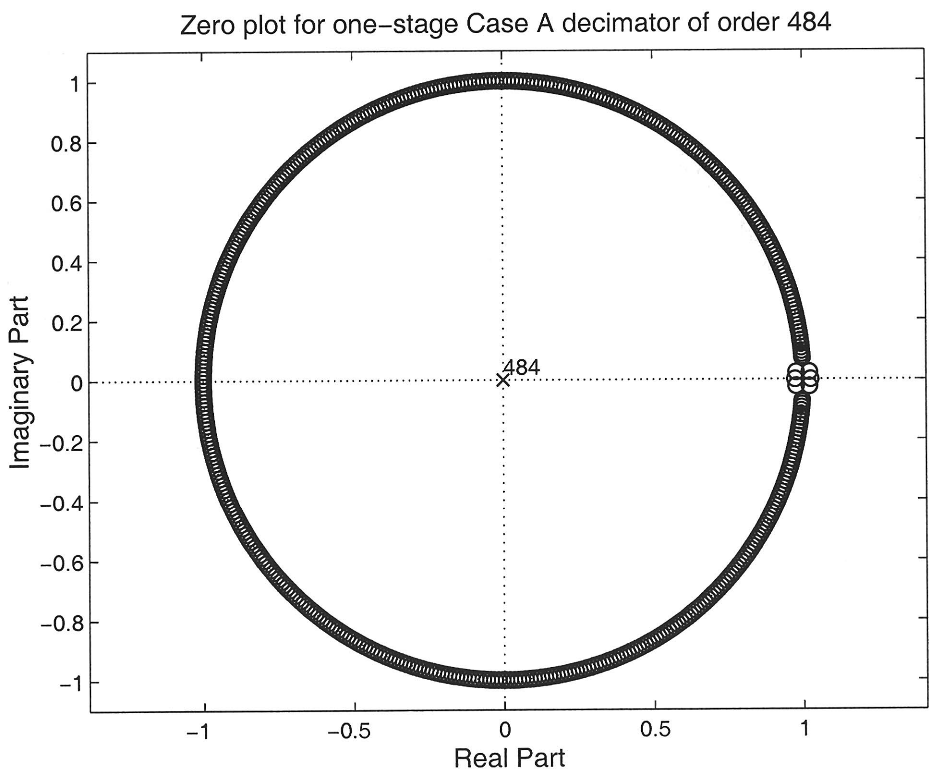
- In /home/ts/matlab/multirate, there is a program for designing multistage Case A, Case B, and Case C designs up to $K = 3$.
- As an example we consider the case with $\delta_p = 0.01$, $\delta_s = 0.001$, $N = 45$, and $\alpha = 0.5$.
- In order to emphasize the usefulness of using multistage implementations, we start with one-stage designs.
- For the one-stage designs, the filter orders are 484, 242, and 256 for Case A, Case B, and Case C, respectively.
- The six following pages illustrate the characteristics of these one-stage designs.

Characteristics of Example Case A One-Stage Decimator

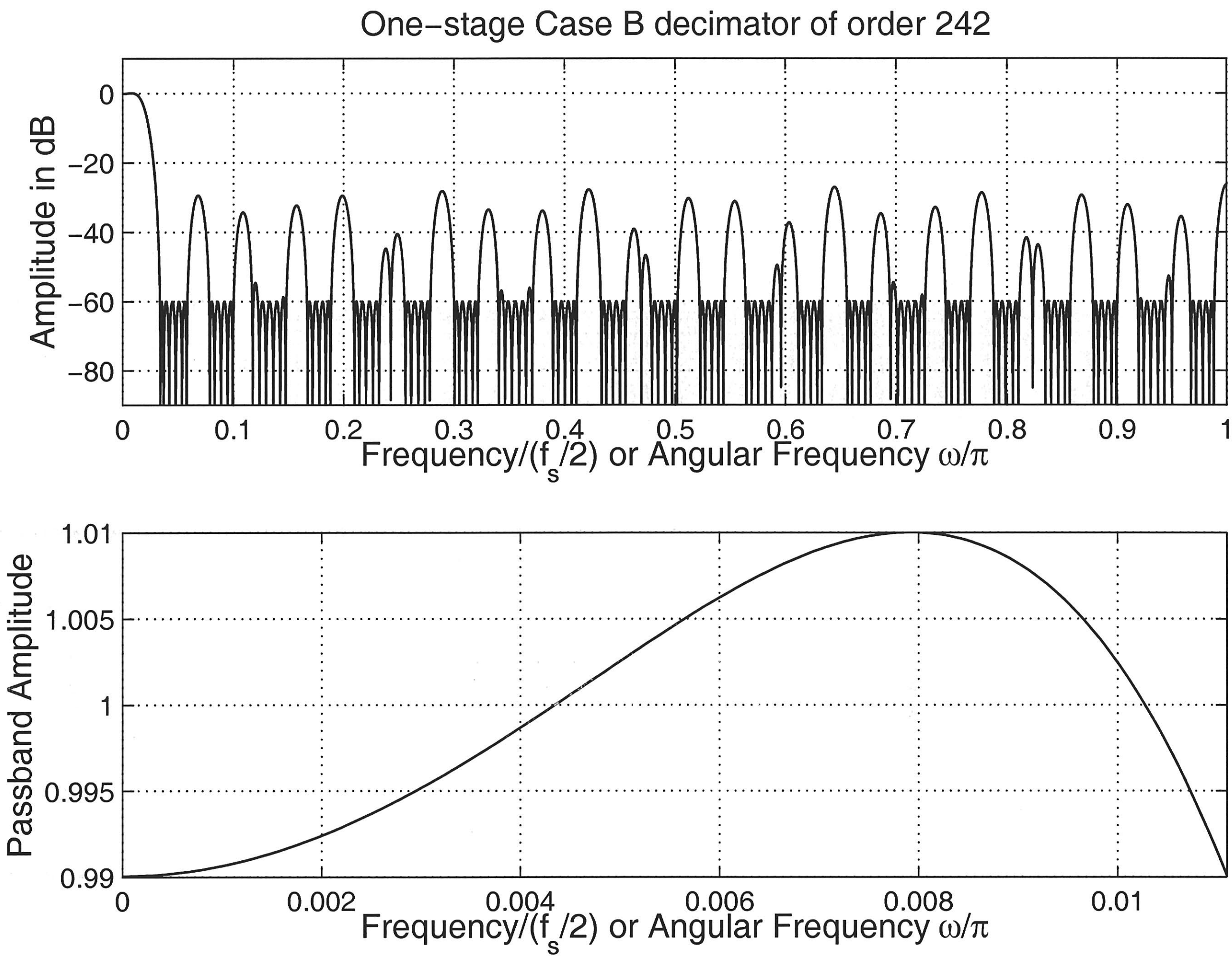


Characteristics of Example Case A One-Stage Decimator

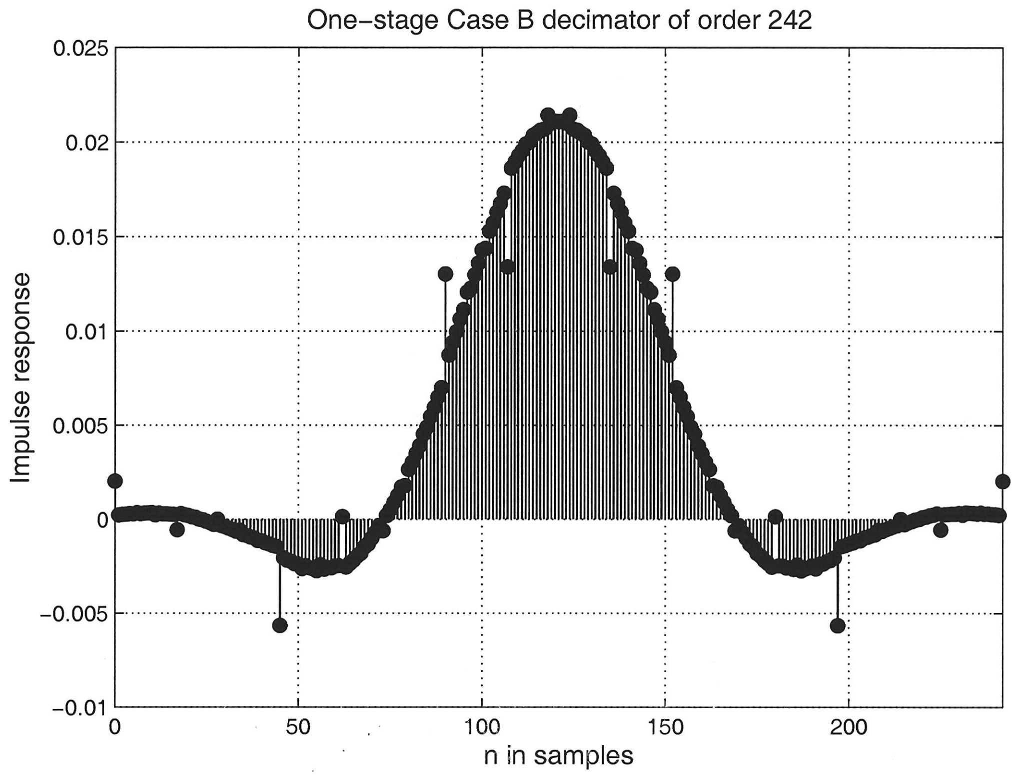


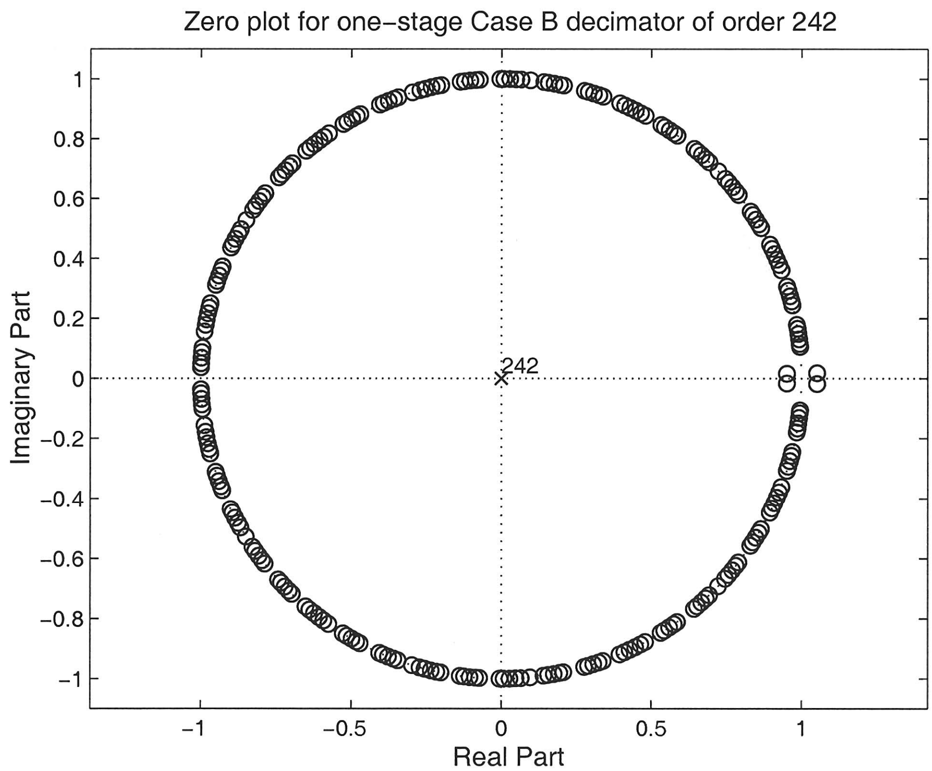


Characteristics of Example Case B One-Stage Decimator

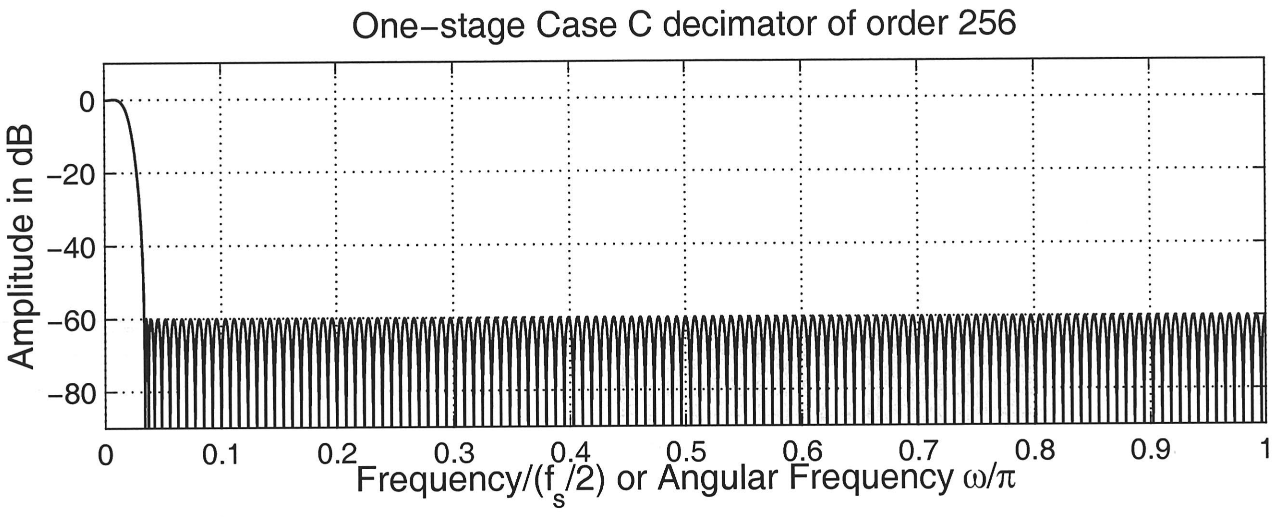


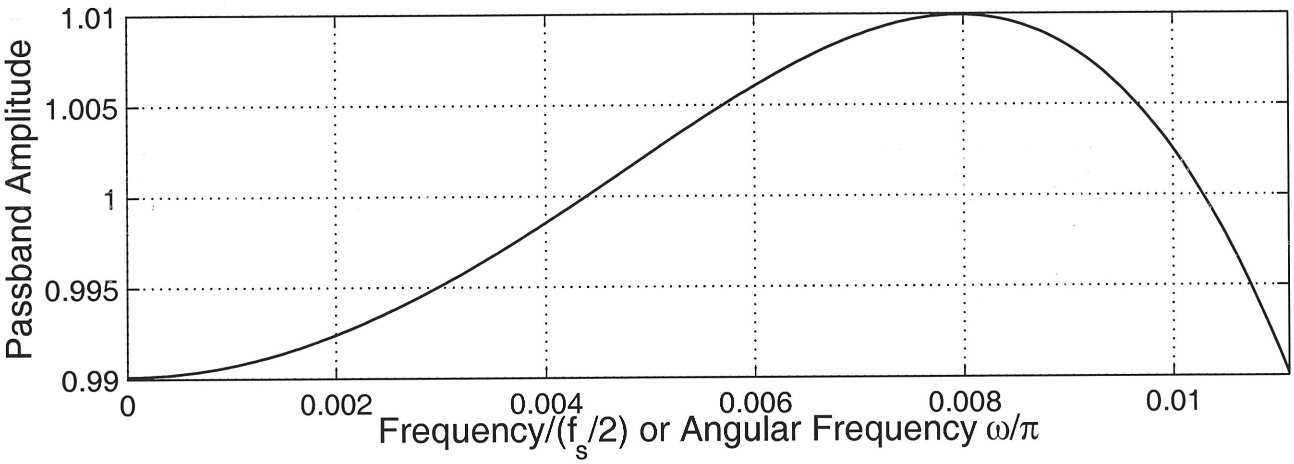
Characteristics of Example Case B One-Stage Decimator



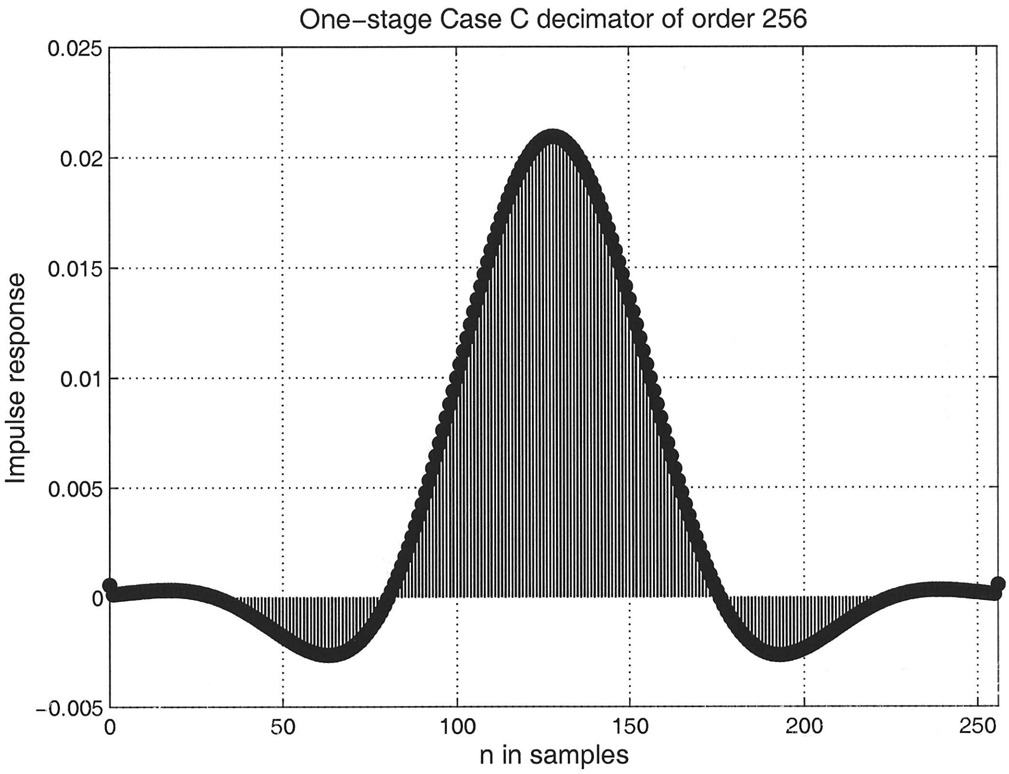


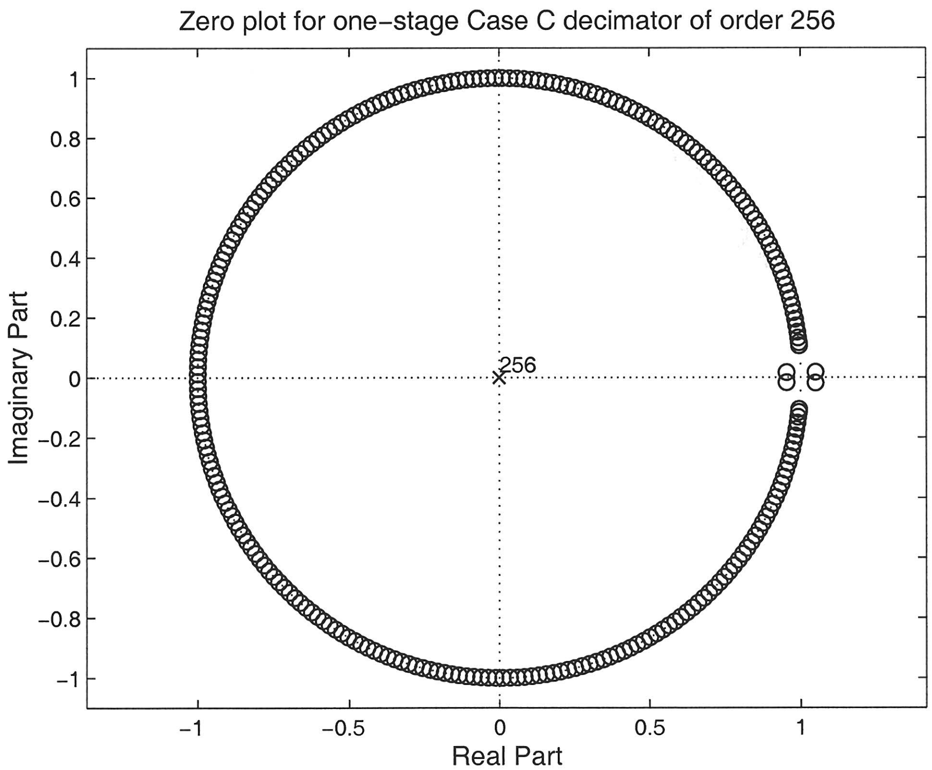
Characteristics of Example Case C One-Stage Decimator





Characteristics of Example Case C One-Stage Decimator





Example Case A Three-Stage Decimator

• For $K = 3$ stages, the best result is obtained by selecting $N_1 = 5$ and $N_2 = N_3 = 3$.

• In this case (see the formulae on Pages 34 and 35), $\alpha = 0.5$, $\widehat{N}_1 = 1$, $\widehat{N}_2 = 5$, $\widehat{N}_1 = 15$.

• The passband regions for $H_1(z)$, $H_2(z)$, and $H_3(z)$ are thus in terms of the angular frequency $[0,\ 0.5\pi/45]$, $[0,\ 0.5\pi/9]$, and $[0,\ 0.5\pi/3]$.

• $\beta_1 = 5/45$ and $\beta_2 = 3/9$.

• The stopband region of $H_1(z)$ is thus a union of bands $[(2 - 5/45)\pi/5,\ (2 + 5/45)\pi/5]$ and $[(4 - 5/45)\pi/5,\ (2 + 5/45)\pi/5]$.

• The stopband regions for $H_2(z)$ and $H_3(z)$ are $[(2 - 3/9)\pi/3,\ (2 + 3/9)\pi/3]$, and $[\pi/3,\ \pi]$, respectively.

• The passband and stopband ripples for the subfilters are $\delta_p/3 = 0.01/3$ and $\delta_s = 0.001$.

• The orders of $H_1(z)$, $H_2(z)$, and $H_3(z)$ to meet the criteria are 10, 10, and 37.

• When using direct-form decimator implementations, the overall number of multipliers is $6 + 6 + 19 = 31$ and the number of multiplications per input sample is $6/5 + 6/15 + 19/45 = 2.022$.

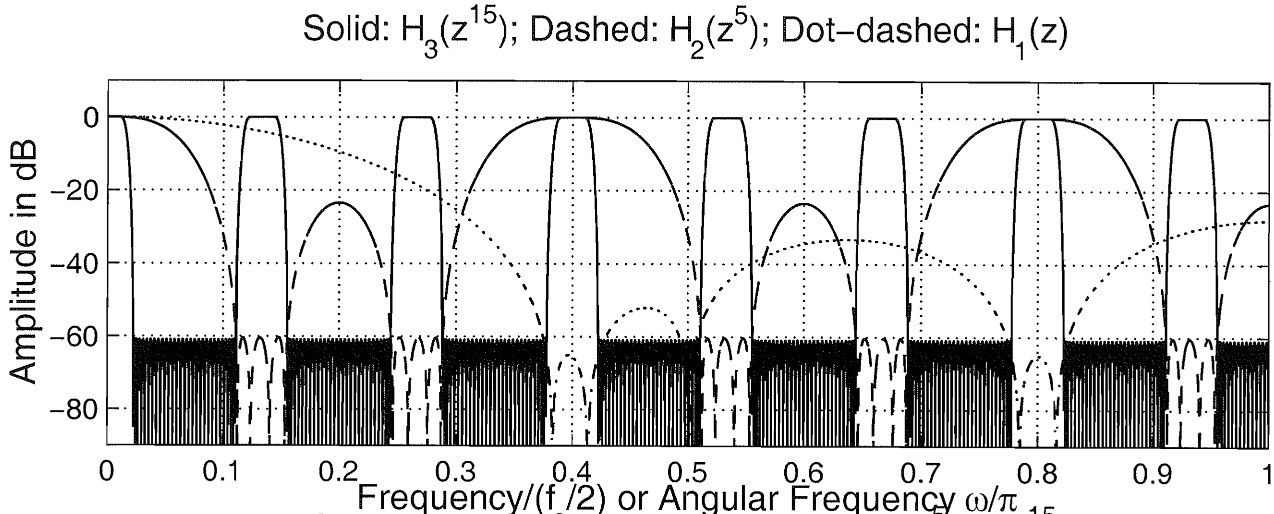
• The corresponding figures for the one-stage design of order 484 are 243 and $243/45 = 5.4$.

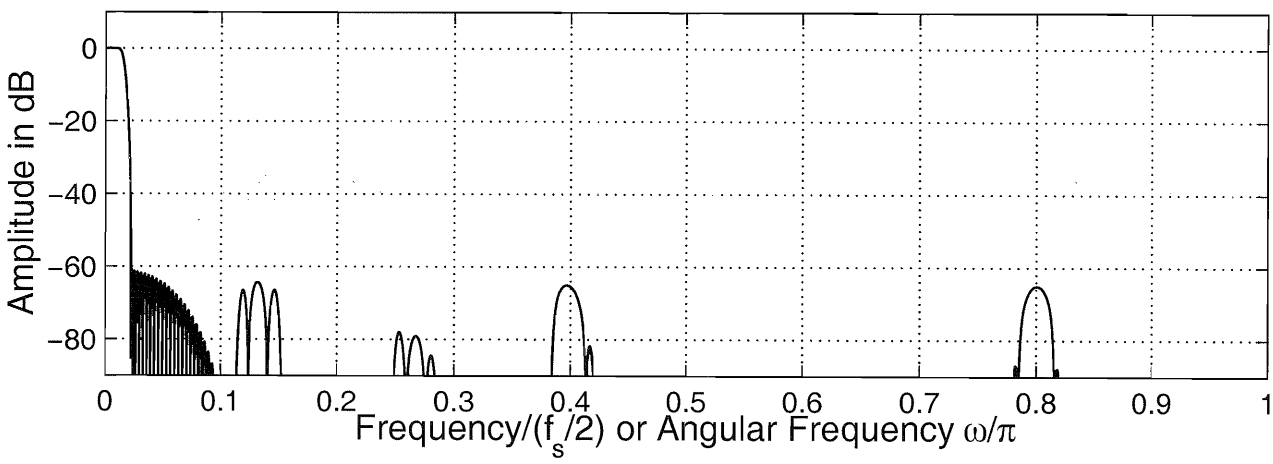
• The transfer function of the single-stage equivalent is given by

$$H(z) = H_1(z)H_2(z^5)H_3(z^{15}).$$

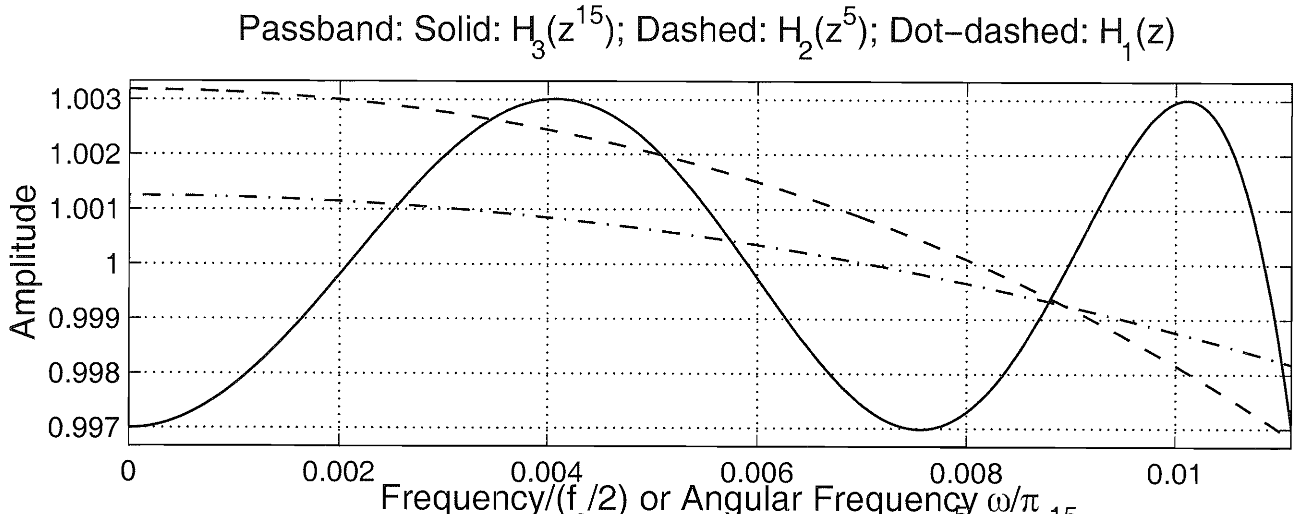
- The order of this equivalent is 615 compared to 484 required by a direct one-stage design.
- In the following there are seven pages illustrating the characteristics of our Case A three-stage design.
- In the first figure page it is seen that because of the periodicity of $H_3(z^{15})$ it takes care of the stopband shaping except for the extra unwanted passband and transition bands around the points $2k\pi/15$ for $k = 1, 2, \cdots, 7$.
- $H_2(z^5)$ attenuates these bands for $k = 1, 2, 4, 5, 7$, whereas $H_1(z)$ takes care of the remaining bands.
- In the second figure page it is seen that $H_3(z^{15})$, $H_2(z^5)$, and $H_1(z)$ have the same passband region $[0,\ 0.5\pi/45]$.
- Since for all of them the passband ripple is less or equal to $\delta_p/3 = 0.01/3$, the passband ripple of the overall single-stage equivalent is less than or equal to $\delta_p = 0.01$.

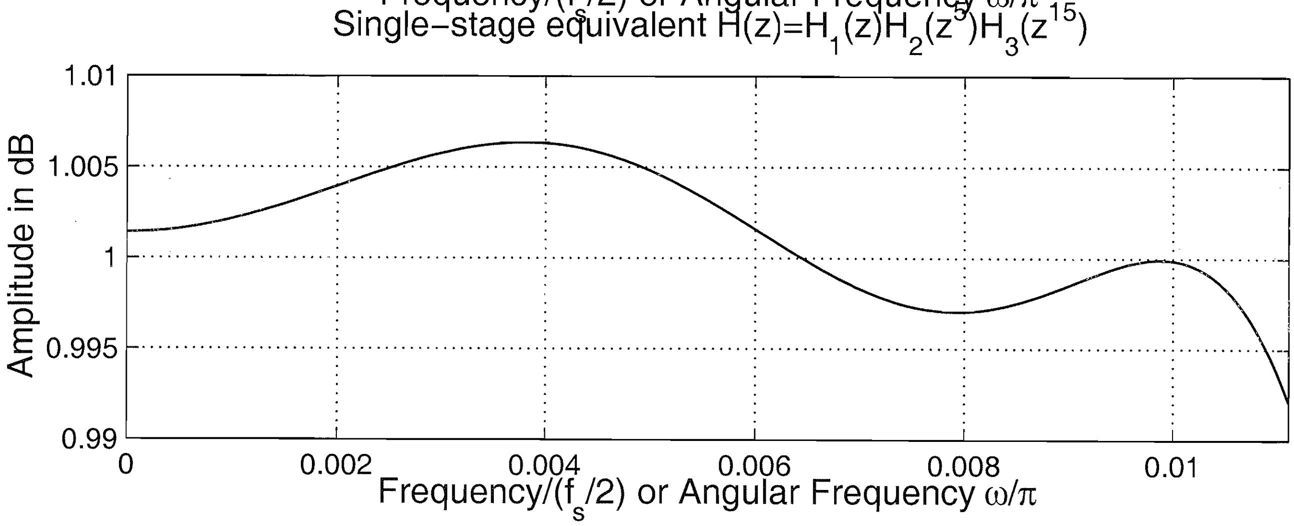
Characteristics of Example Case A Three-Stage Decimator



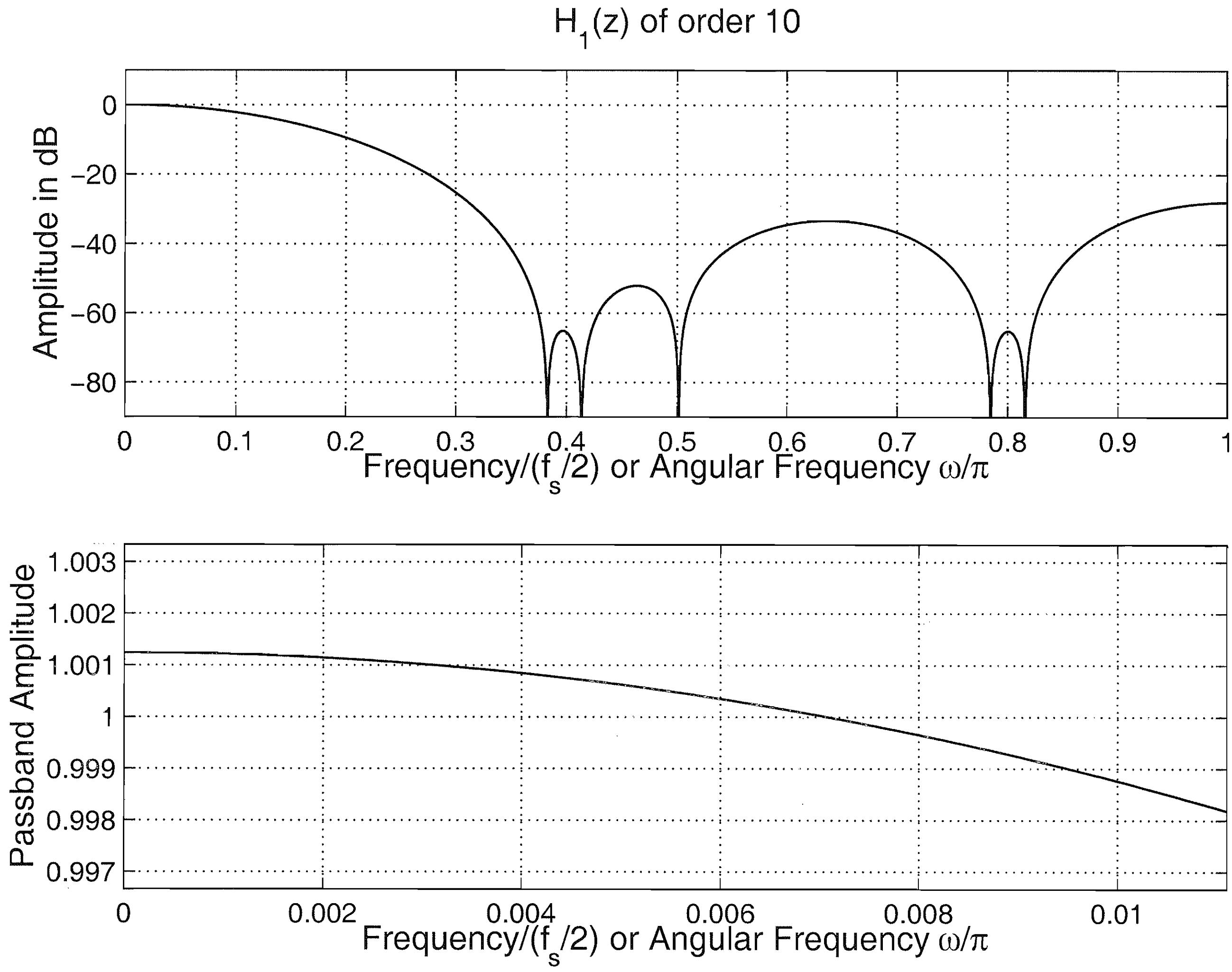


Characteristics of Example Case A Three-Stage Decimator

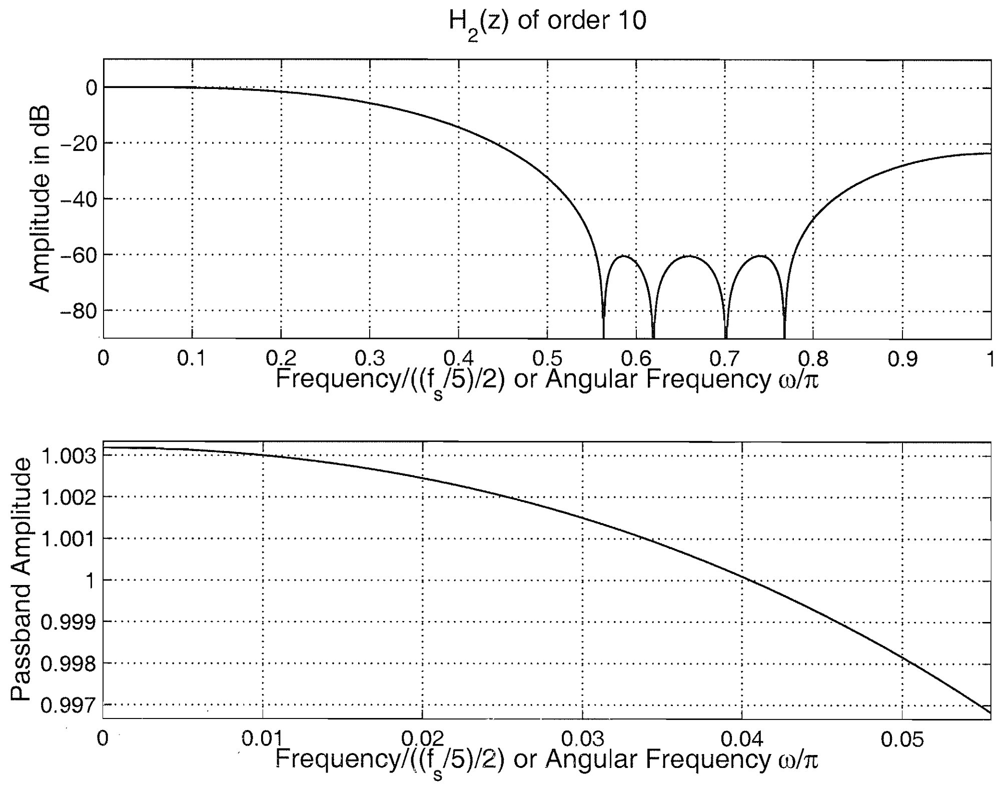


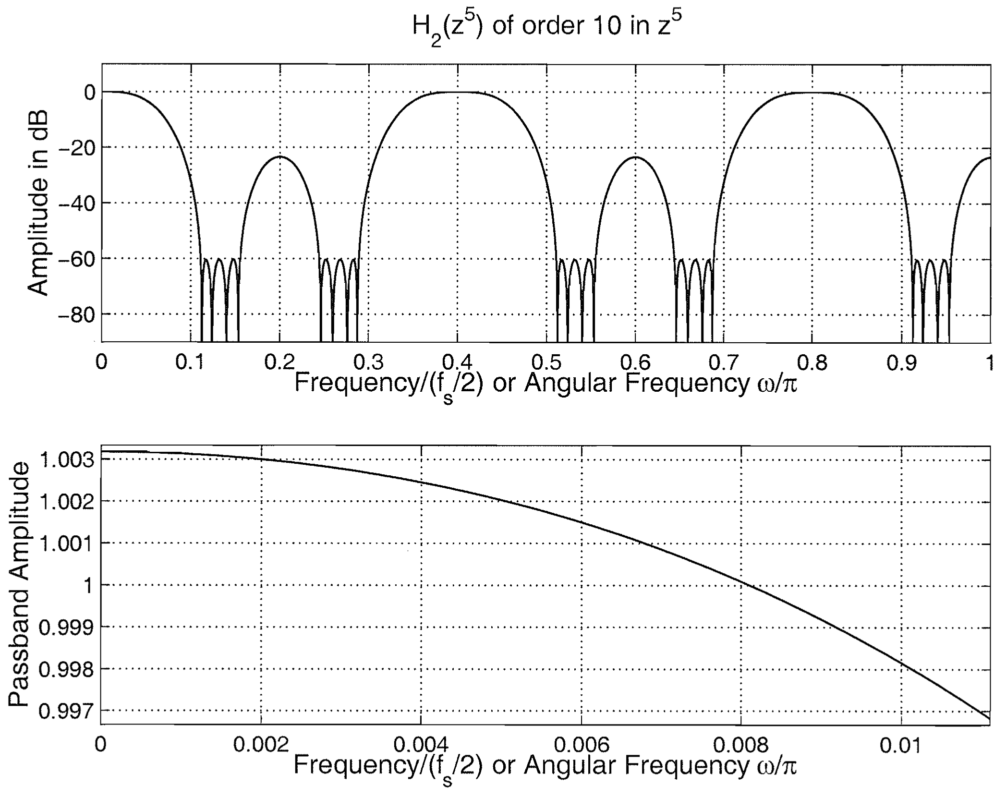


Characteristics of Example Case A Three-Stage Decimator

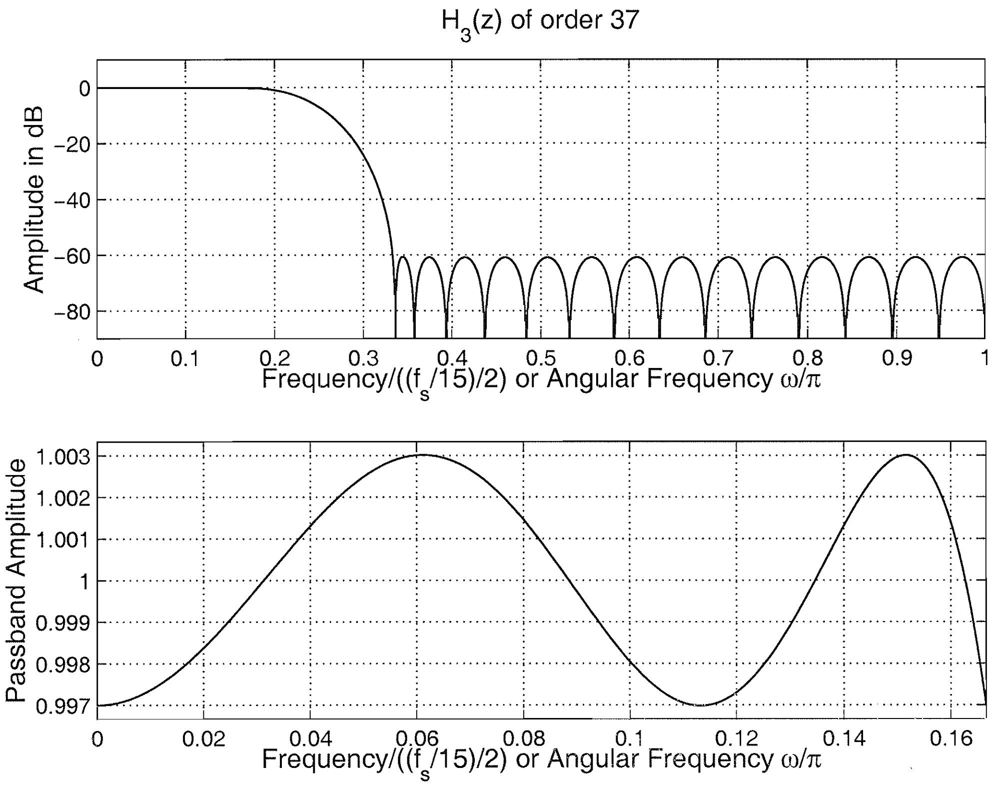


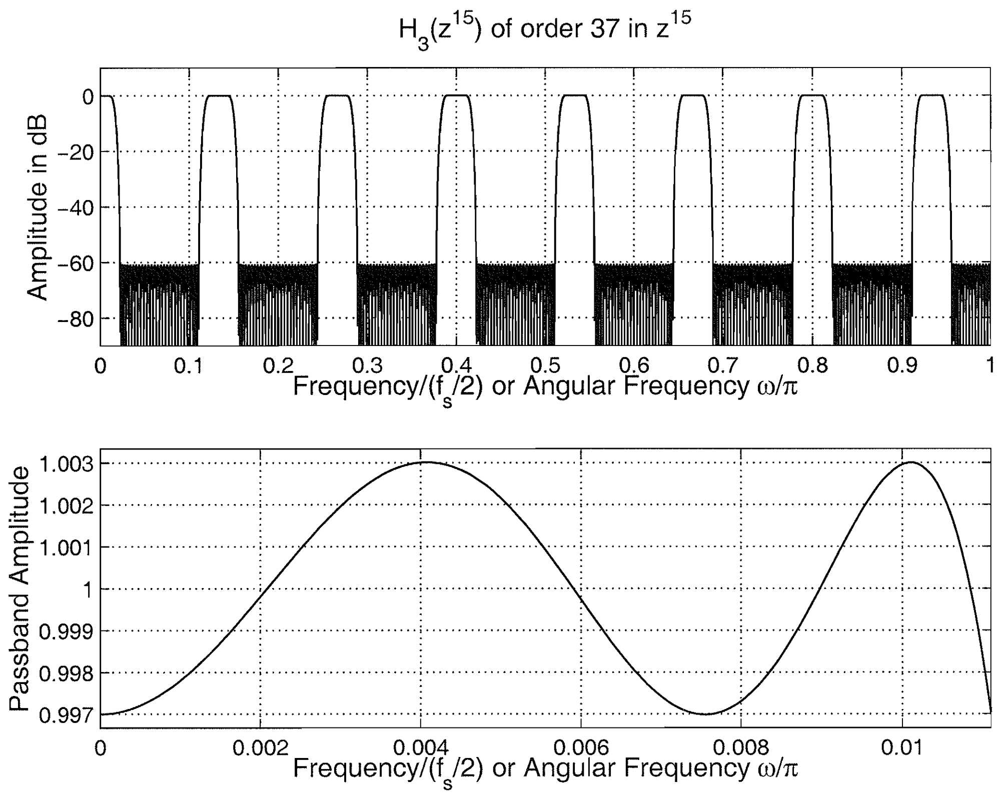
Characteristics of Example Case A Three-Stage Decimator



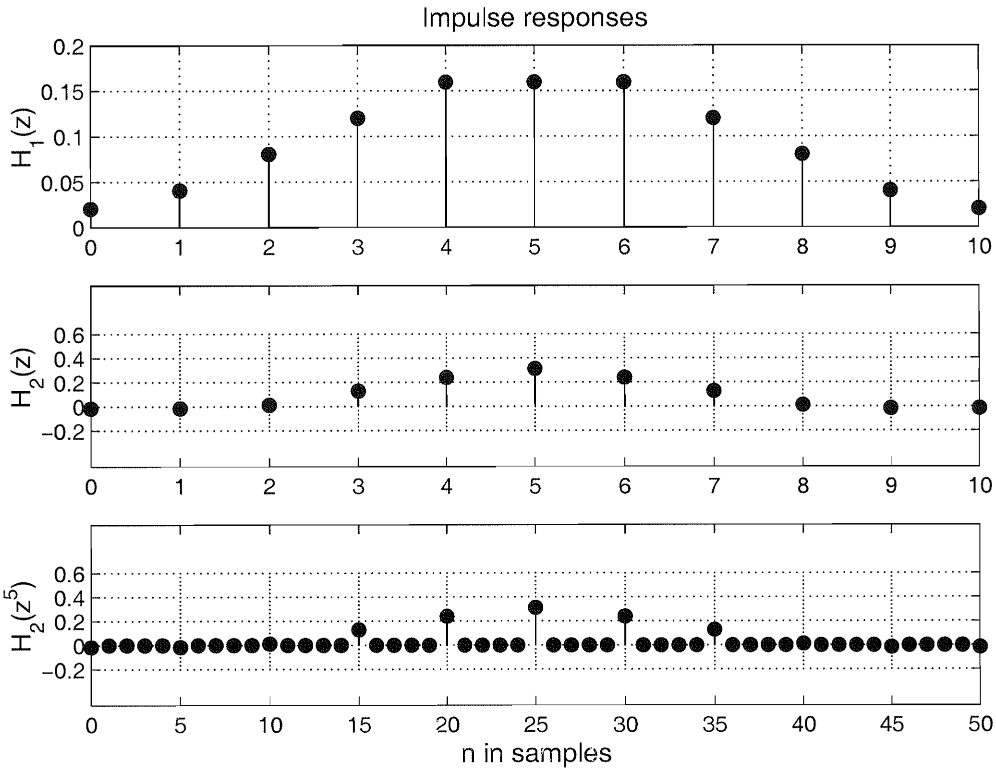


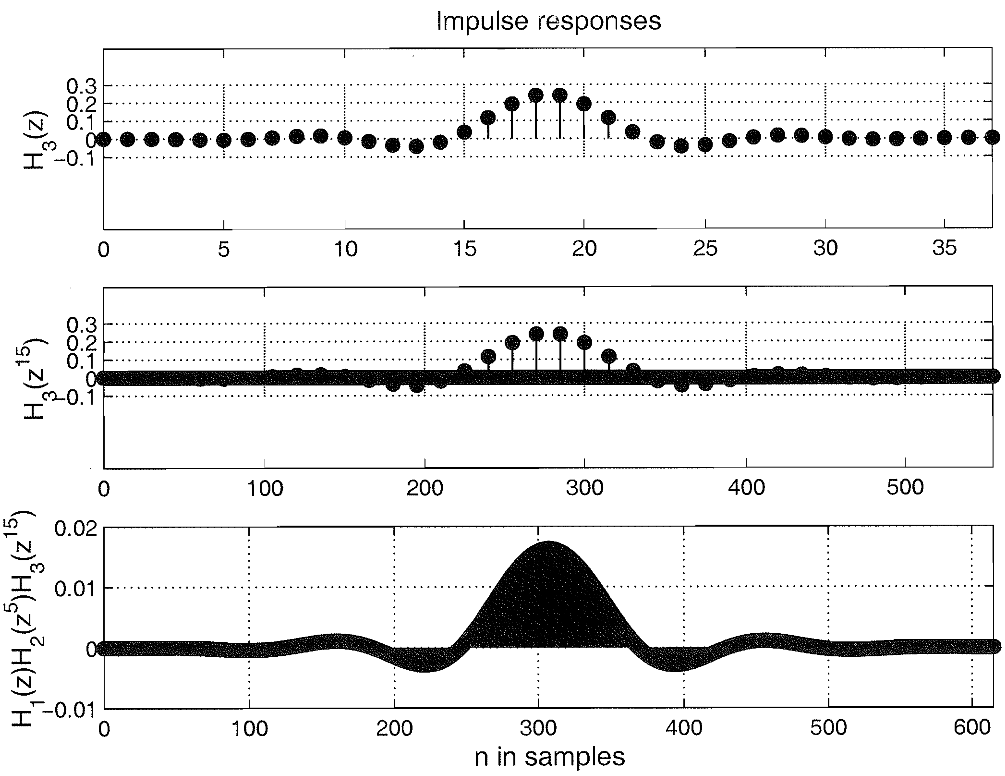
Characteristics of Example Case A Three-Stage Decimator



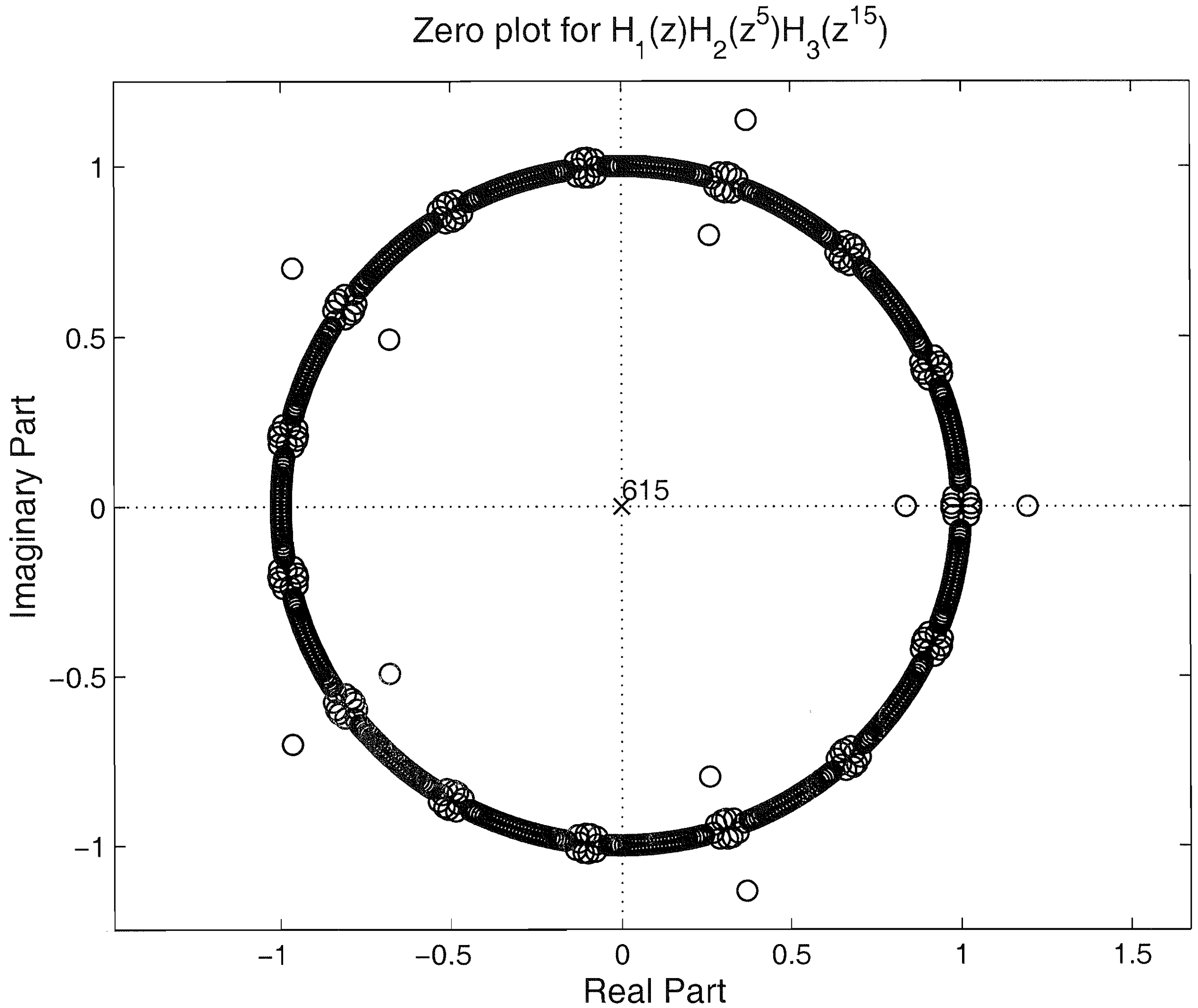


Characteristics of Example Case A Three-Stage Decimator





Characteristics of Example Case A Three-Stage Decimator



Example Case B Three-Stage Decimator

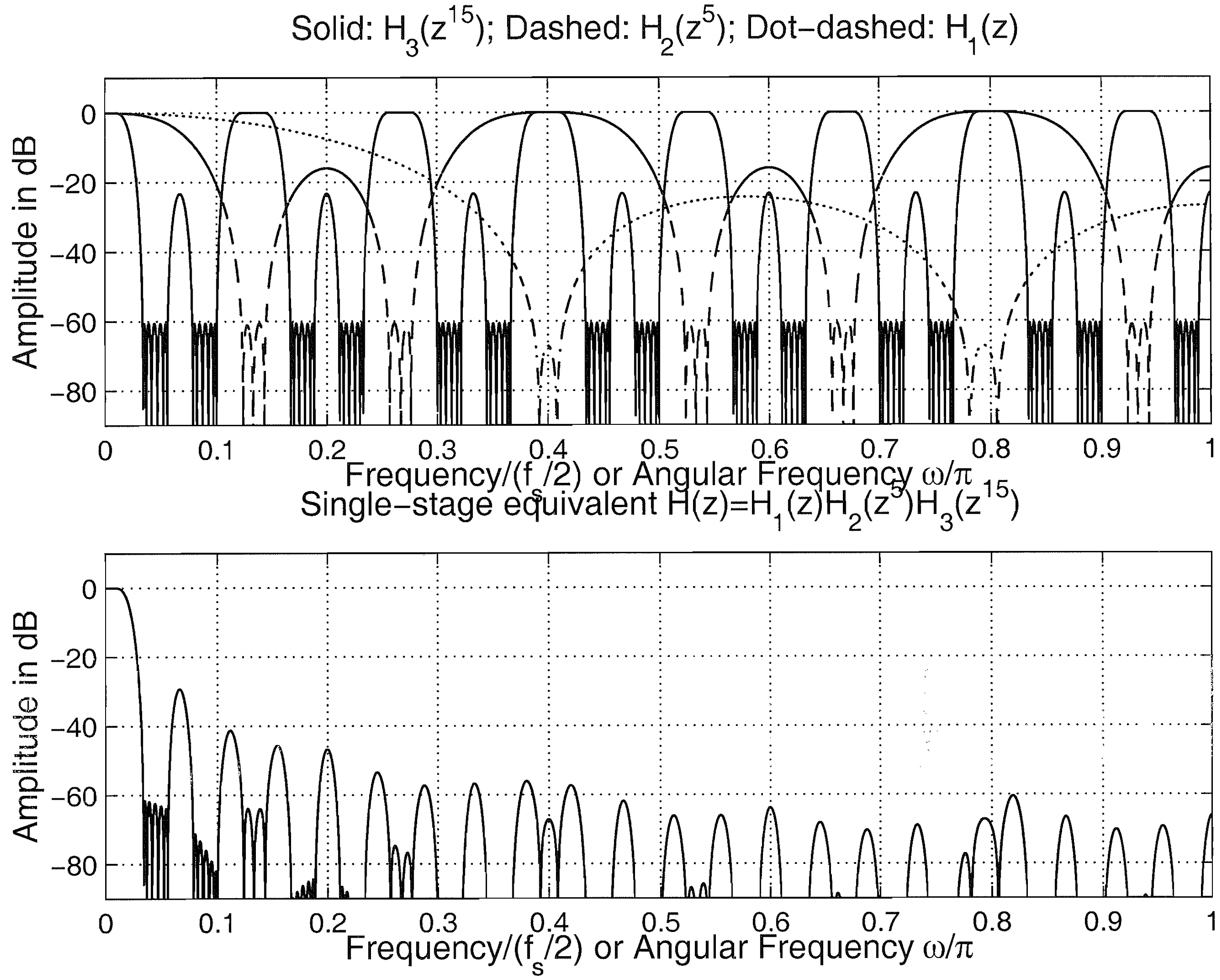
- For $K = 3$ stages, the best result is again obtained by selecting $N_1 = 5$ and $N_2 = N_3 = 3$.
- In this case (see the formulae on Pages 34 and 35), $\alpha = 0.5$, $\widehat{N}_1 = 1$, $\widehat{N}_2 = 5$, and $\widehat{N}_1 = 15$.
- The passband regions for $H_1(z)$, $H_2(z)$, and $H_3(z)$ are thus in terms of the angular frequencys $[0,\ 0.5\pi/45]$, $[0,\ 0.5\pi/9]$, and $[0,\ 0.5\pi/3]$.
- $\beta_1 = 0.5 \cdot 5/45$ and $\beta_2 = 0.5 \cdot 5/9$.
- The stopband region of $H_1(z)$ is thus a union of bands $[(2 - 2.5/45)\pi/5,\ (2 + 2.5/45)\pi/5]$ and $[(4 - 2.5/45)\pi/5,\ (2 + 2.5/45)\pi/5]$.
- The stopband regions for $H_2(z)$ and $H_3(z)$ are $[(2 - 1.5/9)\pi/3,\ (2 + 1.5/9)\pi/3]$, and $[1.5\pi/3,\ 2.5\pi/3]$, respectively.
- The passband and stopband ripples for the subfilters are again $\delta_p/3 = 0.01/3$ and $\delta_s = 0.001$.
- The orders of $H_1(z)$, $H_2(z)$, and $H_3(z)$ to meet the criteria are 8, 8, and 16.
- When using direct-form decimator implementations, the overall number of multipliers is $5 + 5 + 9 = 19$ and the number of multiplications per input sample is $5/5 + 5/15 + 9/45 = 1.5333$.
- The corresponding figures for the one-stage design of order 242 are 122 and $122/45 = 2.7111$.

- The transfer function of the single-stage equivalent is given by

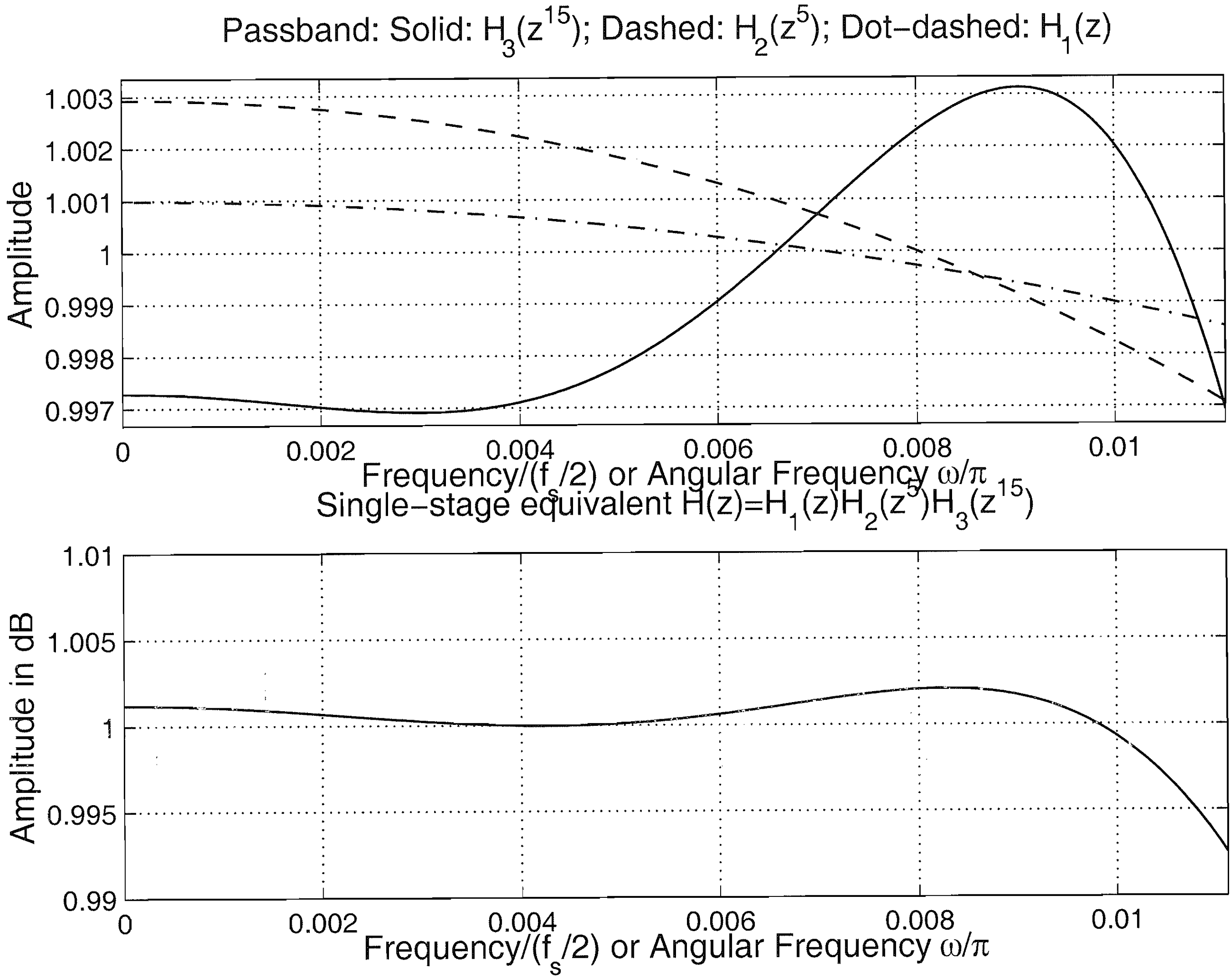
$$H(z) = H_1(z)H_2(z^5)H_3(z^{15}).$$

- The order of this equivalent is 288 compared to 242 required by a direct one-stage design.
- In the following there are seven pages illustrating the characteristics of our Case B three-stage design.
- The main difference compared to Case A is that now those frequency components aliasing in the decimation case into the region $[0,\ 0.5\pi/45]$ are attenuated.
- Aliasing is allowed into the region $[0.5\pi/45,\ \pi/45]$.

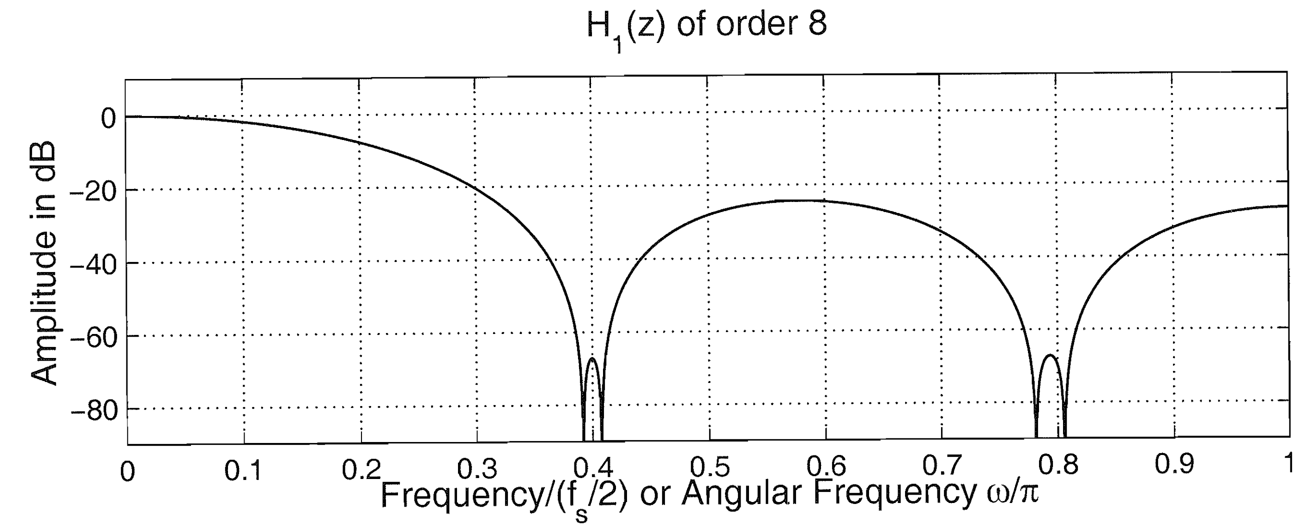
Characteristics of Example Case B Three-Stage Decimator

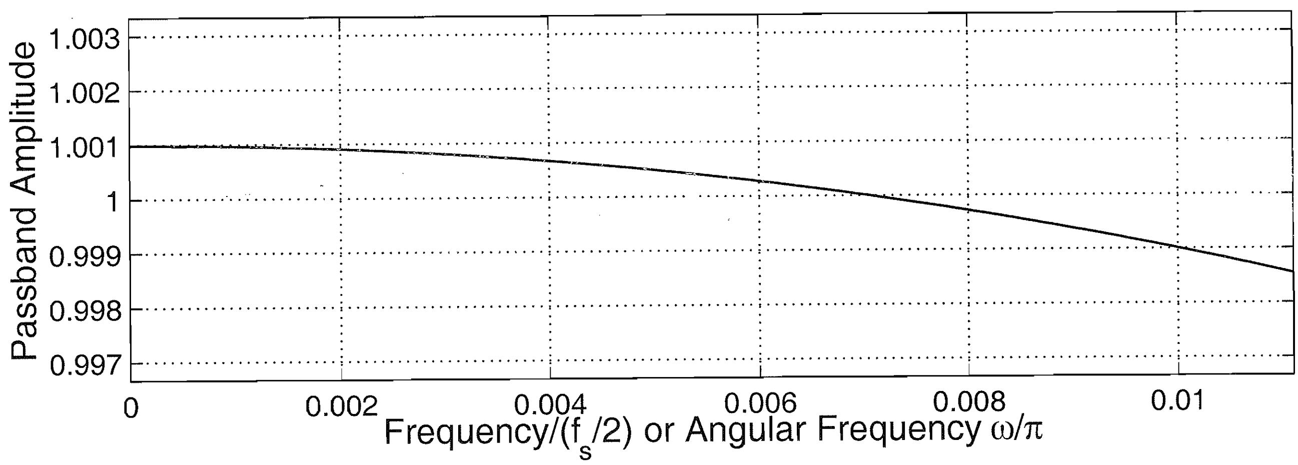


Characteristics of Example Case B Three-Stage Decimator

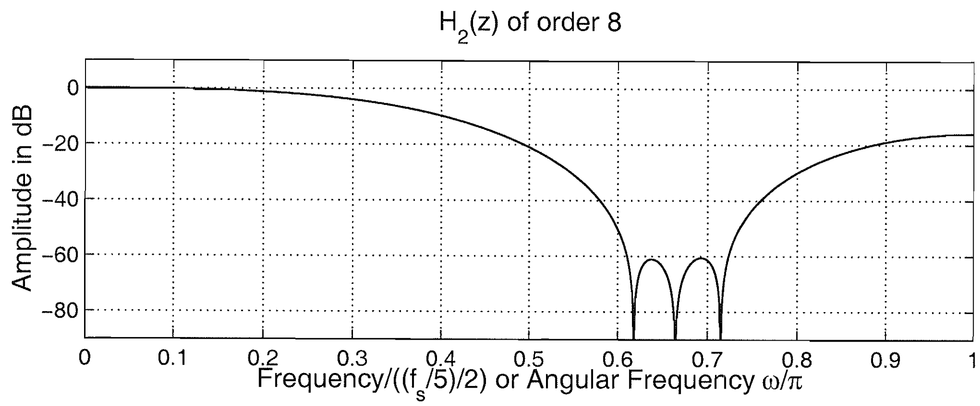


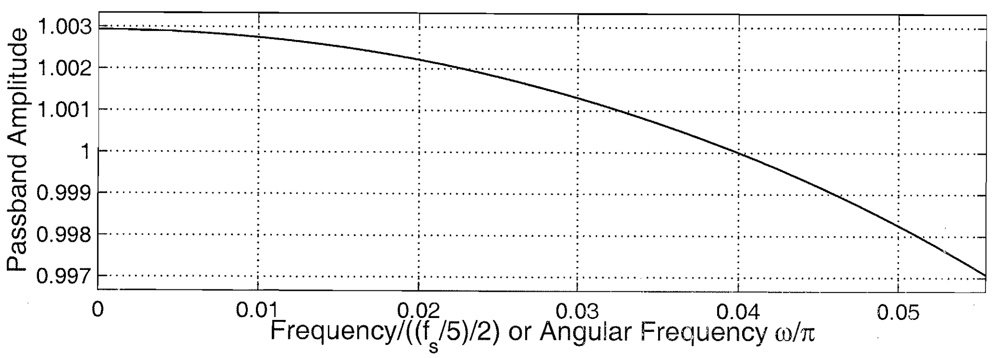
Characteristics of Example Case B Three-Stage Decimator

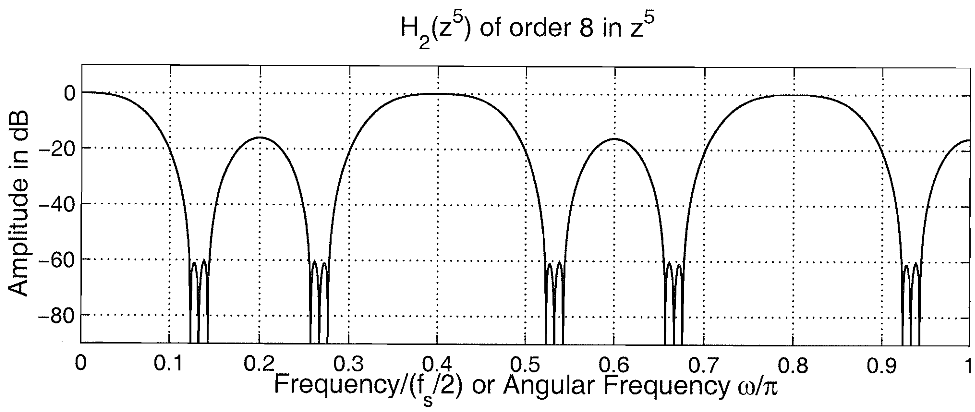


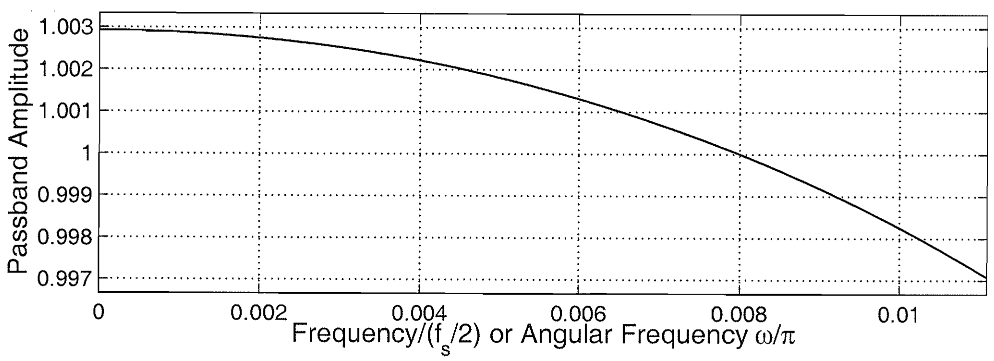


Characteristics of Example Case B Three-Stage Decimator

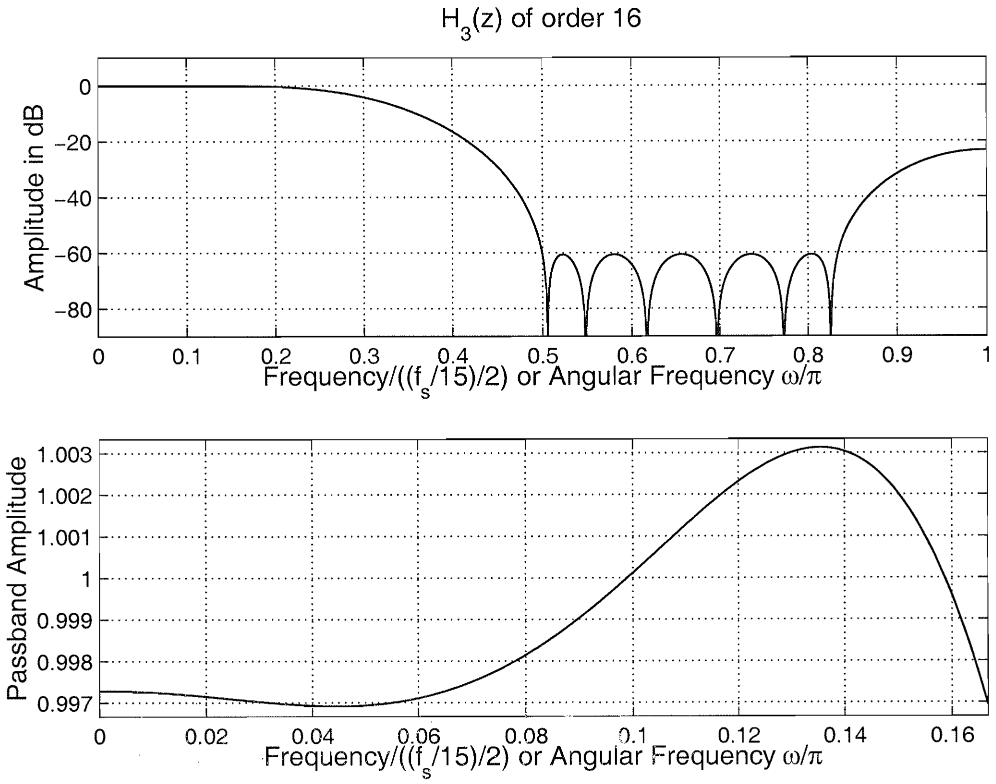


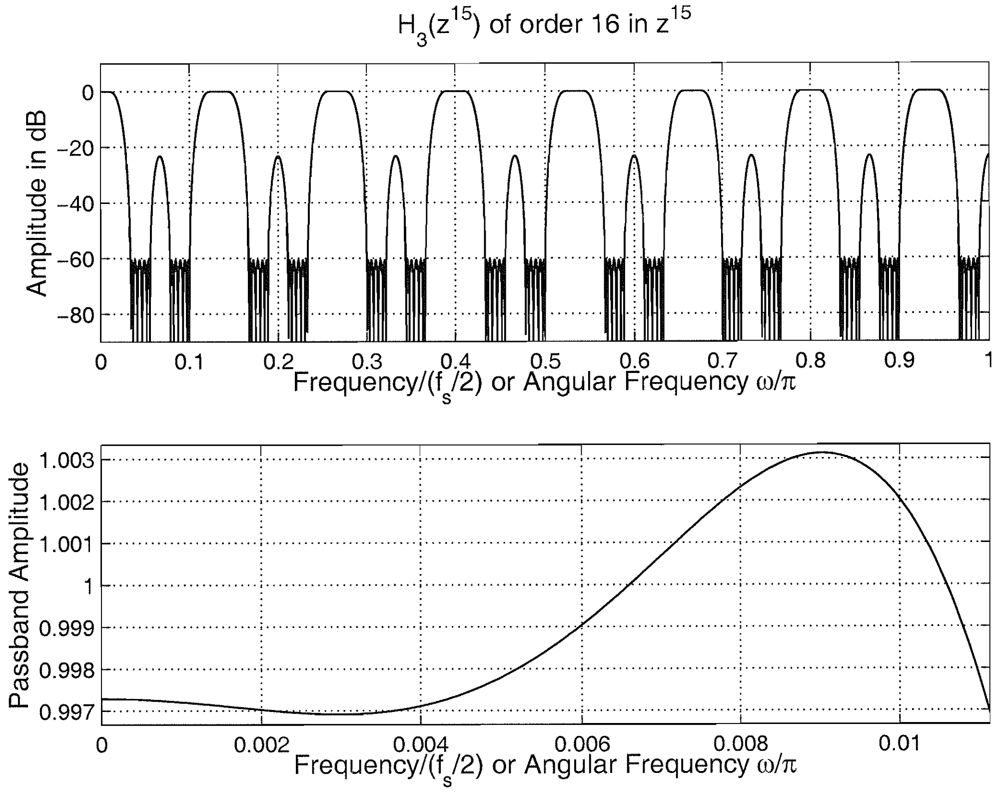




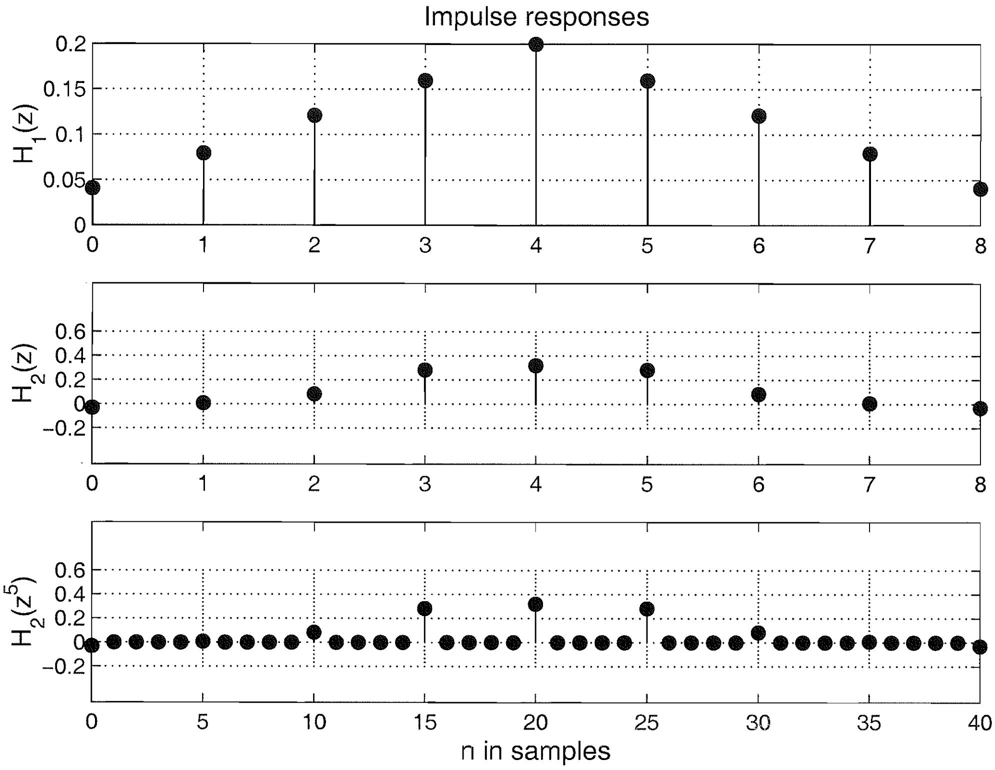


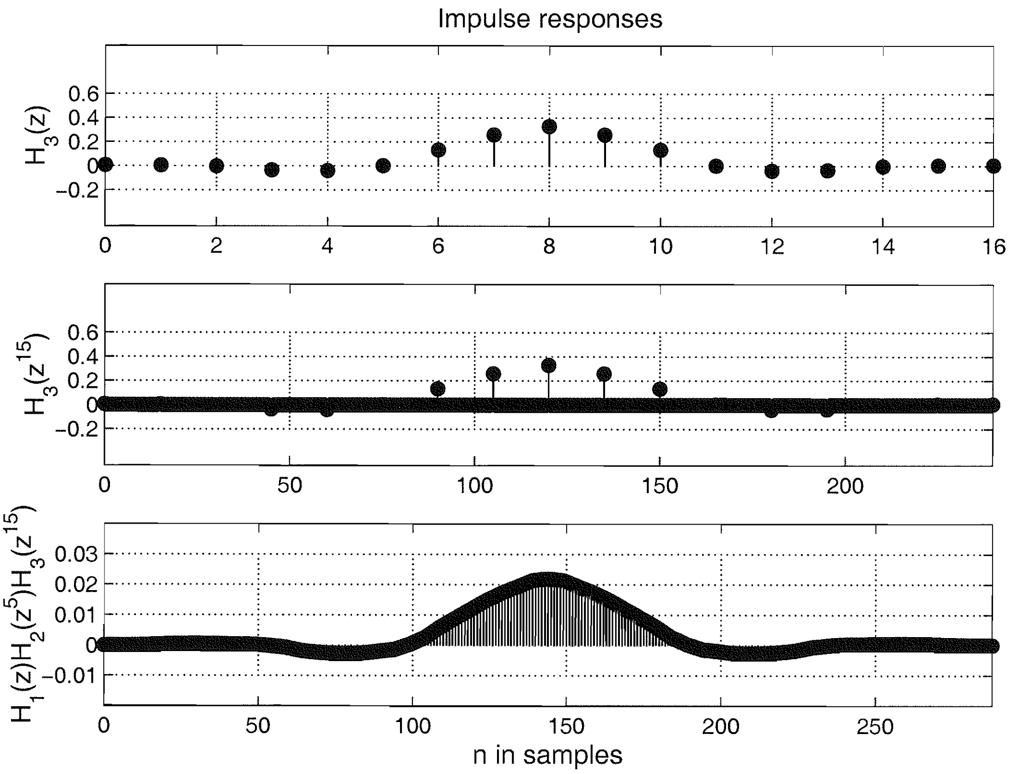
Characteristics of Example Case B Three-Stage Decimator



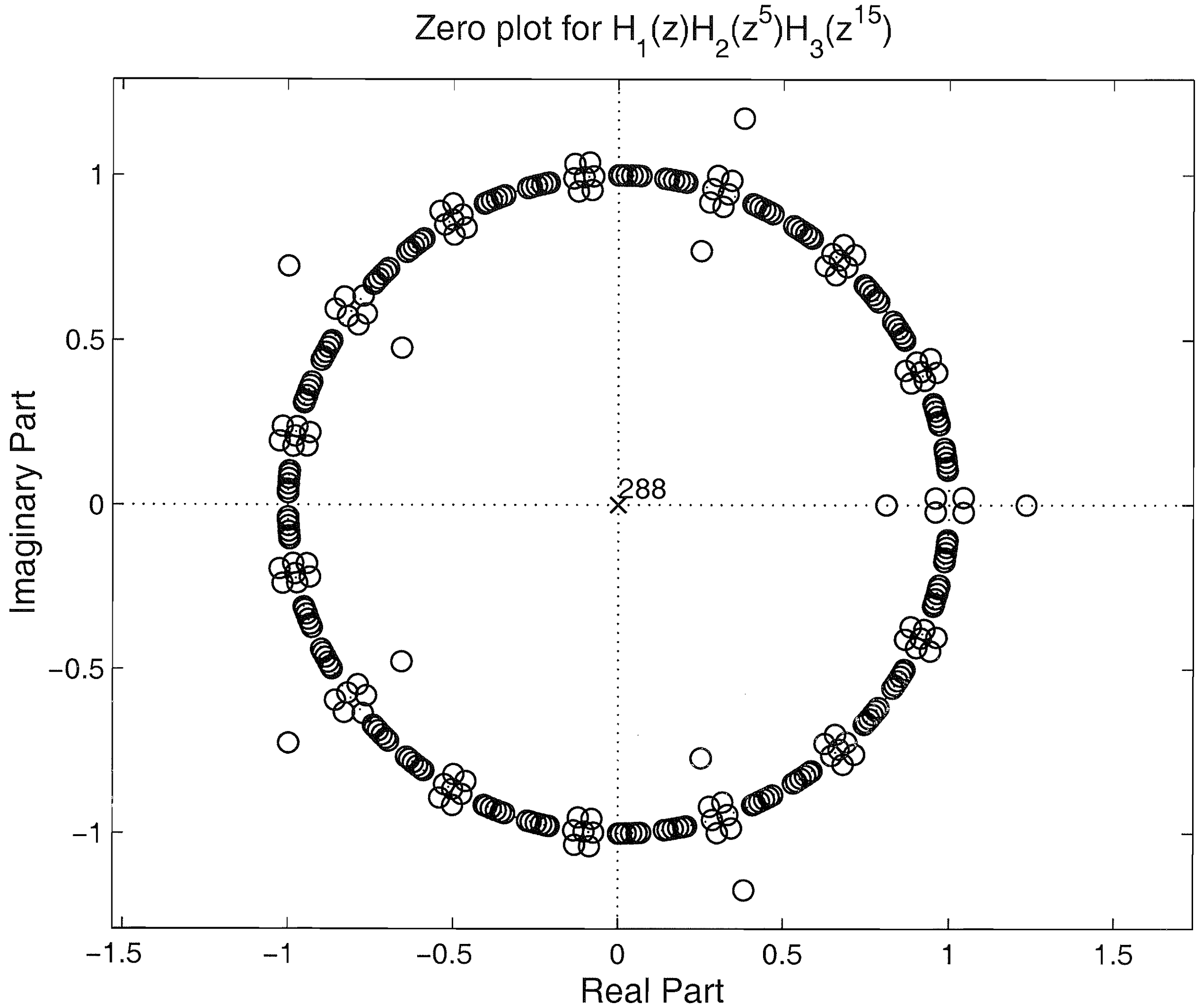


Characteristics of Example Case B Three-Stage Decimator





Characteristics of Example Case B Three-Stage Decimator



Example Case C Three-Stage Decimator

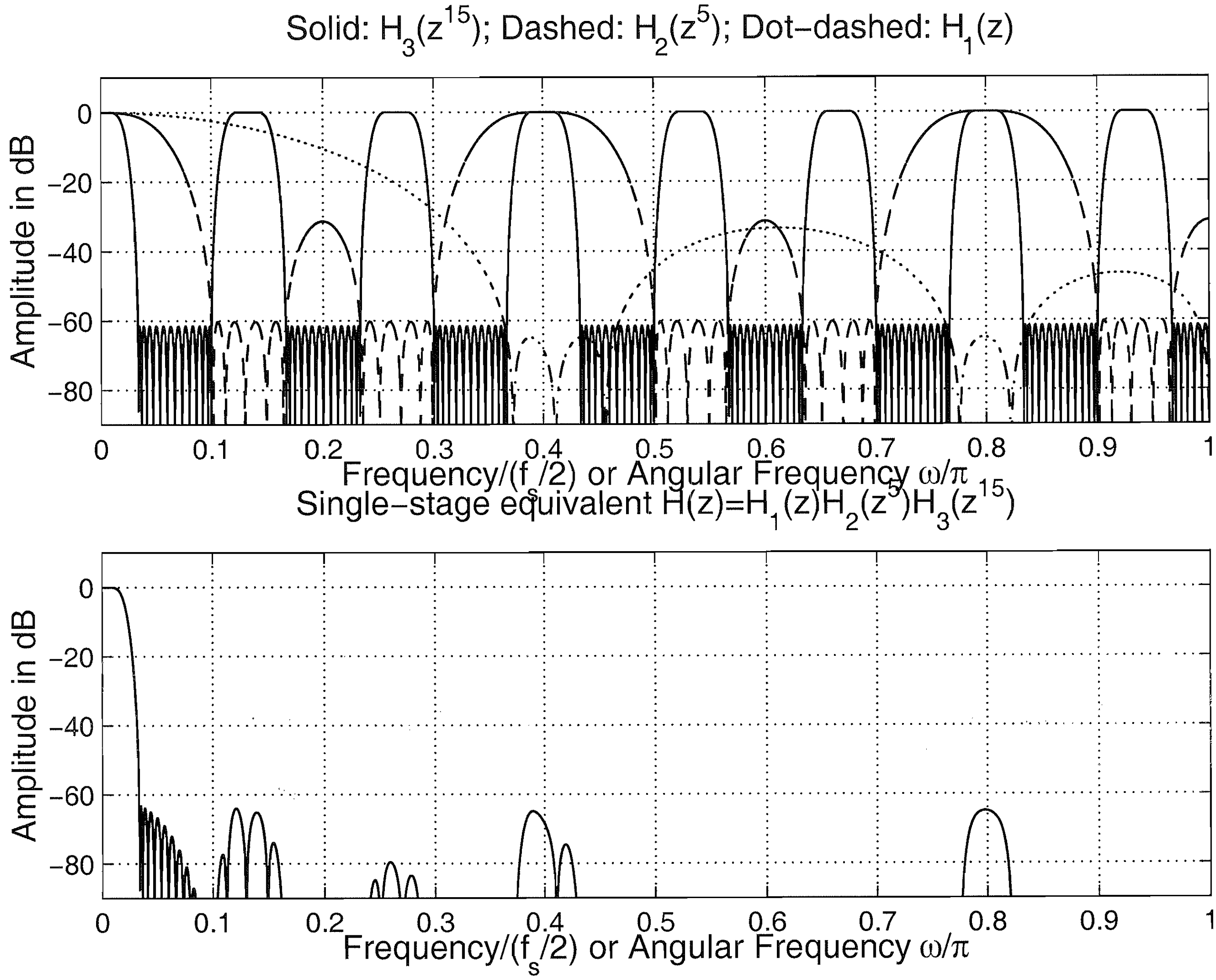
- For $K = 3$ stages, the best result is again obtained by selecting $N_1 = 5$ and $N_2 = N_3 = 3$.
- In this case (see the formulae on Pages 34 and 35), $\alpha = 0.5$, $\widehat{N}_1 = 1$, $\widehat{N}_2 = 5$, $\widehat{N}_1 = 15$.
- The passband regions for $H_1(z)$, $H_2(z)$, and $H_3(z)$ are thus in terms of the angular frequency $[0,\ 0.5\pi/45]$, $[0,\ 0.5\pi/9]$, and $[0,\ 0.5\pi/3]$.
- $\beta_1 = 1.5 \cdot 5/45$ and $\beta_2 = 1.5 \cdot 5/9$.
- The stopband region of $H_1(z)$ is thus a union of bands $[(2 - 7.5/45)\pi/5,\ (2 + 7.5/45)\pi/5]$ and $[(4 - 7.5/45)\pi/5,\ (2 + 7.5/45)\pi/5]$.
- The stopband regions for $H_2(z)$ and $H_3(z)$ are $[(2 - 4.5/9)\pi/3,\ (2 + 4.5/9)\pi/3]$, and $[1.5\pi/3,\ \pi]$, respectively.
- The passband and stopband ripples for the subfilters are again $\delta_p/3 = 0.01/3$ and $\delta_s = 0.001$.
- The orders of $H_1(z)$, $H_2(z)$, and $H_3(z)$ to meet the criteria are 11, 12, and 17.
- When using direct-form decimator implementations, the overall number of multipliers is $6 + 7 + 9 = 22$ and the number of multiplications per input sample is $6/5 +$
$7/15$
$+9/45 = 1.8667$.
- The corresponding figures for the one-stage design of order 256 are 129 and $129/45 = 2.8667$.

- The transfer function of the single-stage equivalent is given by

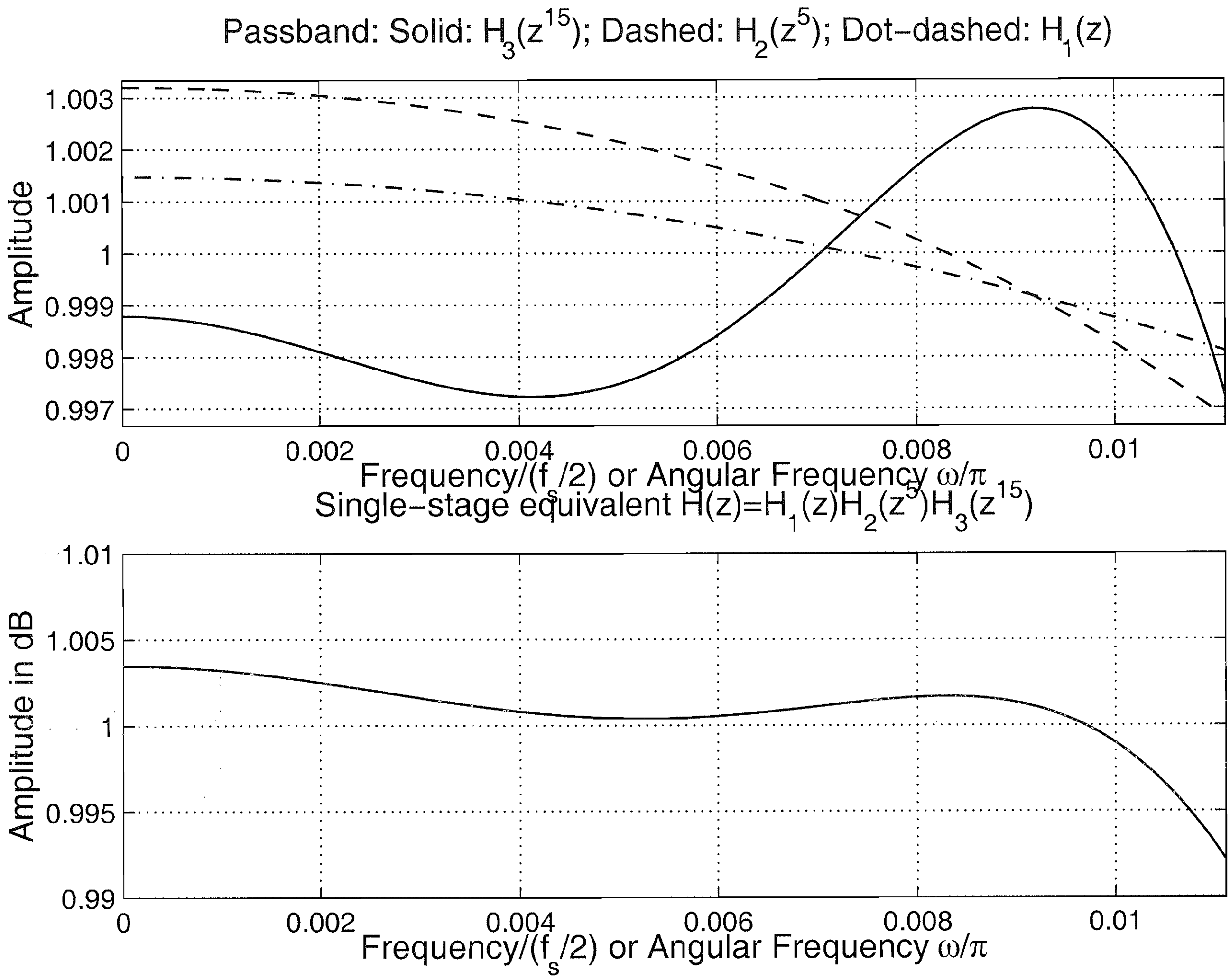
$$H(z) = H_1(z)H_2(z^5)H_3(z^{15}).$$

- The order of this equivalent is 326 compared to 256 required by a direct one-stage design.
- In the following there are seven pages illustrating the characteristics of our Case C three-stage design.
- The main difference compared to Case B is that now aliasing is allowed into the transition band $[0.5\pi/45\ \pi/45]$ only from band $[\pi/45\ 1.5\pi/45]$.
- Note that because the stopband edge of $H_3(z)$ is wider than in Case A, higher filter orders for $H_1(z)$ and $H_2(z)$ are required to attenuate the extra unwanted passband and transtition band regions of $H_3(z^{15})$.

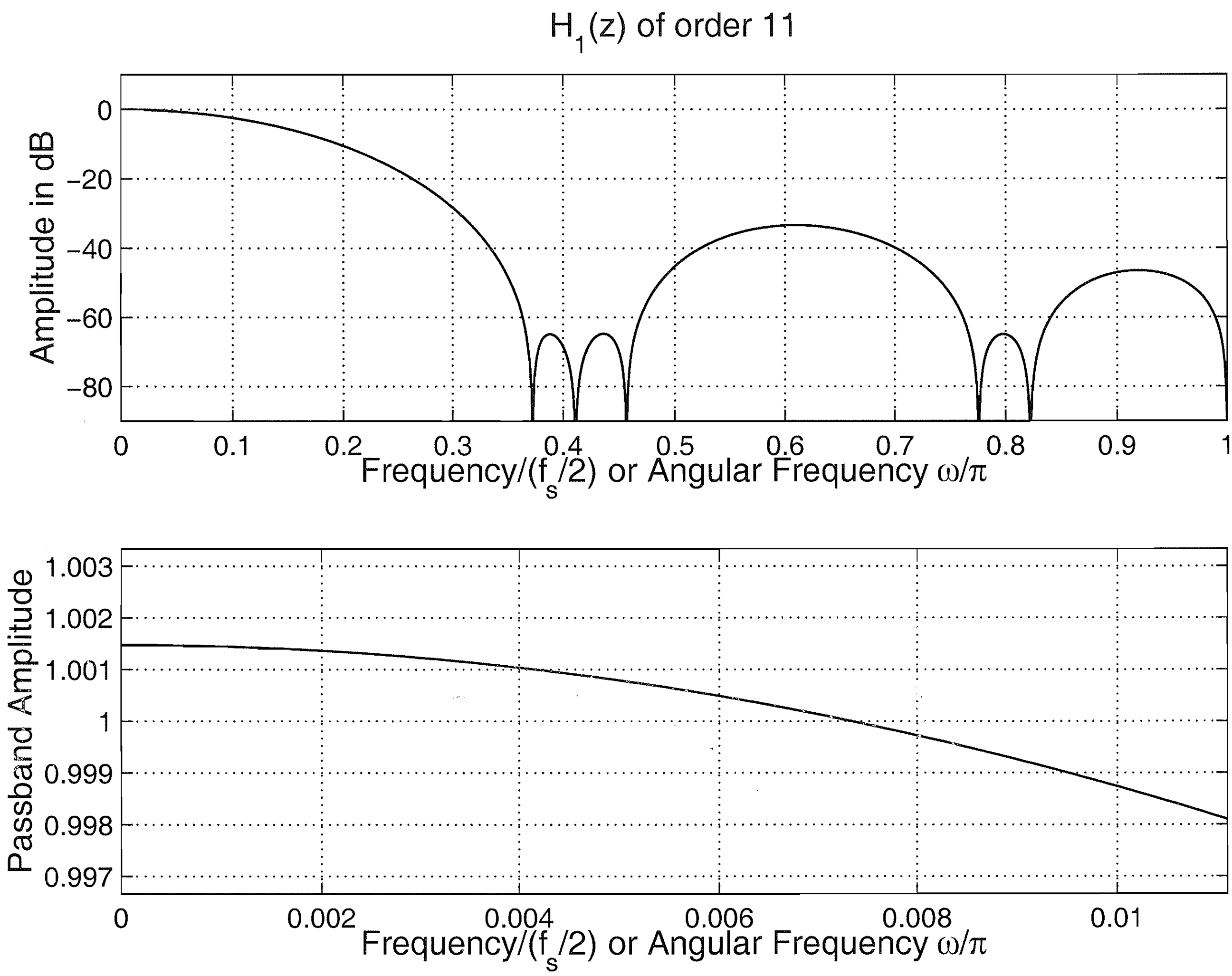
Characteristics of Example Case C Three-Stage Decimator



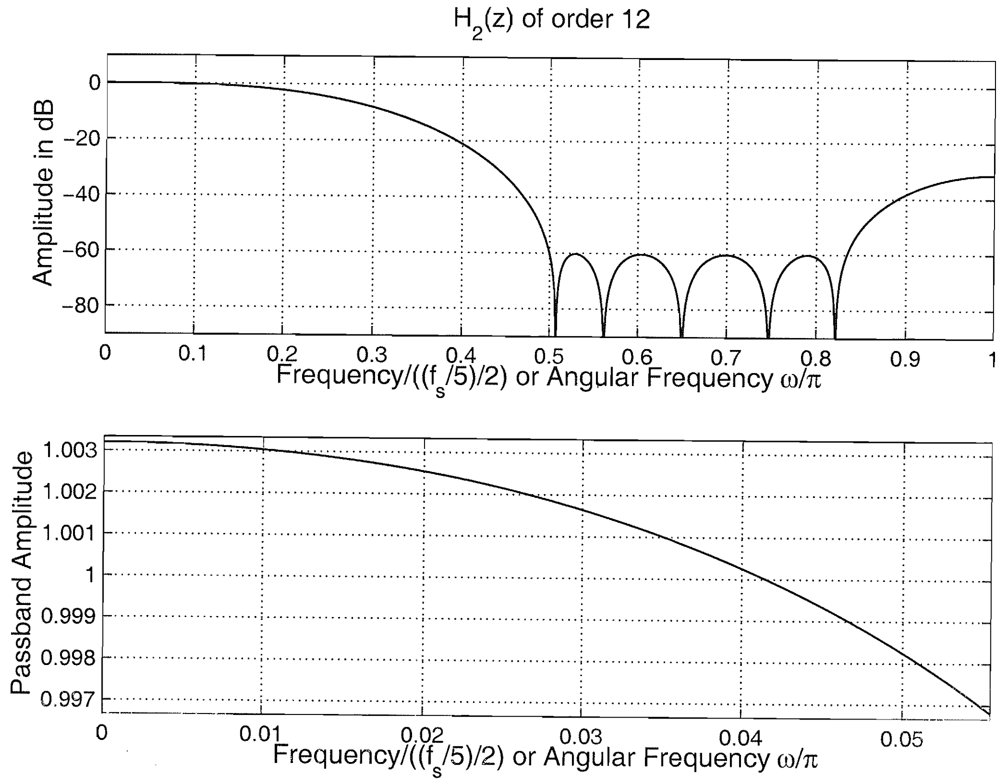
Characteristics of Example Case C Three-Stage Decimator

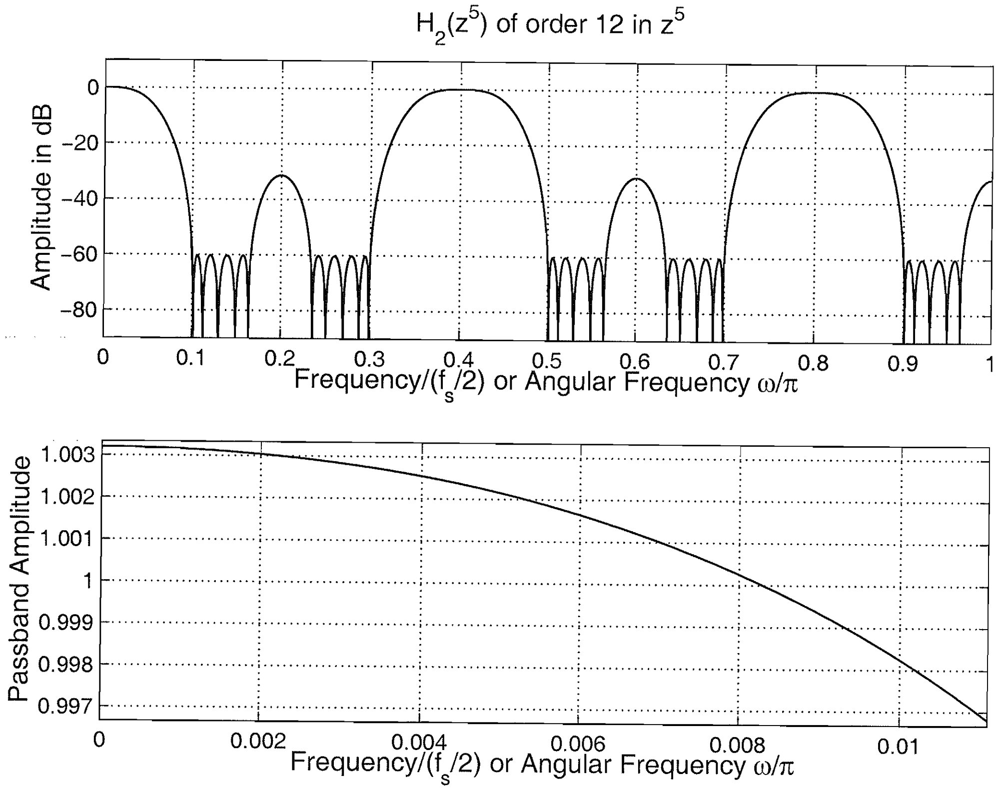


Characteristics of Example Case C Three-Stage Decimator

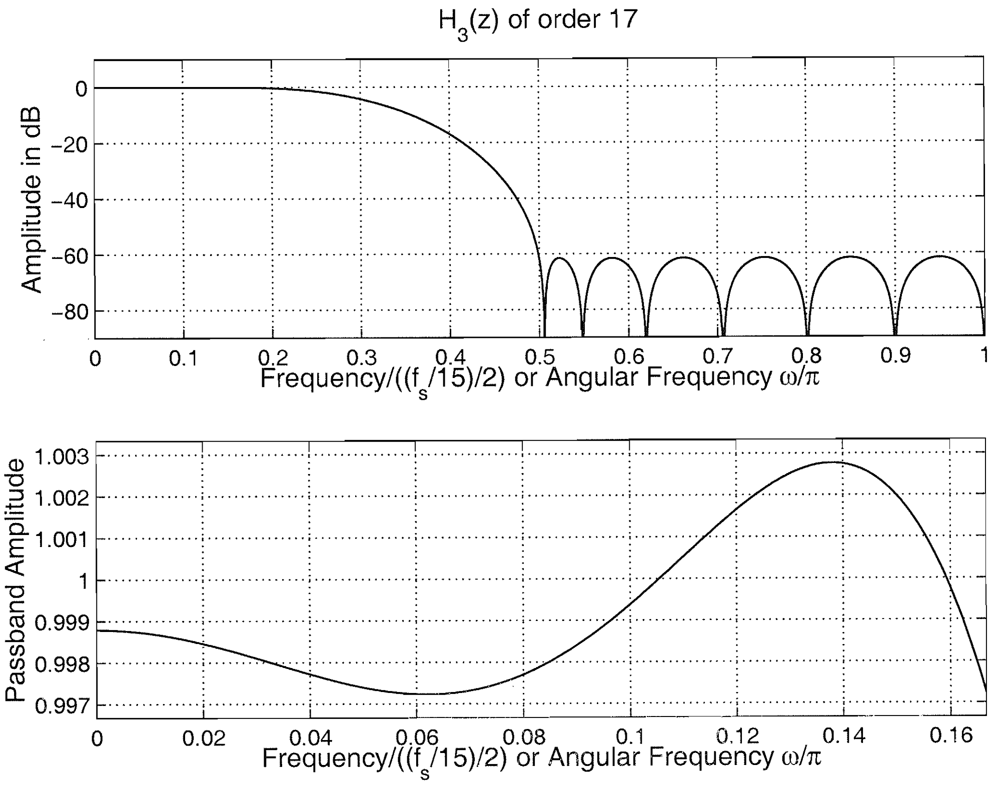


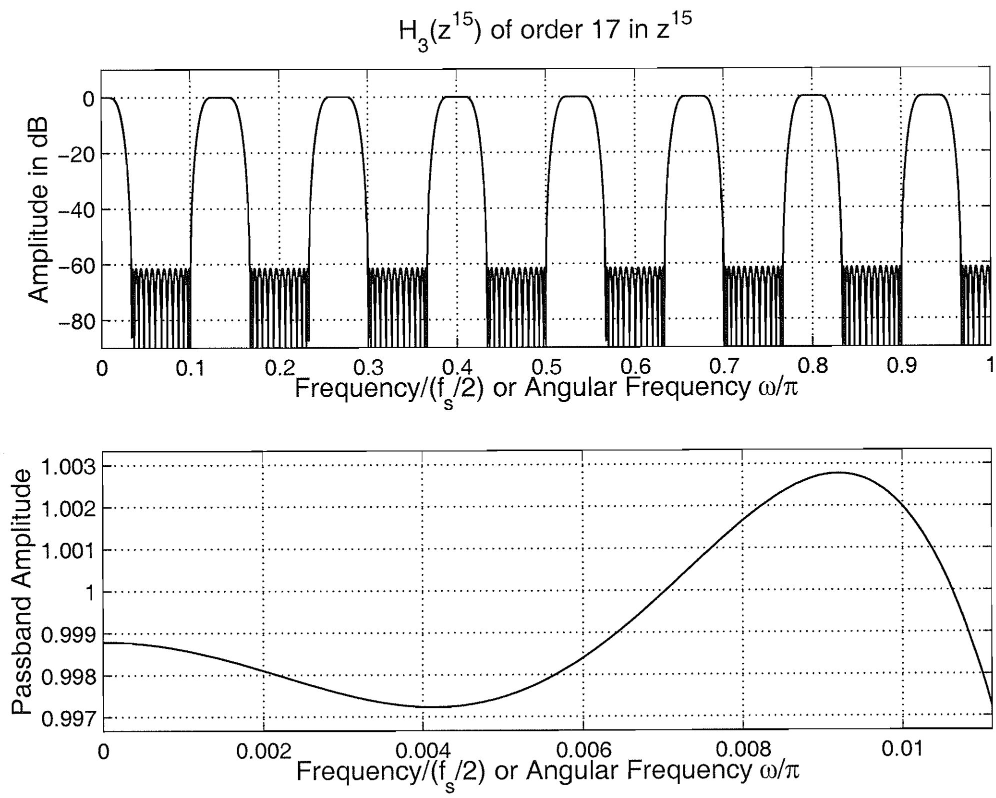
Characteristics of Example Case C Three-Stage Decimator



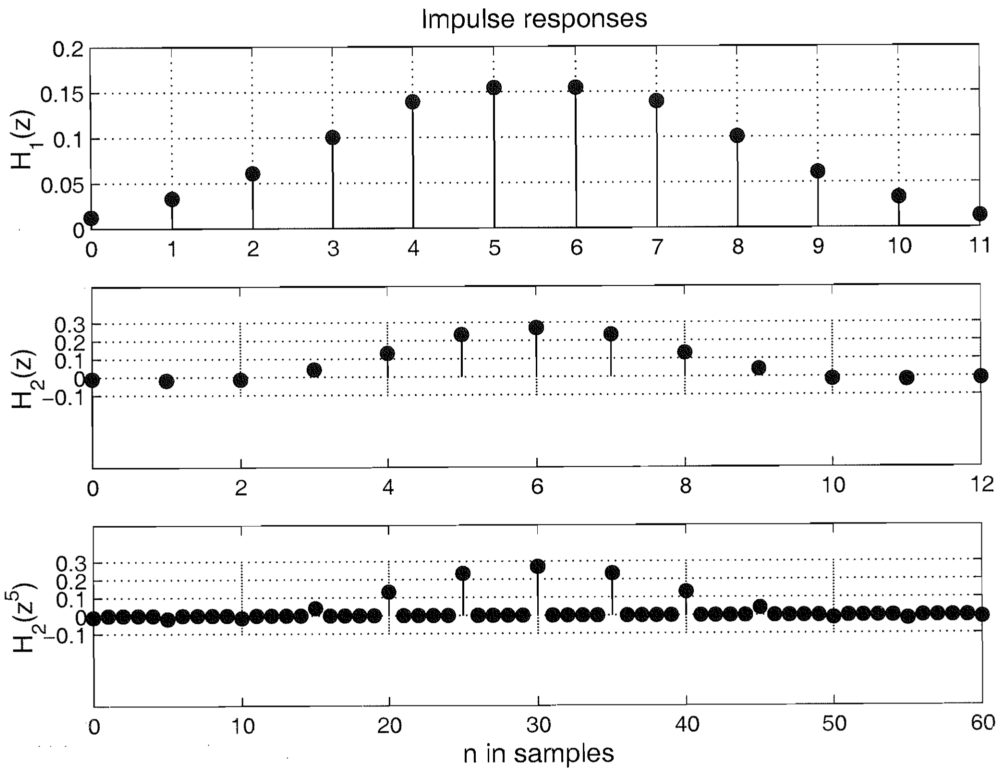


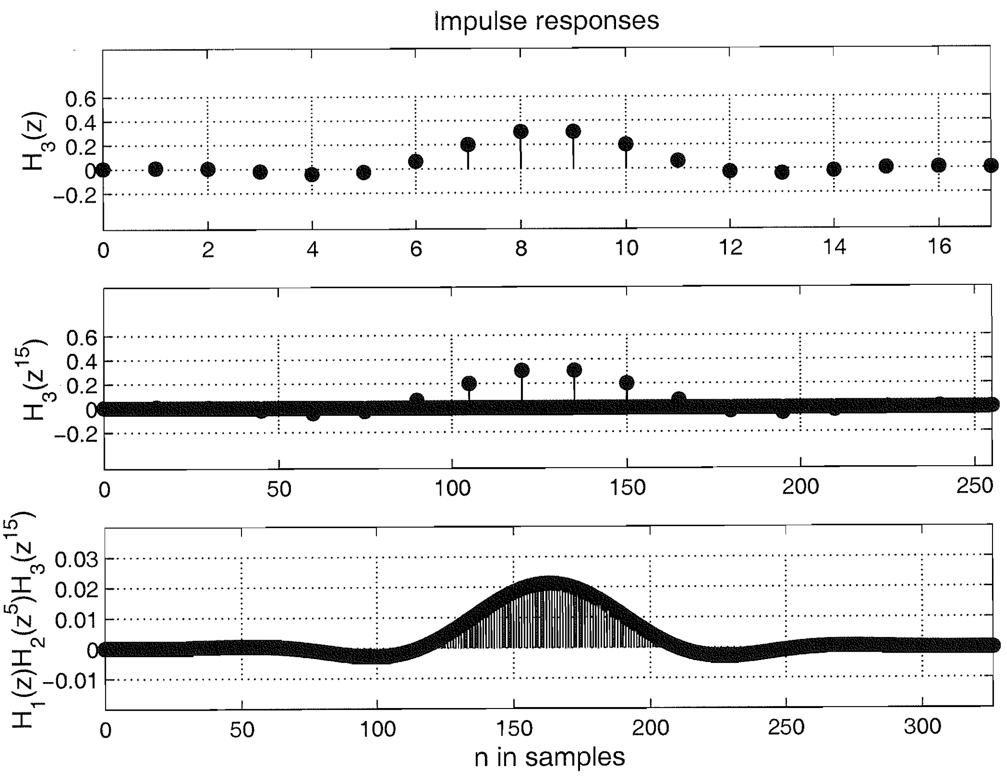
Characteristics of Example Case C Three-Stage Decimator



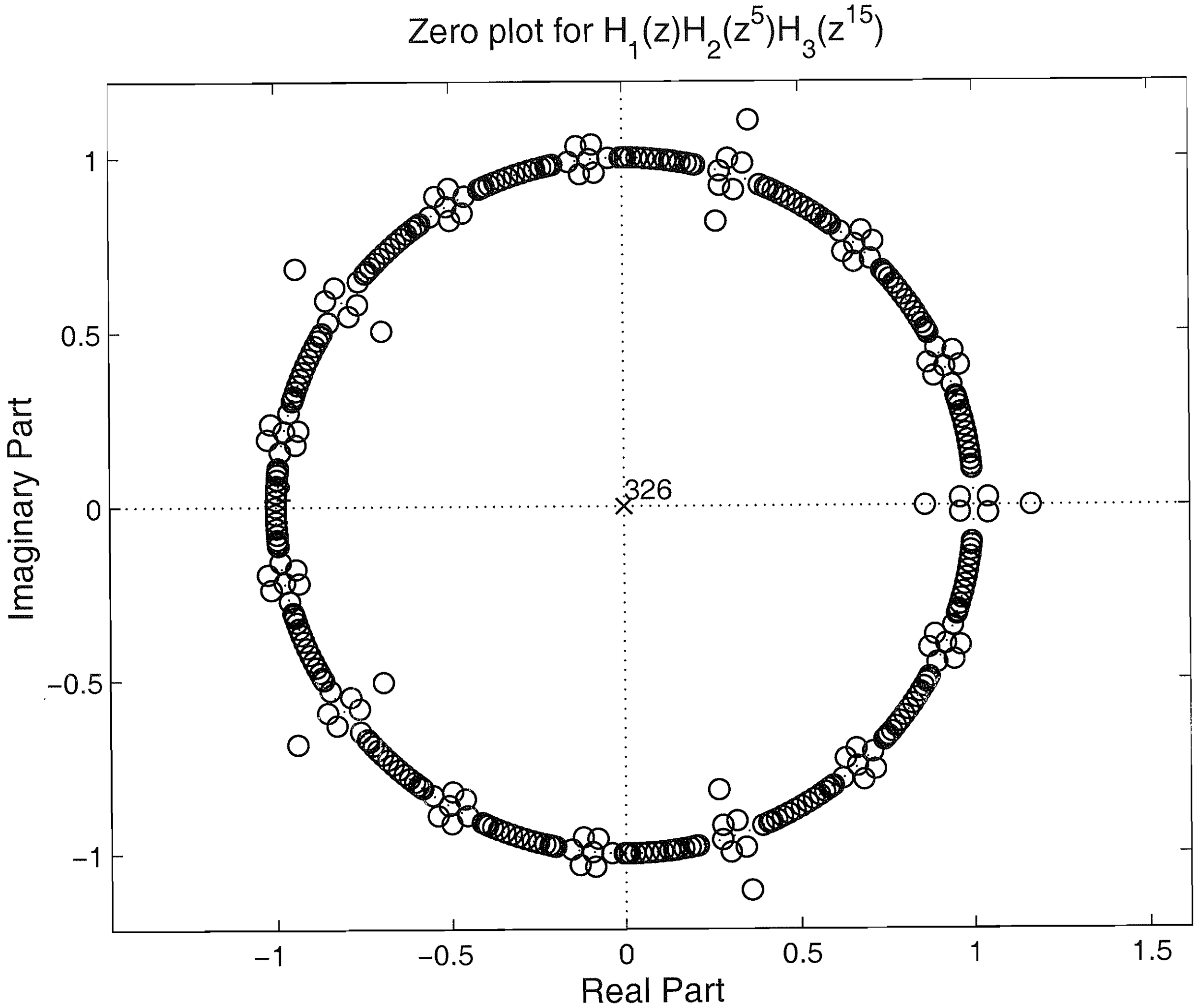


Characteristics of Example Case C Three-Stage Decimator





Characteristics of Example Case C Three-Stage Decimator

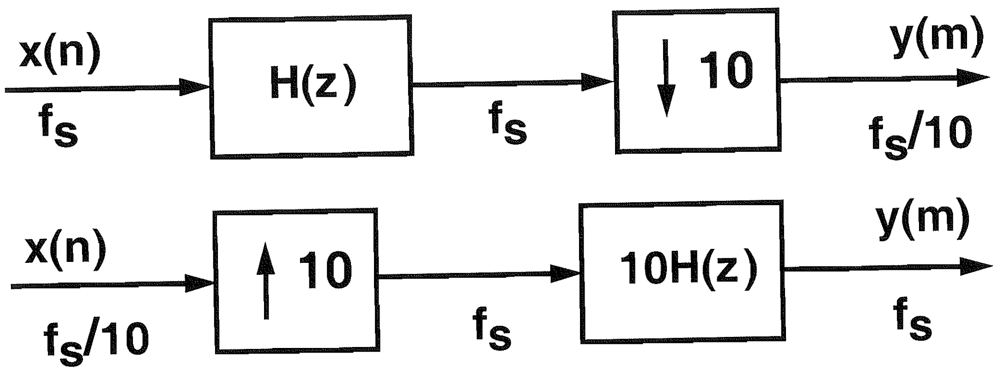


Part II.C: Special Filter Structures

- This subpart introduces some special decimator structures.
- These structures are compared to the conventional one-stage and two-stage linear-phase FIR filters by means of the following Case A example:
- Example specifications: $N = 10$, stopband (angular) edge is at $\omega_s = \pi/10$, passband edge is at $\omega_p = 0.05\pi$, $\delta_p = 0.01$, and $\delta_s = 0.001$.
- In the following, altogether seven different designs are considered. The last four ones are based on the article:
- T. Saramäki, "Design of optimal multistage IIR and FIR filters for sampling rate alteration", in Proc. 1986 IEEE International Symposium on Circuits and Systems (San Jose, CA), pp. 227–230, May 1986.
- At present there are FORTRAN codes for designing these filters. Hopefully, MATLAB codes are coming up next.

One-Stage Conventional FIR Decimator

• The following figures give the implementations for the decimator and the corresponding interpolator.



• In the following, there are three pages illustrating the characteristics of the one-stage decimator design where $H(z)$ is of order 108.

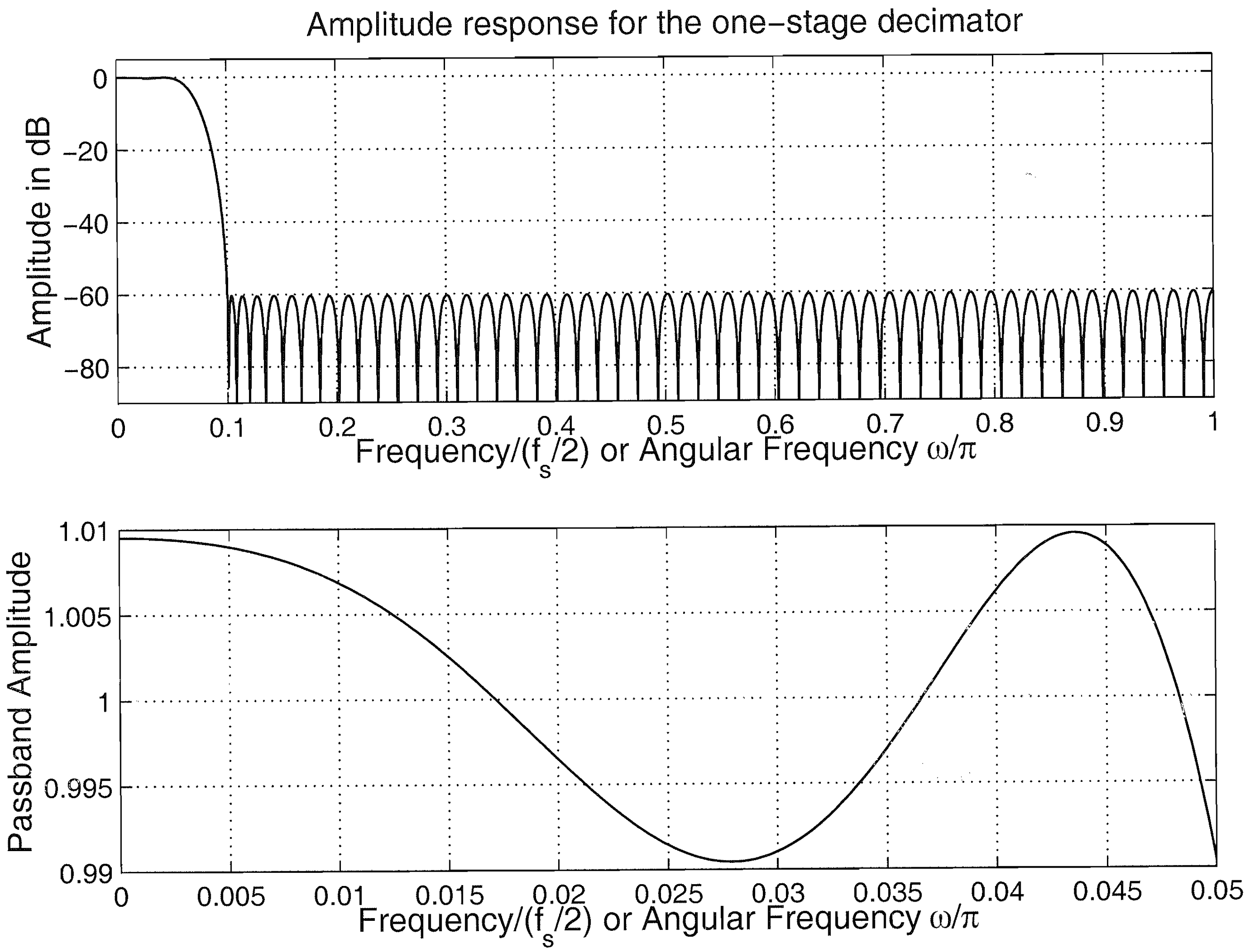
• When implemented in direct-form by exploiting the coefficient symmetry, this design requires 55 multipliers, 108 memory elements, and 5.5 multiplications per input sample for the decimator (per output sample for the interpolator).

• When implemented using the polyphase structure, there are ten branches with $G_k(z)$ consisting of samples occuring at k, $k+10$, $k+20$, $\cdots$ (see Page 74).

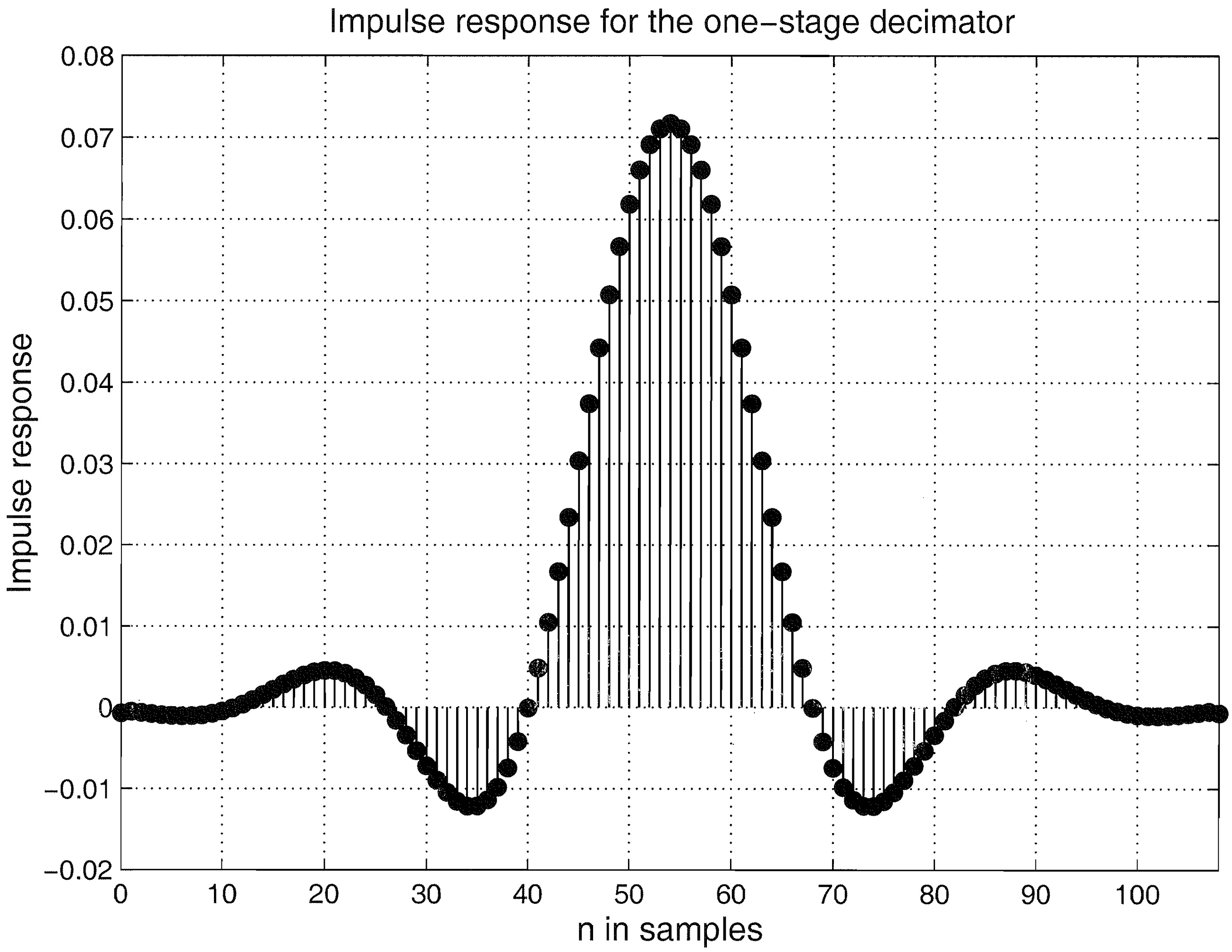
• $H_k(z)$ for $k = 0, 1, \cdots, 8$ have 11 samples (the order is 10), whereas $H_9(z)$ has 10 samples (the order is 9).

- Only $H_4(z)$ is symmetric and requires 6 multipliers when exploiting the coefficient symmetry. $H_k(z)$ for $k = 0, 1, \cdots, 3$ and for $k = 4, 5, \cdots, 8$ require 11 multipliers and $H_9(z)$ requires 10 multipliers.
- The overall number of multipliers is 104 and the number of multiplications per input (ouput) sample is 10.4 for the decimator (interpolator).
- When all the branch filters are implemented using the same delays, the number of these elements is only 11.

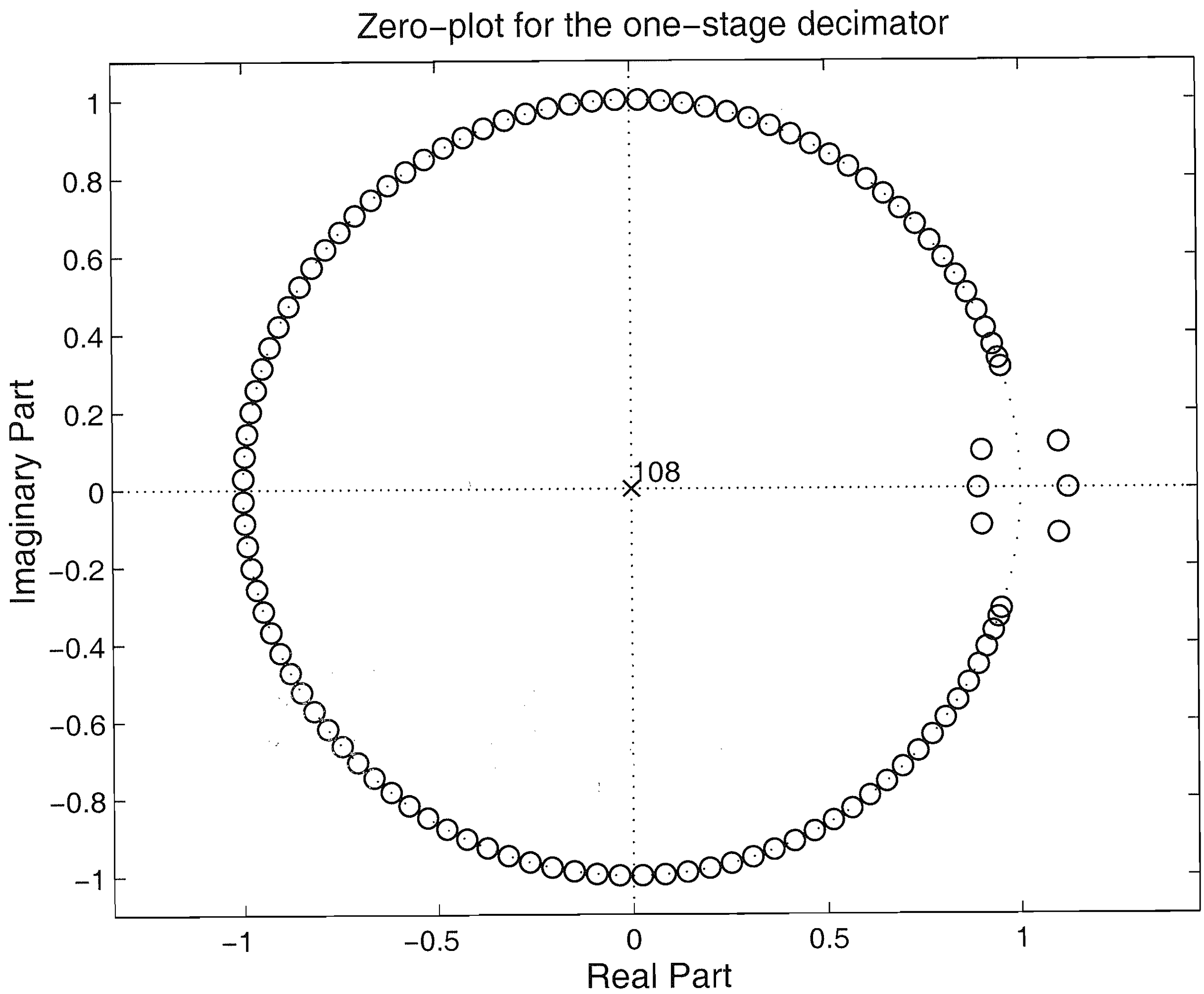
Characteristics of One-Stage Conventional FIR Decimator



Characteristics of One-Stage Conventional FIR Decimator



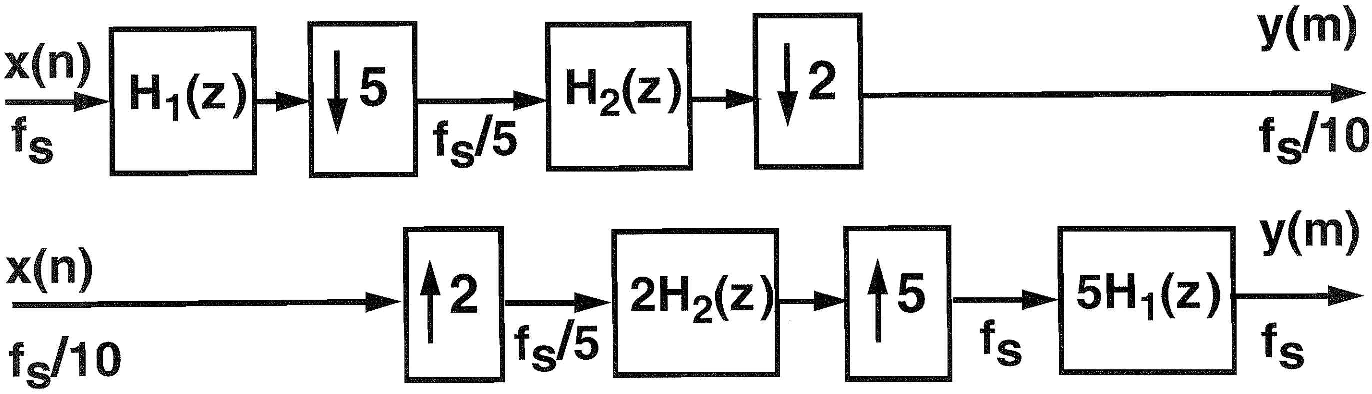
Characteristics of One-Stage Conventional FIR Decimator



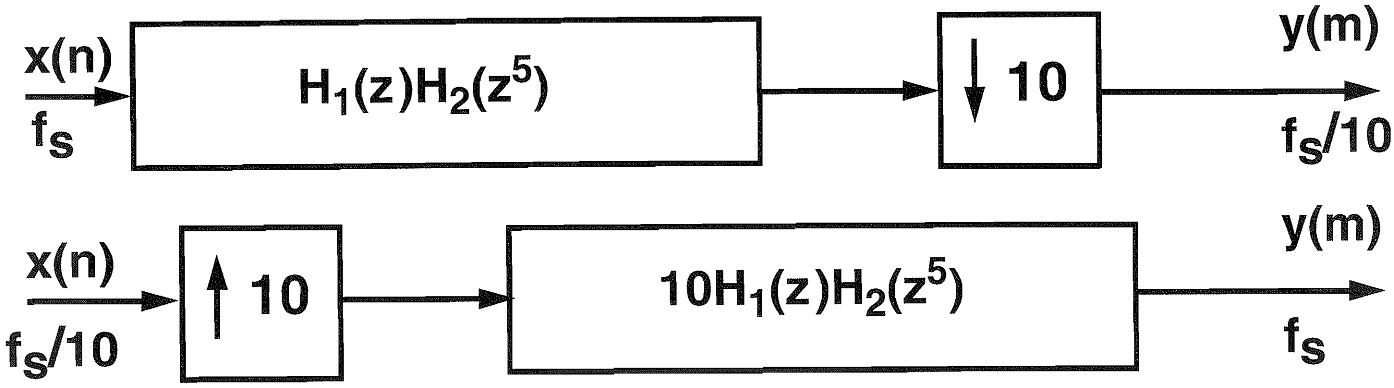
Two-Stage Conventional FIR Decimator

• The best result is obtained by selecting $N_1 = 5$ and $N_2 = 2$.

• The following figures give the implementation forms for both the decimator and intepolator as well as for the corresponding single-stage equivalents.



Single-Stage Equivalents

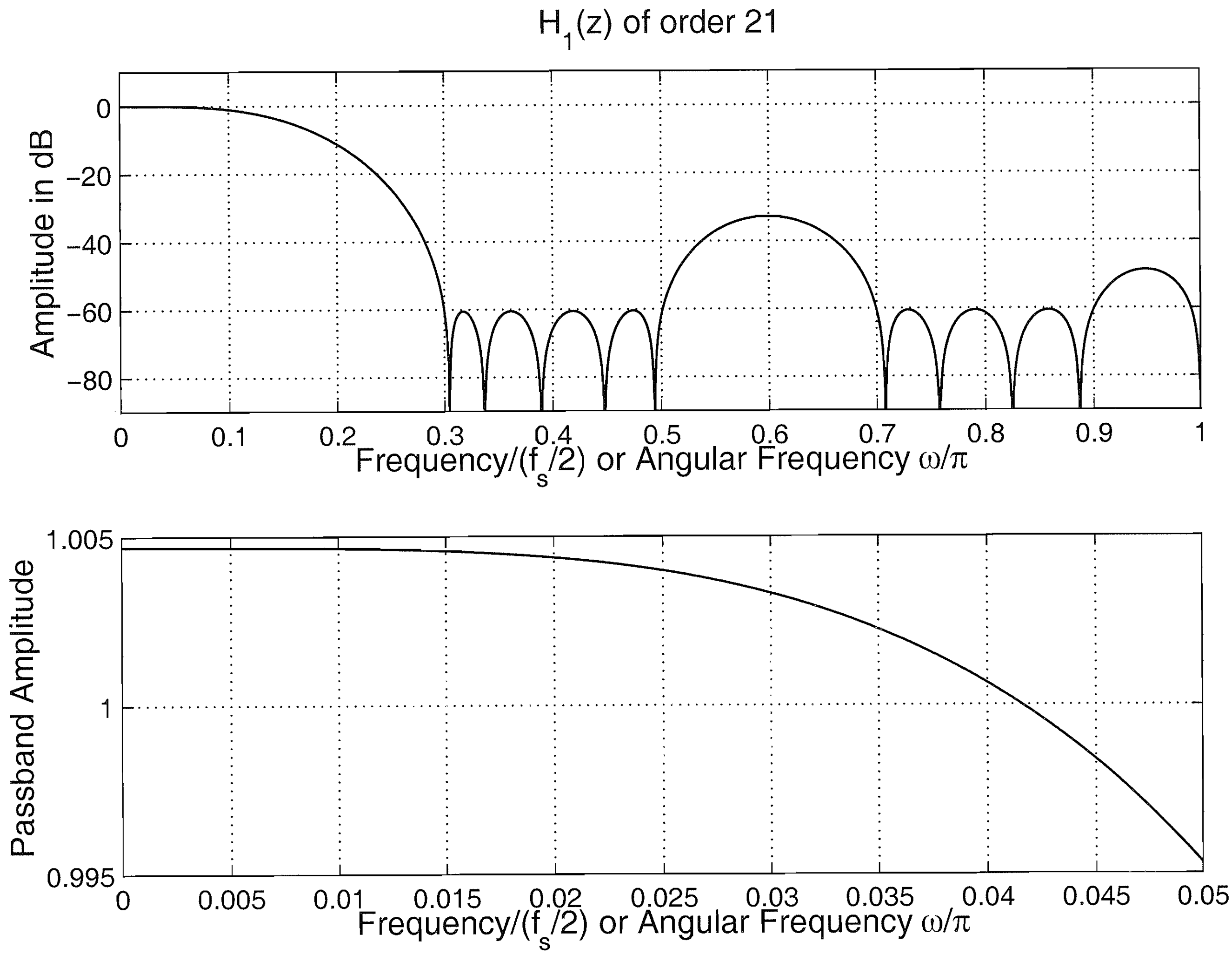


• When using the conventional Case A synthesis, as described earlier, the orders of $H_1(z)$ and $H_2(z)$ become 21 and 24.

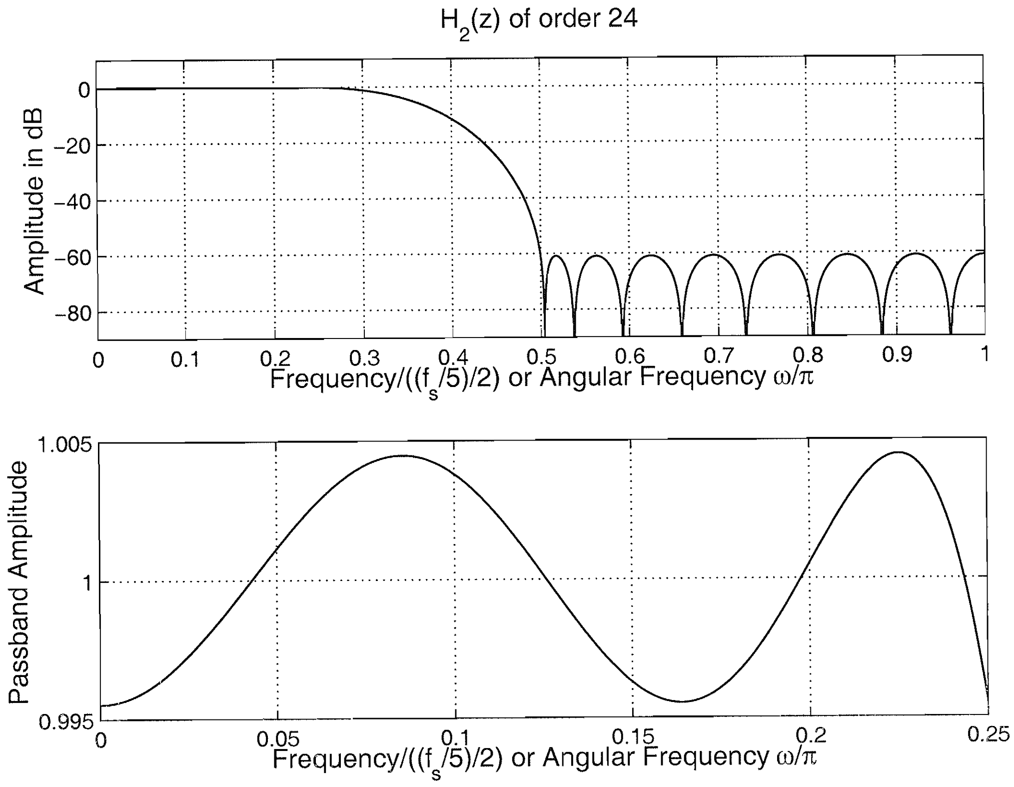
• In the following, there are six pages illustrating the characteristics of the overall design.

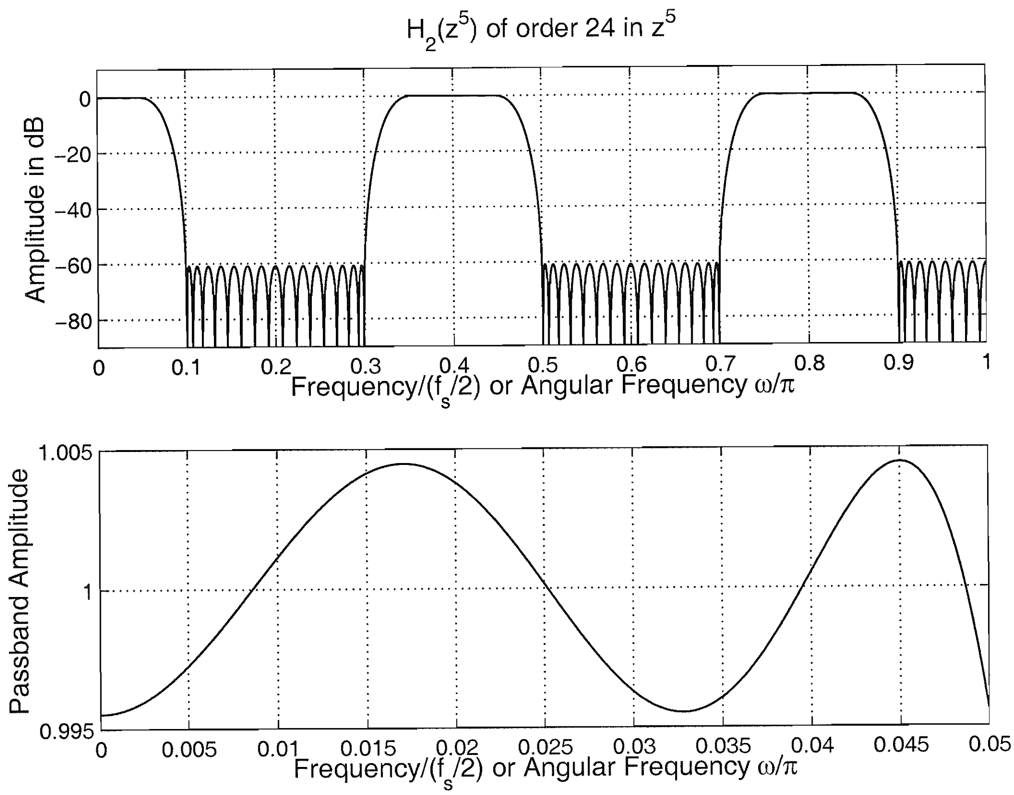
- When exploiting the coefficient symmetries and using direct-form structures, the overall design requires $11 + 13 = 24$ multipliers, $11/5 + 13/20 = 3.5$ multiplications per input sample, and 45 delay elements.
- For the one-stage design, the corresponding figures were 55, 5.5 and 108.
- The delay for the single-stage equivalent $H(z) = H_1(z)H_2(z^5)$ is $21 + 5 \cdot 24 = 141$.

Characteristics of Two-Stage Conventional FIR Decimator

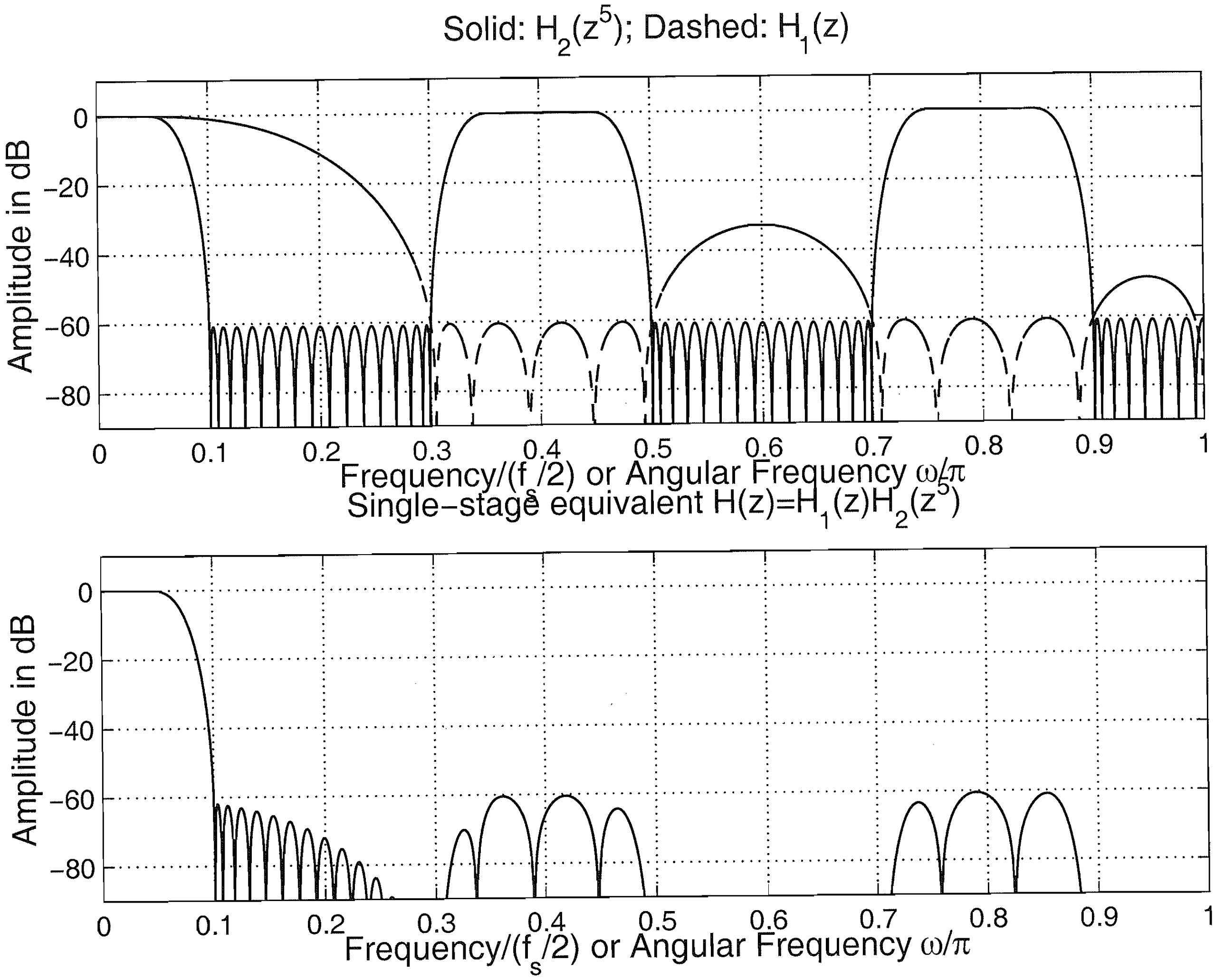


Characteristics of Two-Stage Conventional FIR Decimator

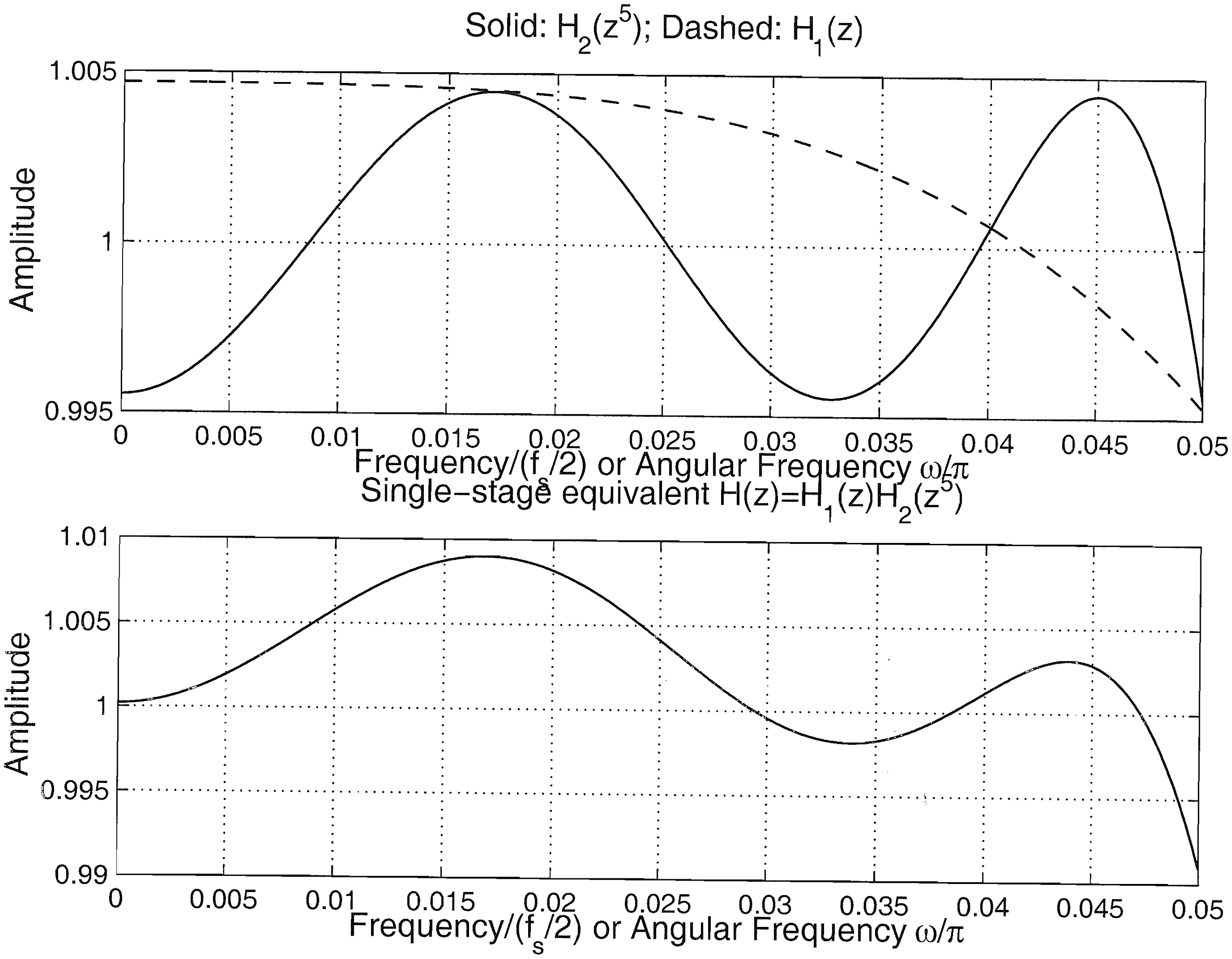




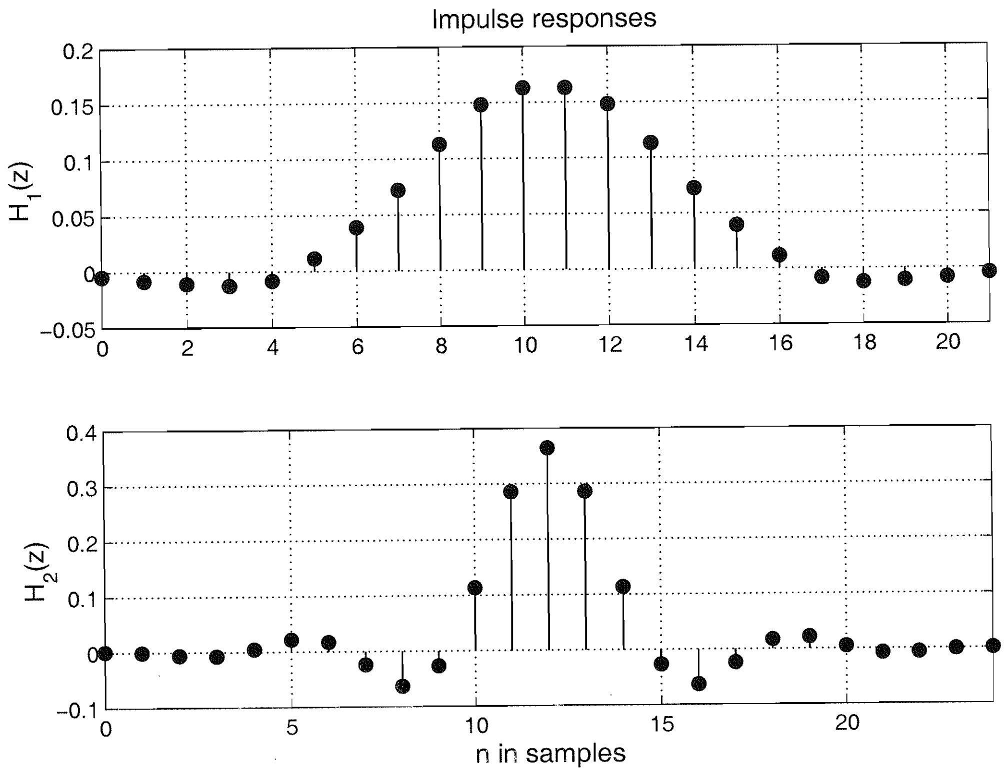
Characteristics of Two-Stage Conventional FIR Decimator

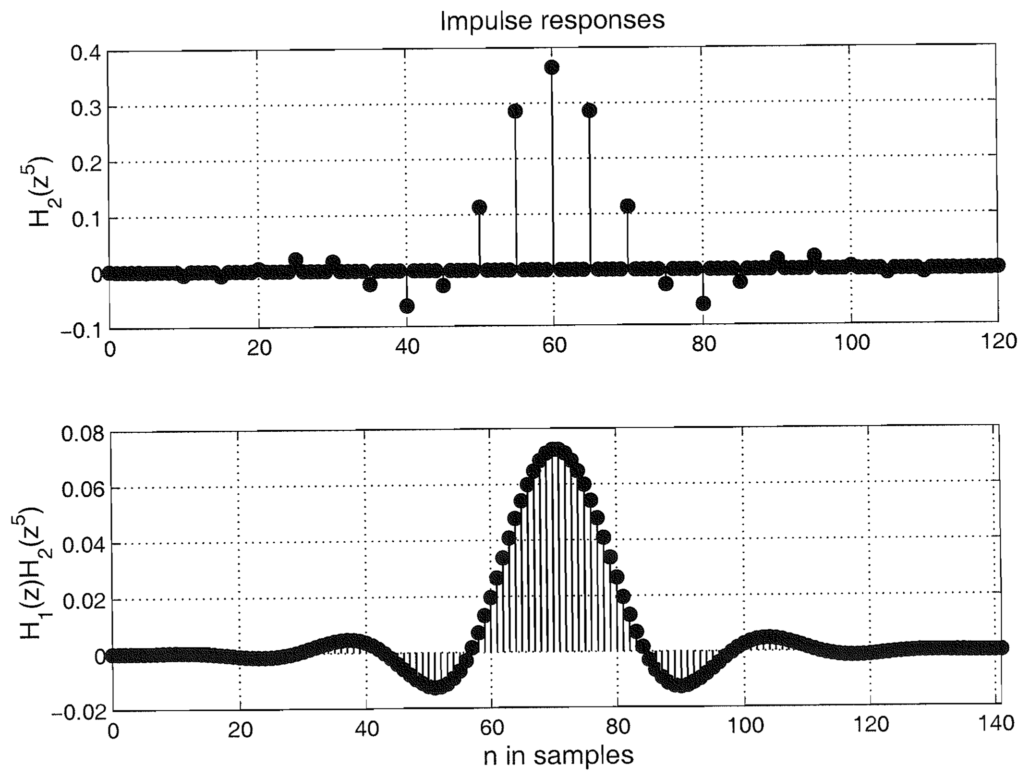


Characteristics of Two-Stage Conventional FIR Decimator

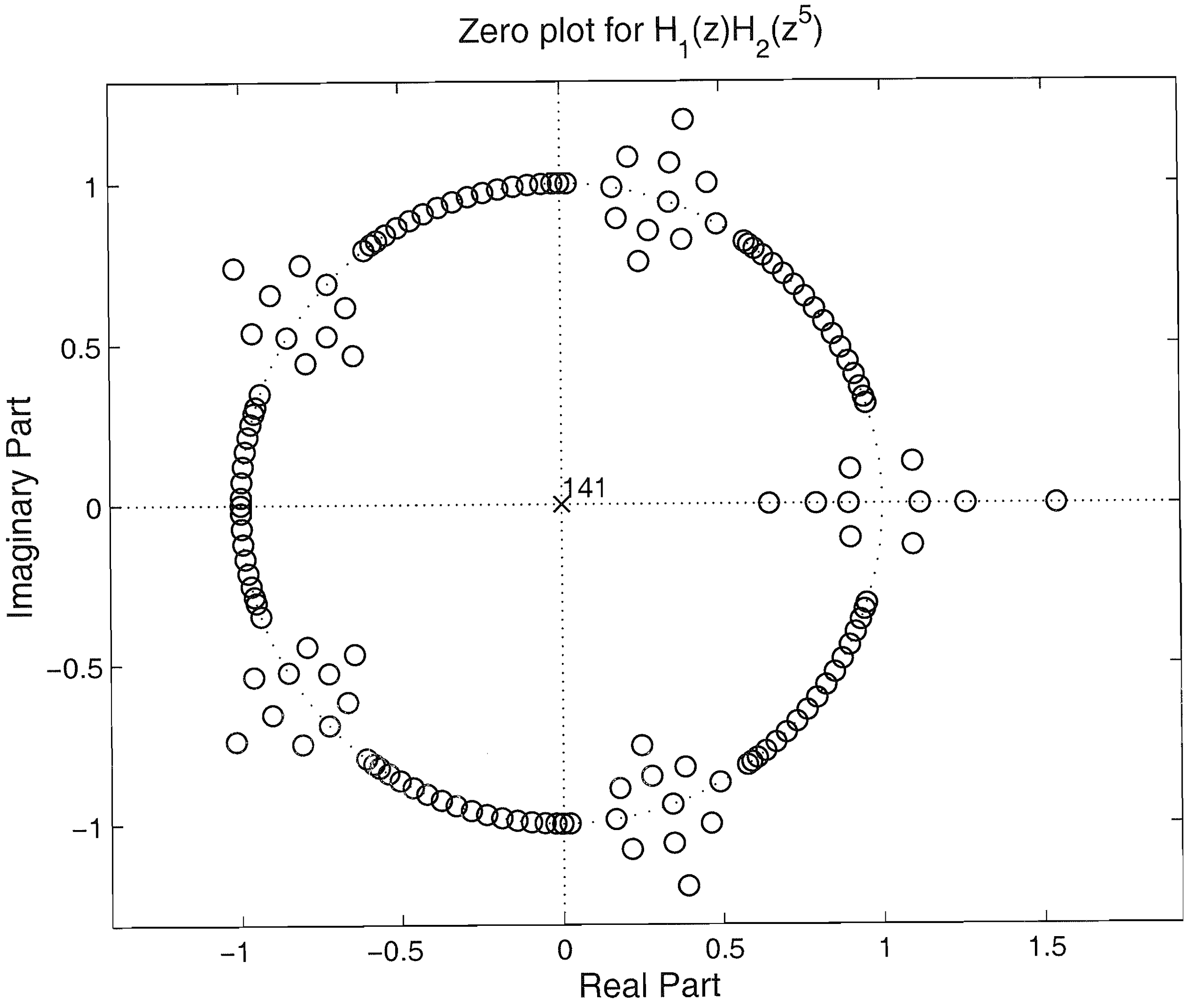


Characteristics of Two-Stage Conventional FIR Decimator





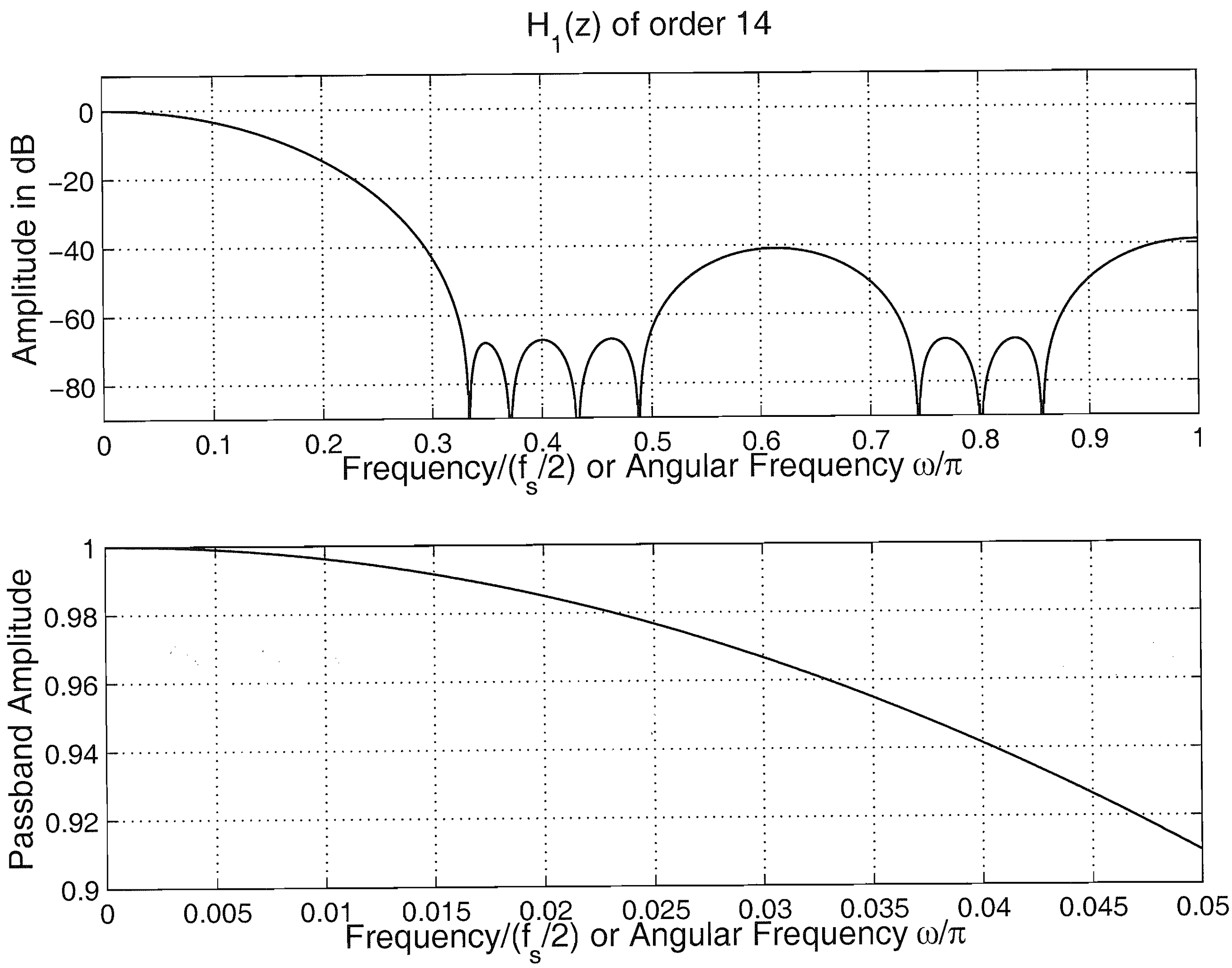
Characteristics of Two-Stage Conventional FIR Decimator



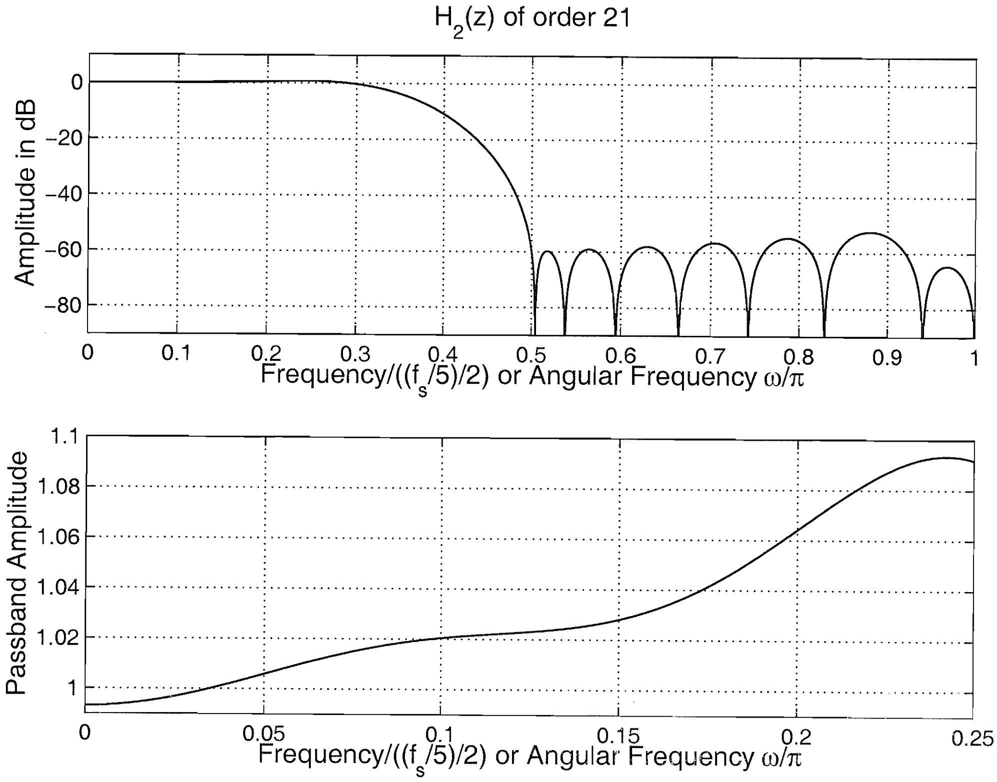
Two-stage 'Optimized' FIR Decimator

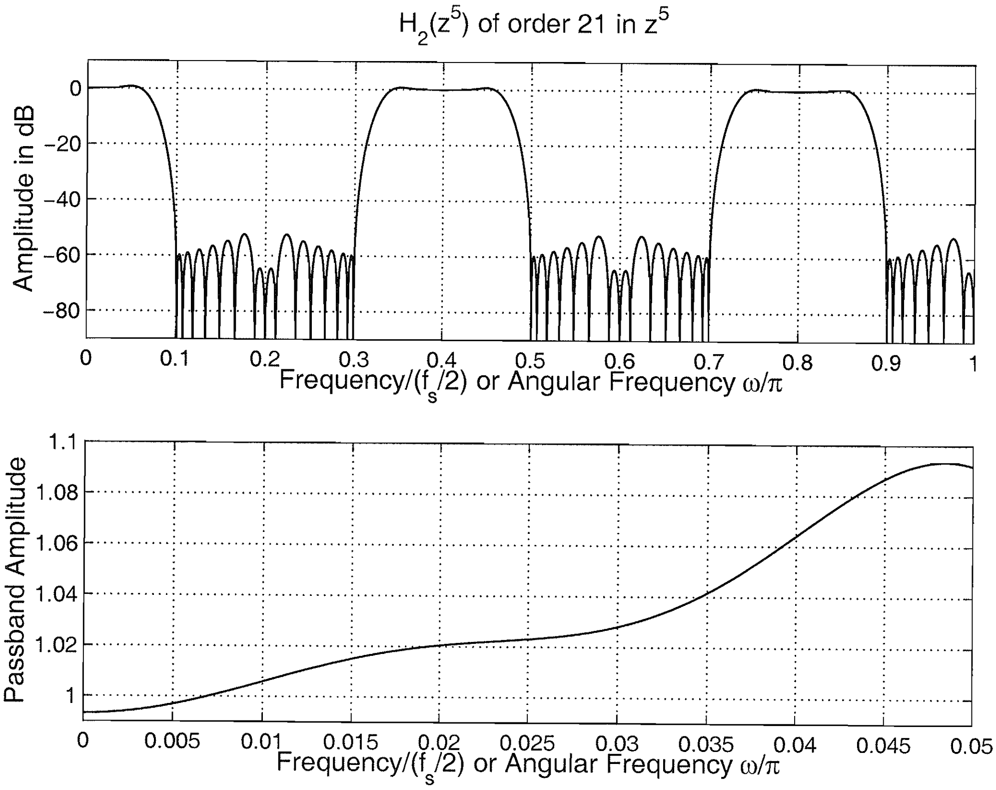
- $H_2(z^5)$ takes care of the overall response in $[0,\ 0.2\pi]$.
- $H_1(z)$ has all its zeros on the unit circle and attenuates the extra passbands and transition bands of $H_2(z^5)$.
- The order of $H_1(z)$ reduces from 21 to 14 and the order of $H_2(z)$ from 24 to 21.
- Compared to the previous design, the number of multipliers reduces from 24 to 19, the number of multiplications per input sample from 3.5 to 2.7, and the number of delay elements from 45 to 35.
- The order of the single-stage equivalent $H(z) = H_1(z)H_2(z^5)$ reduces from 141 to 119 compared to the previous two-stage design.
- The above filter has been optimized by using one the techniques described in:
- T. Saramäki, "Finite impulse response filter design", Chapter 4 in Handbook for Digital Signal Processing, edited by S. K. Mitra and J. F. Kaiser, John Wiley and Sons, New York, 1993, pp. 155-277.
- T. Saramäki, "Design of computationally efficient FIR filters using periodic subfilters as building blocks" in The Circuits and Filters Handbook, edited by W.-K. Chen, CRC Press, Inc., 1995, pp. 2578–2601.
- In the following, there are six pages illustrating the characteristics of the overall design.

Characteristics of Two-Stage 'Optimized' FIR Decimator

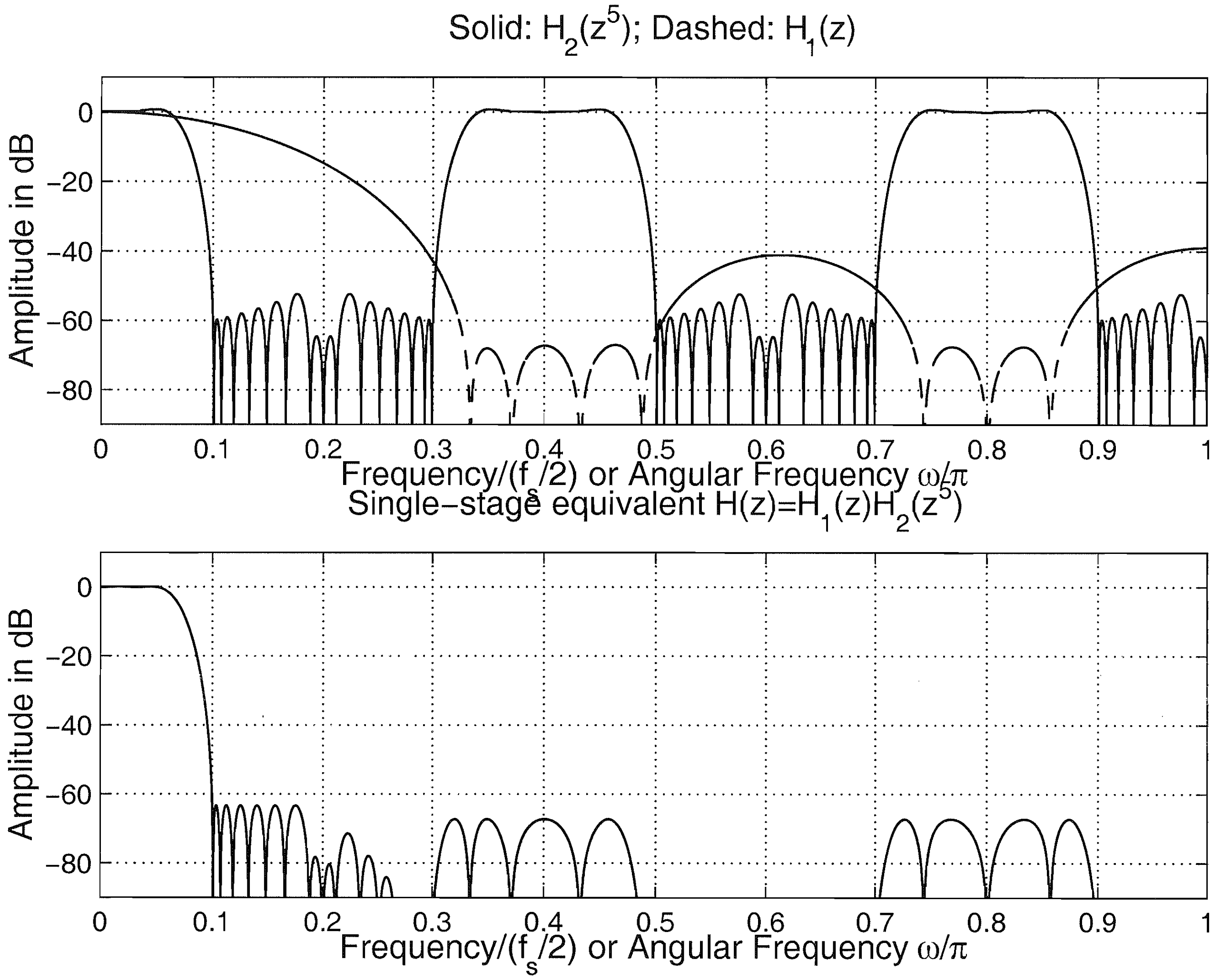


Characteristics of Two-Stage 'Optimized' FIR Decimator

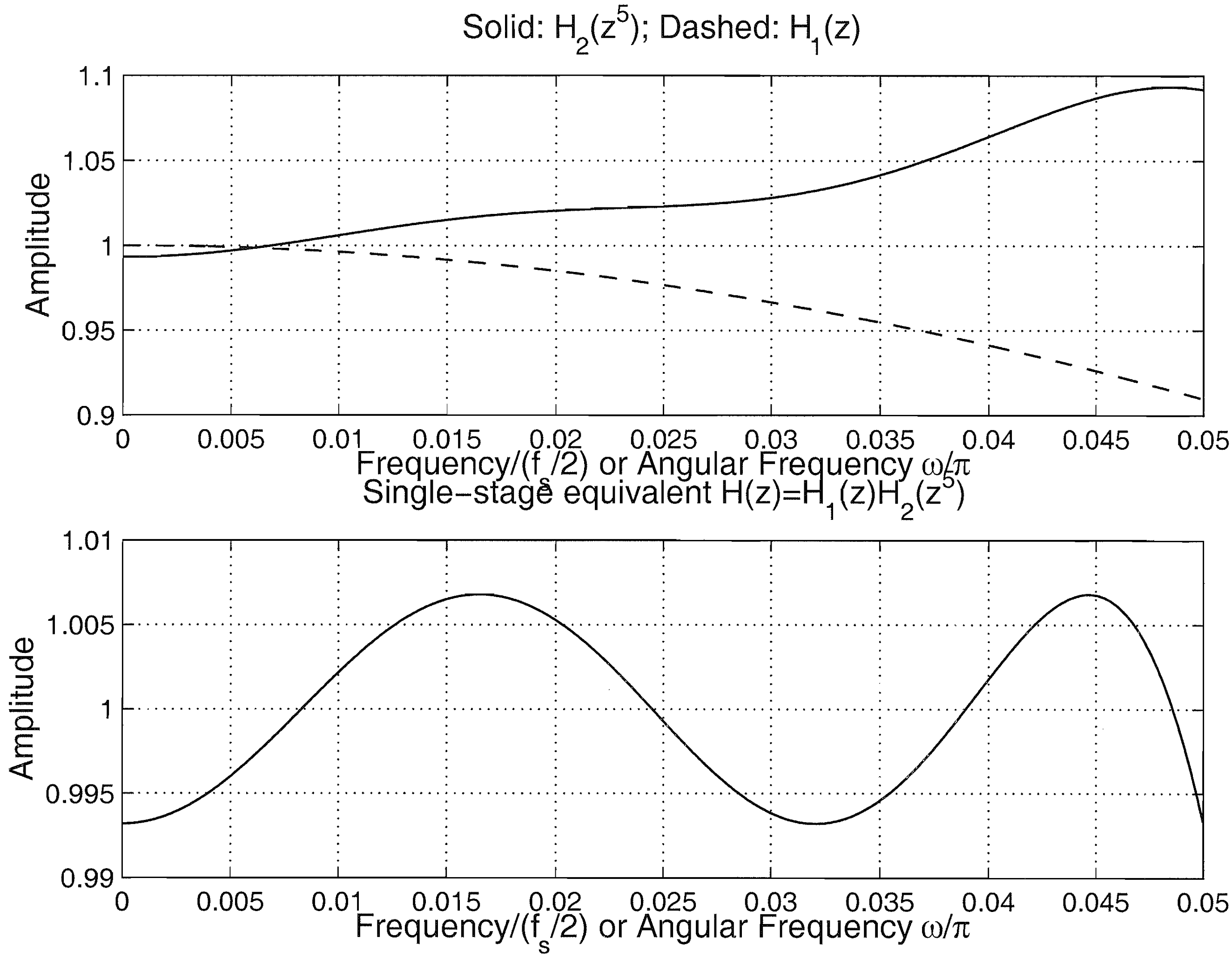




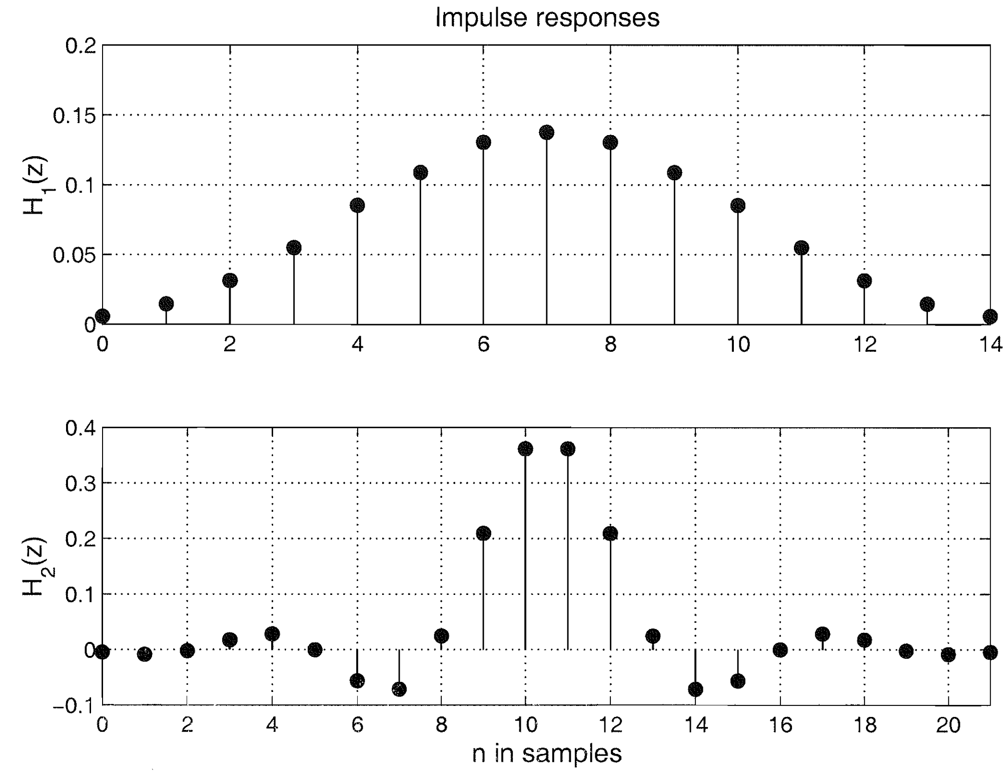
Characteristics of Two-Stage 'Optimized' FIR Decimator

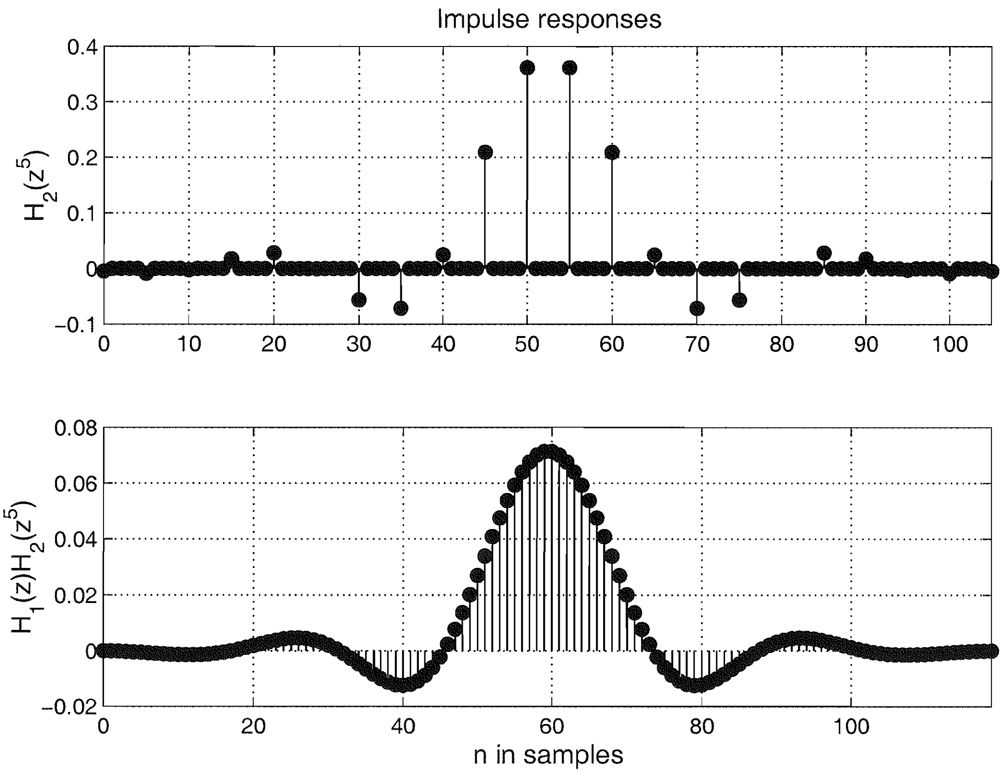


Characteristics of Two-Stage 'Optimized' FIR Decimator

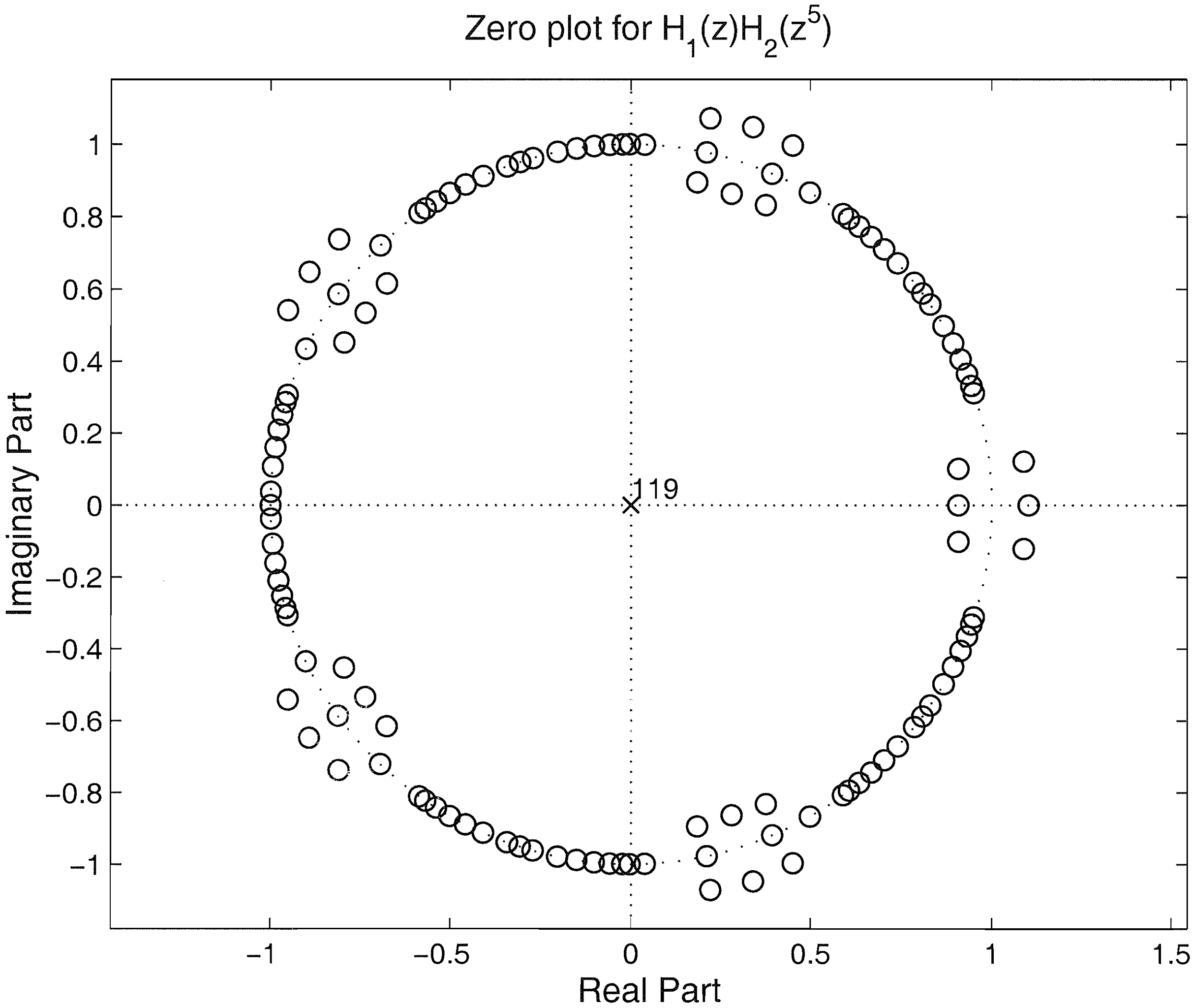


Characteristics of Two-Stage 'Optimized' FIR Decimator



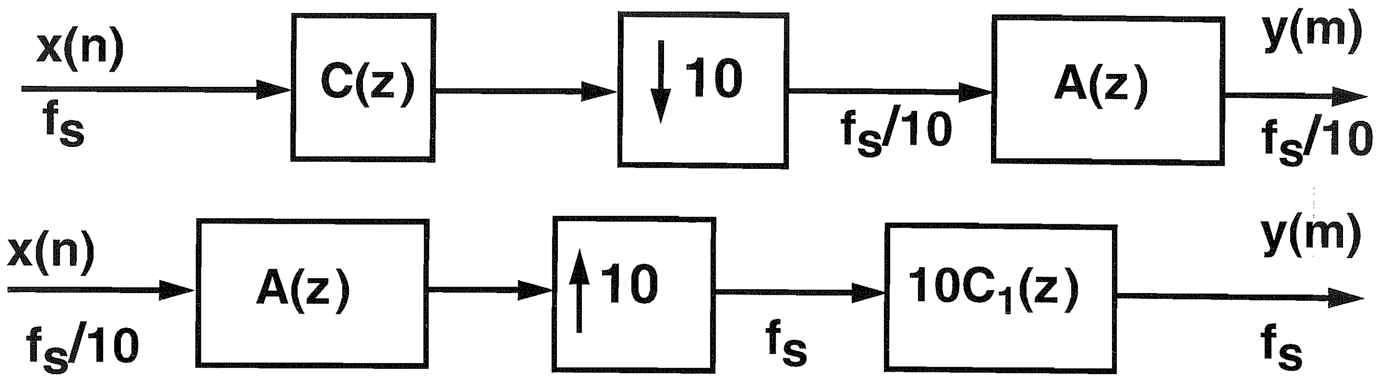


Characteristics of Two-Stage 'Optimized' FIR Decimator

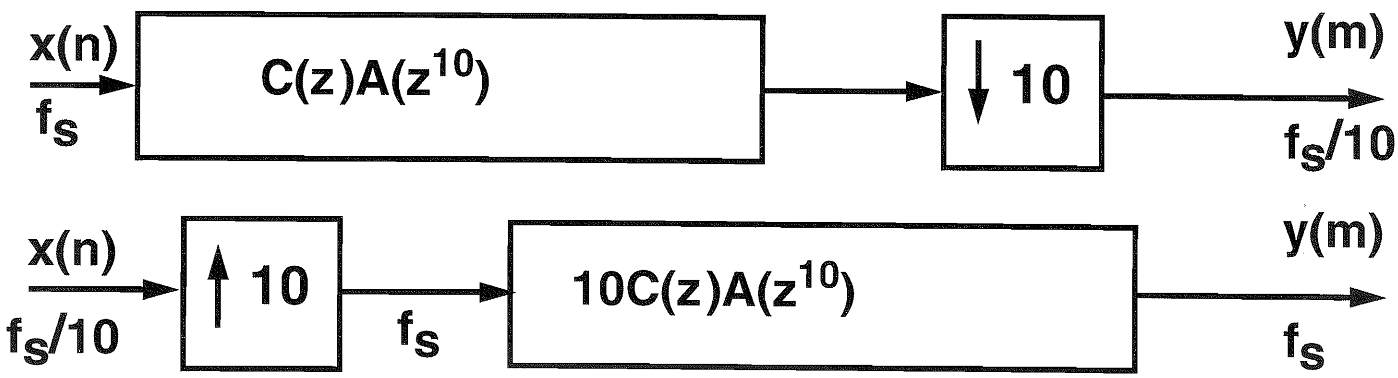


Optimized One-Stage Design of the Form $H(z) = C(z)A(z^{10})$

• The following figures give the implementation forms for both the decimator and intepolator as well as for the corresponding single-stage equivalents.



Single-Stage Equivalents

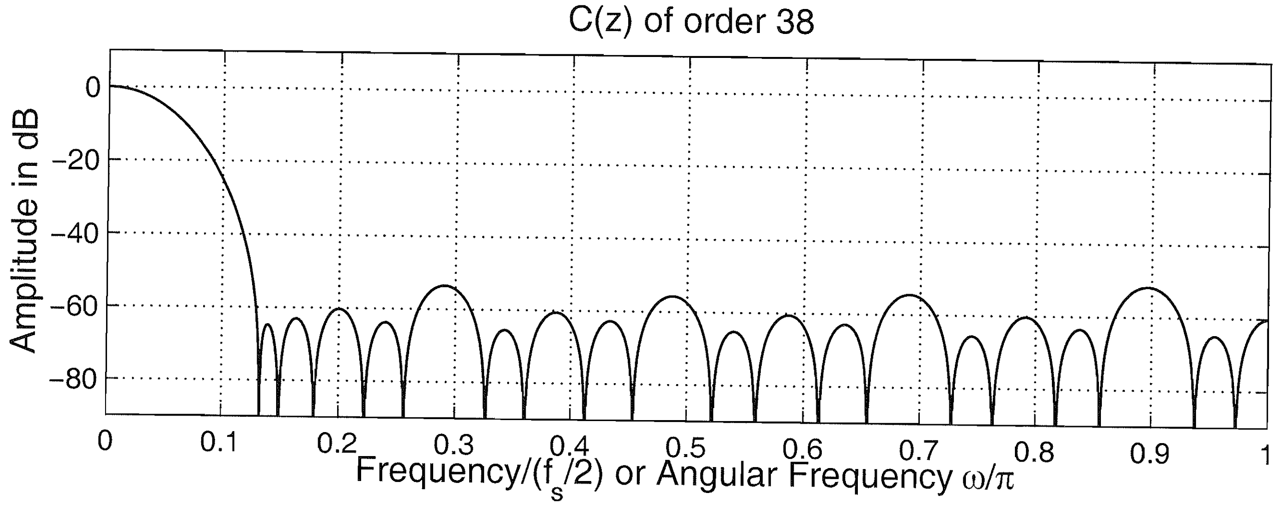


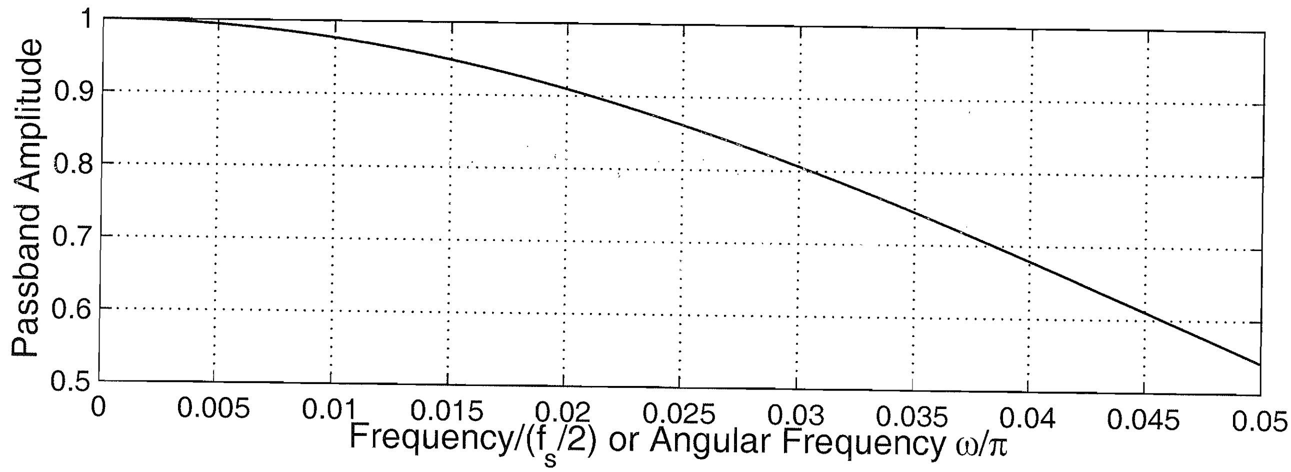
• The multiplication rate is minimized when $A(z)$ and $C(z)$ are of orders 13 and 38, respectively.

• When exploiting the coefficient symmetries and using direct-form structures, the overall design requires $7 + 10 = 27$ multipliers, $7/10 + 10/10 = 2.7$ multiplications per input sample, and 51 delay elements.

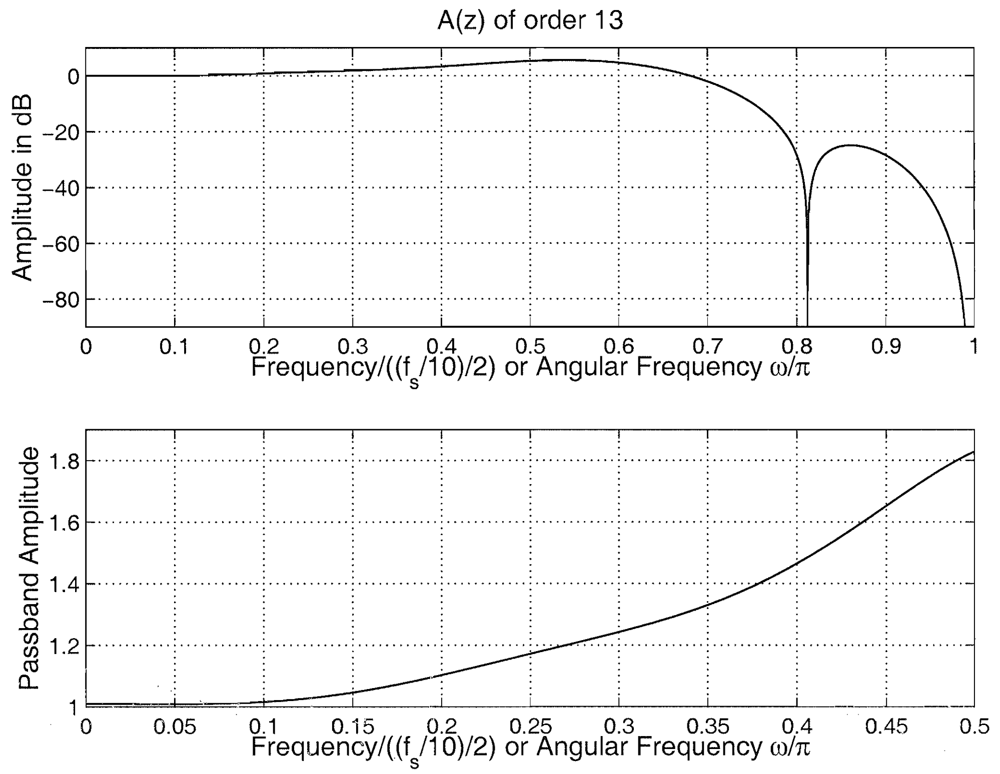
- $A(z^{10})$ takes care of the passband shaping as well as the stopband shaping on $[0.1\pi, 0.1245\pi]$ and on $[(2k+1)\pi/10 - 0.0245,\ (2k+1)\pi/10 + 0.0245]$ for $k = 1, 2, 3, 4$.
- $C(z)$ has all its zeros on the unit circle and takes care of the rest of the stopband region.
- Compared to the conventional one-stage decimator, the use of an additional filter at the output reduces the order of the filter before the decimation from 108 to 38.
- The order of the single-stage equivalent is $10 \cdot 13 + 38 = 168$.
- In the following, there are six pages illustrating the characteristics of the overall design.

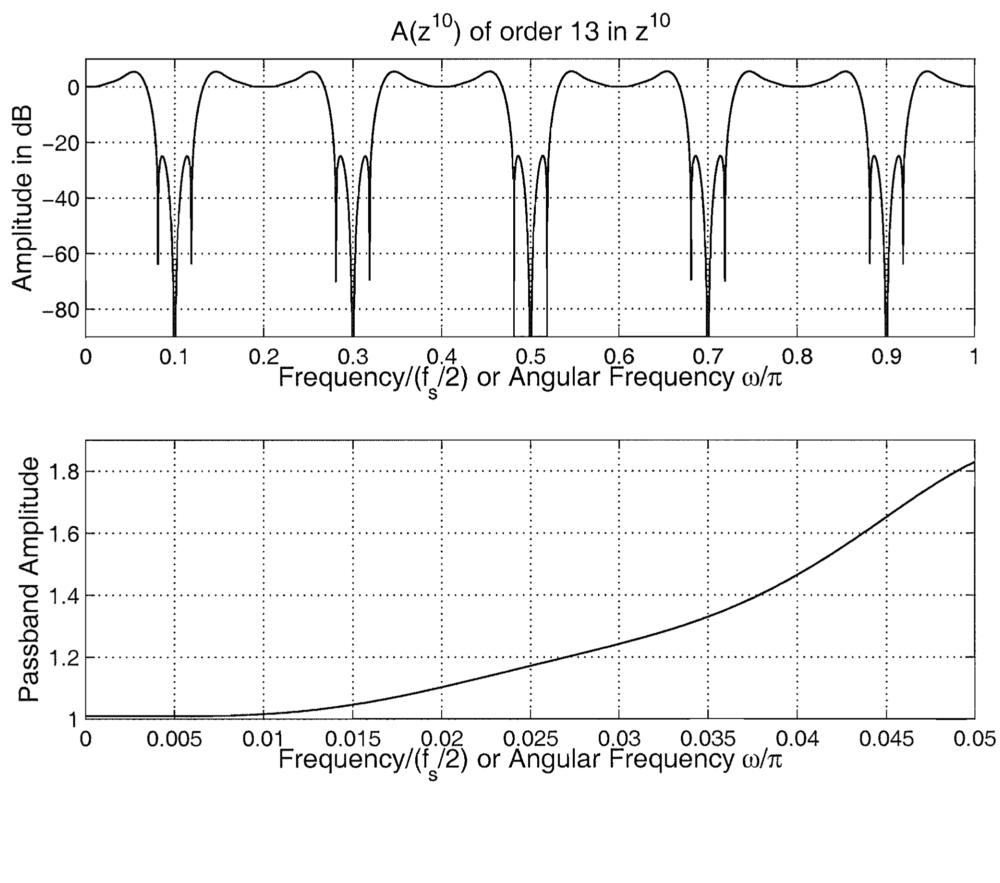
Characteristics of FIR Decimator of the Form $H(z) = C(z)A(z^{10})$



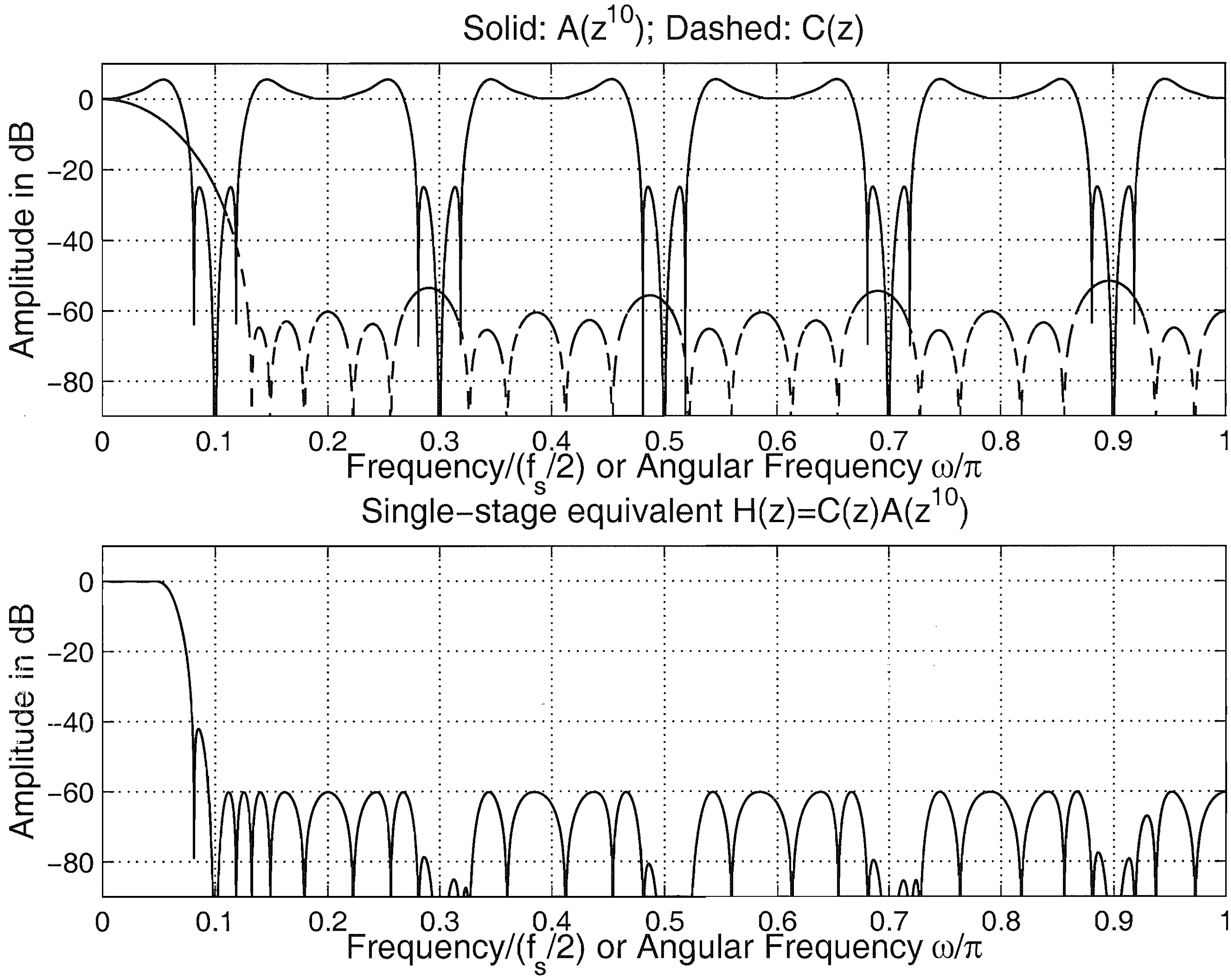


Characteristics of FIR Decimator of the Form $H(z) = C(z)A(z^{10})$

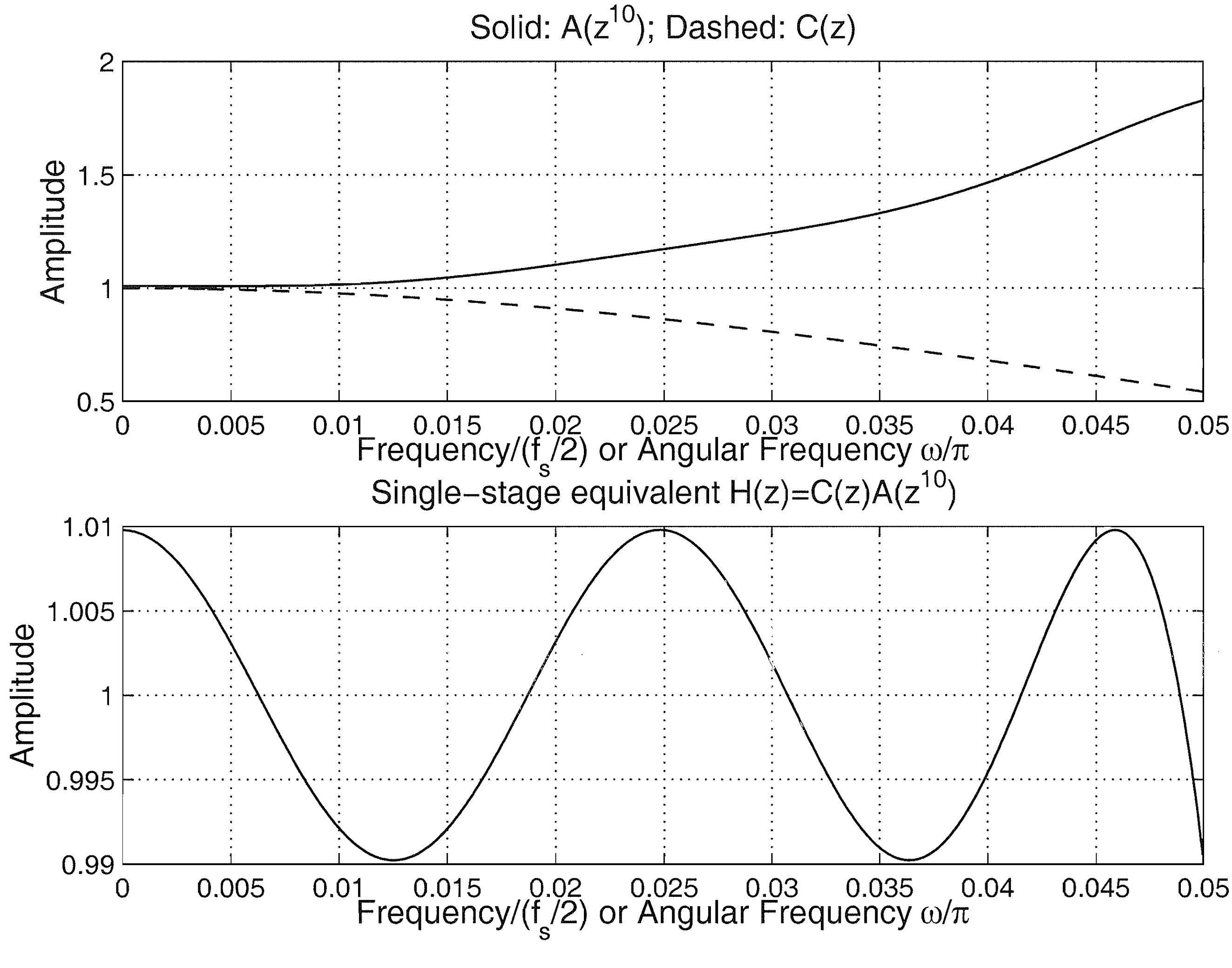




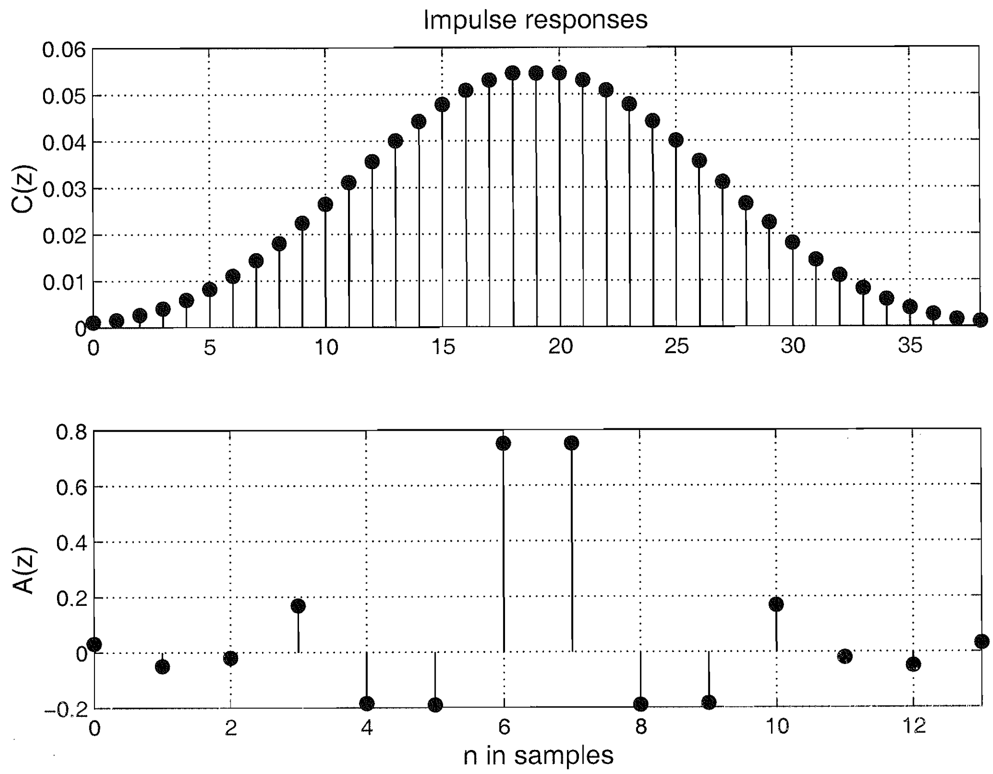
Characteristics of FIR Decimator of the Form $H(z) = C(z)A(z^{10})$

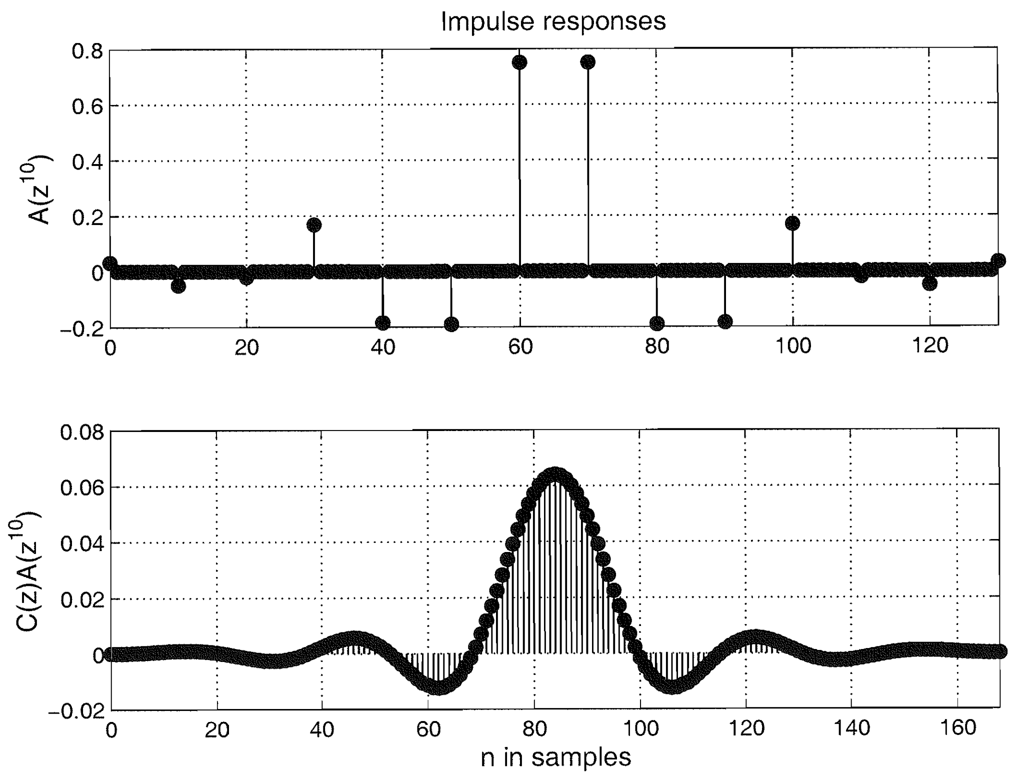


Characteristics of FIR Decimator of the Form $H(z) = C(z)A(z^{10})$

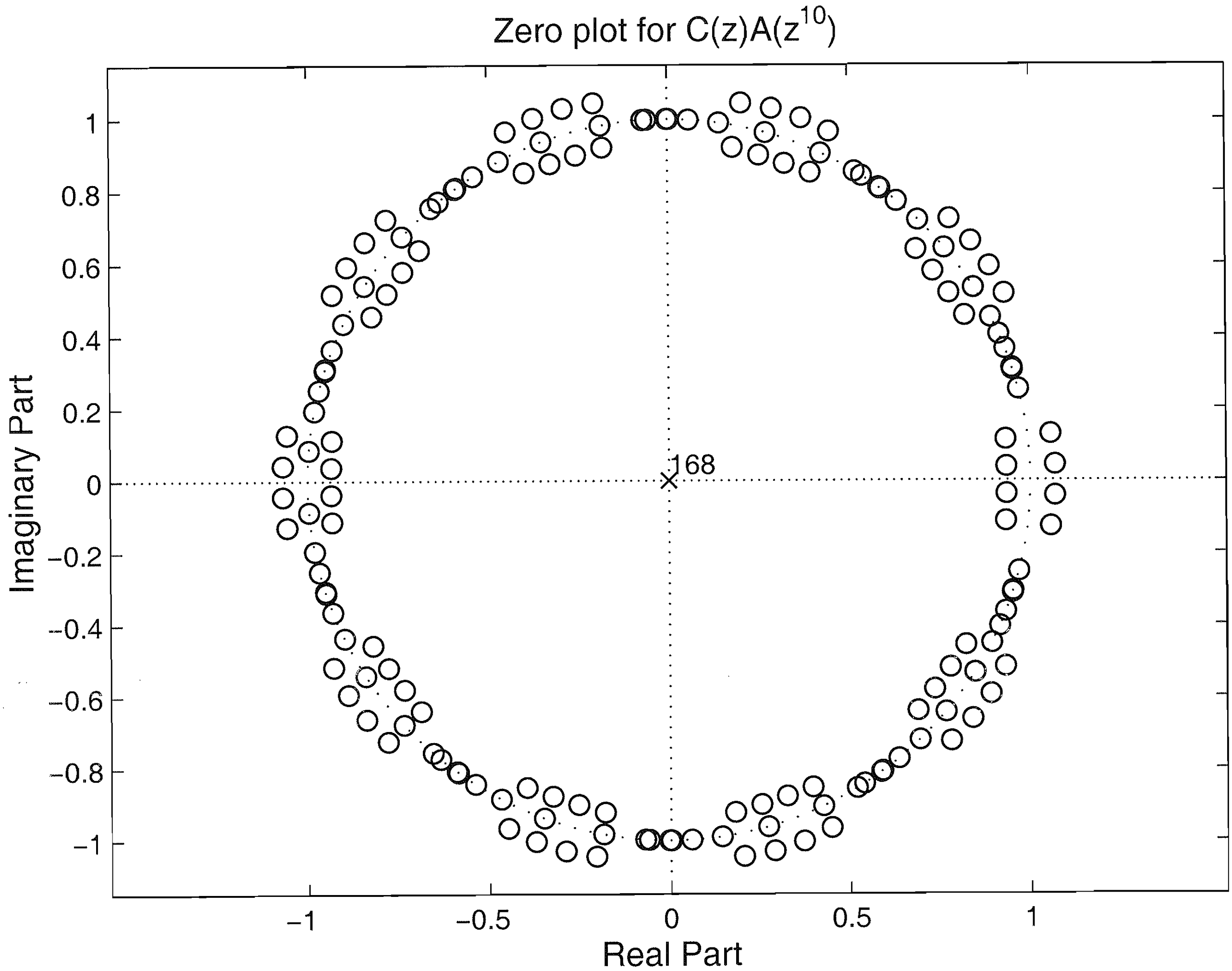


Characteristics of FIR Decimator of the Form $H(z) = C(z)A(z^{10})$



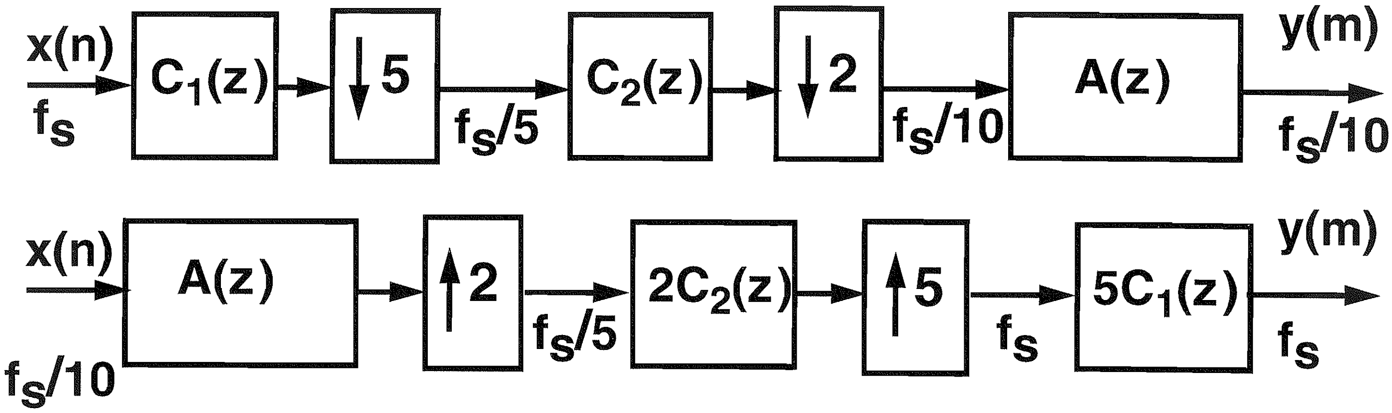


Characteristics of FIR Decimator of the Form $H(z) = C(z)A(z^{10})$

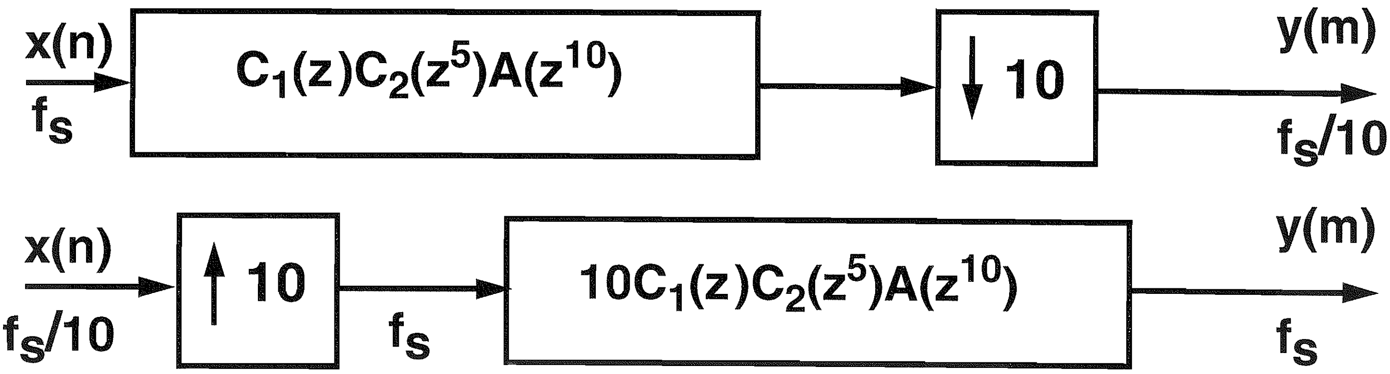


Optimized Two-Stage Design of the Form $H(z) = C_1(z)C_2(z^5)A(z^{10})$

• The following figures give the implementation forms for both the decimator and intepolator as well as for the corresponding single-stage equivalents.



Single-Stage Equivalents



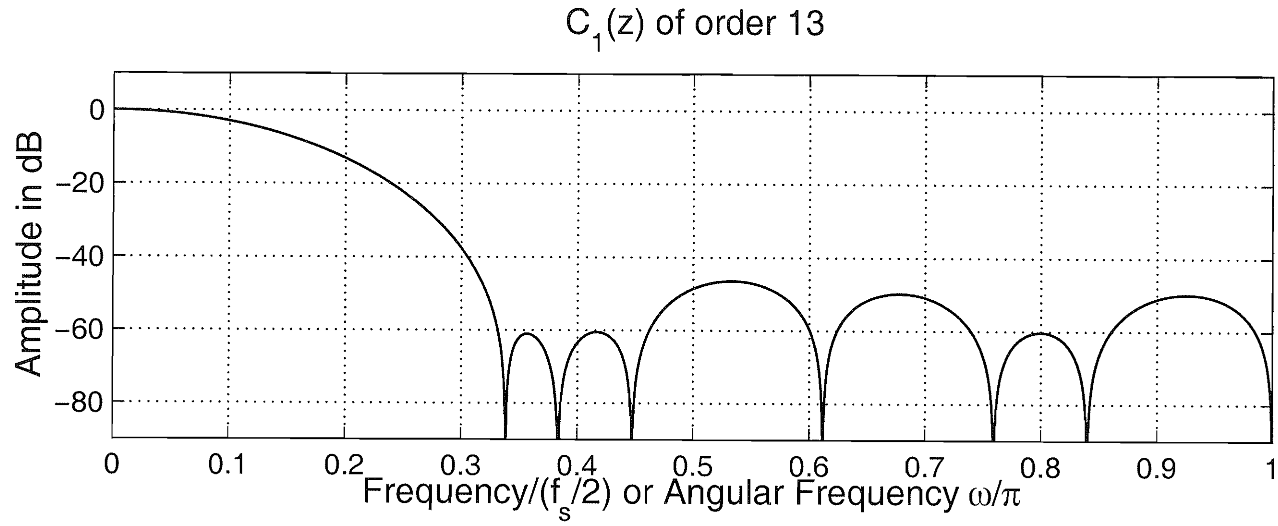
• The multiplication rate is minimized when $A(z)$, $C_2(z)$, and $C_1(z)$ are of orders 13, 9, and 7, respectively.

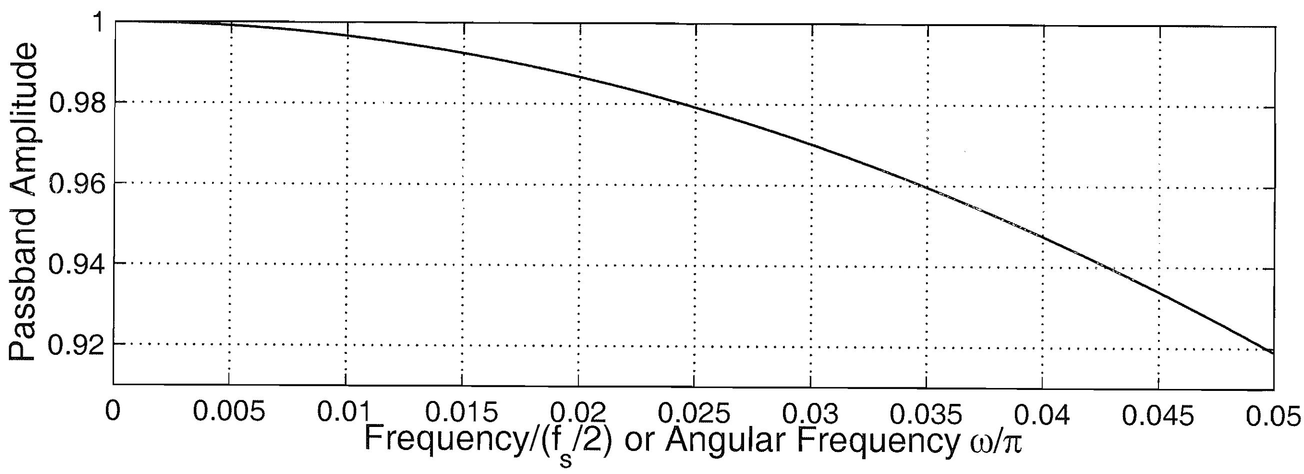
• When exploiting the coefficient symmetries and using direct-form structures, the overall design requires $4+5+7=16$ multipliers, $4/5+5/10+7/5=2.3$ multiplications per input sample, and 29 delay elements.

• The order of the single-stage equivalent is 128.

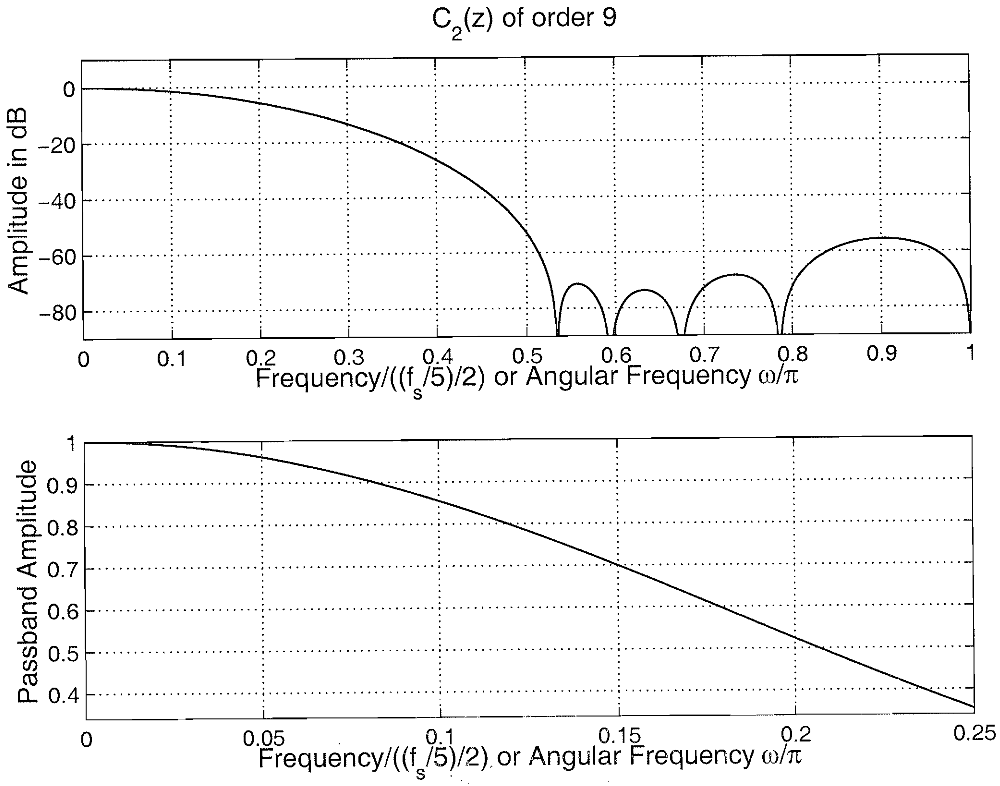
• In the following, there are seven pages illustrating the characteristics of the overall design.

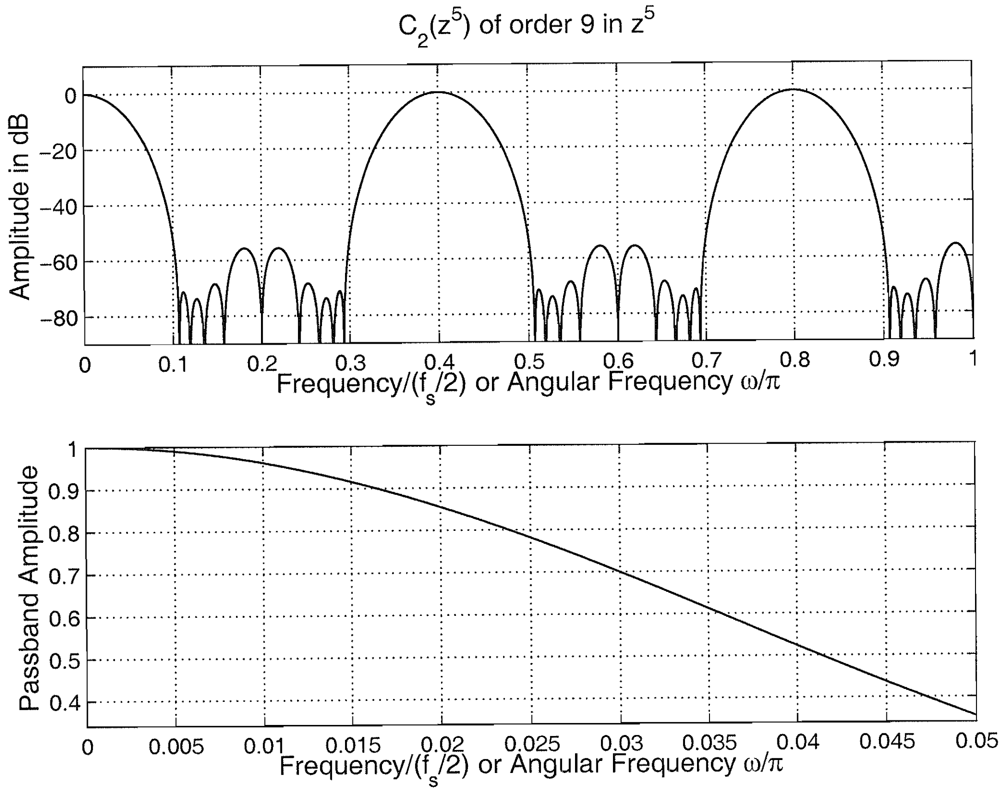
Characteristics of FIR Decimator of the Form $H(z) = C_1(z)C_2(z^5)A(z^{10})$



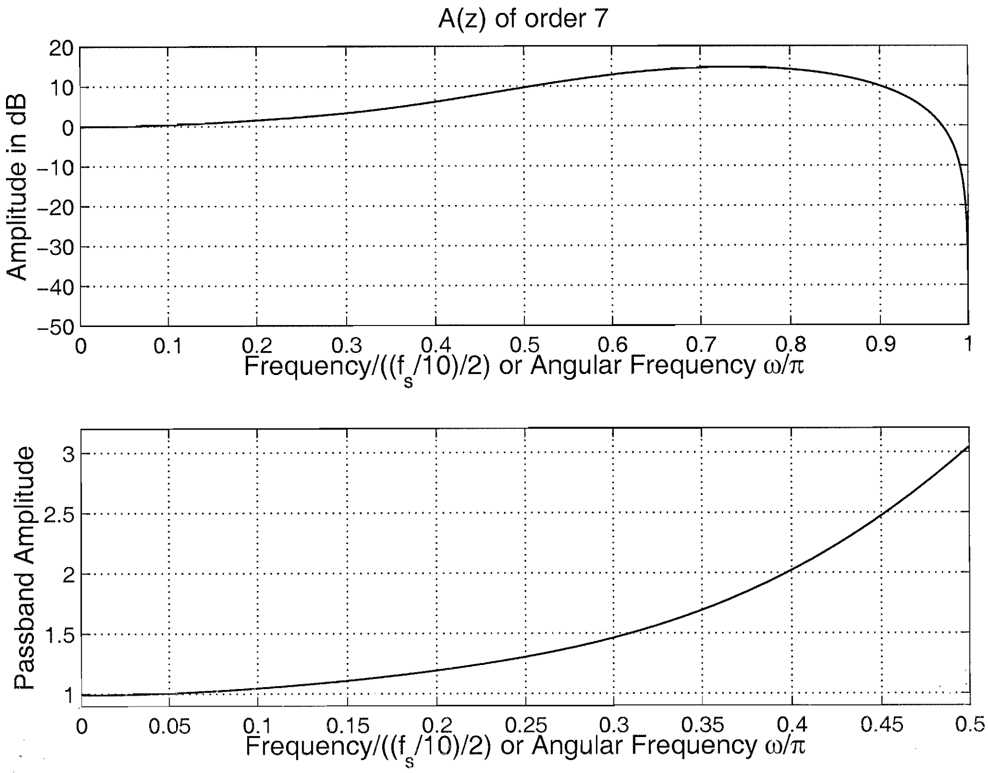


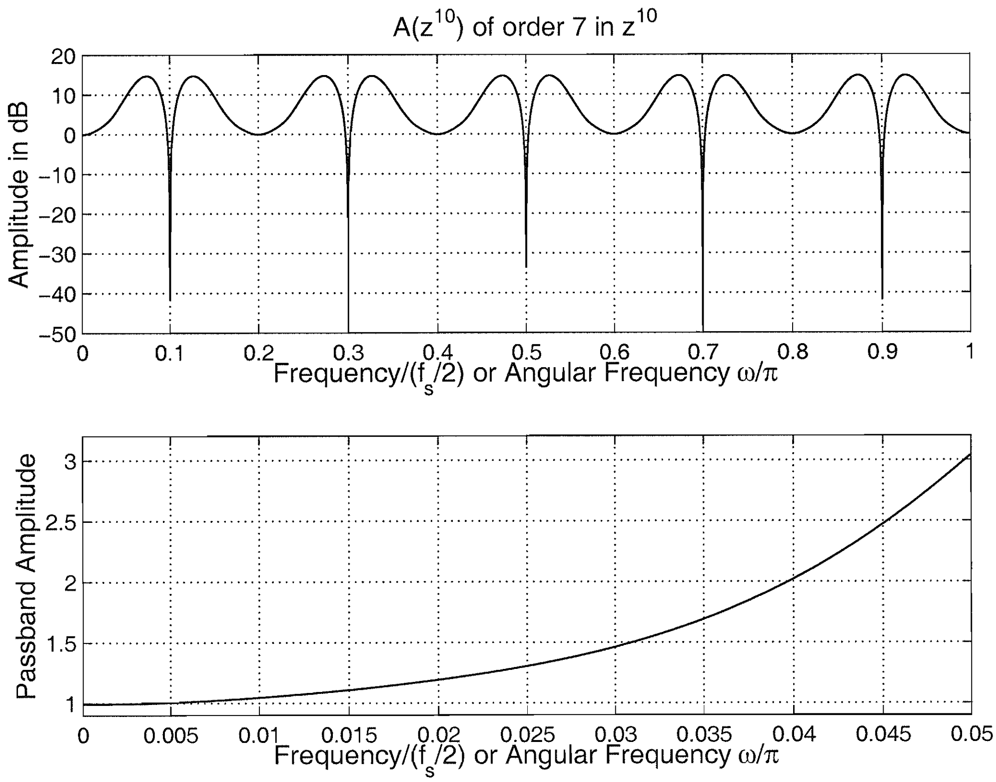
Characteristics of FIR Decimator of the Form $H(z) = C_1(z)C_2(z^5)A(z^{10})$



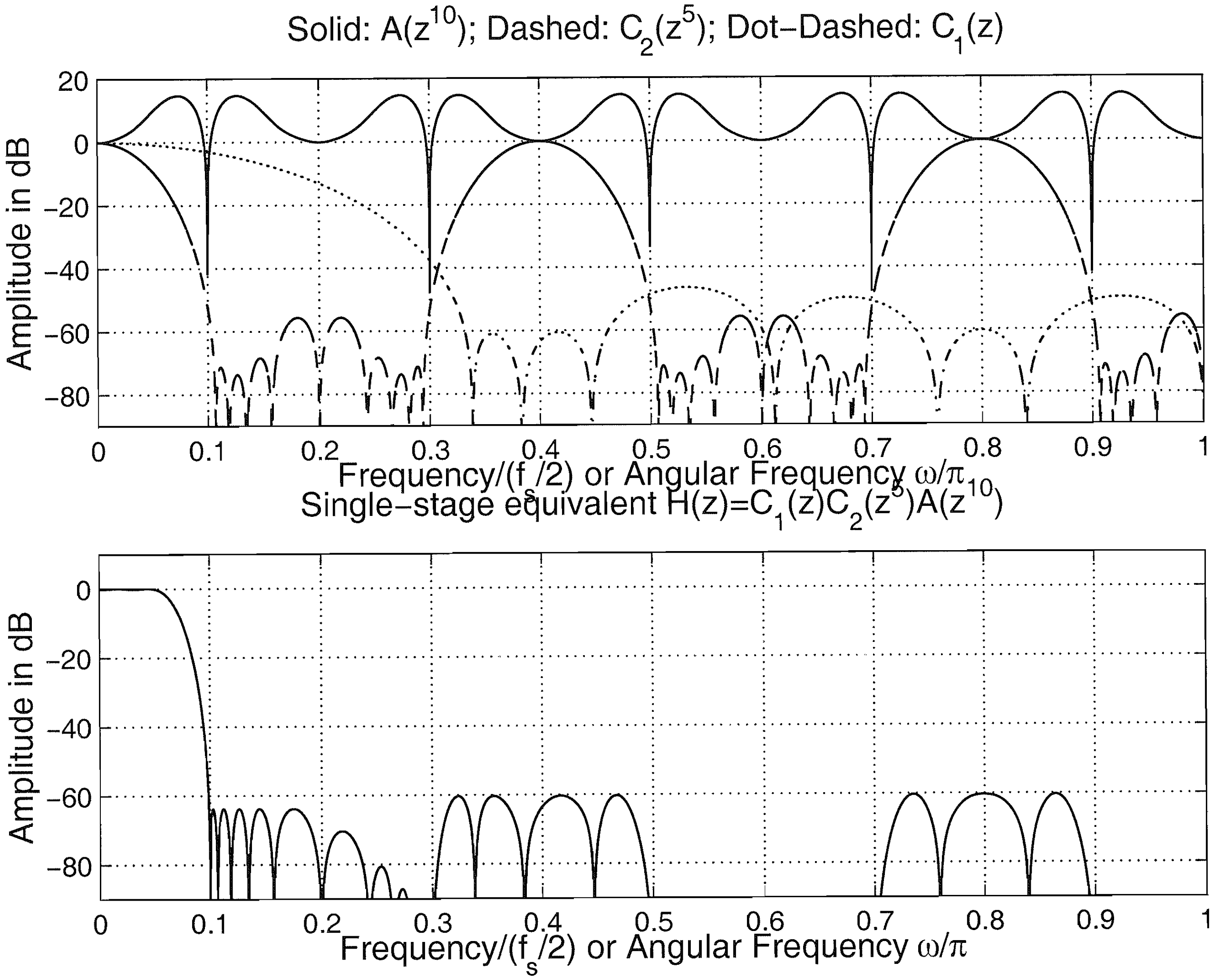


Characteristics of FIR Decimator of the Form $H(z) = C_1(z)C_2(z^5)A(z^{10})$

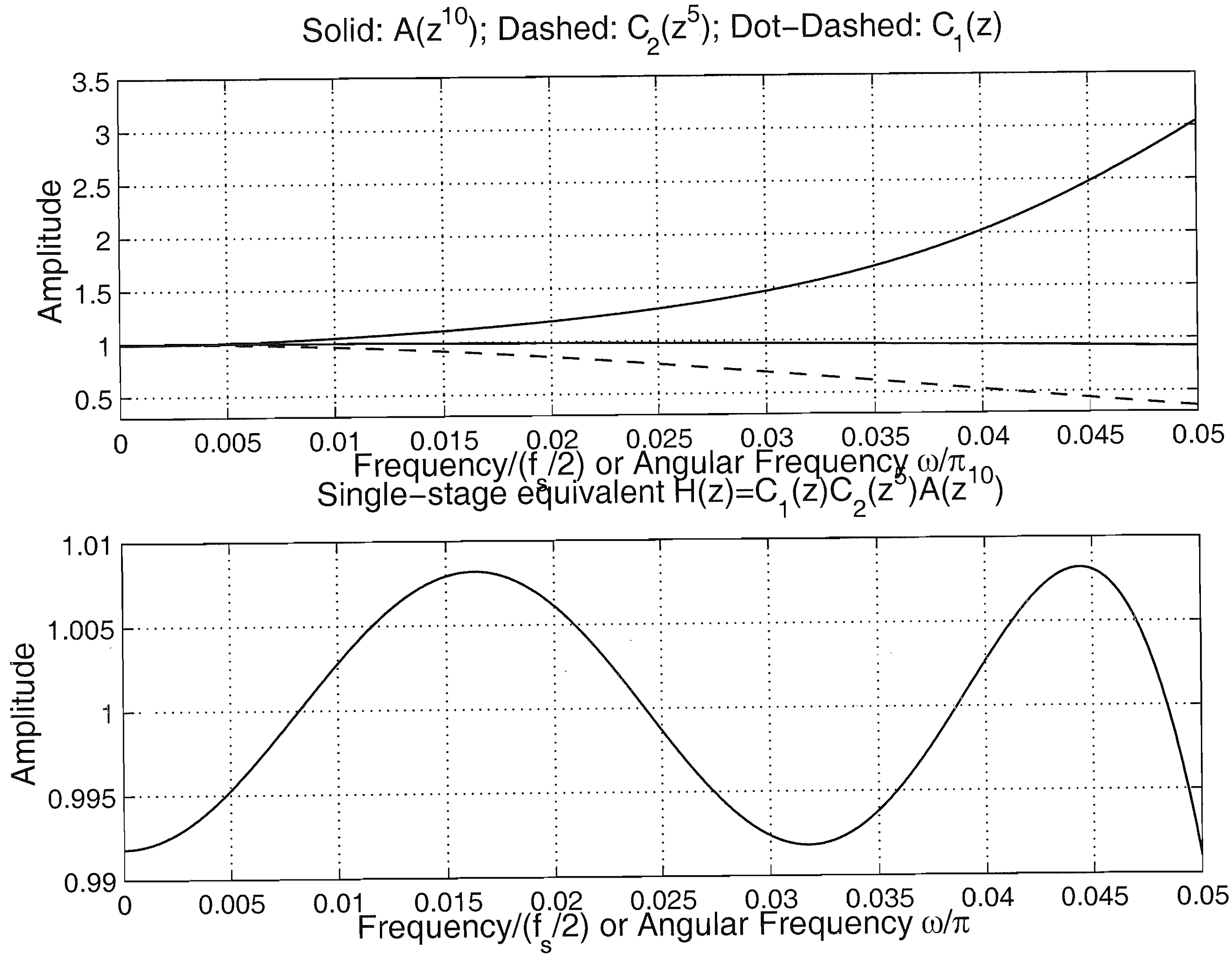




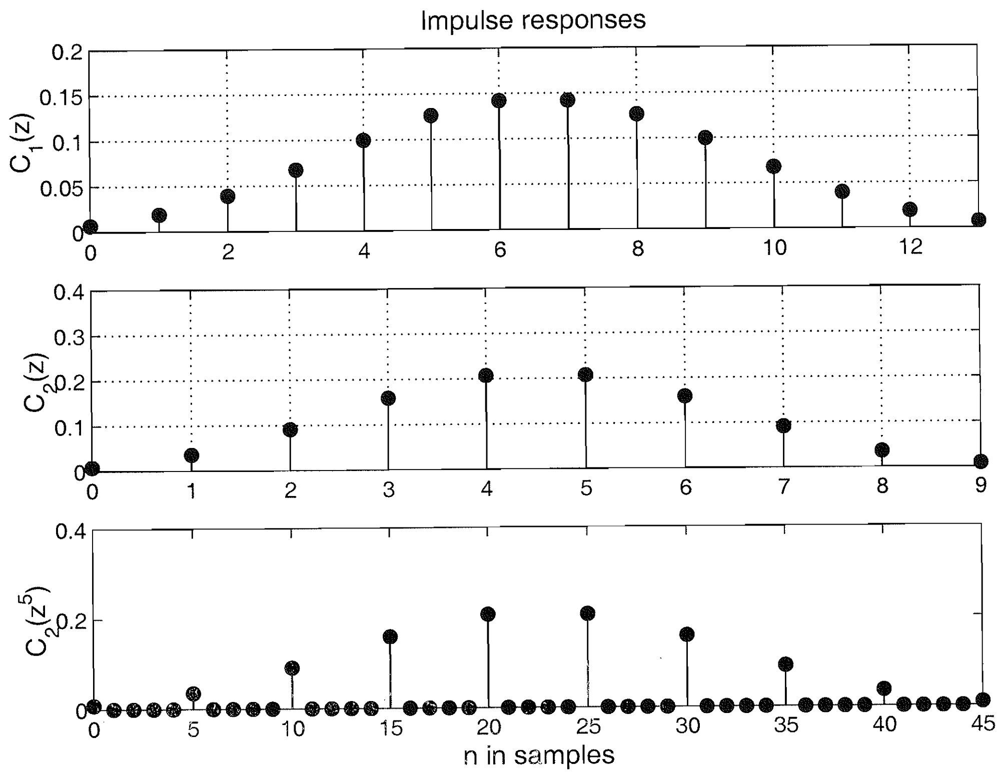
Characteristics of FIR Decimator of the Form $H(z) = C_1(z)C_2(z^5)A(z^{10})$

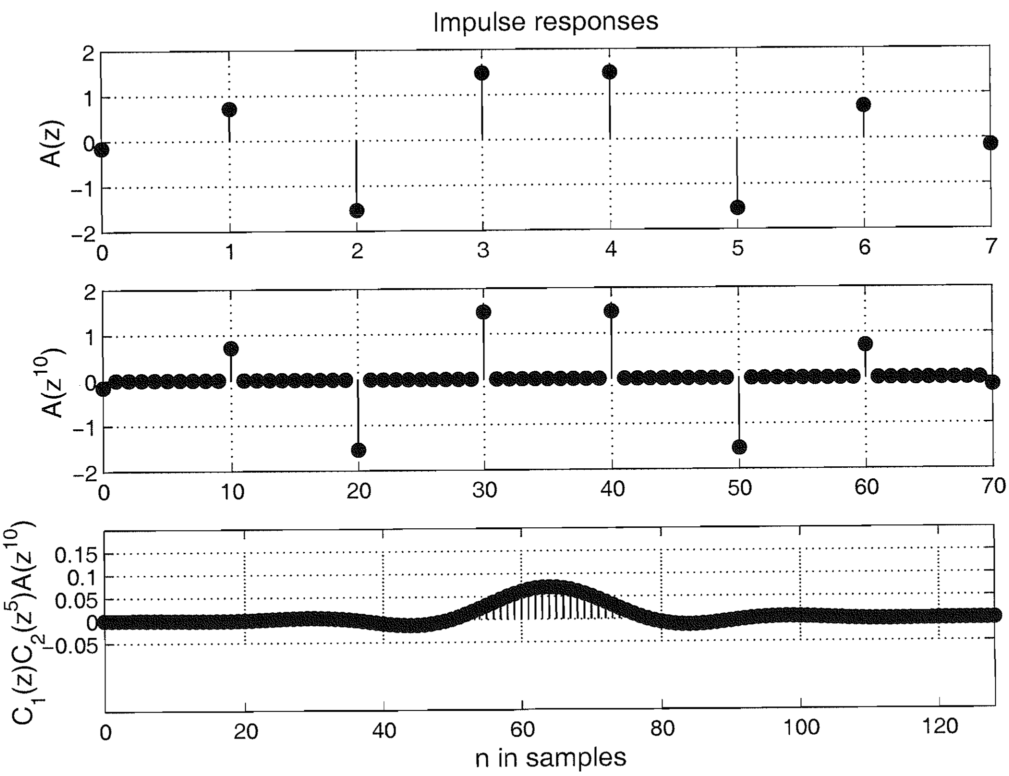


Characteristics of FIR Decimator of the Form $H(z) = C_1(z)C_2(z^5)A(z^{10})$

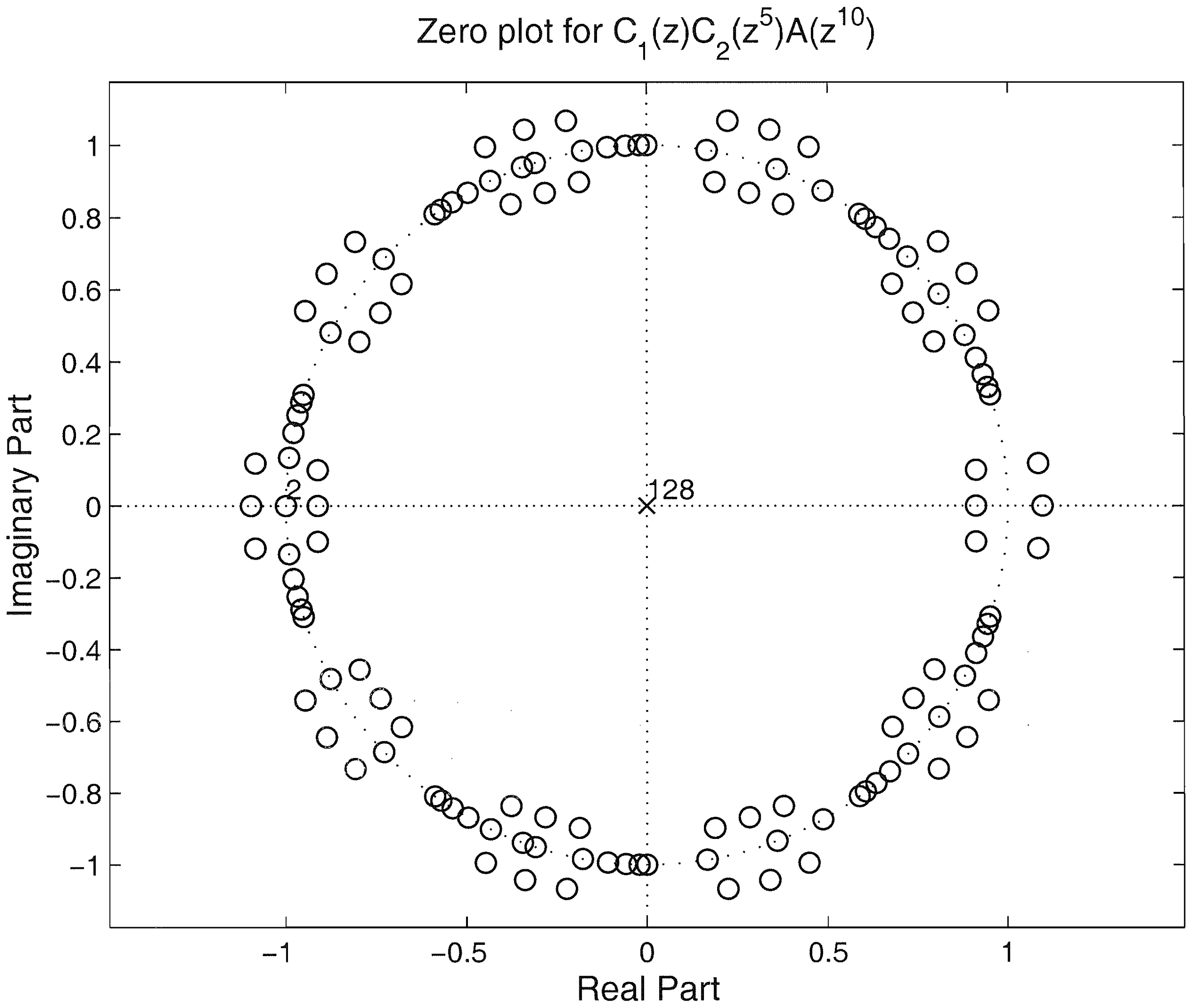


Characteristics of FIR Decimator of the Form $H(z) = C_1(z)C_2(z^5)A(z^{10})$



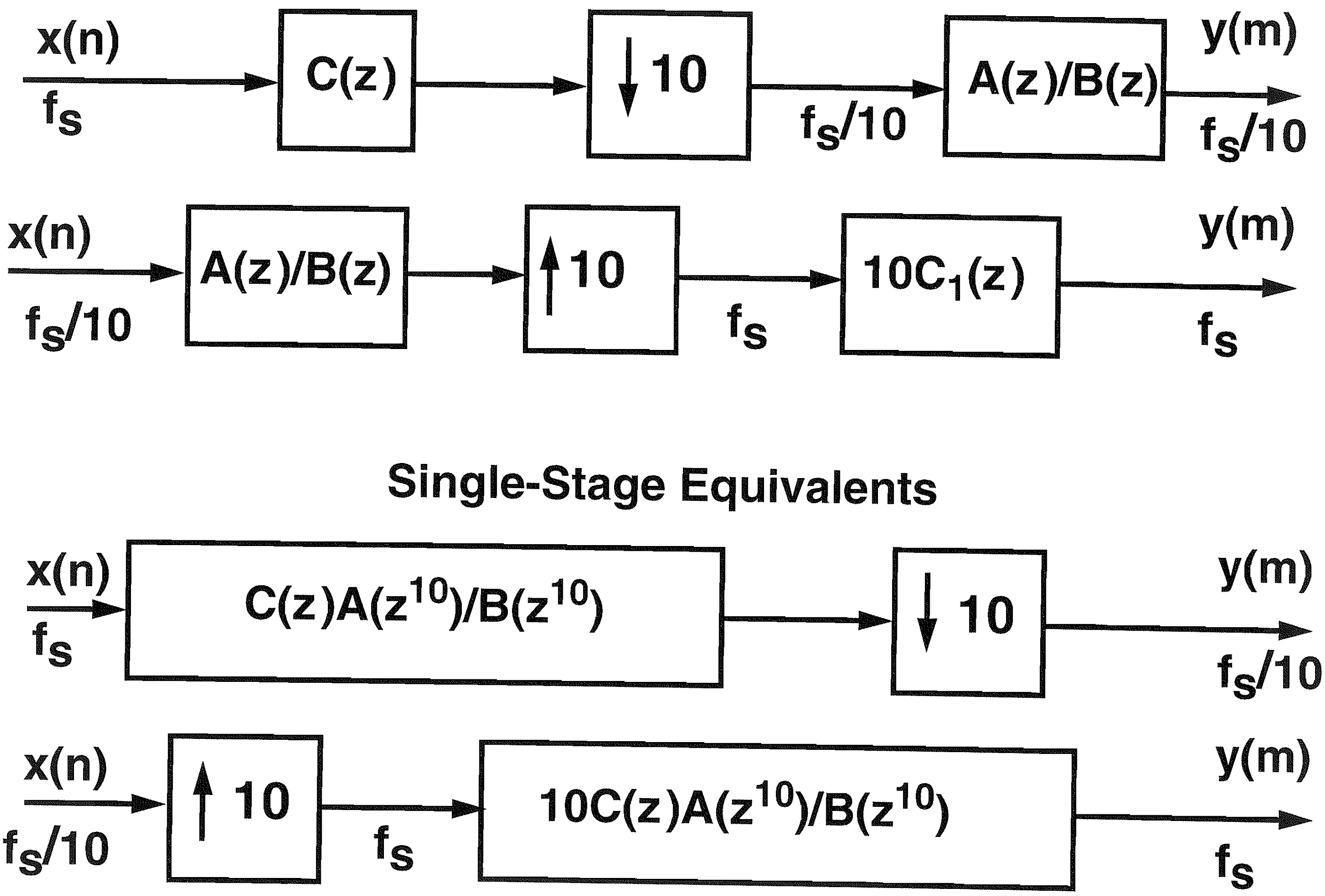


Characteristics of FIR Decimator of the Form $H(z) = C_1(z)C_2(z^5)A(z^{10})$



Optimized One-Stage IIR Design of the Form $H(z) = C(z)A(z^{10})/B(z^{10})$

- The following figures give the implementation forms for both the decimator and intepolator as well as for the corresponding single-stage equivalents.



- The multiplication rate is minimized when $A(z)$, $B(z)$, and $C(z)$ are of orders 4, 2, and 36, respectively.
- $A(z)$ consisits of the following two parts:
- A third-order linear-phase FIR filter part having all the zeros on the unit circle.

• One zero inside (or if desired, outside) the unit circle.

• It should be pointed out that if $A(z)$ does not have a single zero outside the unit circle, then the order of $B(z)$ should be four.

• When exploiting the coefficient symmetries when inplementing $C(z)$ and the linear phase part of $A(z)$ (the outside the unit-circle-zero of $A(z)$ is implemented separately), the overall design requires $19+2+2+2=25$ multipliers, $25/10=2.5$ multiplications per input sample, and 40 delay elements.

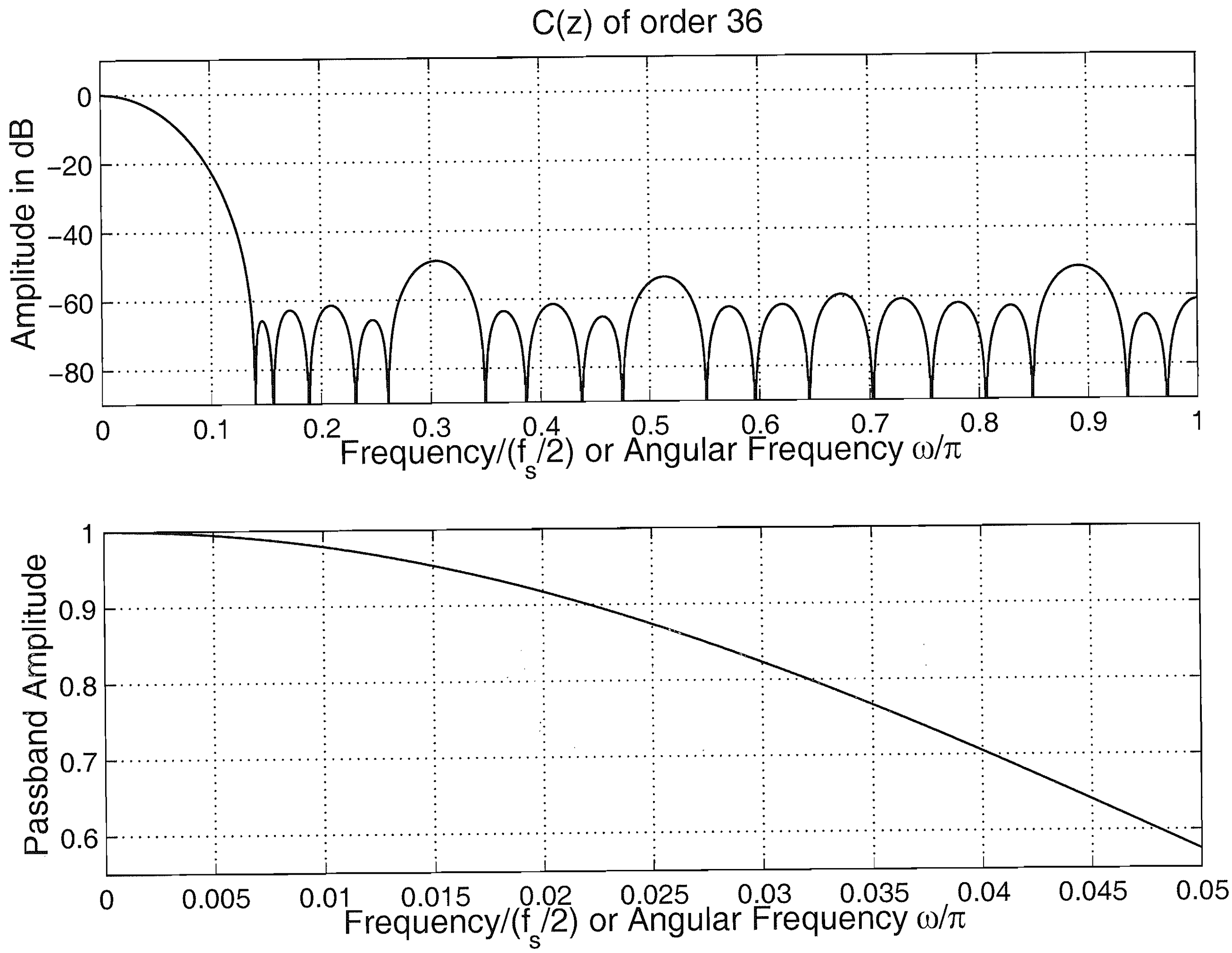
• $A(z^{10})/B(z^{10})$ takes care of the passband shaping as well as the stopband shaping on $[0.1\pi, 0.13\pi]$ and on $[(2k+1)\pi/10-0.03,\ (2k+1)\pi/10+0.03]$ for $k=1,2,3,4$.

• $C(z)$ has all its zeros on the unit circle and takes care of the rest of the stopband region.

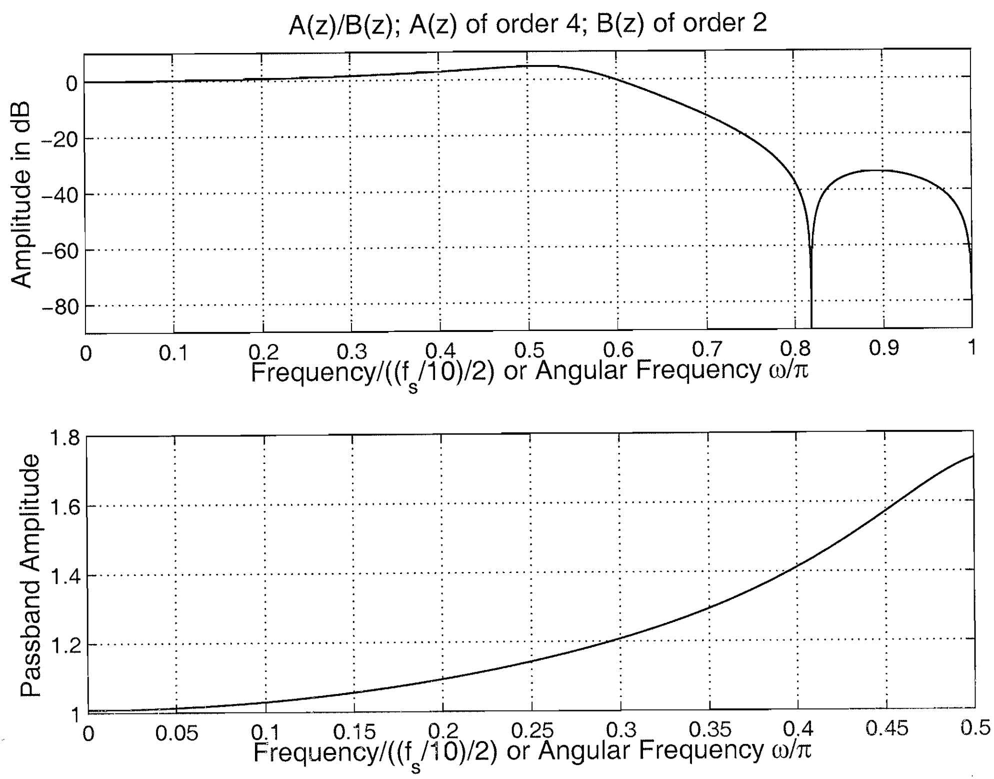
• Compared to the conventional one-stage FIR decimator, the use of an additional filter at the output reduces the order of the filter before the decimation from 108 to 36.

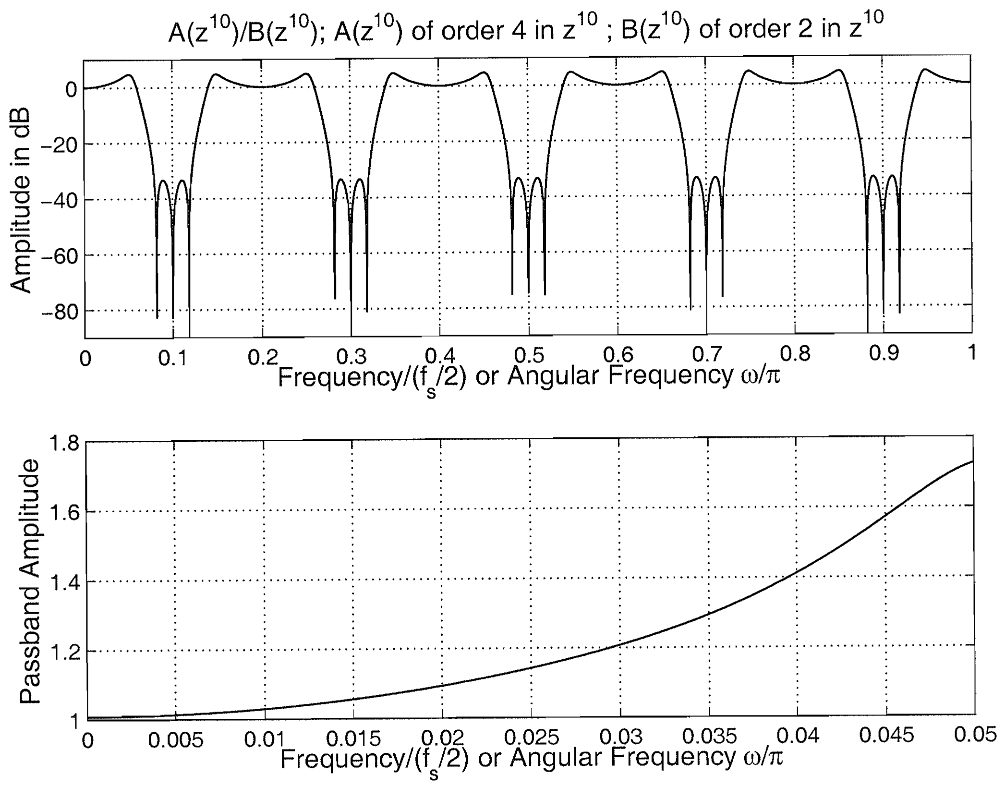
• In the following, there are seven pages illustrating the characteristics of the overall design.

Characteristics of IIR Decimator of the Form $H(z) = C(z)A(z^{10})/B(z^{10})$

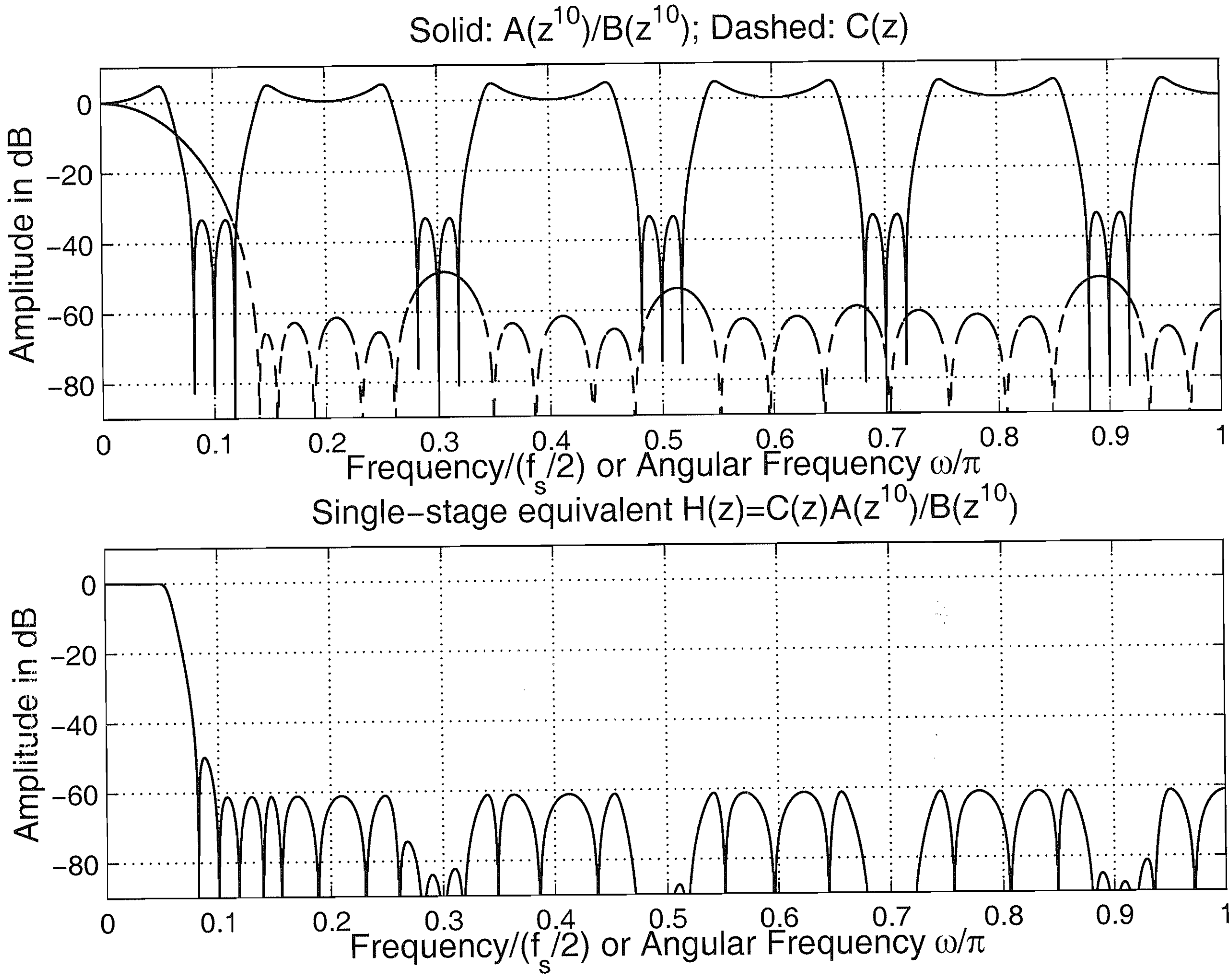


Characteristics of IIR Decimator of the Form $H(z) = C(z)A(z^{10})/B(z^{10})$

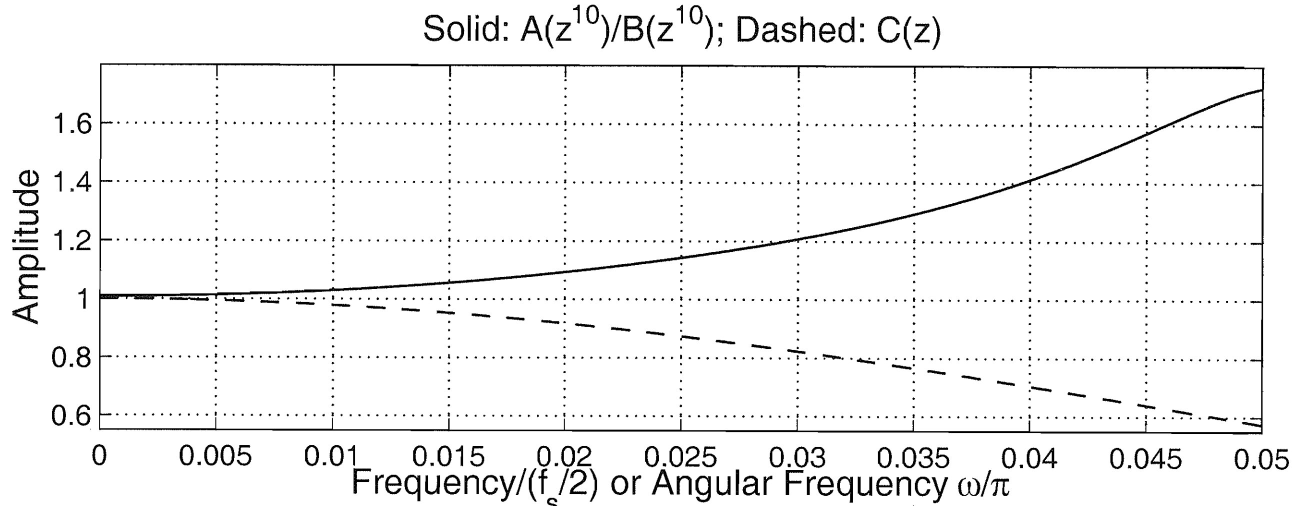


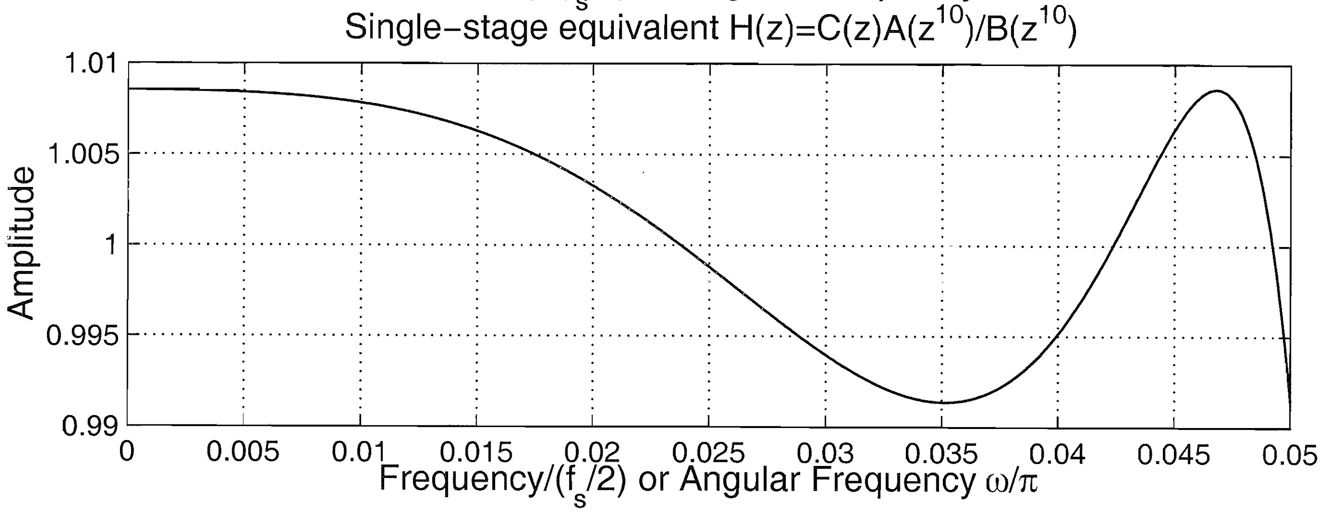


Characteristics of IIR Decimator of the Form $H(z) = C(z)A(z^{10})/B(z^{10})$

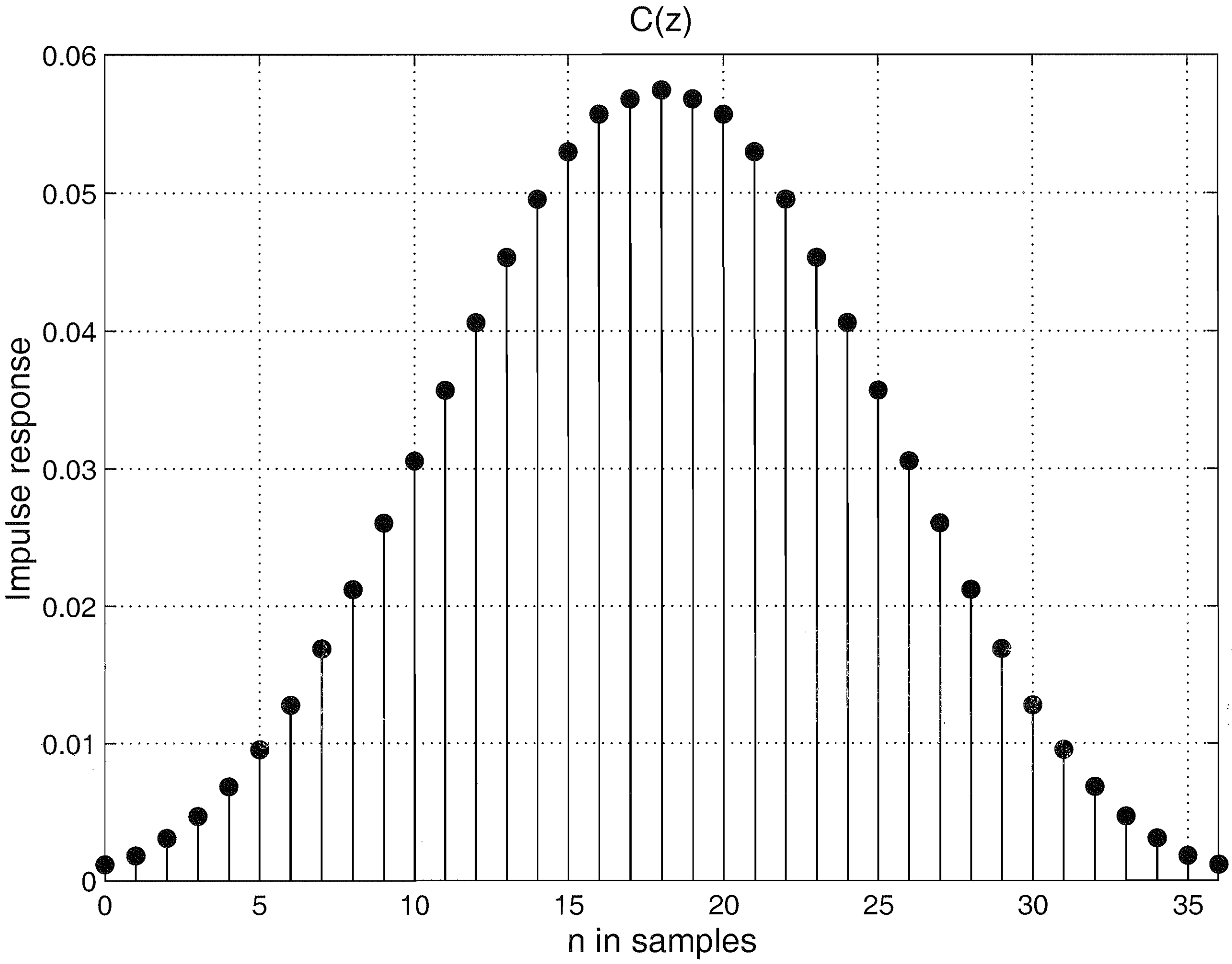


Characteristics of IIR Decimator of the Form $H(z) = C(z)A(z^{10})/B(z^{10})$

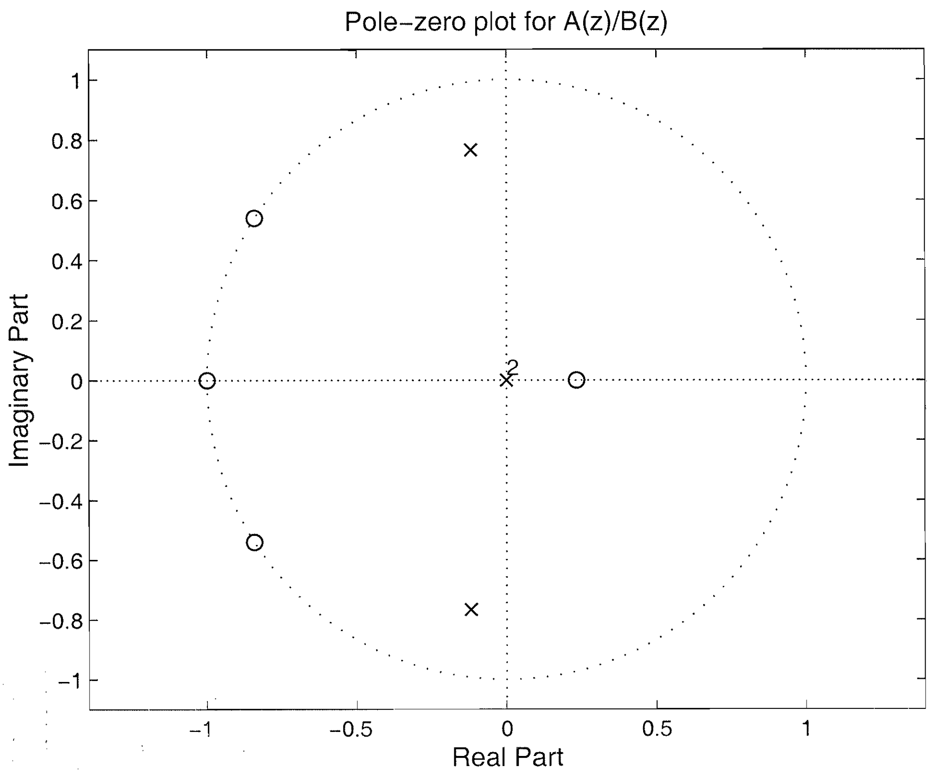


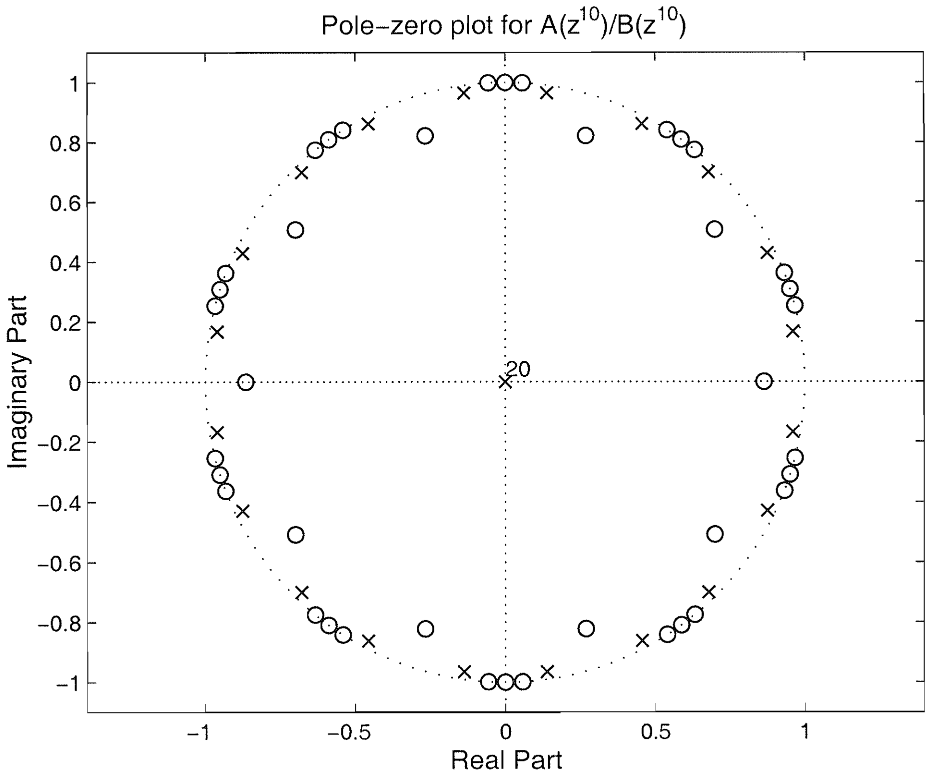


Characteristics of IIR Decimator of the Form $H(z) = C(z)A(z^{10})/B(z^{10})$

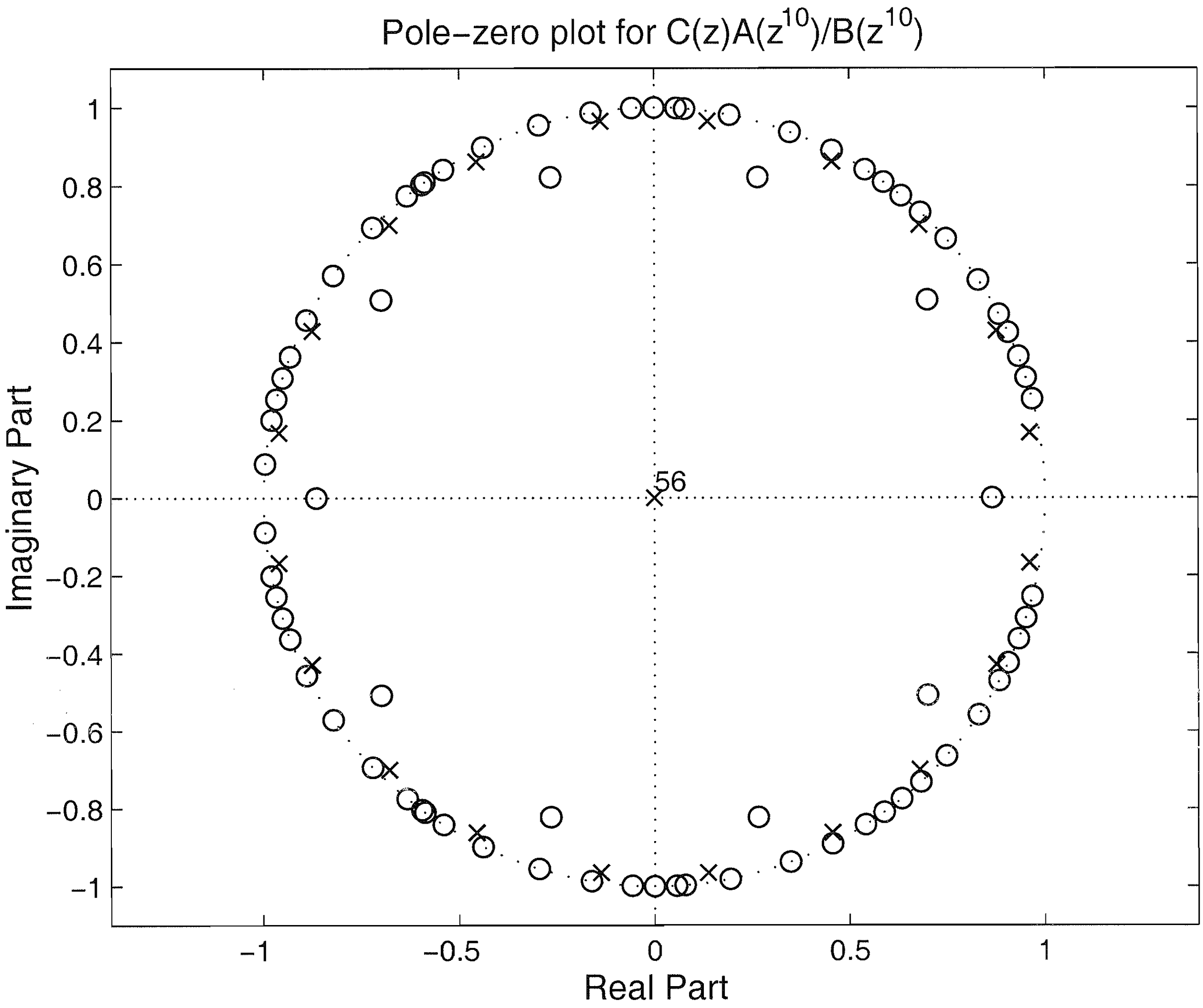


Characteristics of IIR Decimator of the Form $H(z) = C(z)A(z^{10})/B(z^{10})$



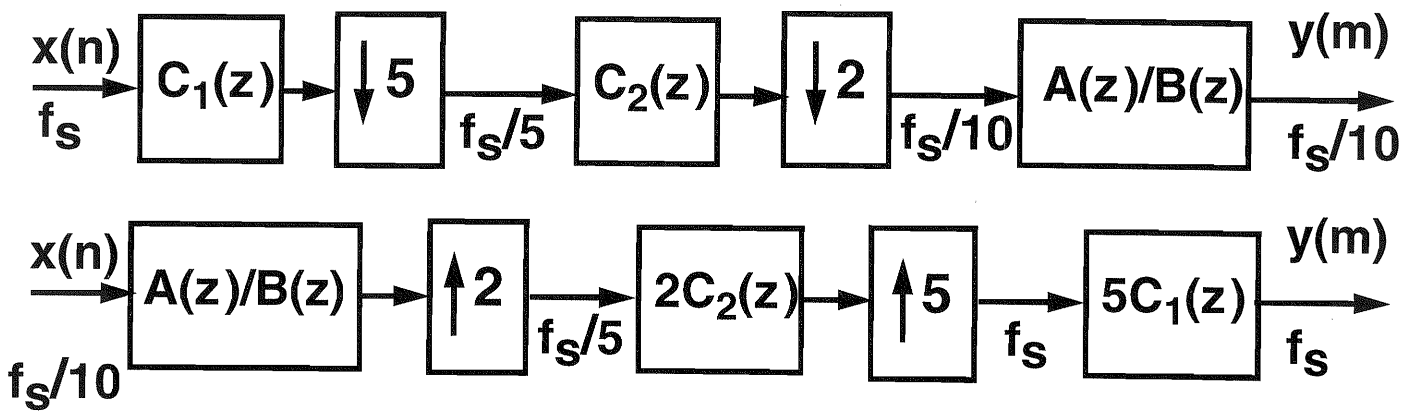


Characteristics of IIR Decimator of the Form $H(z) = C(z)A(z^{10})/B(z^{10})$

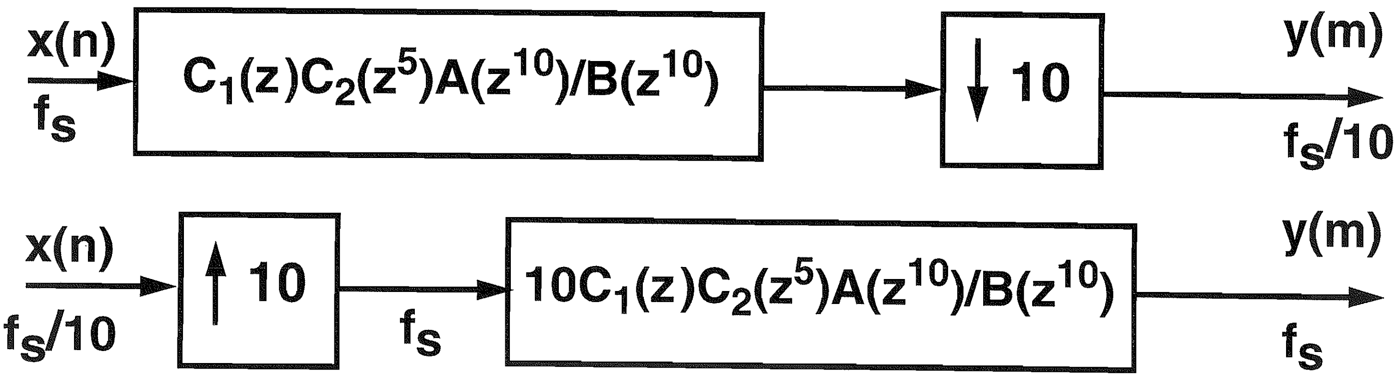


Optimized Two-Stage IIR Design of the Form $H(z) = C_1(z)C_2(z^5)A(z^{10})/B(z^{10})$

- The following figures give the implementation forms for both the decimator and intepolator as well as for the corresponding single-stage equivalents.



Single-Stage Equivalents

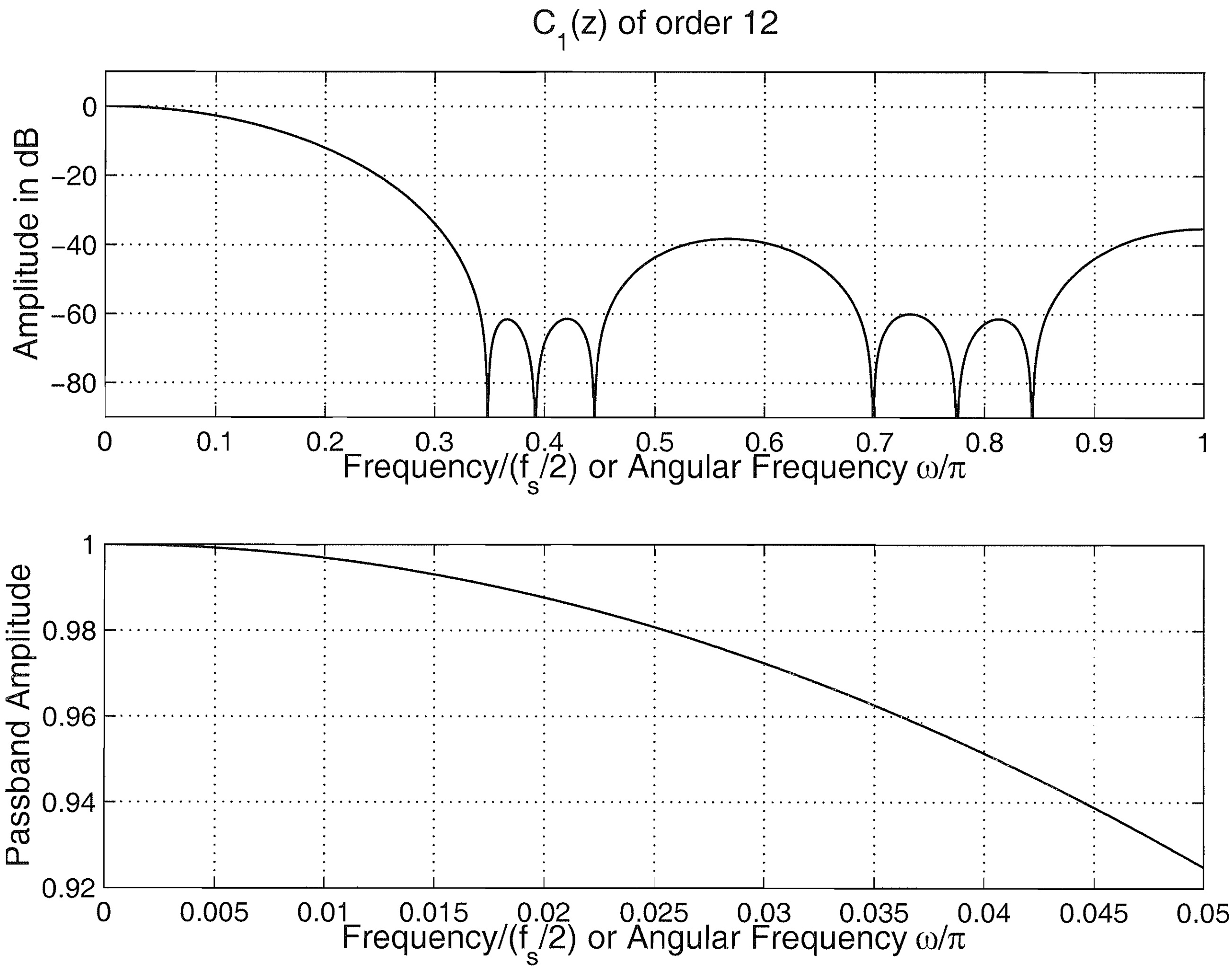


- The multiplication rate is minimized when $A(z)$, $B(z)$, $C_2(z)$, and $C_1(z)$ are of orders 3, 2, 6 and 12, respectively.
- $A(z)$ consisits of the following two parts:
- A second-order linear-phase FIR filter part having all

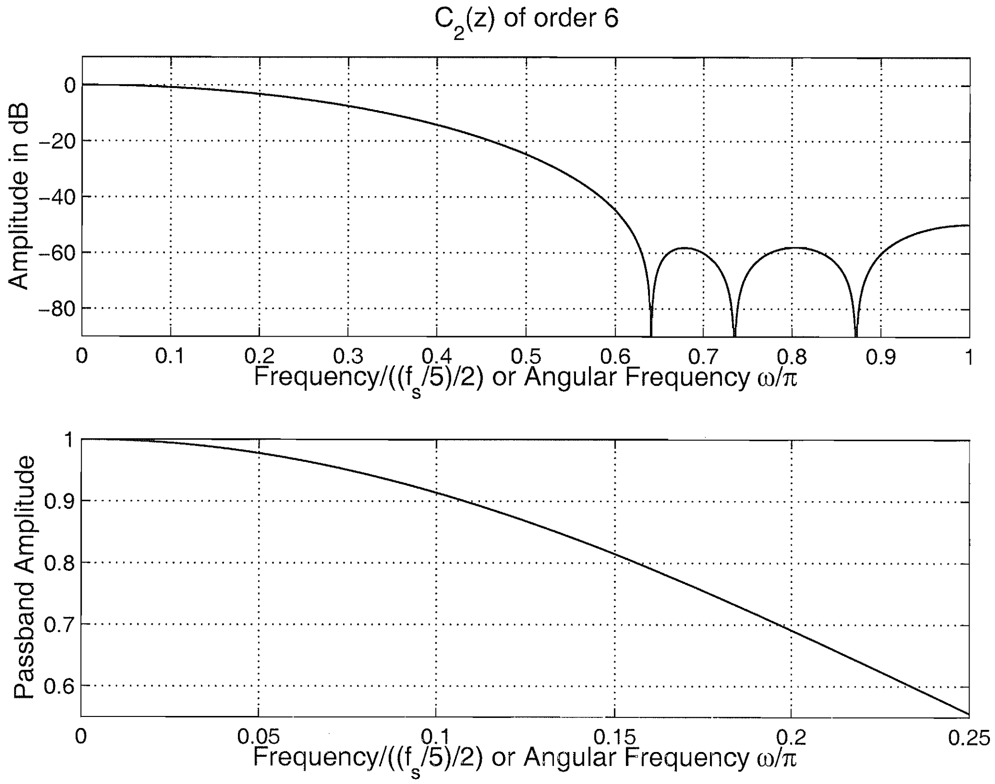
the zeros on the unit circle.

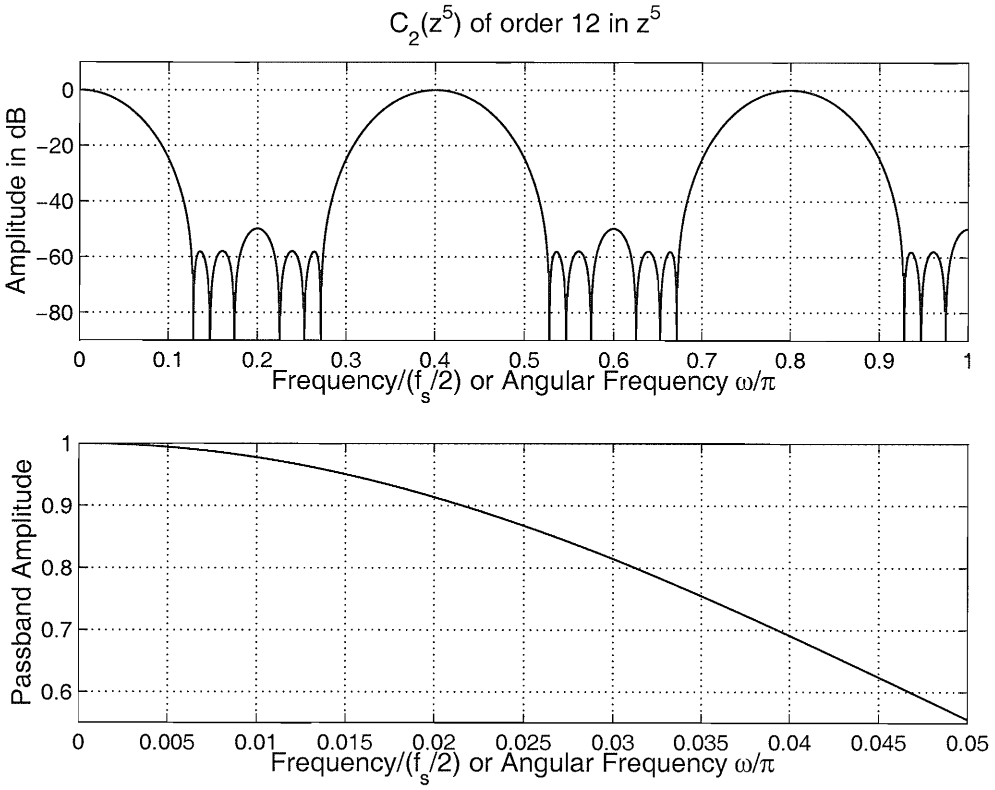
- One zero inside (or if desired, outside) the unit circle.
- When exploiting the coefficient symmetries when inplementing $C_1(z)$, $C_2(z)$, and the linear phase part of $A(z)$ (the outside the unit-circle-zero of $A(z)$ is implemented separately), the overall design requires $7+4+2+2+2=17$ multipliers, $7/5+10/10=2.4$ multiplications per input sample, and 21 delay elements.
- In the following, there are seven pages illustrating the characteristics of the overall design.

Characteristics of IIR Decimator of the Form $H(z) = C_1(z)C_2(z^5)A(z^{10})/B(z^{10})$

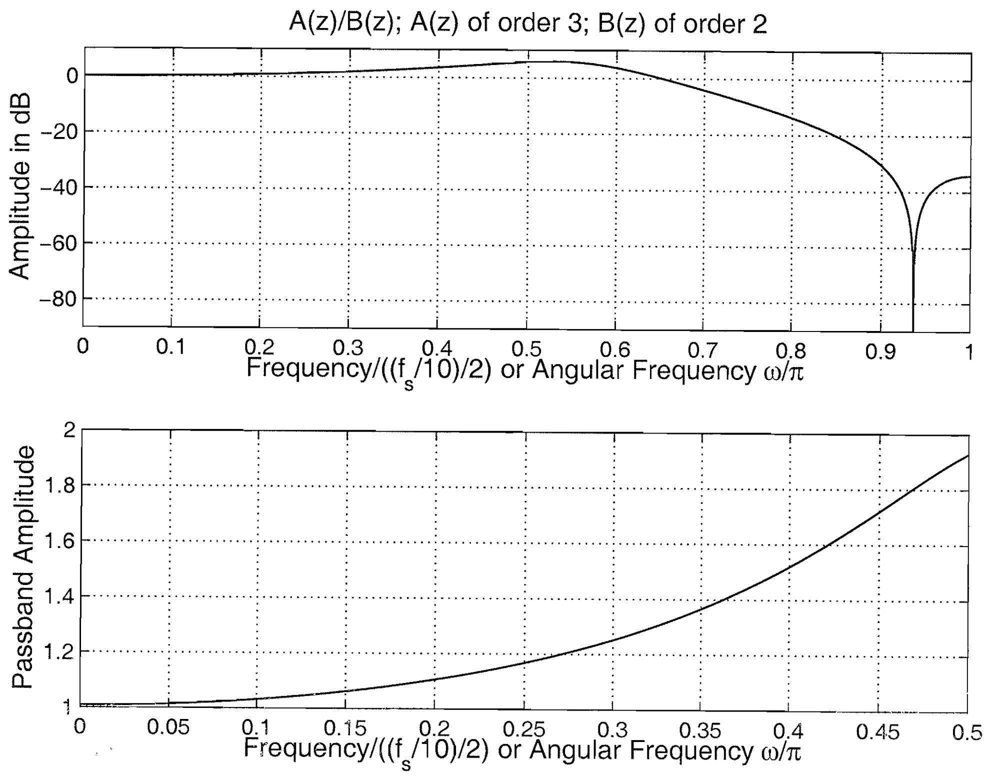


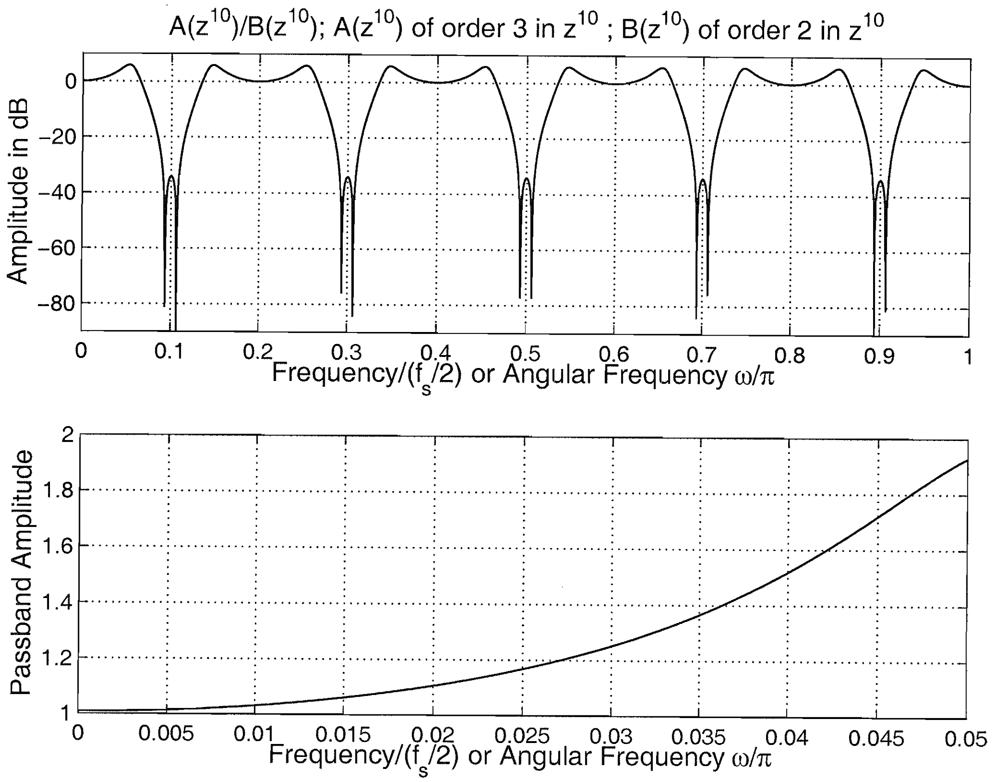
Characteristics of IIR Decimator of the Form $H(z) = C_1(z)C_2(z^5)A(z^{10})/B(z^{10})$



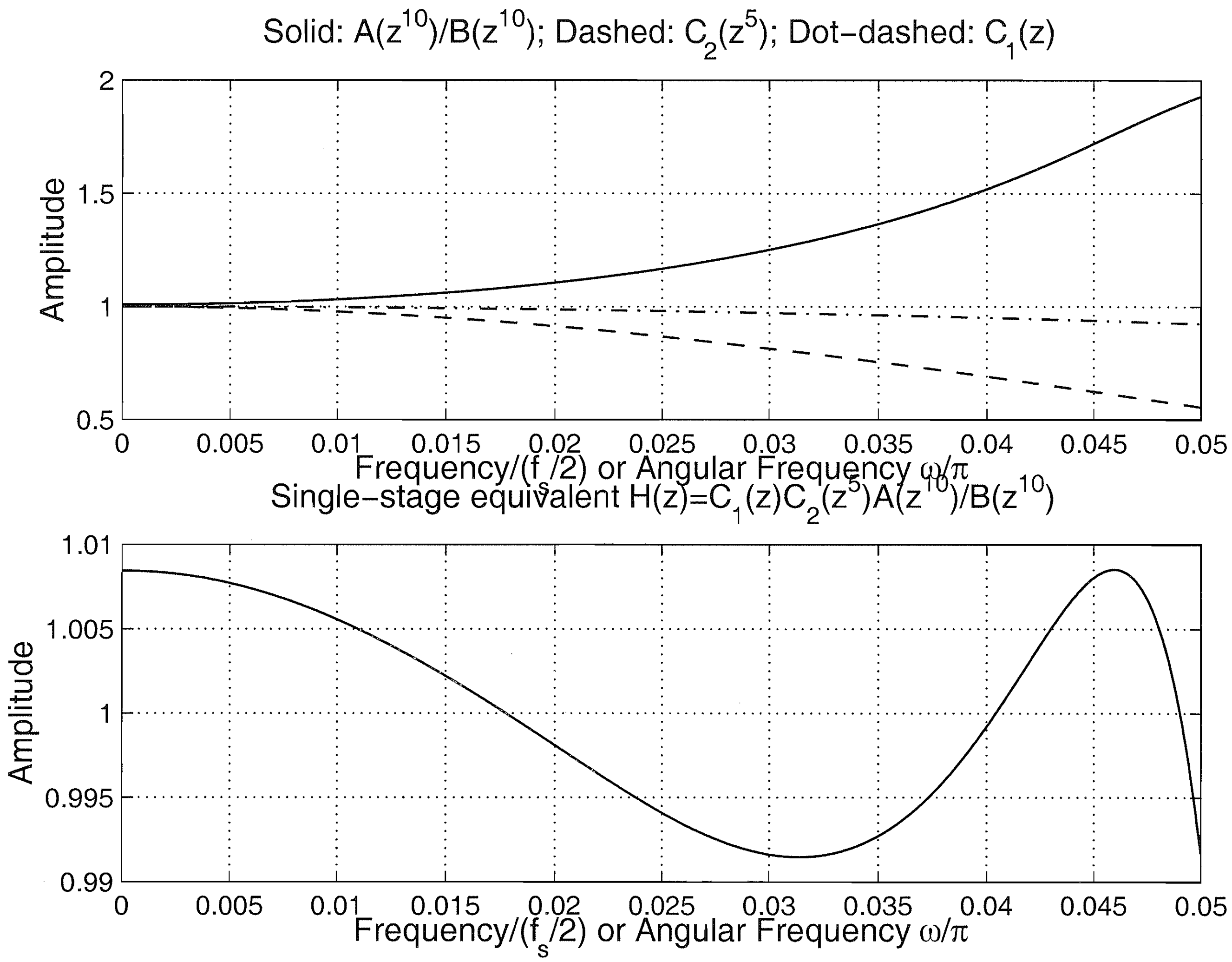


Characteristics of IIR Decimator of the Form $H(z) = C_1(z)C_2(z^5)A(z^{10})/B(z^{10})$

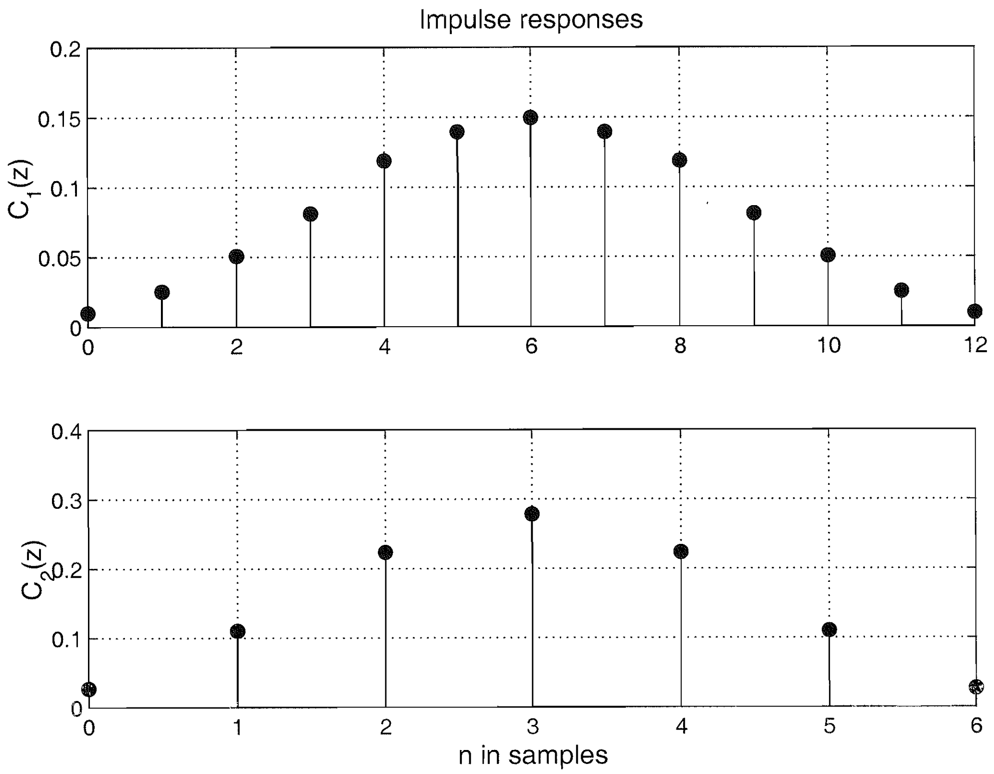


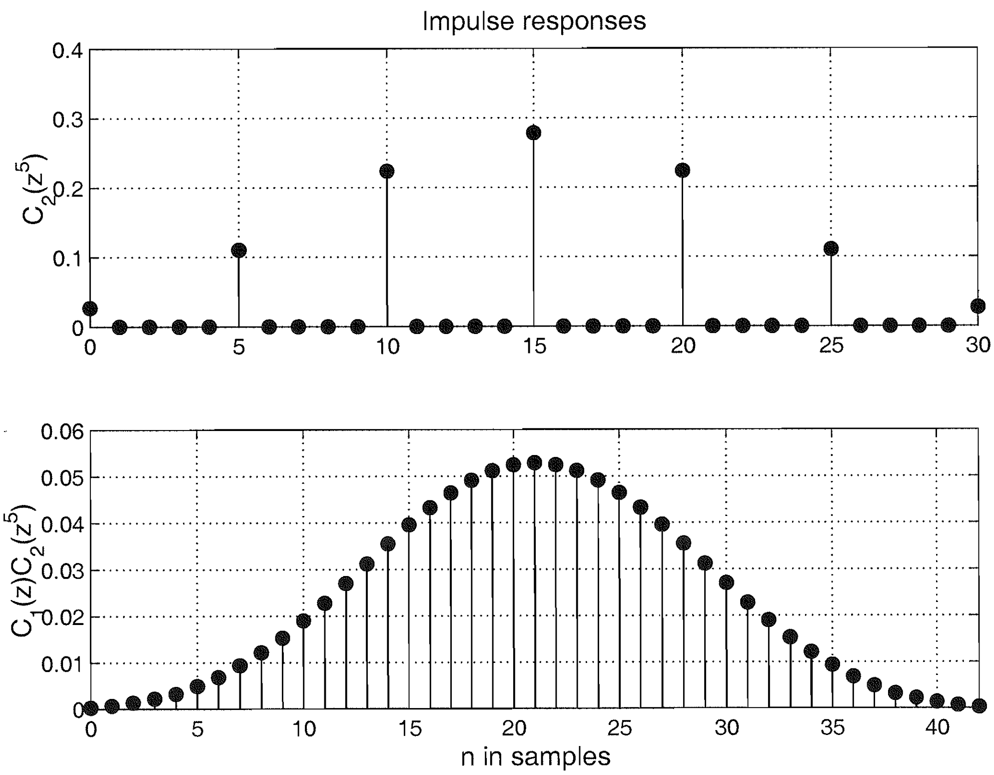


Characteristics of IIR Decimator of the Form $H(z) = C_1(z)C_2(z^5)A(z^{10})/B(z^{10})$

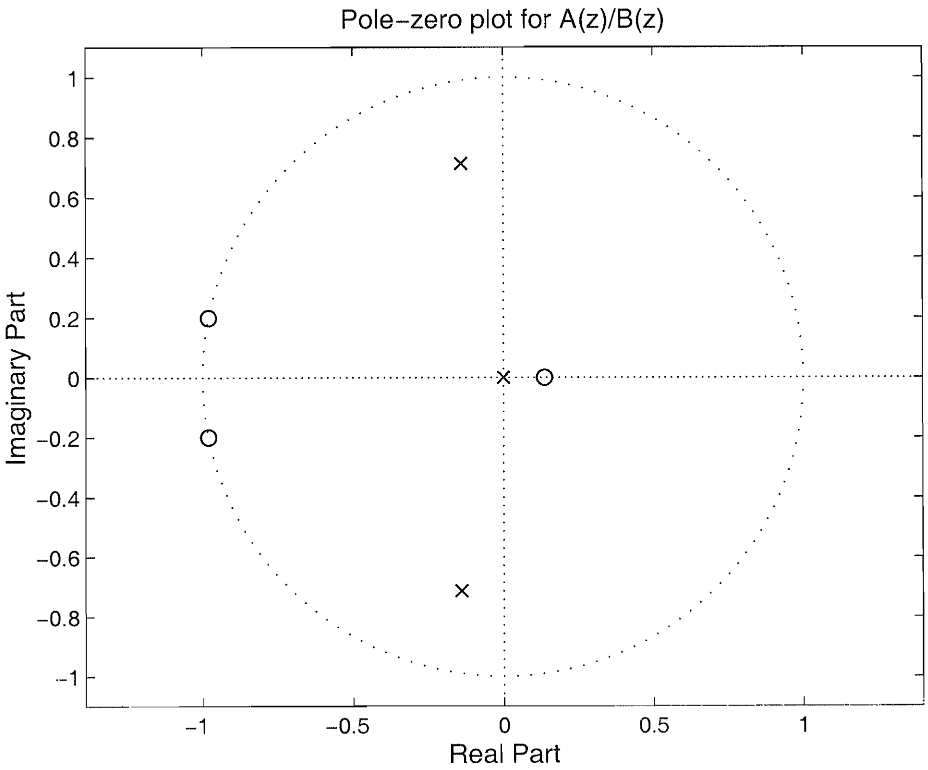


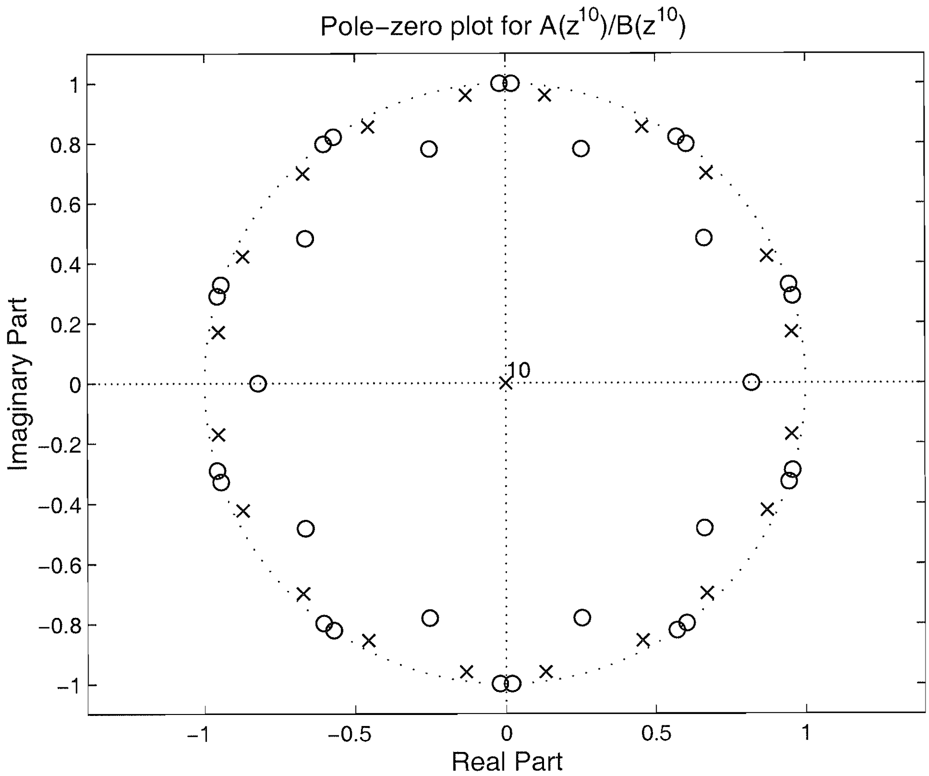
Characteristics of IIR Decimator of the Form $H(z) = C_1(z)C_2(z^5)A(z^{10})/B(z^{10})$



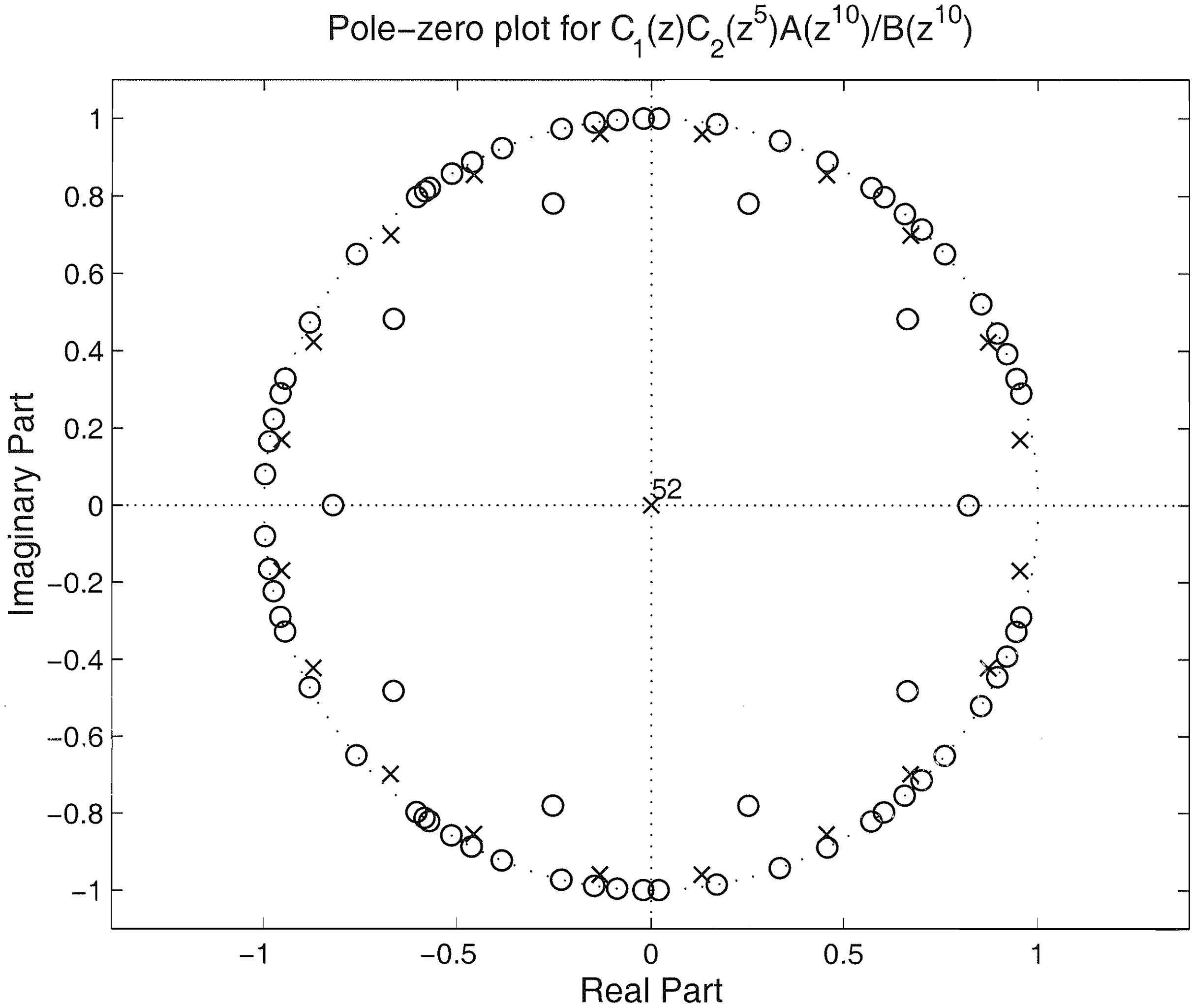


Characteristics of IIR Decimator of the Form $H(z) = C_1(z)C_2(z^5)A(z^{10})/B(z^{10})$





Characteristics of IIR Decimator of the Form $H(z) = C_1(z)C_2(z^5)A(z^{10})/B(z^{10})$



Part II.D: Nth-Band Recursive Digital Filters

- This is a very short pile of lecture notes on the design and properties of Nth-band recursive digital filters.
- An interested reader should read the articles:

M. Renfors and T. Saramäki, "Recursive Nth-band digital filters—Part I: Design and properties," IEEE Transactions on Circuits and Systems, vol. CAS-34, pp. 24–39, January 1987.

M. Renfors and T. Saramäki, "Recursive Nth-band digital filters—Part II: Design of multistage decimators and interpolators," IEEE Transactions on Circuits and Systems, vol. CAS-34, pp. 40–51, January 1987.

- There exists an efficient FORTRAN routine for designing these filters. A MATLAB program is coming up next.

• For these filters, the overall transfer function is of the following polyphase form:

$$H(z) = \frac{1}{N}\sum_{n=0}^{N-1} z^{-n} G_n(z^N), \tag{23a}$$

where

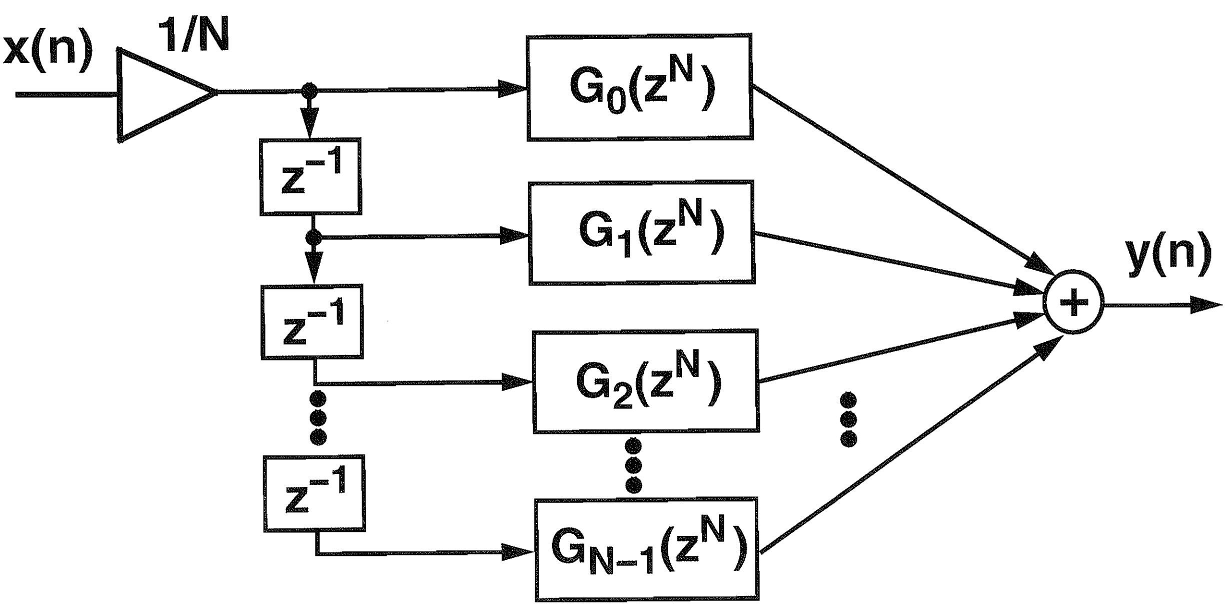
$$G_n(z^N) = z^{-k_n N}\frac{z^{-K_n N} D_n(z^{-N})}{D_n(z^N)}. \tag{23b}$$

• Here, $G_n(z^N)$ is obtained from

$$G_n(z) = z^{-k_n} A_n(z), \quad A_n(z) = \frac{z^{-K_n} D_n(z^{-1})}{D_n(z)} \tag{24}$$

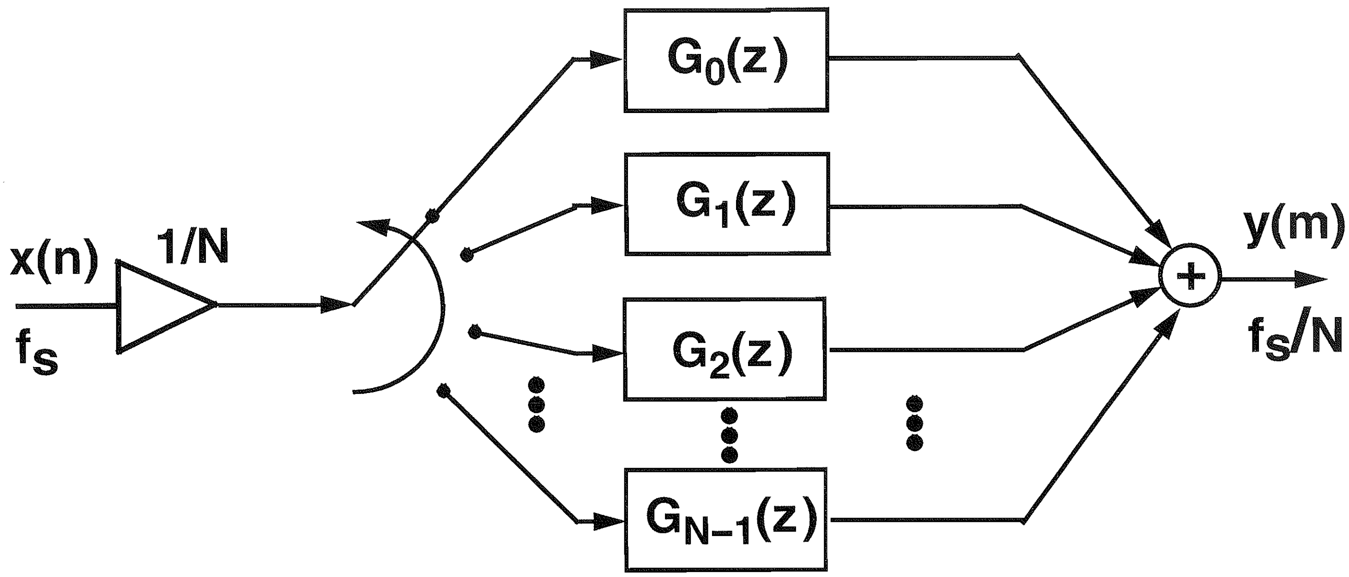
by replacing each unit delay by N delays. $A_n(z)$ is an allpass filter of order K_n.

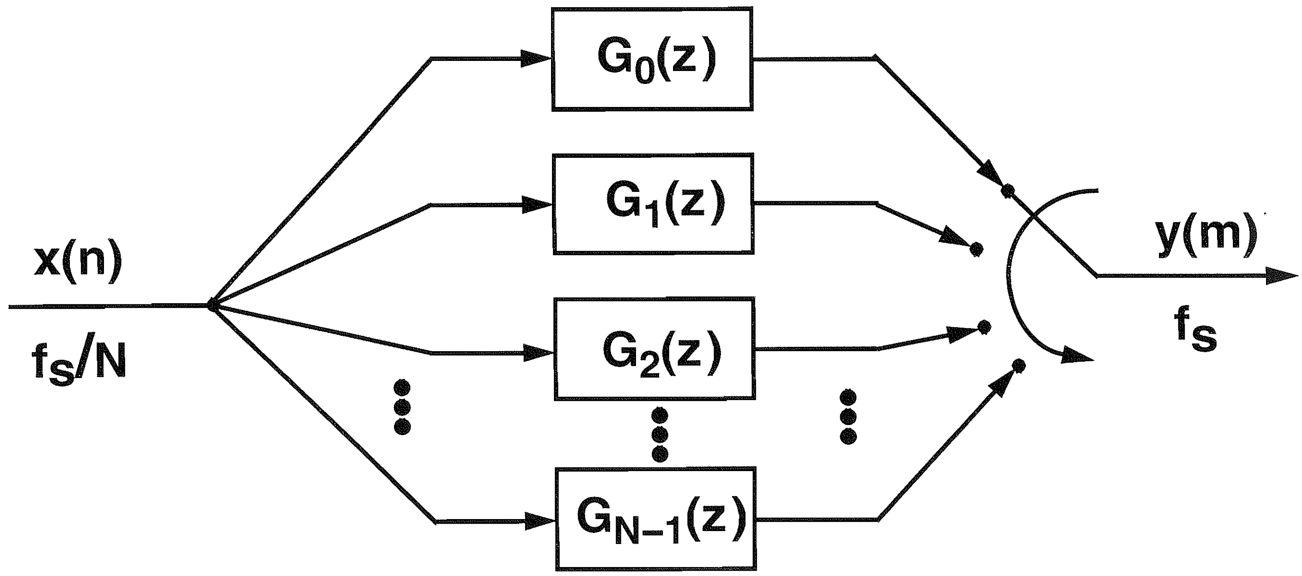
• An implementation of the above transfer function is shown below.



- The next page shows commutative models for the overall transfer function in the decimation and interpolation cases.
- In these structures, the branch filters $G_n(z)$ work at the lower sampling rate.
- According to the consideration of the previous page, $G_n(z)$ consists of an allpass filter and a possible extra delay term z^{-k_n}.
- As we shall see later on, if there are no phase constraints, then each $G_n(z)$ is simply a cascade of first-order allpass filters.
- An approximately linear phase performance can be achieved in the passband by selecting one branch filter to be a pure delay. In this case, the other branch filters are cascades of first- and second-order allpass filters.

Commutative Models for Nth-Band Recursive Filters in Decimation and Interpolation Cases





Advantages of Nth-Band Recursive Filters

1. Lowest multiplication rate among known recursive decimators and interpolator.
2. Approximately linear phase designs possible by selecting one of the all-pass filters to be a pure delay.
3. Low noise and sensitivity, limit cycles can be supressed.
4. Very modular structures, can be constructed using first- and second-order allpass filters.
5. If phase distortion is not of interest, only first-order blocks are present.

- Since the $G_n(z)$'s are allpass filters, the frequency response of the overall filter is expressible as

$$H(e^{j\omega}) = \frac{1}{N}\sum_{n=0}^{N-1} e^{j\phi_n(\omega)}, \qquad (25a)$$

where

$$\phi_n(\omega) = -n\omega + \arg[G_n(e^{jN\omega})]. \qquad (25b)$$

- It can be shown that this frequency response satisfies

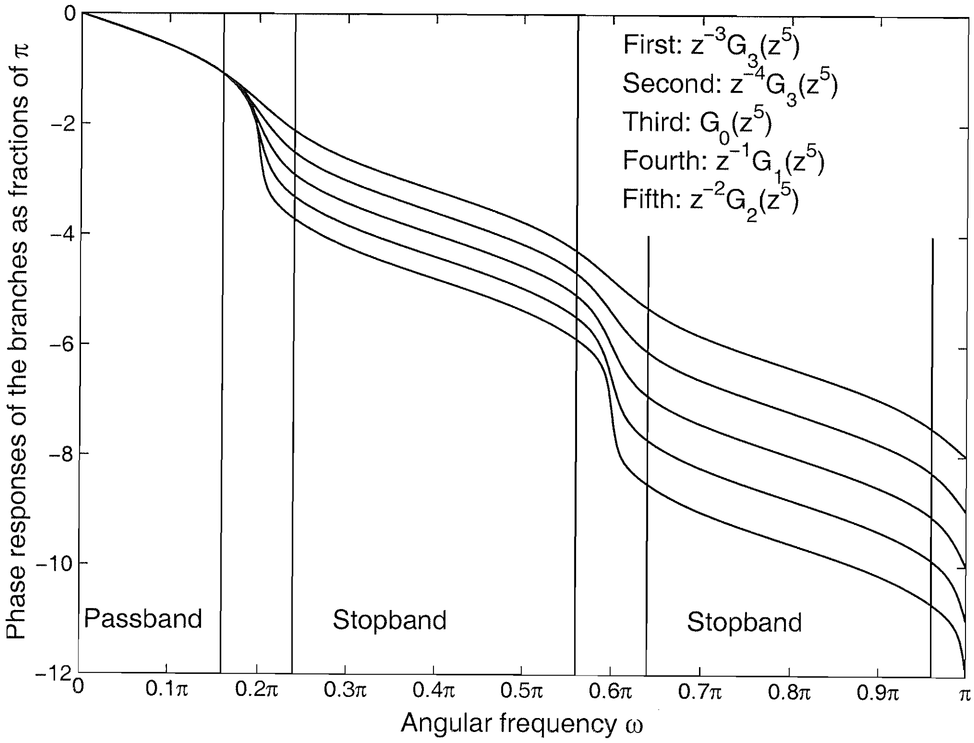
$$\sum_{r=0}^{N-1} |H(e^{j(\omega+2\pi r/N)})|^2 = 1. \qquad (26)$$

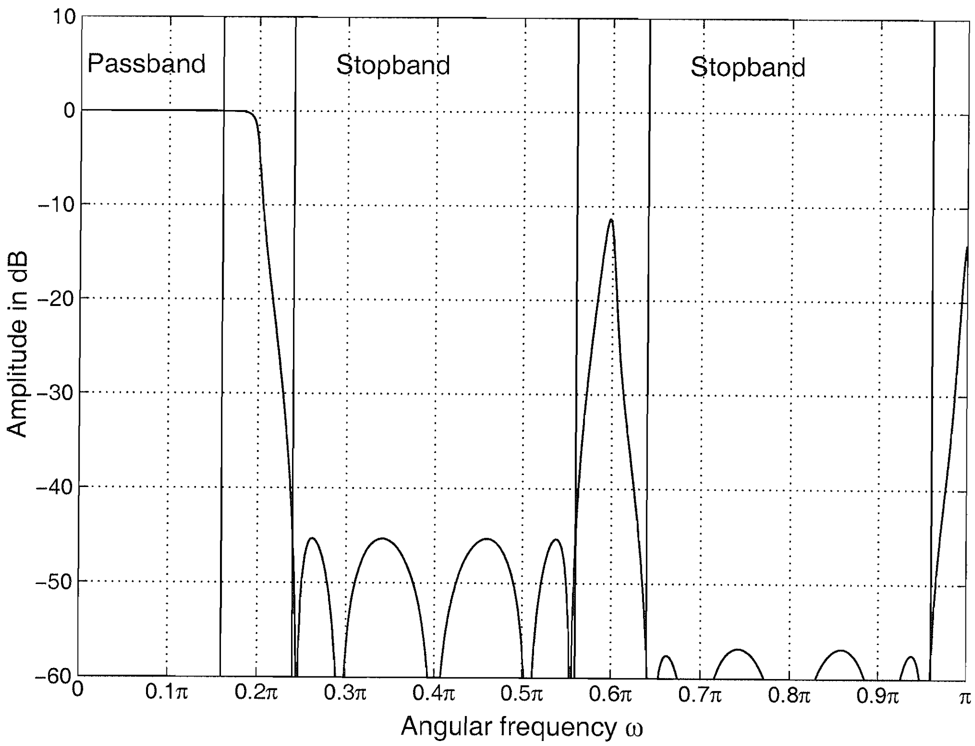
- This states some limitations on the frequency-domain behavior of the filter as we shall see later on.
- In order to arrive at a filter with good amplitude characteristics, the orders of the branch filters have to be selected properly (see the above-mentioned articles for details)
- We start with an example and, then, generalize the results.

Example Filter: $N = 5$, $G_0(z)$, $G_1(z)$, and $G_2(z)$ are Cascades of Two First-Order Allpass Sections, whereas $G_3(z)$ and $G_4(z)$ are First-Order Allpass Sections

- $G_0(z)$: poles at $z = -0.06064045371, \quad -0.7032070490$
- $G_1(z)$: poles at $z = -0.1465783514, \quad -0.8191991118$
- $G_2(z)$: poles at $z = -0.2502545637, \quad -0.9210697554$
- $G_3(z)$: a poles at $z = -0.3770169094$
- $G_4(z)$: a pole at $z = -0.5665661715$
- The following page shows the phase responses of the branches and the resulting amplitude response.
- In the region $[0, 0.8\pi/5]$, the phases are almost the same. $\Rightarrow |H(e^{j\omega})| = \frac{1}{5}|\sum_{n=0}^{4} e^{j\phi_n(\omega)}| \approx 1$.
- In the region $[1.2\pi/5, 2.8\pi/5]$, the differences between the consecutive phases are approximately $2\pi/5$. $\Rightarrow H(e^{j\omega}) = \frac{1}{5}\sum_{n=0}^{4} e^{j\phi_n(\omega)} \approx 0$.
- In the region $[3.2\pi/5, 4.8\pi/5]$, the differences between the consecutive phases are approximately $4\pi/5$. $\Rightarrow H(e^{j\omega}) = \frac{1}{5}\sum_{n=0}^{4} e^{j\phi_n(\omega)} \approx 0$.

Responses for an Example Filter





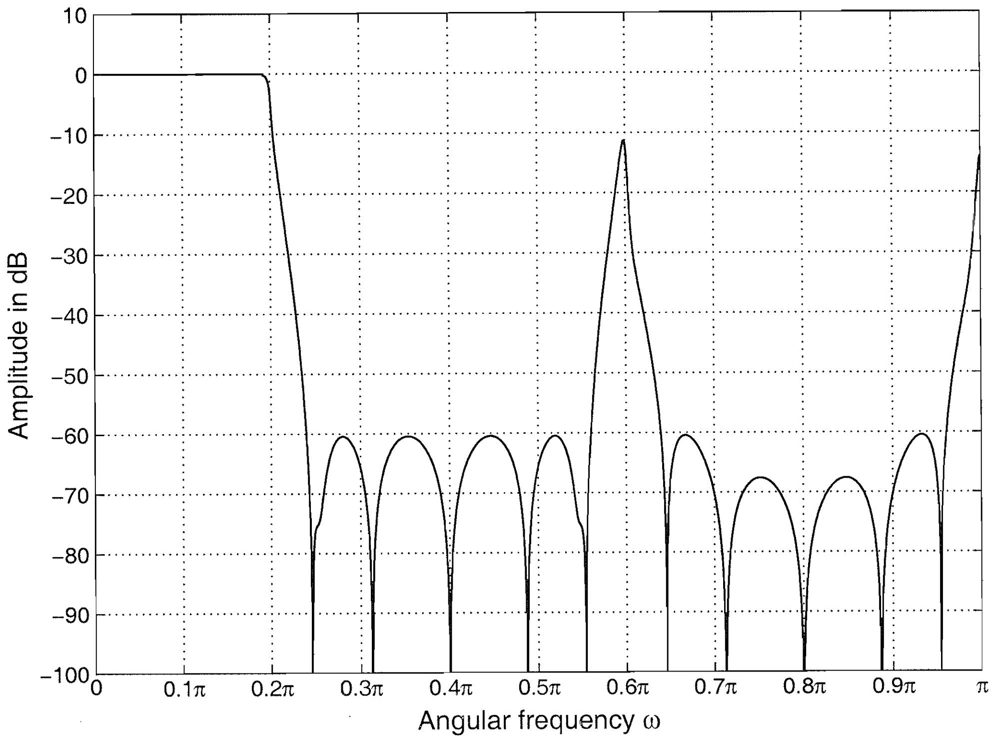
Filter Properties in General

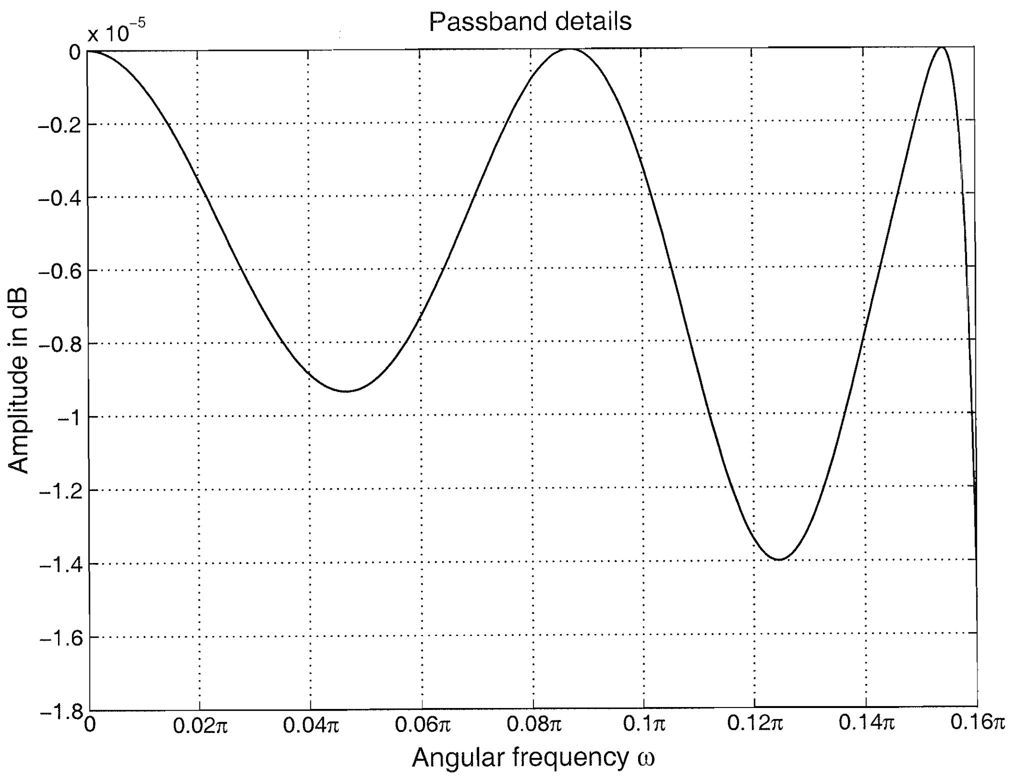
- The passband edge must satisfy $\omega_p < \pi/N$.
- The filter has stopbands of width $2\omega_p$ around the points $k2\pi/N$ for $k = 1, 2, ...$
- The filter has always peaks around the points $(2k+1)\pi/N$ for $k = 1, 2, ...$ and the stopband edge cannot be located at π/N.
- The filter satisfies automatically the Case B specifications considered earlier.
- The Case A specifications can be met by using in the decimation (interpolations) case at the filter output (input) an extra filter stage, as we shall see later on.

Example Nonlinear-Phase Filter: $N = 5$, $\omega_p = 0.8\pi/N$ and at Least a 60-dB Attenuation in the Stopbands

- These criteria are met as follows:
- $G_0(z)$ consists of three first-order allpass filters with poles at $z = -0.03247627480$, $z = -0.4519480048$, $z = -0.9477051753$.
- $G_1(z)$ has two first-order allpass filters with poles at $z = -0.08029157130$ and $z = -0.5548998293$.
- $G_2(z)$ has two first-order allpass filters with poles at $z = -0.1417079348$ and $z = -0.6883346404$.
- $G_3(z)$ has two first-order allpass filters with poles at $z = -0.2320513100$ and $z = -0.7961481351$.
- $G_4(z)$ has two first-order allpass filters with poles at $z = -0.3532045984$ and $z = -0.8755417392$.
- The next page shows the responses for this design.

Responses for the Nonlinear-Phase Design





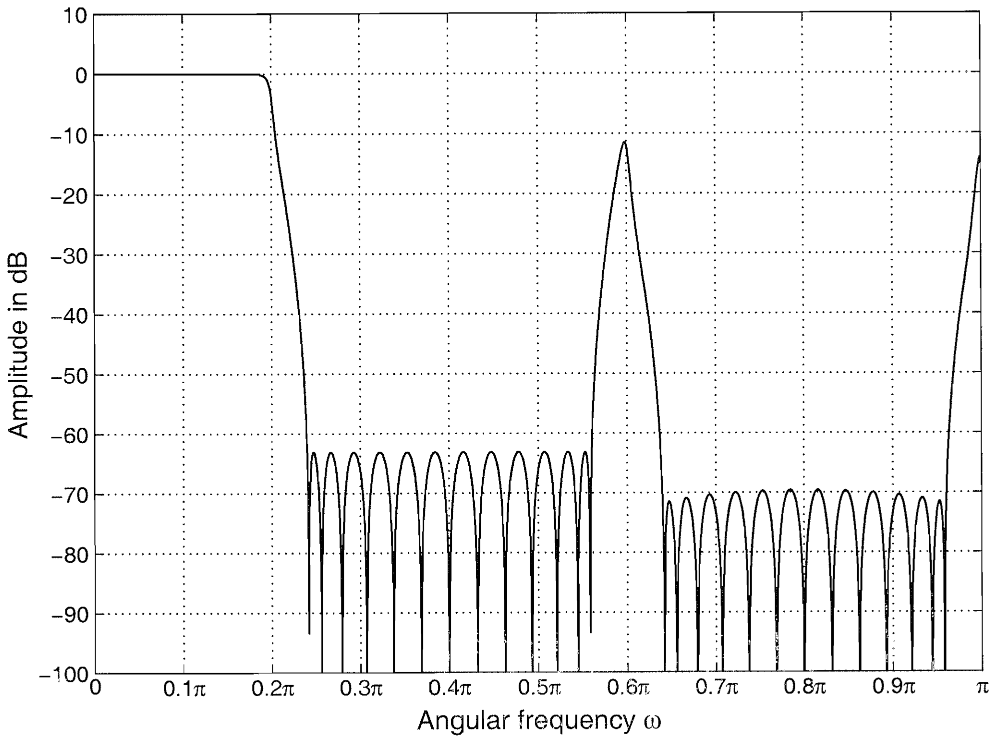
Comments

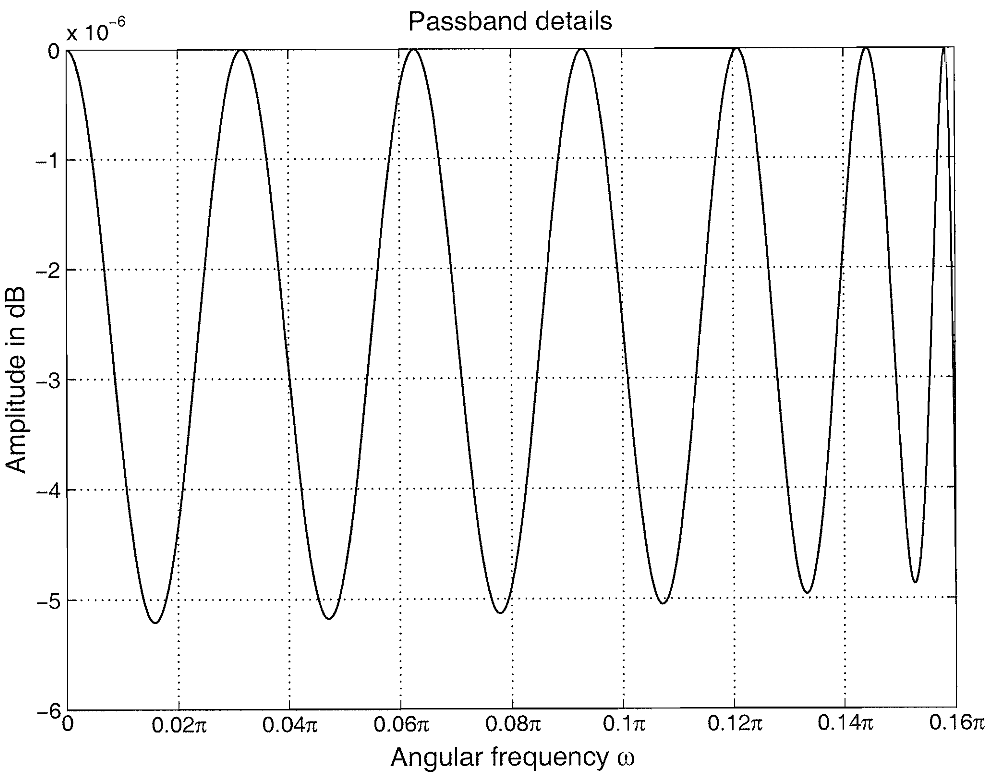
- If aliasing is allowed into the transition band $[0.8\pi/5, \pi/5]$, then this design can be directly used.
- The peaks in the stopband region occur in the regions that alias into this band, whereas regions aliasing into the passband $[0, 0.8\pi/5]$ are well attenuated.
- The Case B specifications are thus met.

- These criteria are met as follows:
- $G_0(z)$ consists of two first-order allpass filters with poles at $z = 0.3539559551$ and $z = -0.600954816$; and two second-order sections with poles at $z = 0.36045530\exp(\pm j0.32083089\pi)$ and at $z = 0.39292469\exp(\pm j0.63853828\pi)$.
- $G_1(z)$ consists of two first-order allpass filters with poles at $z = 0.3362484694$ and $z = -0.7296322306$; and two second-order sections with poles at $z = 0.34509052\exp(\pm j0.32657981\pi)$ and at $z = 0.38698306\exp(\pm j0.65132553\pi)$.
- $G_2(z)$ consists of two first-order allpass filters with poles at $z = 0.2969941345$ and $z = -0.8290479$; and two second-order sections with poles at $z = 0.30739674\exp(\pm j0.33448924\pi)$ and at $z = 0.35542291\exp(\pm j0.67048455\pi)$.

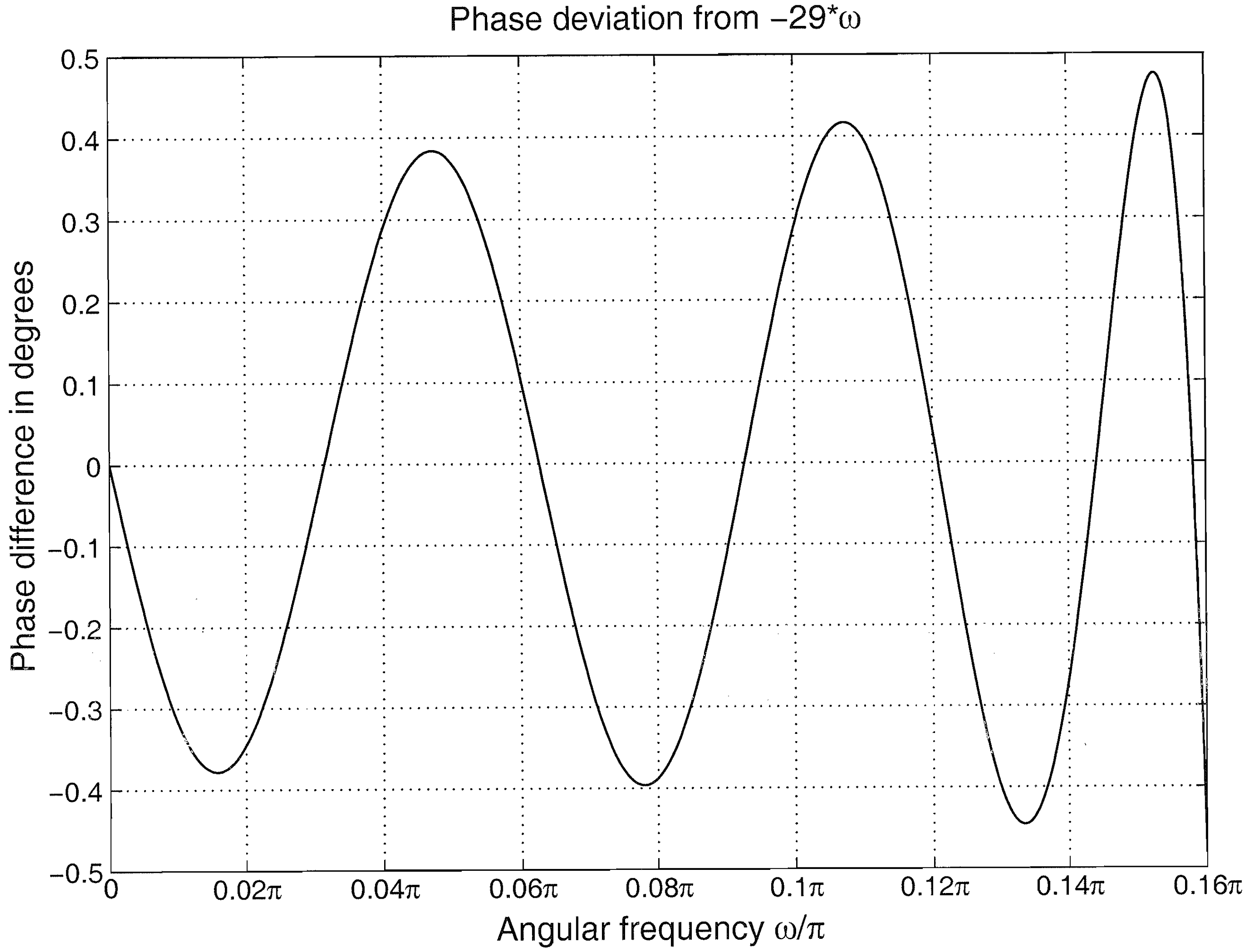
- $G_3(z)$ consists of two first-order allpass filters with poles at $z = 0.2422225263$ and $z = -0.9169533323$; and two second-order sections with poles at $z = 0.2528221 \exp(\pm j0.34583305\pi)$ and at $z = 0.30084497 \exp(\pm j0.6997103\pi)$.
- $G_4(z) = z^{-5}$
- The following two pages show the responses of this design.
- Note that by selecting one branch filter to be a pure delay term makes the phase response very linear in the passband at the expense of increased orders for other filter branches.

Responses for the Approximately Linear-Phase Design





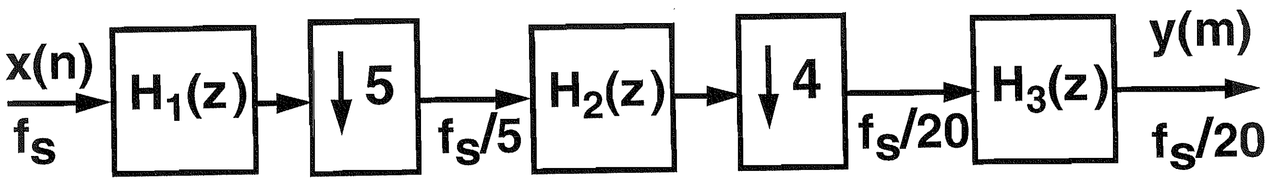
Responses for the Approximately Linear-Phase Design

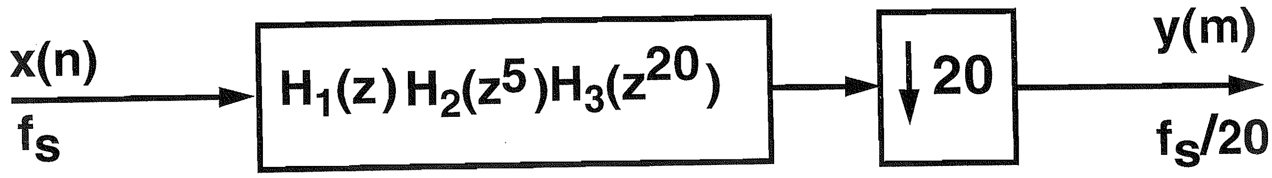


- Note the extremaly small phase error.

Example: $N = 20$, $\omega_p = 0.9\pi/20$, $\omega_s = \pi/20$, **maximum deviation from unity in the passband=0.1, stopband ripple=46 dB.**

- Best two-stage design is obtained by using $N_1 = 5$ and $N_2 = 4$.
- The unwanted peaks in the transition band can be attenuated by using an extra filter stage at the output (input) sampling rate in the decimation (interpolation) case, as shown below.
- The second figure below shows the single-stage equivalent.

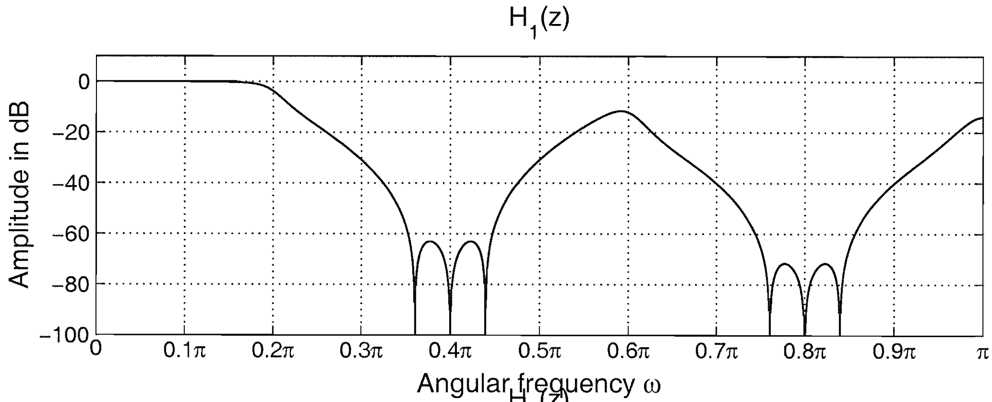


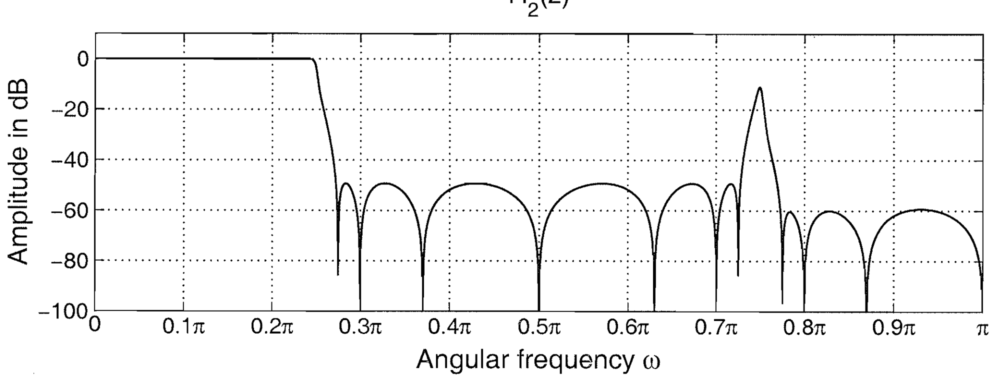


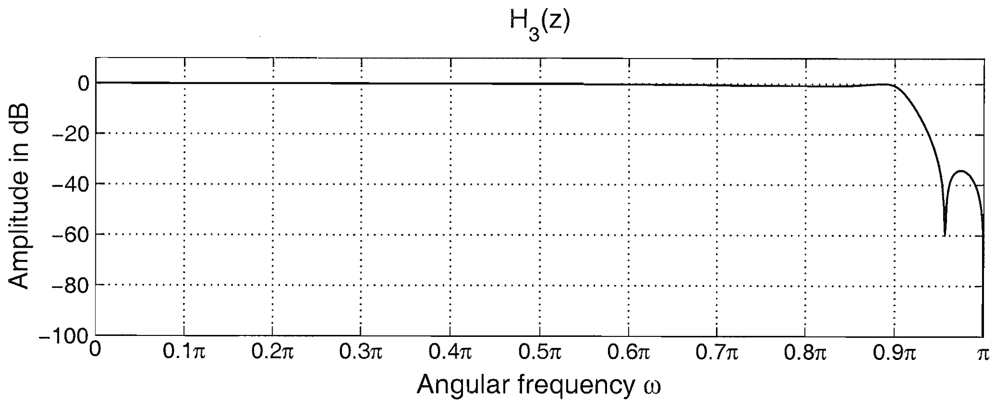
- The given criteria are met as follows:
- First stage: $G_0(z)$ has one pole at $z = -0.1170248386$, $G_1(z)$ has one pole at $z = -0.2598957560$, $G_2(z)$ has one pole at $z = -0.4397925668$, $G_3(z)$ has one pole at $z = -0.6753309237$, and $G_4(z) = 1$.

- Second stage: $G_0(z)$ has three real poles at $z = -0.05703249116$, -0.5752126667, and -0.9604894401; $G_1(z)$ has two real poles at $z = -0.1490762168$ and -0.6955946084; $G_2(z)$ has two real poles at $z = -0.2706050144$ and -0.821873825; and $G_3(z)$ has two real poles at $z = -0.4395465071$ and -0.9012967887.
- The filter stage $H_3(z)$ working at the output sampling rate is a parallel connection of first-order allpass filter with pole at $z = -0.56377338$ and a second-order allpass filter with poles at $z = 0.93383553 \exp(\pm j0.90404521\pi)$.
- In the following, there are altohether four pages illustrating the performance of the overall design.
- The design of the second figure page with the last stage absent can be directly used for Case B specifications: no aliasing into the passband region $[0,\ 0.9\pi/20]$.
- As seen from the third and fourth figure pages, the last stage attenuates the unwanted peaks in the stopband and starts the stopband region at $\omega = \pi/20$, as is desired.

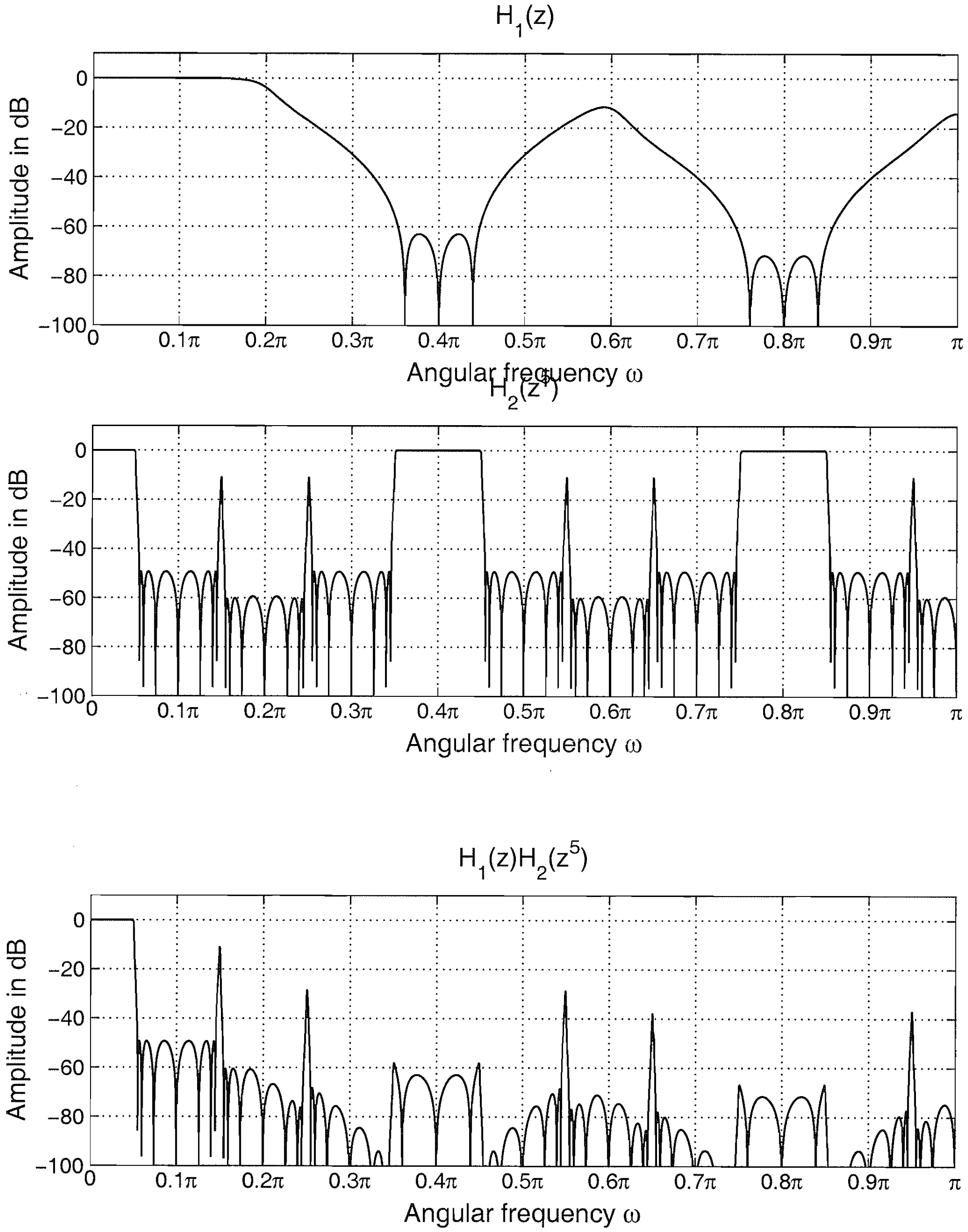
Responses for the Subfilters



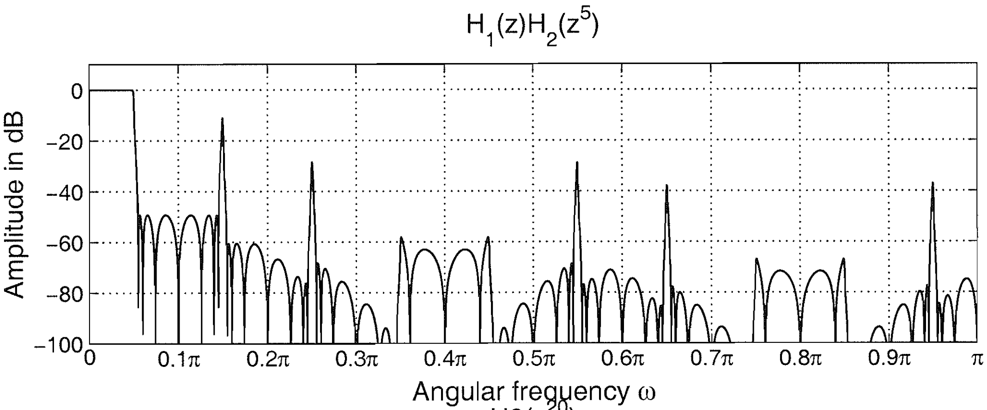


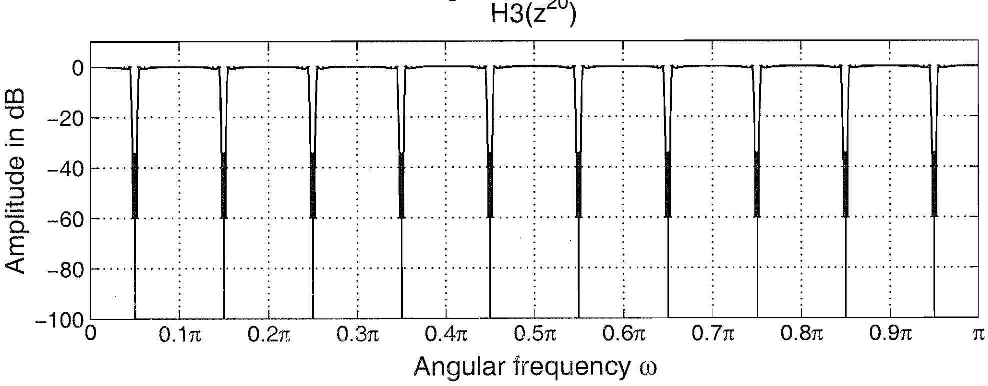


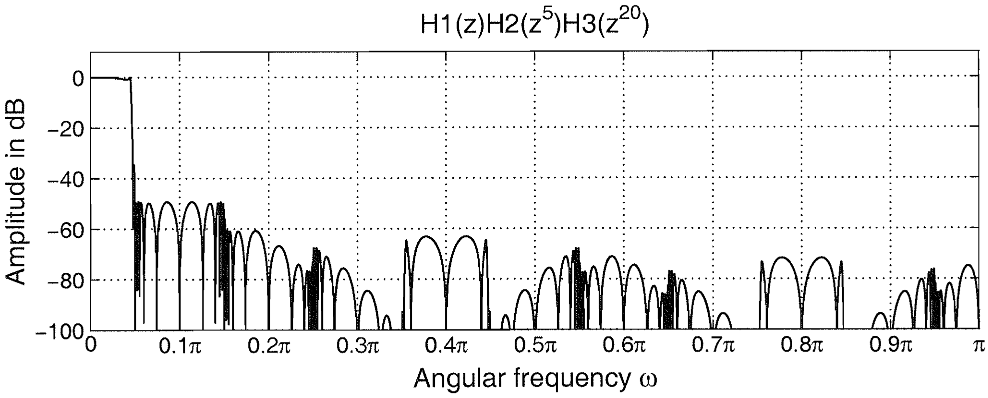
Responses for $H_1(z)$, $H_2(z^5)$ and $H_1(z)H_2(z^5)$



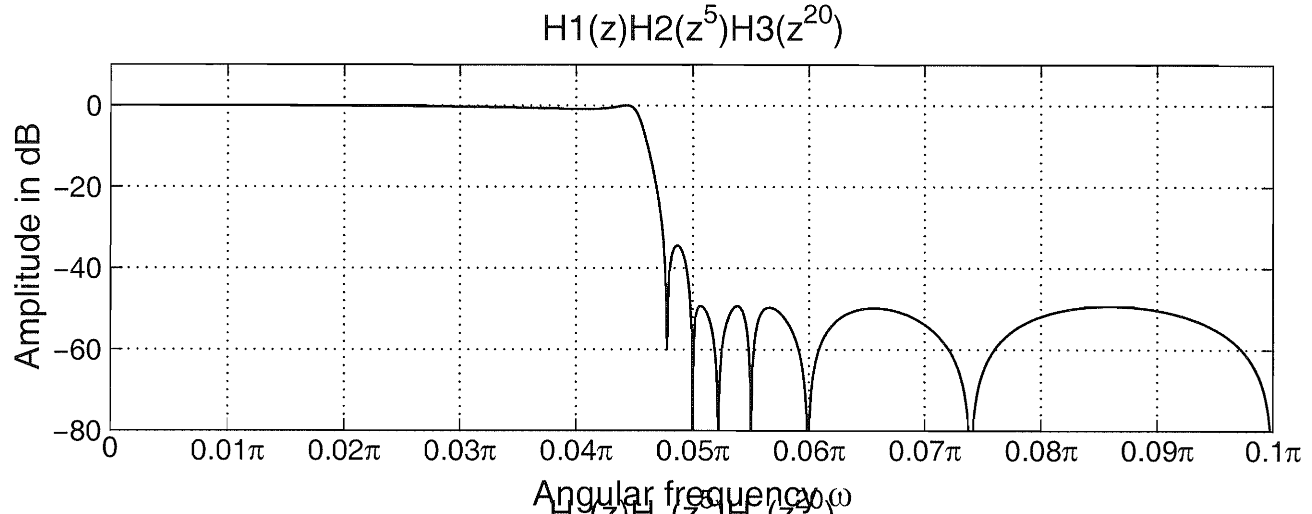
Responses for $H_1(z)H_2(z^5)$, $H_3(z^{20})$, and $H_1(z)H_2(z^5)H_3(z^{20})$

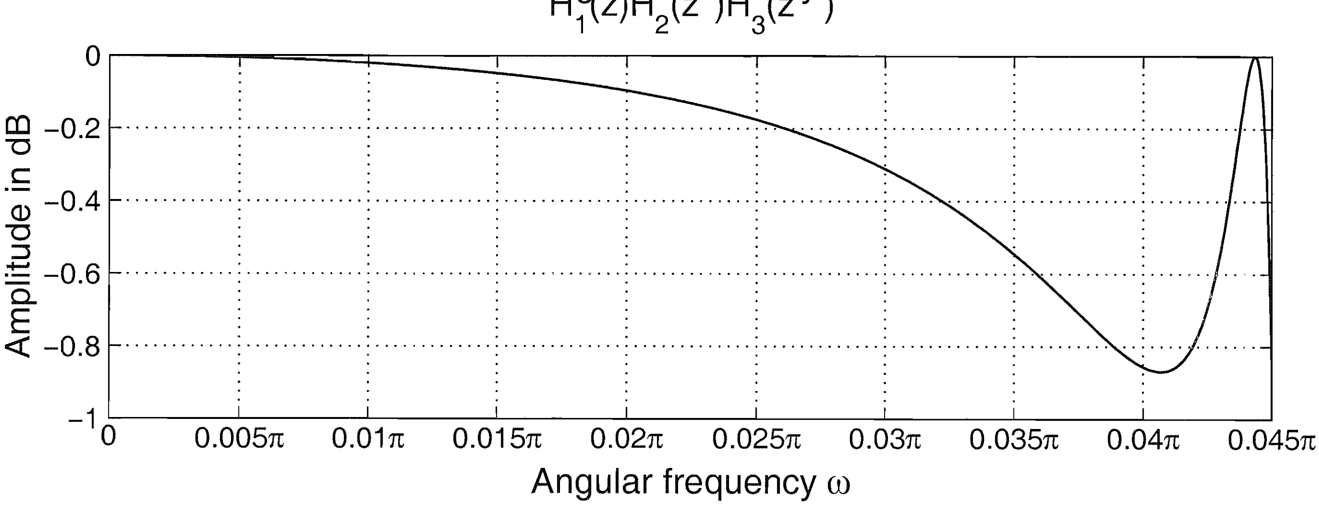






Low-Frequency Details for the overall filter





Comparison Between Different IIR Decimators

- The table of the next page compares different IIR decimators with each other.
- In Case 1 aliasing is allowed into the transition band $[0.9\pi/20, \pi/20]$ (Case B considered in these lecture notes), whereas in Case 3 no aliasing is allowed into the band $[0, \pi/20]$ (Case A considered in these lecture notes).
- New filters mean the Nth-band IIR filters. K stands for the sum of the orders of the branch filters, whereas K_{COR} is the order of the last filter stage.
- The comparison includes elliptic filters as well as those filters considered on Pages 108 – 125.

Comparison Between Different Decimator Types

	Case, Conversion Ratios	Orders	Multiplication Rate	Number of Multipliers	Data Storage
One-Stage Elliptic	*Case 3* $N = 20$	$M = 7$	7.20	11	7
	Case 1 $N = 20$	$M = 6$	6.20	10	6
One-Stage Generalized Martinez-Parks Filter [13]	*Case 3* $N = 20$	$M_B = 4$ $M_A = 3$ $M_{C_1} = 92$	2.65	53	96
	Case 1 $N = 20$	$M_B = 3$ $M_A = 0$ $M_{C_1} = 90$	2.45	49	93
One-Stage New Filter	*Case 3* $N = 20$	$K = 42$ $K_{COR} = 3$	2.25	45	45
	Case 1 $N = 20$	$K = 42$	2.10	42	42
Two-Stage Elliptic	*Case 3* $N_1 = 10$ $N_2 = 2$	$M_1 = 3$ $M_2 = 7$	4.10	16	10
	Case 1 $N_1 = 10$ $N_2 = 2$	$M_1 = 3$ $M_2 = 6$	4.00	15	9
Two-Stage Generalized Martinez-Parks Filter [13]	*Case 3* $N_1 = 10$ $N_2 = 2$	$M_B = 4$ $M_A = 1$ $M_{C_1} = 24$ $M_{C_2} = 9$	1.75	22	37
	Case 1 $N_1 = 10$ $N_2 = 2$	$M_B = 3$ $M_A = 0$ $M_{C_1} = 24$ $M_{C_2} = 8$	1.70	21	35
Two-Stage New Filter	*Case 3* $N_1 = 5$ $N_2 = 4$	$K_1 = 4$ $K_2 = 9$ $K_{COR} = 3$	1.40	16	16
	Case 1 $N_1 = 5$ $N_2 = 4$	$K_1 = 4$ $K_2 = 9$	1.25	13	13

Part II.E: Nth-Band FIR Digital Filters

- This is a pile of lecture notes on the design and properties of Nth-band FIR digital filters.
- An interested reader should read the following articles:

T. Saramäki and Y. Neuvo, "A class of FIR (Nth-band) Nyquist filters with zero intersymbol interference", IEEE Transactions on Circuits and Systems, vol. CAS-34, pp. 1182-1190, October 1987.

M. Renfors and and T. Saramäki, "Pulse-shaping filters for digital transmission systems,", in Proc. 1992 IEEE Global Telecommunications Conference (Orlando, FL), pp. 467-471, Dec. 1992.

T. Saramäki and M. Renfors, "Nth band filter design," in IX European Signal Processing Conference (Island of Rhodes, Greece), pp. 1943–1947, September 1998.

Division of Nth-Band Linear-Phase FIR Filters into Subclasses

• Nth-band linear-phase FIR filters can be divided into the following subclasses:

I. Nonseparable single-stage filters

II. Separable single-stage filters

III. Nonseparable multistage filters

IV. Separable multistage filters

• In the sequel, all these filter classes are considered.

• For designing these filters, there exists a MATLAB routine, nykki.m, in /home/ts/matlab/multirate.

What are Nonseparable Single-Stage Nth-Band Linear-Phase FIR Filters ?

• Consider a Type I linear-phase FIR filter with transfer function $[h(2M-n) = h(n)$ for $n = 0, 1, \cdots, M-1]$

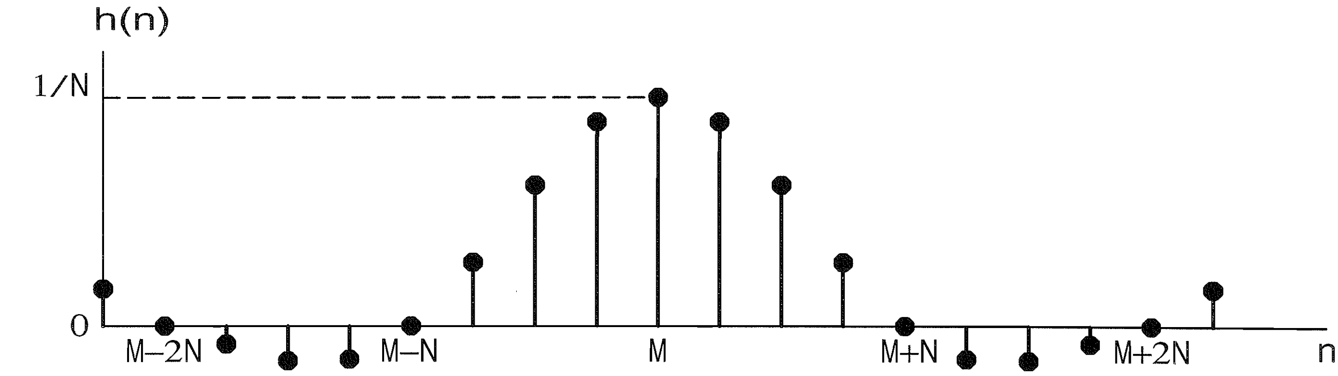
$$H(z) = \sum_{n=0}^{2M} h(n) z^{-n} \tag{27}$$

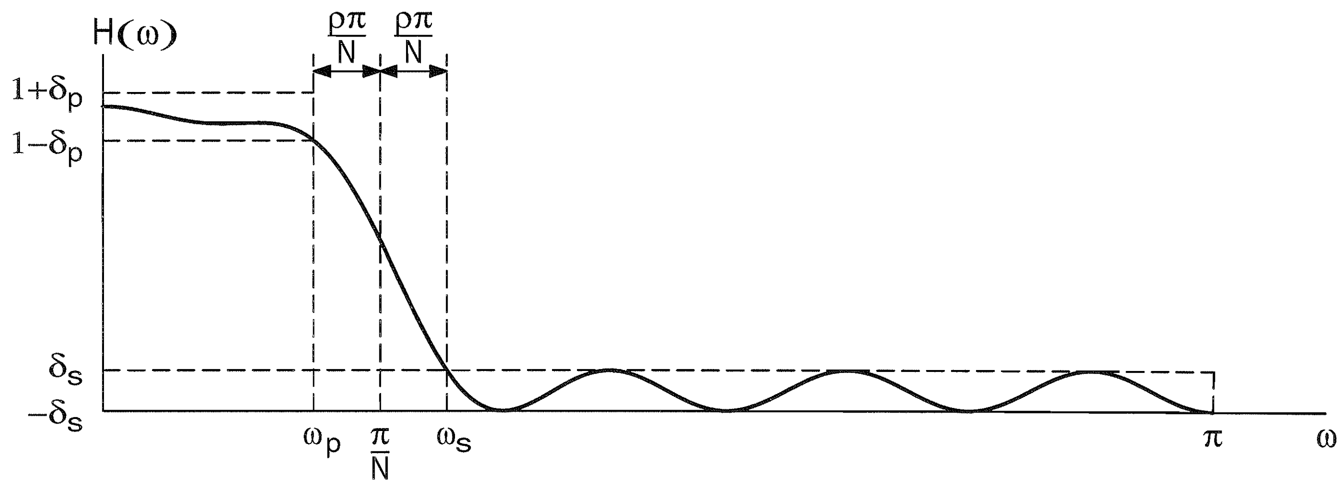
• This filter is defined to be an N-th band filter if its coefficients (see the next page for $N = 4$) satisfy

$$h(M) = 1/N \tag{28a}$$

$$h(M + rN) = 0 \quad \text{for} \quad r = \pm 1, \pm 2, \ldots, \lfloor M/L \rfloor. \tag{28b}$$

Time-Domain and Frequency-Domain Conditions for Lowpass Nth-Band Linear-Phase FIR Filters





Frequency-Domain Conditions for Lowpass Nth-Band Linear-Phase FIR Filters

• The time-domain conditions imply that the the zero-phase frequency response as given by

$$H(\omega) = 1/N + 2\sum_{n=1}^{M} h(M-n)\cos(n\omega) \quad (29)$$

satisfies

$$\sum_{r=0}^{N-1} H(\omega + 2\pi r/N) = 1. \quad (30)$$

• This means that in the lowpass case the passband and stopband edges are related through (see the previous page for $N = 4$)

$$\omega_p = (1-\rho)\pi/N, \quad \omega_s = (1+\rho)\pi/N, \quad (31)$$

where $\rho > 0$.

• Furthermore, $\delta_p \leq (N-1)\delta_s$ and a 6-dB point is approximately at $\omega = \pi/N$.

• This means that for a small stopband ripple δ_s, δ_p, the maximum deviation of $H(\omega)$ from unity in the passband, is guaranteed to be small.

• Therefore, in many cases, the filter optimization can concentrate on shaping the stopband response.

Approximation Criteria

- There exist the following two criteria:

Minimax approximation: Find the filter unknowns to minimize

$$\epsilon_\infty = \max_{\omega \in [(1+\rho)\pi/N,\ \pi]} |H(\omega)|. \tag{32}$$

Least-squared approximation: Find the filter unknowns to minimize

$$\epsilon_2 = \int_{(1+\rho)\pi/N}^{\pi} |H(\omega)|^2 d\omega. \tag{33}$$

Example: $\rho = 0.2$ and $N = 8$

Minimax design: The minimum stopband atteunuation is at least 40 dB, that is, δ_s is less than or equal to 0.01.

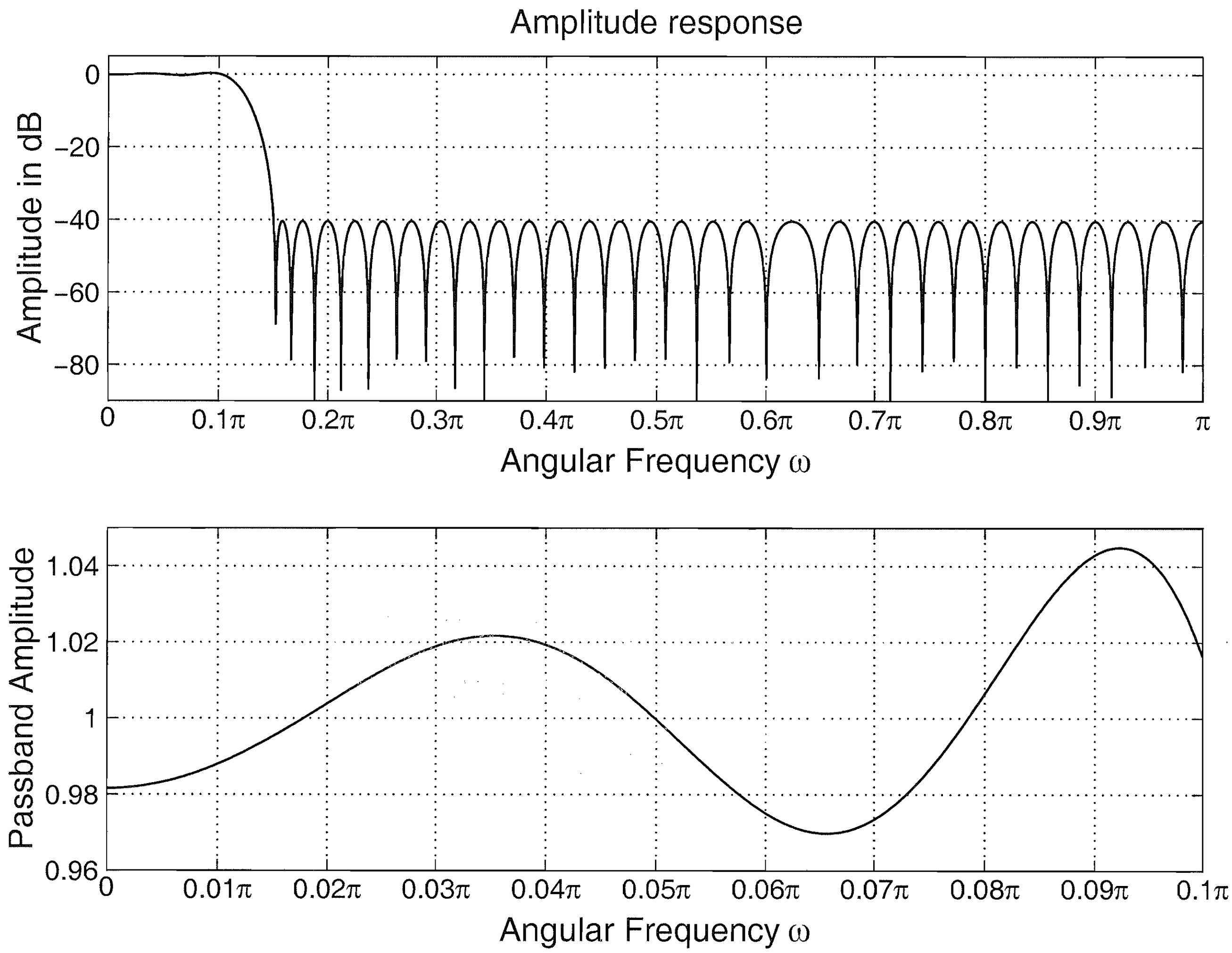
• The minimum even order to meet the given criteria is $2M = 74$.

• By exploiting the coefficient symmetry and the fact that $h(37 \pm 8r) = 0$ for $r = 1, 2, 3$ and assuming that the implementation of $h(37) = 1/8 = 2^{-3}$ requires no multipliers, the overall number of multipliers is 32.

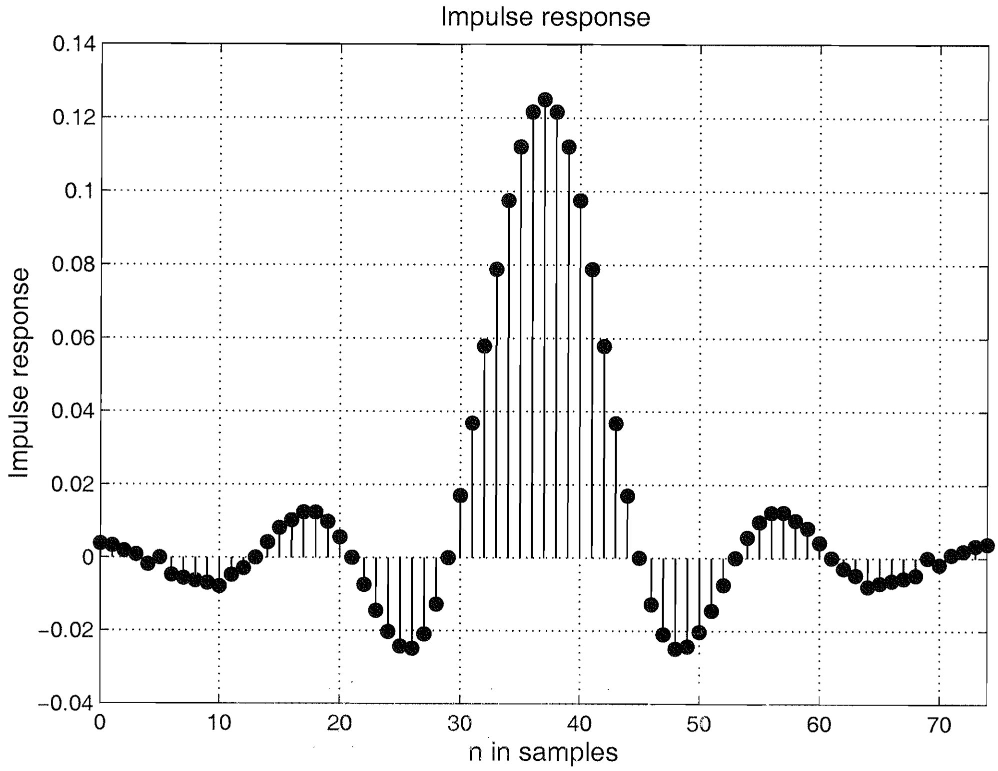
Least-squared design: It is desired to minimize the filter stopband energy for $2M = 74$.

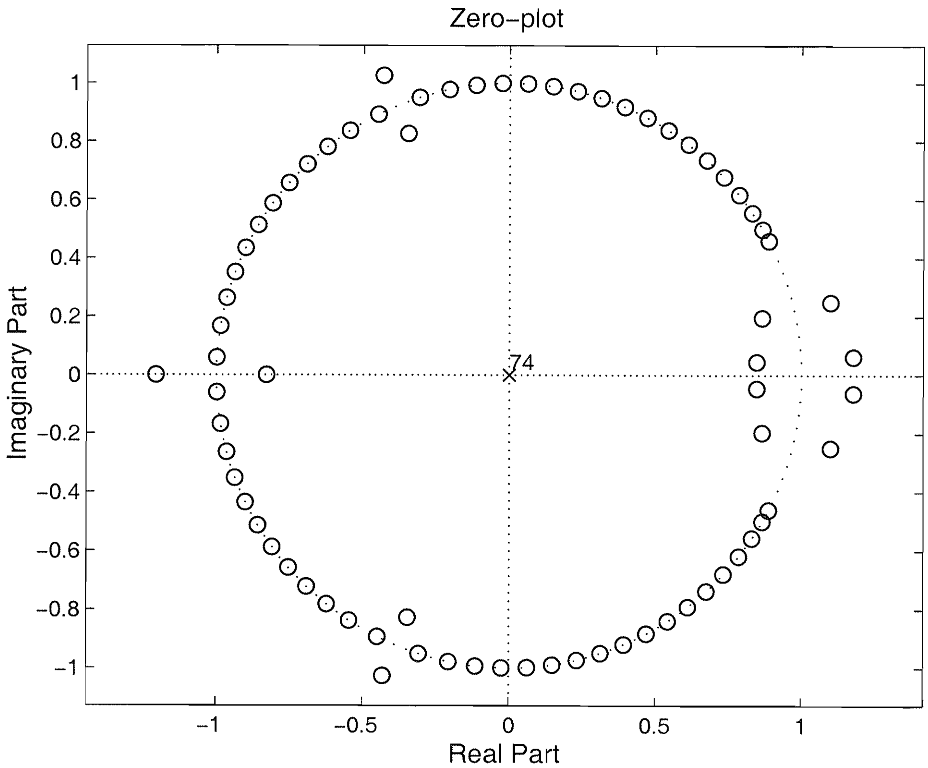
• The following four pages show the characteristics of the optimized minimax and least-squared designs.

Example Nonseparable Minimax Eighth-Band ($N = 8$) FIR filter

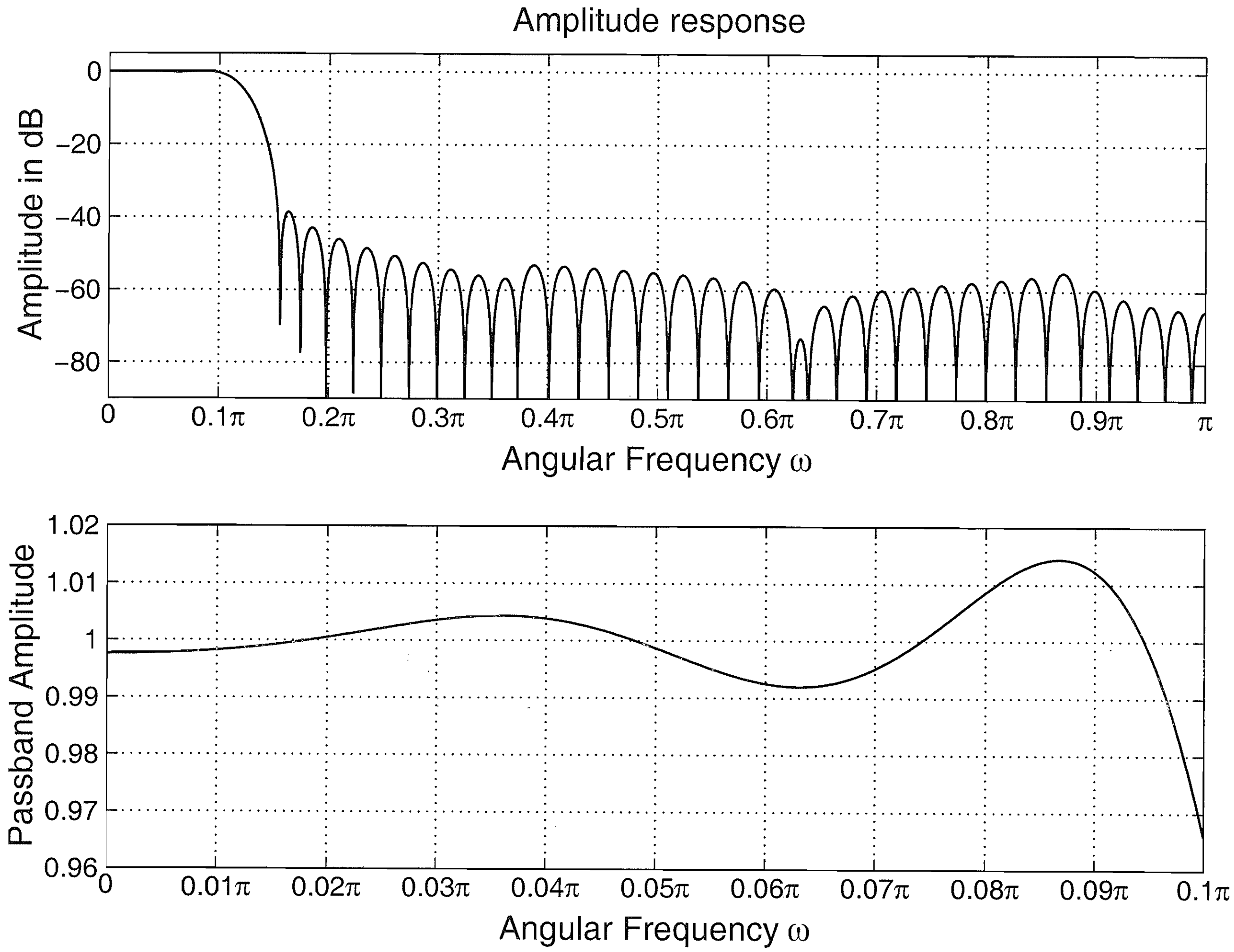


Example Nonseparable Minimax Eighth-Band ($N = 8$) FIR filter

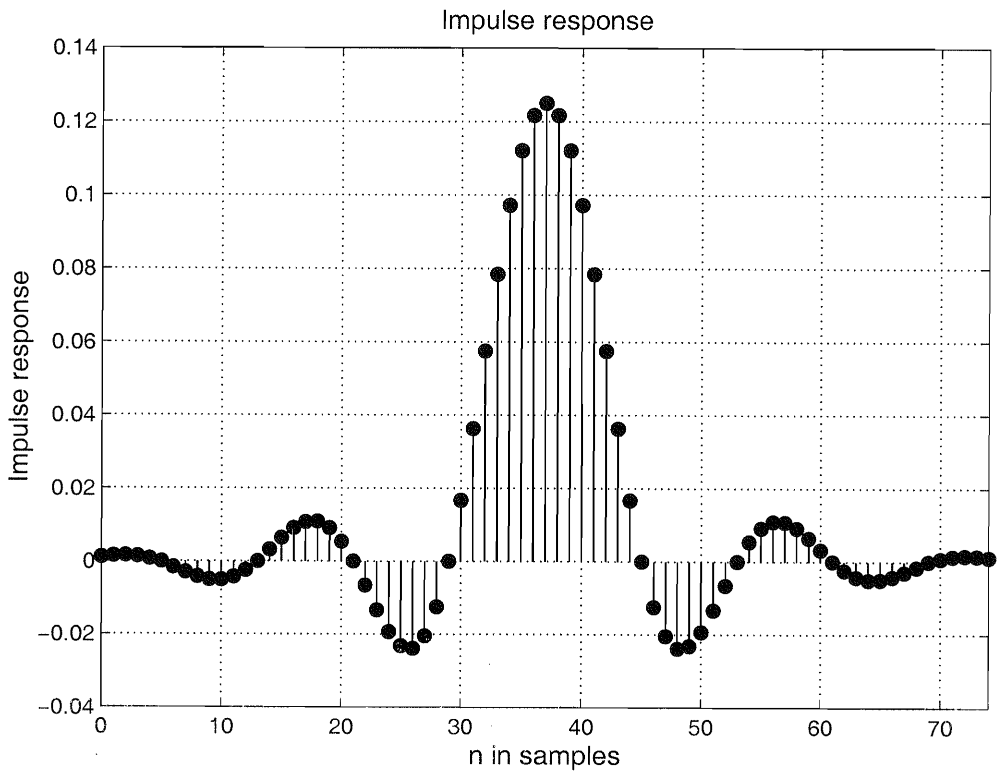


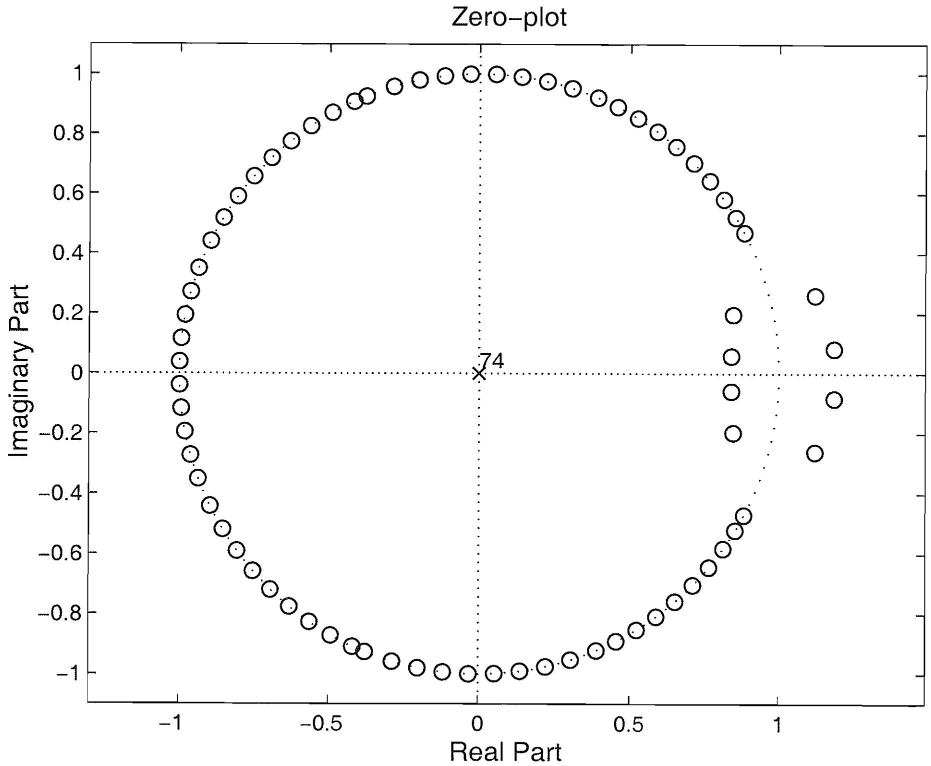


Example Nonseparable Least-Squared Eighth-Band ($N = 8$) FIR filter



Example Nonseparable Least-Squared Eighth-Band ($N = 8$) FIR filter





What are Separable Single-Stage Nth-Band Linear-Phase FIR Filters?

• In pulse shaping in telecommunication applications, it is desired that the overall Nth-band (Nyquist) filter is factorizable as

$$H(z) = T(z)R(z), \tag{34}$$

where the amplitude responses of $T(z)$ and $R(z)$ are identical.

• Here, the half-Nyquist filters $T(z)$ and $R(z)$ are used in the transmitter and in the receive, respectively.

• In order to make the desired factorizable possible, it is required that the impulse responses of $T(z)$ and $R(z)$ are time-reversed versions of each other, that is, $R(z) = z^{-M}T(z^{-1})$, where M is half the order of $H(z)$.

• In this case, the impulse responses of $T(z)$ and $R(z)$ satisfy $r(M-n) = t(n)$ for $n = 0, 1, \cdots, M$.

• In order to make $H(z)$ separable into the terms $T(z)$ and $R(z)$, it is required that the zeros of $H(z)$ occurring on the unit circle are double zeros.

• Alternatively, it is required that the zero-phase frequency response of the linear-phase $H(z)$, as given by Eq. (29), is non-negative.

Approximation Criteria

• There exist the following two criteria:

Minimax approximation: Find the filter unknowns to minimize

$$\epsilon_\infty = \max_{\omega \in [(1+\rho)\pi/N,\ \pi} |H(\omega)| \tag{35}$$

subject to

$$H(\omega) \geq 0 \quad \text{for} \quad \omega \in [0,\ \pi]. \tag{35}$$

Least-squared approximation: Find the filter unknowns to minimize

$$\epsilon_2 = \int_{(1+\rho)\pi/N}^{\pi} |H(\omega)| d\omega \tag{37}$$

subject to

$$H(\omega) \geq 0 \quad \text{for} \quad \omega \in [0,\ \pi]. \tag{38}$$

• Note that if $H(\omega)$ is nonnegative, then, after factorization, $H(\omega) = |T(e^{j\omega})|^2 = |R(e^{j\omega})|^2$. Hence, the stopband energies of $T(z)$ and $R(z)$ are minimized.

Example: $\rho = 0.2$ and $N = 8$

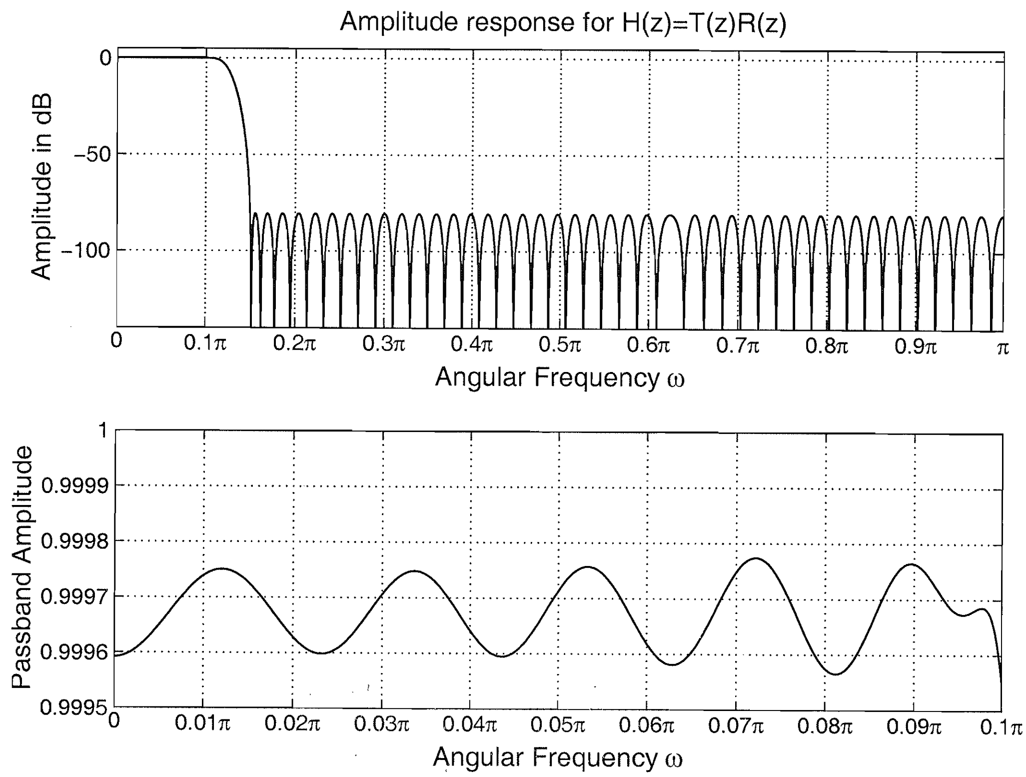
Minimax design: The minimum stopband atteunuations of both $T(z)$ and $R(z)$ is at least 40 dB, that is, δ_s is less than or equal to 0.01. For the separable transfer function $H(z) = T(z)R(z)$, the required attenuation is thus 80 dB.

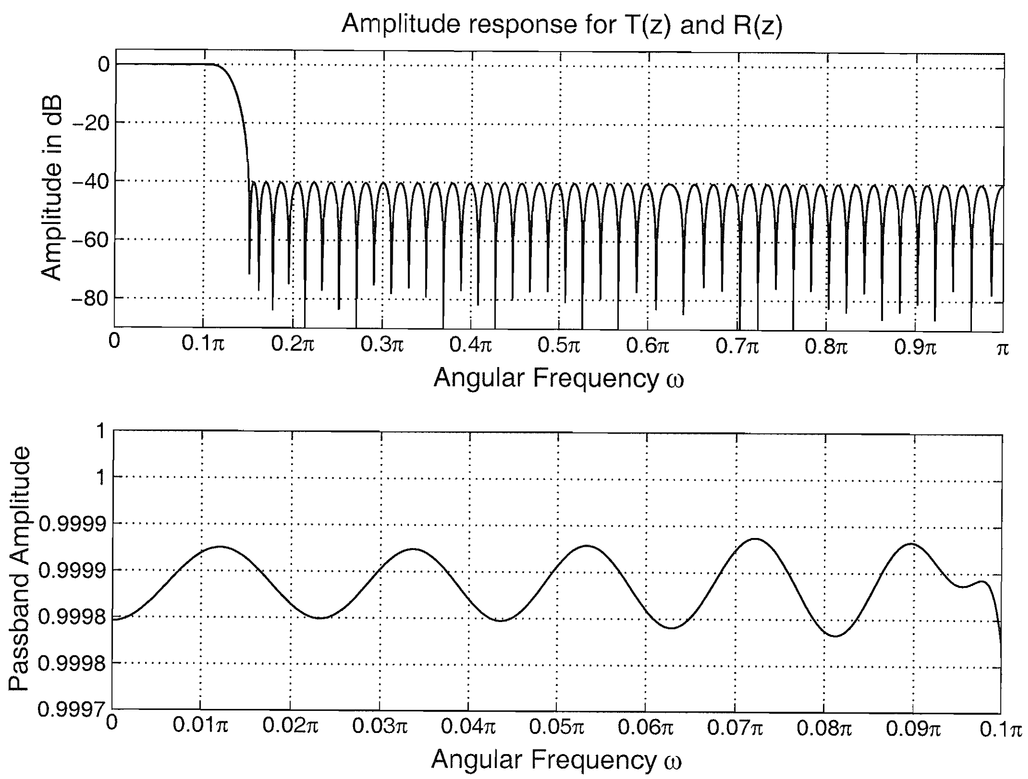
- The minimum even order of $H(z)$ to meet the given criteria is $2M = 202$.
- The orders of the minimum-phase $T(z)$ and maximum-phase $R(z)$ are thus $M = 101$ and they require 102 multipliers.

Least-squared design: It is desired to minimize the filter stopband energies of $T(z)$ and $R(z)$ for $2M = 202$.

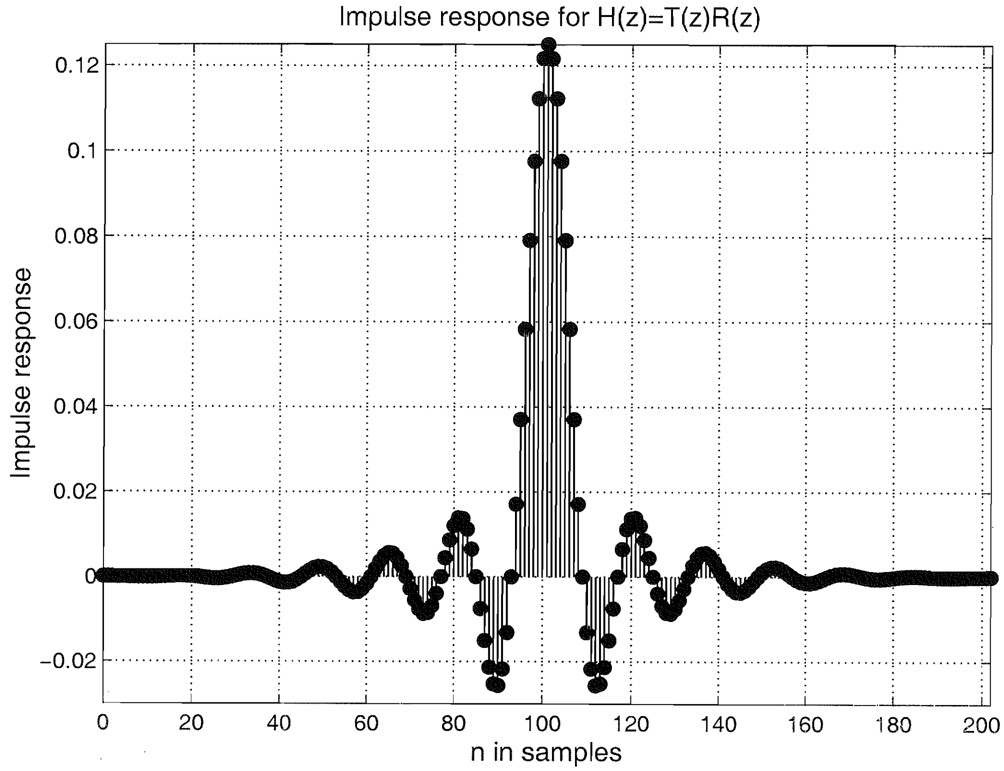
- The following eight pages show the characteristics of the optimized minimax and least-squared designs.
- For simplicity, the factorization of $H(z)$ into terms $T(z)$ and $R(z)$ has been performed such that $T(z)$ and $R(z)$ are minimum-phase and maximum-phase filters.
- Mixed-phase designs can also be obtained by sharing the off-the-unit circle zeros of $H(z)$ between $T(z)$ and $R(z)$ in different manners.

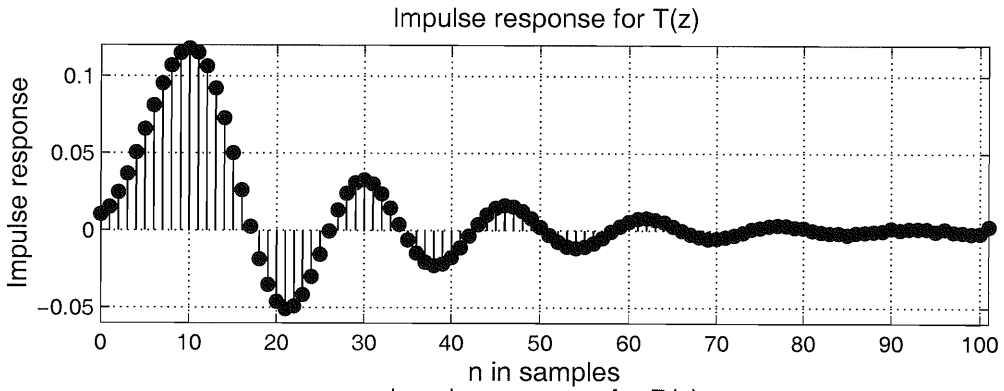
Example Separable Minimax Eighth-Band $(N = 8)$ FIR filter

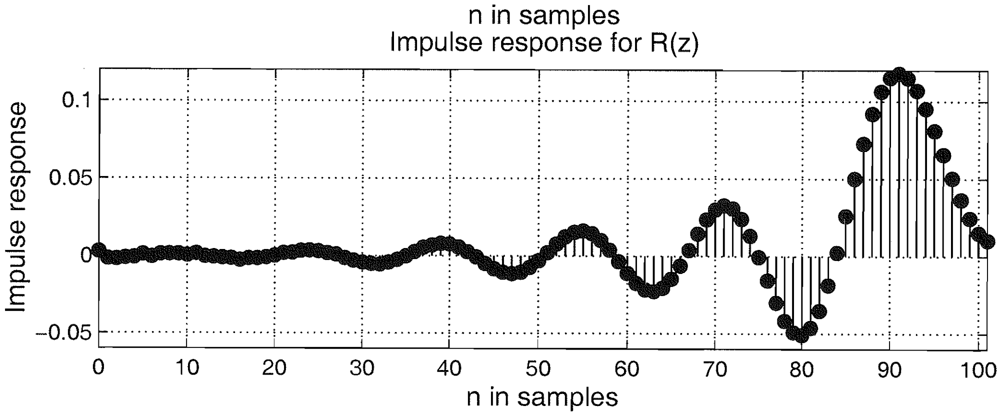




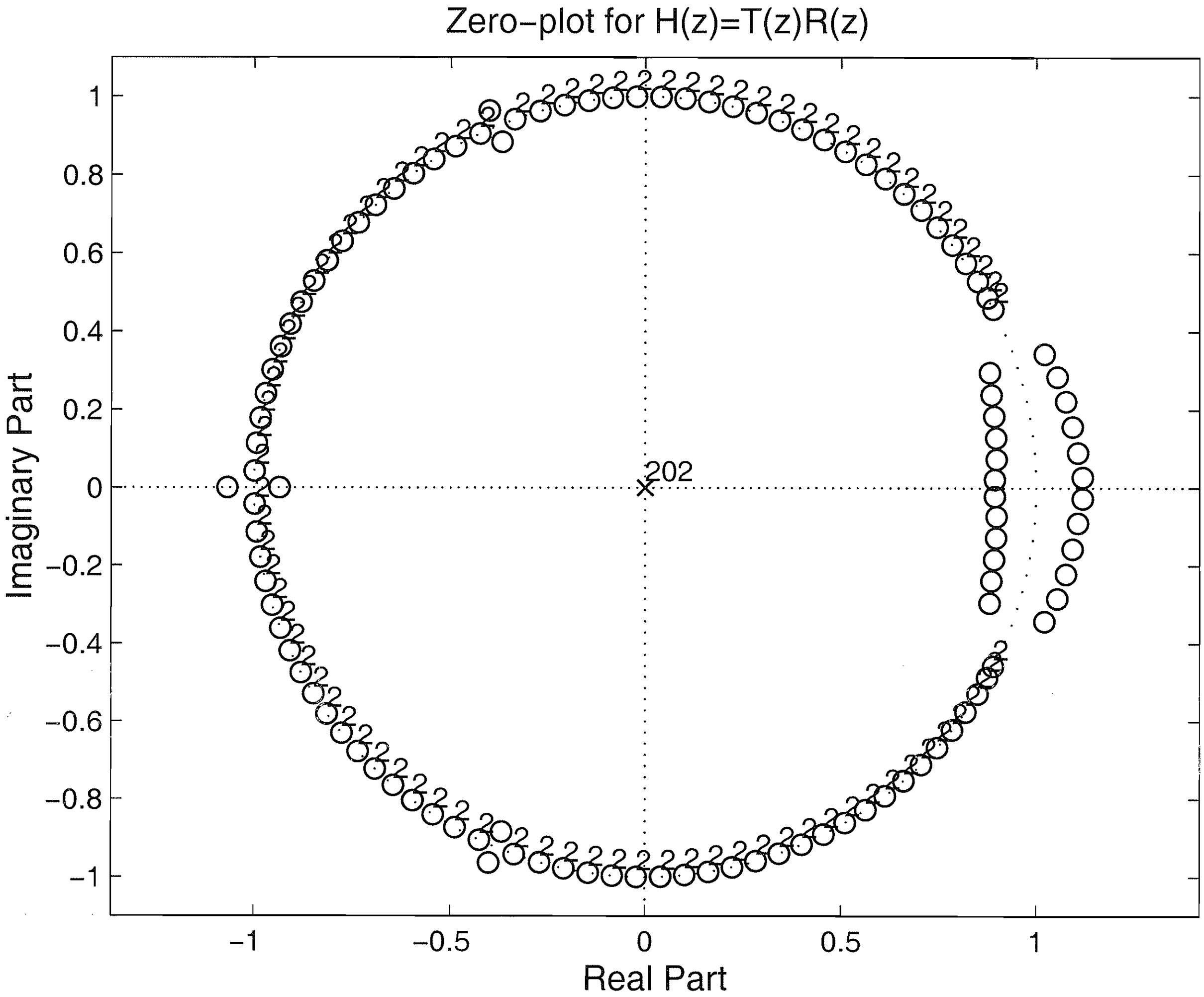
Example Separable Minimax Eighth-Band $(N = 8)$ FIR filter



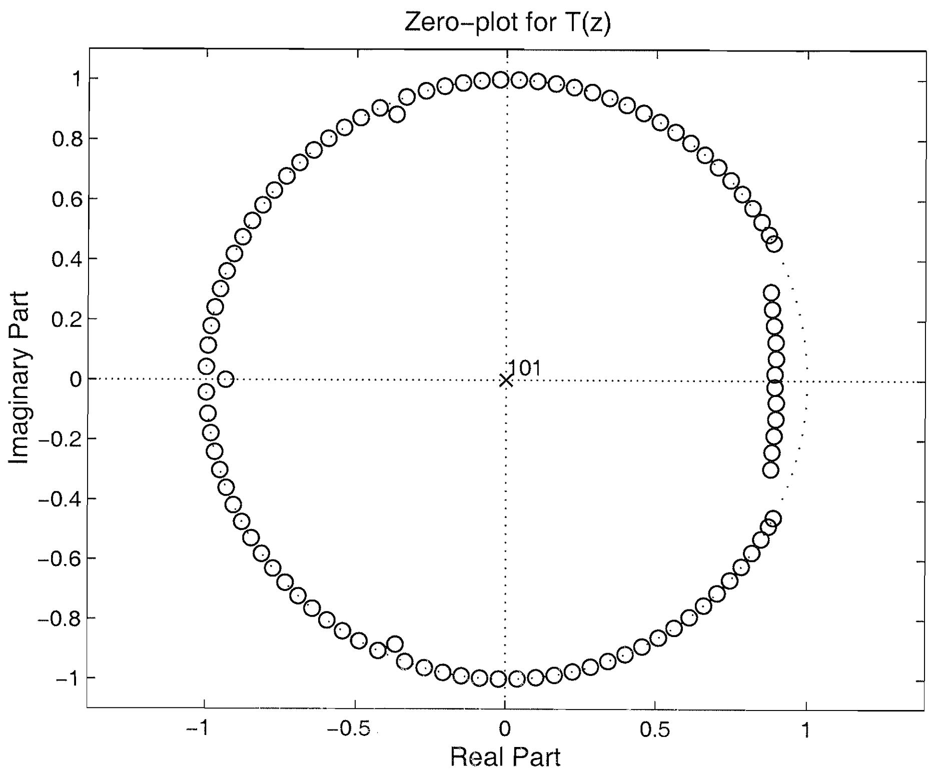


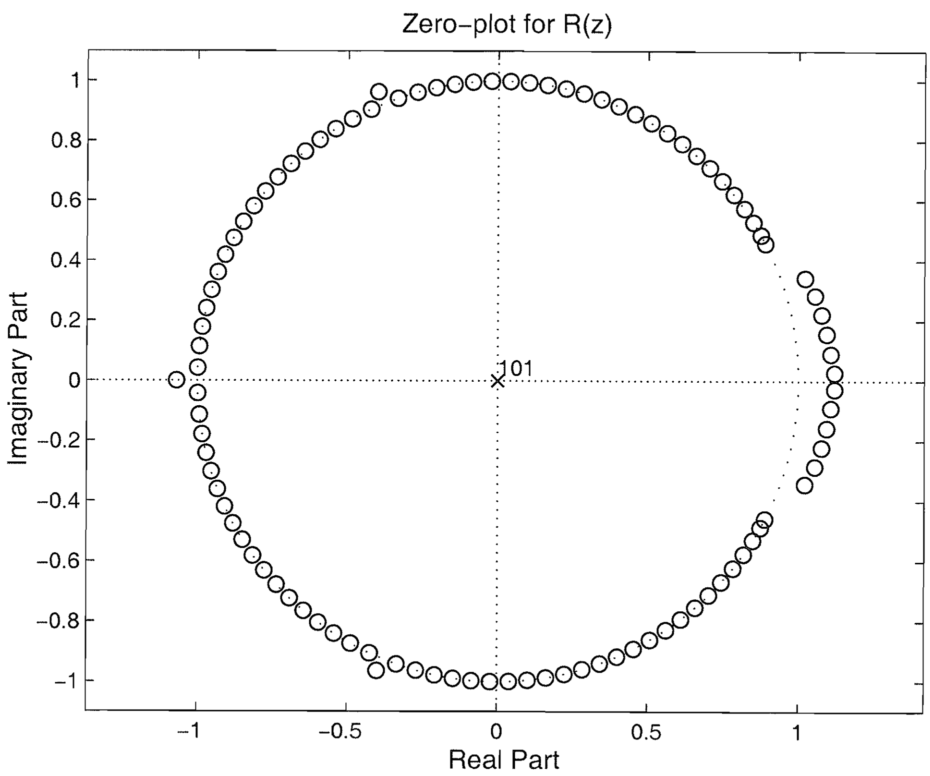


Example Separable Minimax Eighth-Band ($N = 8$) FIR filter

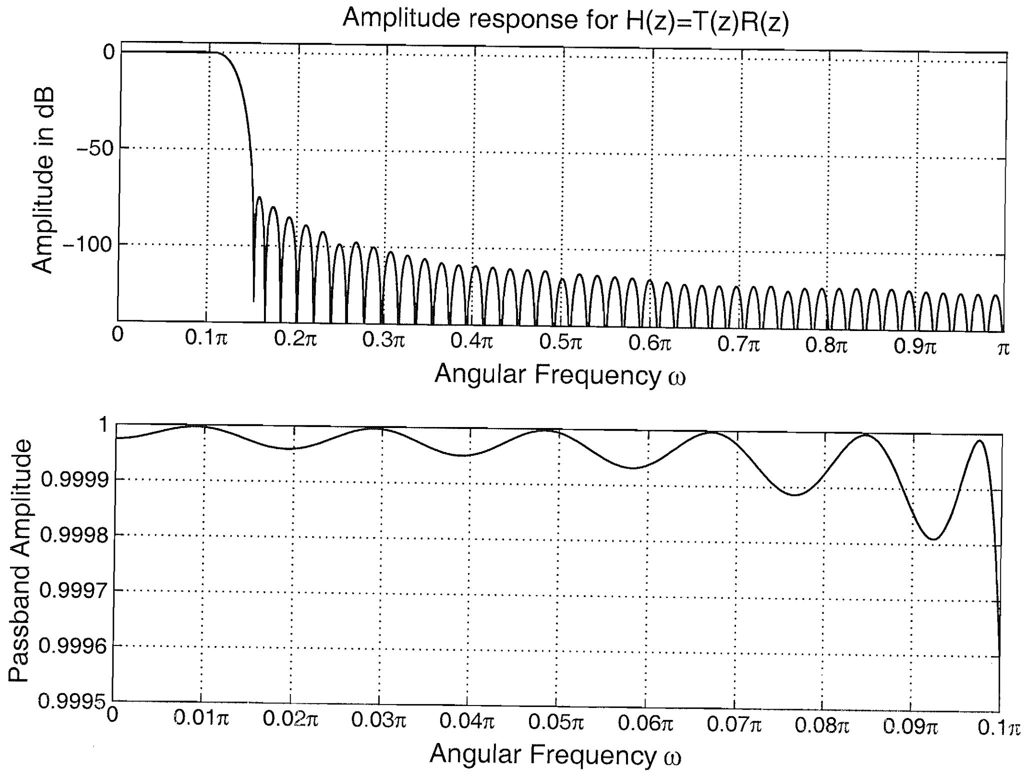


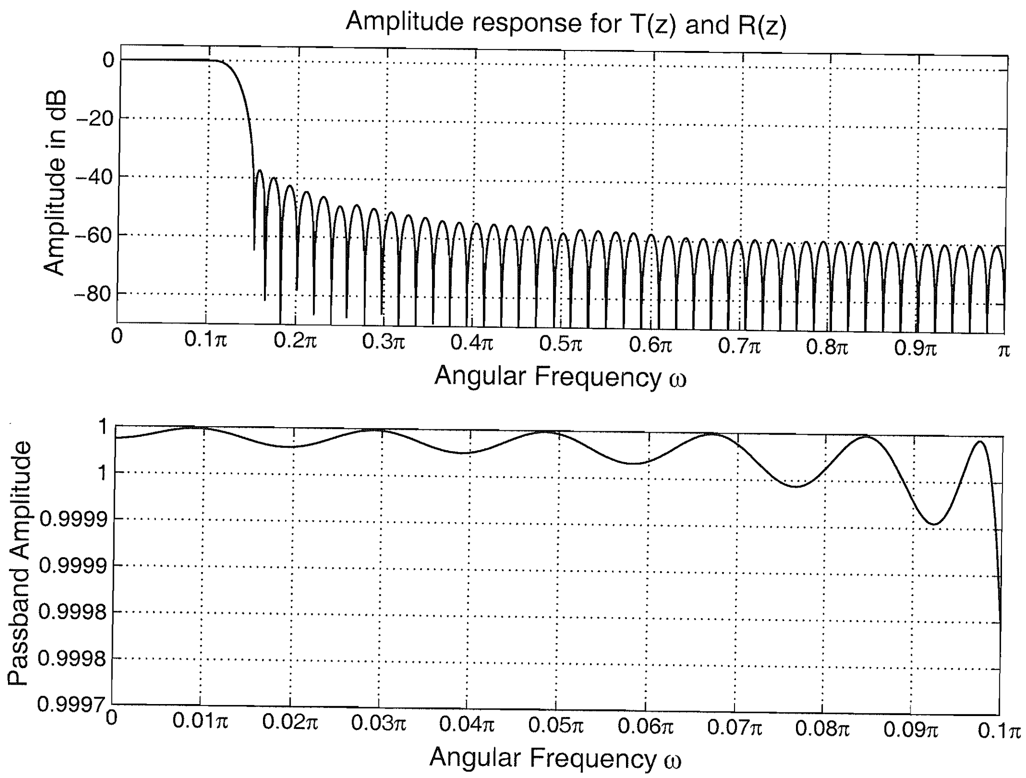
Example Separable Minimax Eighth-Band ($N = 8$) FIR filter



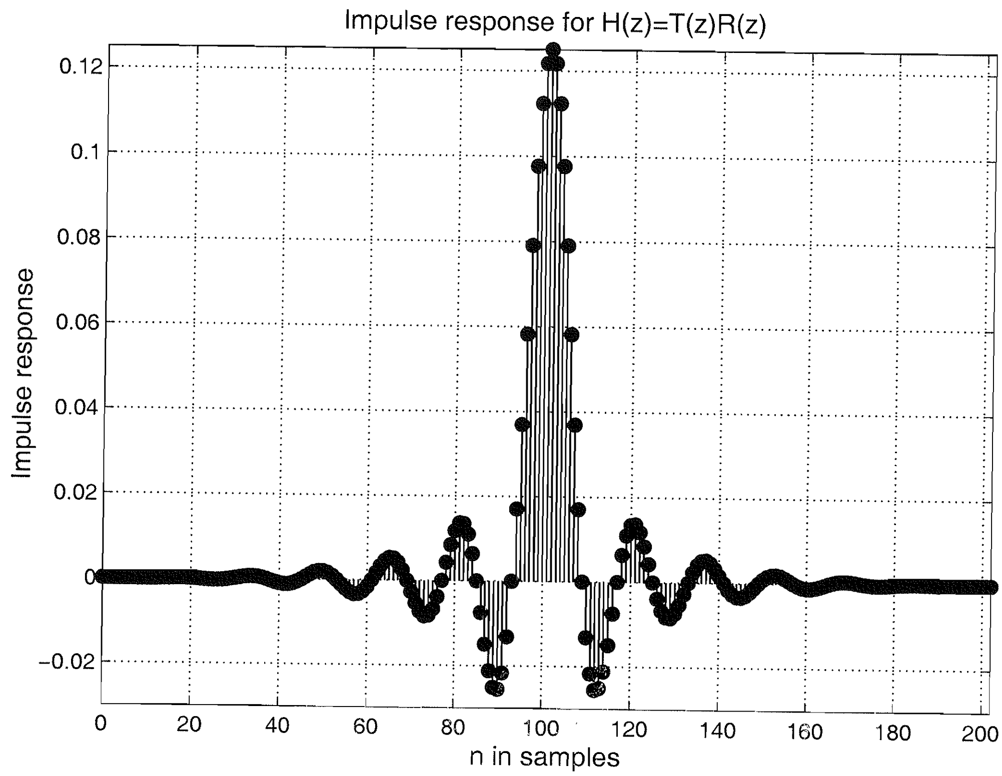


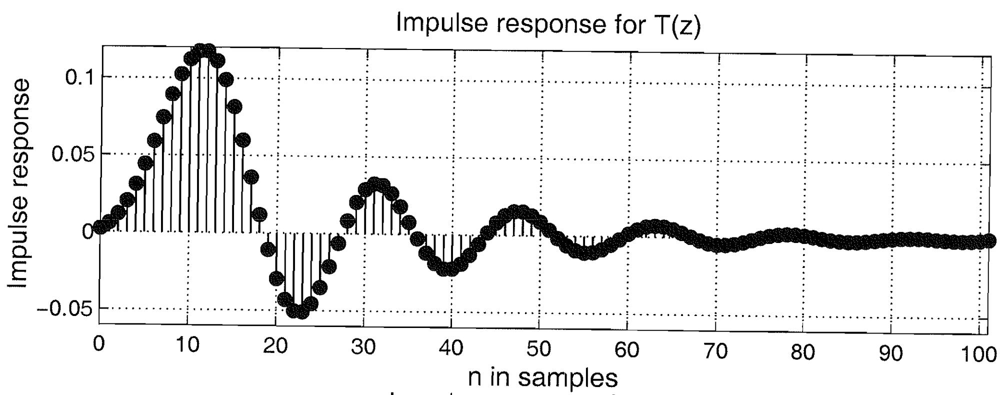
Example Separable Least-Squared Eighth-Band ($N = 8$) FIR filter

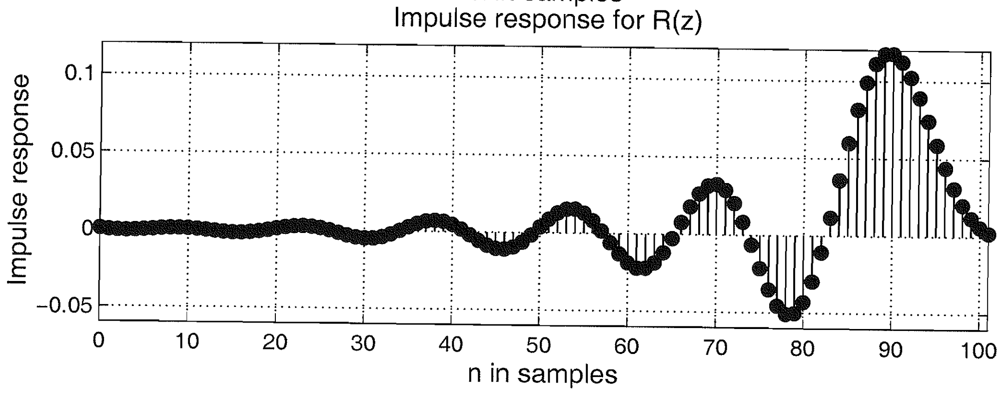




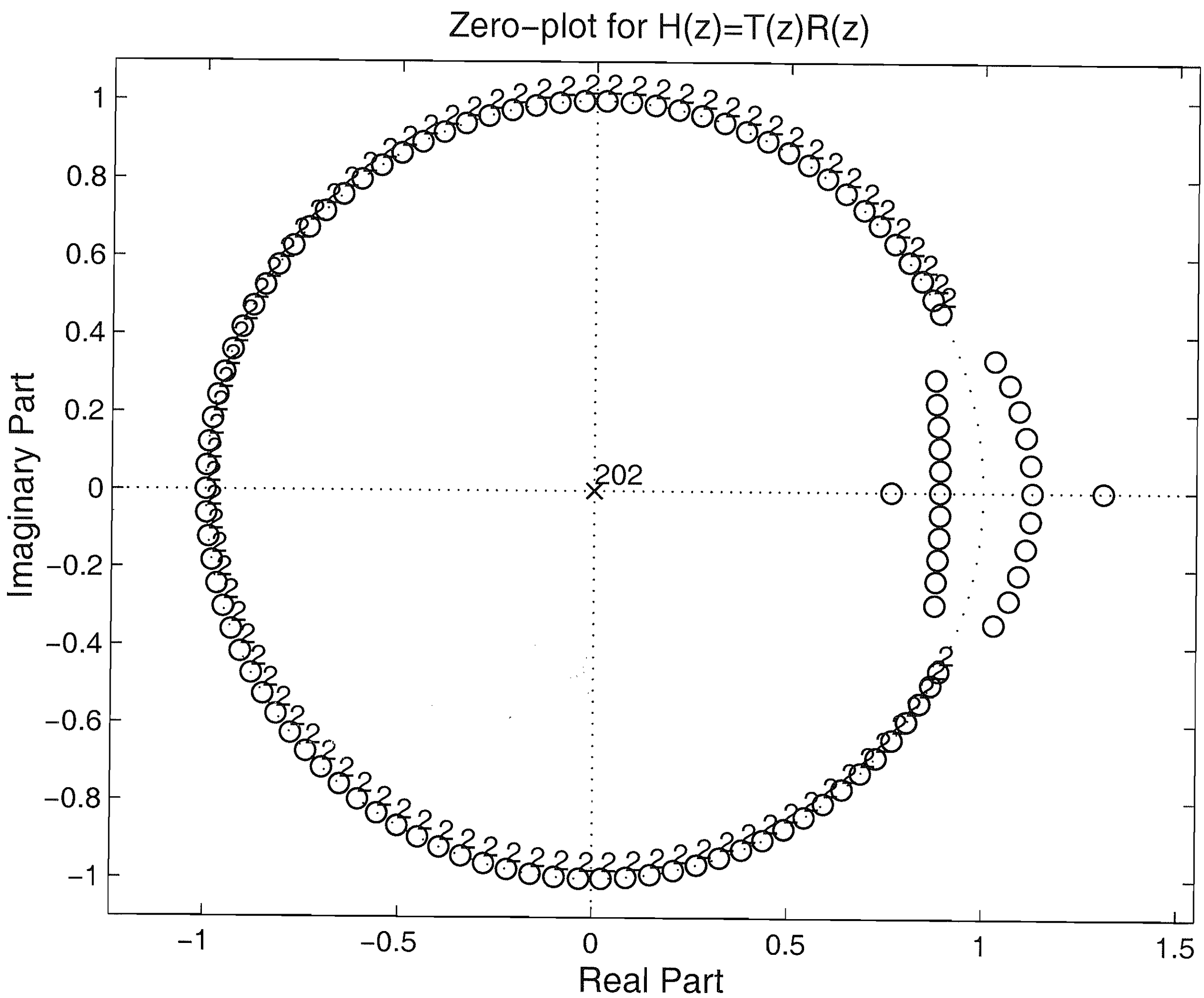
Example Separable Least-Squared Eighth-Band ($N = 8$) FIR filter



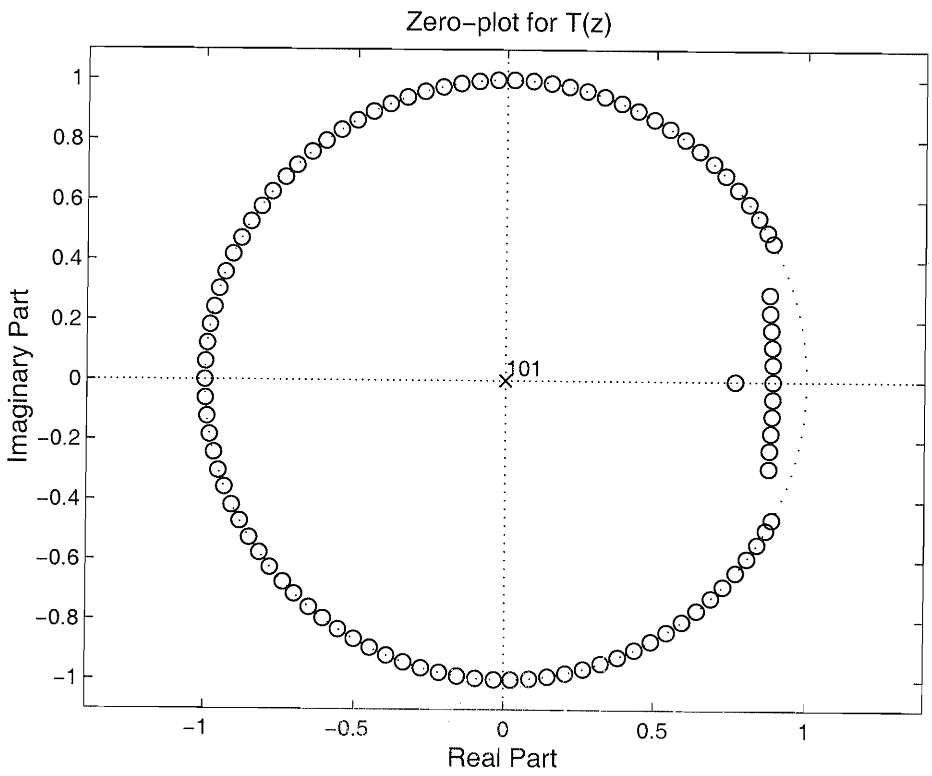


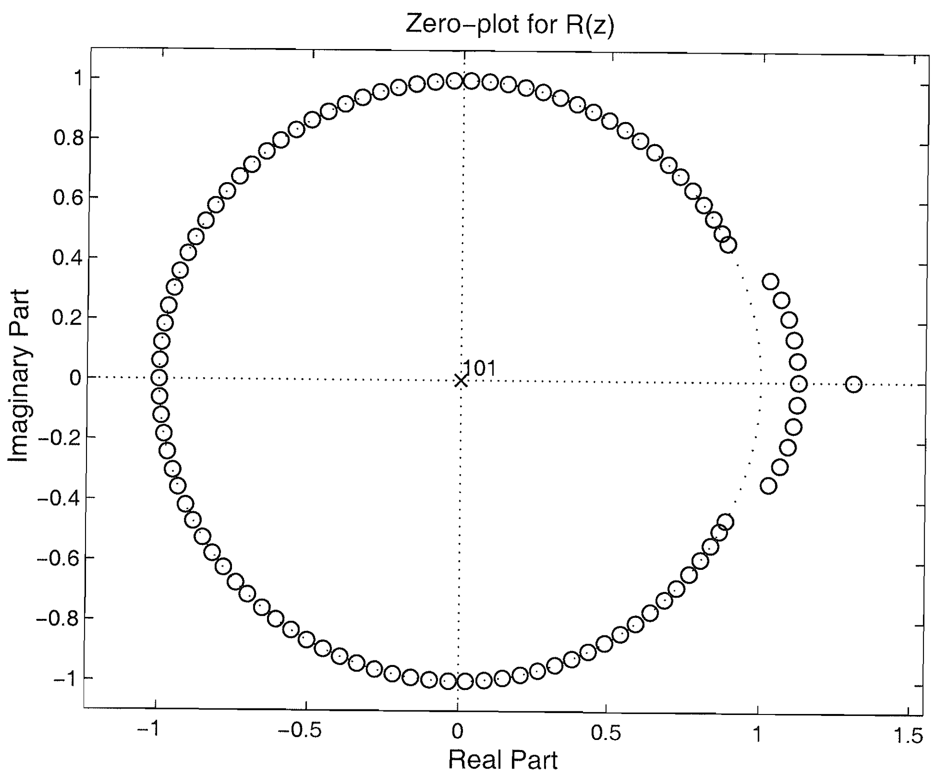


Example Separable Least-Squared Eighth-Band ($N = 8$) FIR filter



Example Separable Least-Squared Eighth-Band ($N = 8$) FIR filter





What are Multistage Nth-Band FIR Filters?

- **Let** N be factorizable as

$$N = N_1 \cdot N_2 \cdots N_K, \tag{39}$$

where the N_k's are integers and the $H_k(z)$'s for $k = 1, 2, \cdots, K$ be linear-phase N_kth-band filters, that is,

$$H_k(z) = \sum_{n=0}^{2M_k} h_k(n) z^{-n}, \tag{40a}$$

where

$$h_k(M_k) = 1/N_k \tag{40b}$$

and

$$h_k(M_k \pm rN_k) = 0 \quad \text{for} \quad r = 1, 2, \cdots, \lfloor M_k/N_k \rfloor. \tag{40c}$$

- **Then**

$$H(z) = \prod_{k=1}^{K} H_k(z^{\widehat{N}_k}), \tag{41a}$$

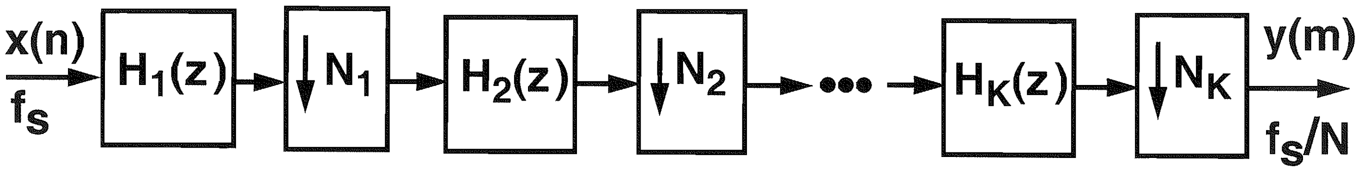
where

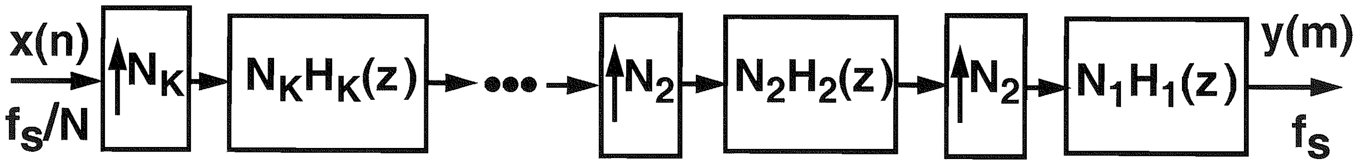
$$\widehat{N}_1 = 1, \quad \widehat{N}_k = \prod_{l=1}^{k-1} N_k, \quad k = 2, 3, \cdots, K \tag{41b}$$

is an Nth-band linear-phase FIR filter.

- The overall order of this filter is $2M = 2(\widehat{N}_1 M_1 + \widehat{N}_2 M_2 + \cdots + \widehat{N}_K M_K)$
- The main advantage of the above decomposition is that the number of multipliers is significantly reduced when compared with the direct-form implementation.

- Furthermore, if the overall filter is used for decimation or interpolation by a factor of N, then it can be implemented as shown below.
- Note that in these implementations unit delays are used.





Nonseparable Multistage Nth-Band Linear-Phase FIR Filters

- In this case, $H(z)$ is synthesized directly in the form of Eq. (41).
- The corresponding zero-phase frequency response is expressible as

$$H(\omega) = \prod_{k=1}^{K} H_k(\widehat{N}_k\omega), \qquad (42a)$$

where

$$H_k(\omega) = 1/N_k + 2\sum_{n=1}^{M_k} h(M_k - n)\cos(n\omega). \qquad (42b)$$

- It can be shown that the maximum deviation of $H(\omega)$ from zero is less than or equal to δ_s if each $H_k(\omega)$ is designed in such a way that

$$\epsilon_\infty^{(k)} = \max_{\omega\in\Omega_s^{(k)}} |W_k(\omega)H_k(\omega)|, \qquad (43a)$$

where

$$W_k(\omega) = \prod_{\substack{l=1\\ l\neq k}}^{K} H_l(\widehat{N}_l\omega/\widehat{N}_k), \qquad (43b)$$

and

$$\Omega_s^{(k)} = \begin{cases} [(1+\rho)\pi/N_K,\ \pi] & \text{for } k = K \\ \bigcup_{l=1}^{\lfloor N_k/2\rfloor} [\frac{(2l-\alpha_k)\pi}{N_k},\ \min(\frac{(2l+\alpha_k)\pi}{N_k},\ \pi)] & \\ & \text{for } k < K. \end{cases} \qquad (43c)$$

with

$$\alpha_k = (1+\rho)\widehat{N}_{k+1}/N \tag{43d}$$

becomes less than or equal to δ_s.

• Note that $W_k(\omega)H_k(\omega)$ is obtained from $H(\omega)$ by dividing ω by $\widehat{N}_k$.

• The overall filter can be designed in the minimax sense as follows:

Step 1: Set $H_k(\omega) \equiv 1$ for $k = 1, 2, \cdots K$.

Step 2: Design successively $H_k(\omega)$ for $k = 1, 2, \cdots K$ to minimize $\epsilon_\infty^{(k)}$.

Step 3: Repeat Step 2 until the difference between the successive overall solutions is within the given tolerance limits.

• Typically, 3 to 5 interations of the above algorithm is required to arrive at the desired overall solution. What is left is to find the minimum orders $2M_k$ required to make all the $\epsilon_\infty^{(k)}$'s less than or equal to the given stopband ripple of δ_s.

• In the least-mean-square case, the basic difference is that the quantity to be minimized at Step 2 is

$$\epsilon_2^{(k)} = \int_{\Omega_s^{(k)}} [W_k(\omega)H_k(\omega)]^2 d\omega. \tag{44}$$

Example: $\rho = 0.2$ and $N = 8$

Minimax design with $K = 2$: The given criteria are met by $N_1 = 4$, $N_2 = 2$, $2M_1 = 14$, and $2M_2 = 18$. The minimum stopband atteunuation is at least 40 dB, that is, δ_s is less than or equal to 0.01.

- By exploiting the coefficient symmetries and the facts that $h_1(3) = h_1(11) = 0$ and $h_2(9 \pm 2r) = 0$ for $r = 1, 2, 4$ and assuming that the implementations of $h_1(7) = 1/4 = 2^{-2}$ and $h_1(9) = 1/2 = 2^{-1}$ require no multipliers, the overall design has $6 + 5 = 11$ multipliers.
- The number of delays is $2(M_1 + N_1 M_2) = 86$.
- In the following, there are 5 pages illustrating the characteristics of this design.
- The corresponding one-stage design requires $2M = 74$, having 32 multipliers and 74 delays.

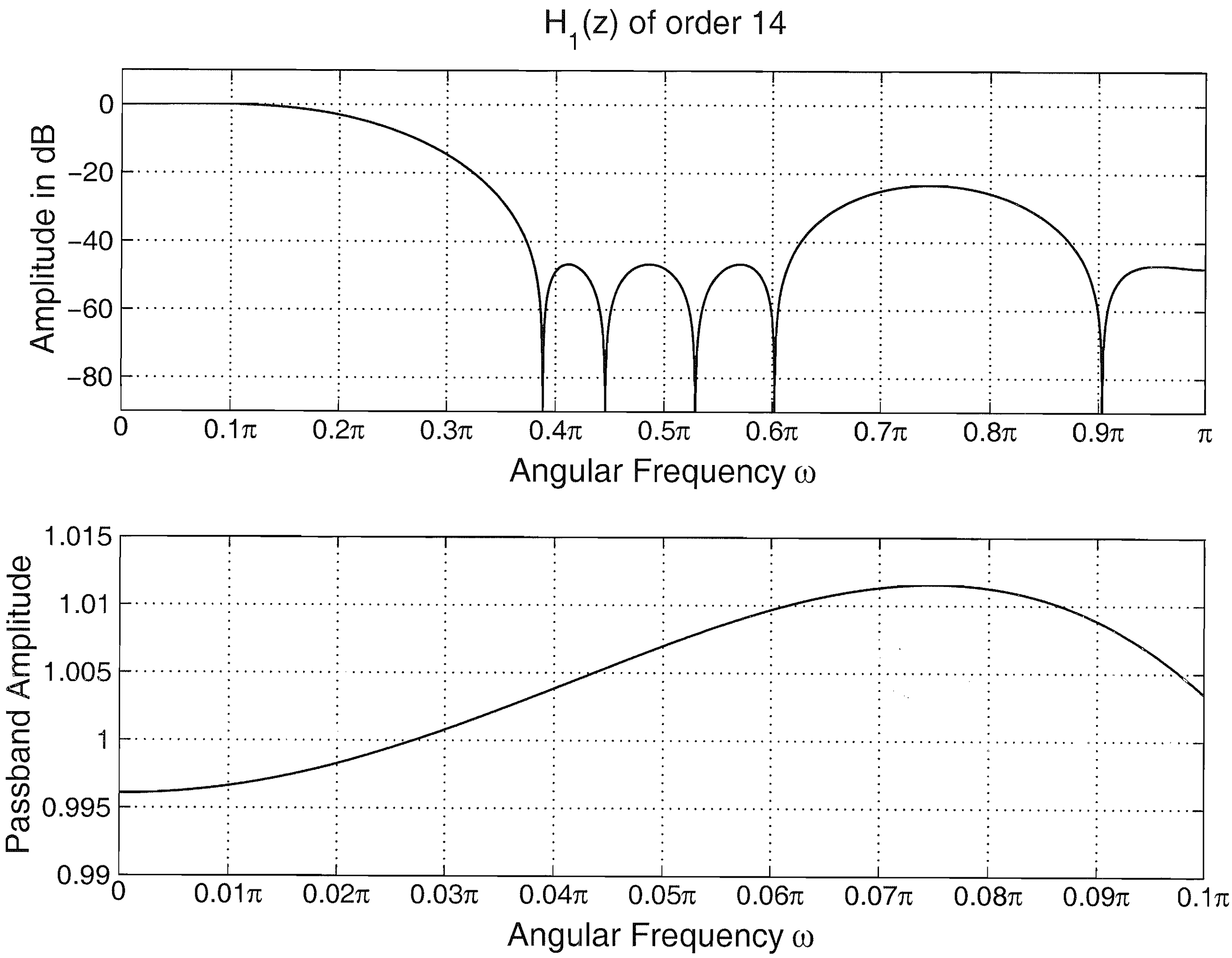
Minimax design with $K = 3$: The criteria are met by $N_1 = N_2 = N_3 = 2$ $2M_1 = 2M_2 = 6$, and $2M_3 = 18$.

- In the following, there are 6 pages illustrating the characteristics of this design.
- The number of non-trivial coefficients is only $2 + 2 + 5 = 7$ and the overall number of delay is 90.
- When used for decimation (interpolation) by 8, then the number of multiplications per input sample (per output sample) are for the one-stage, two-stage, and

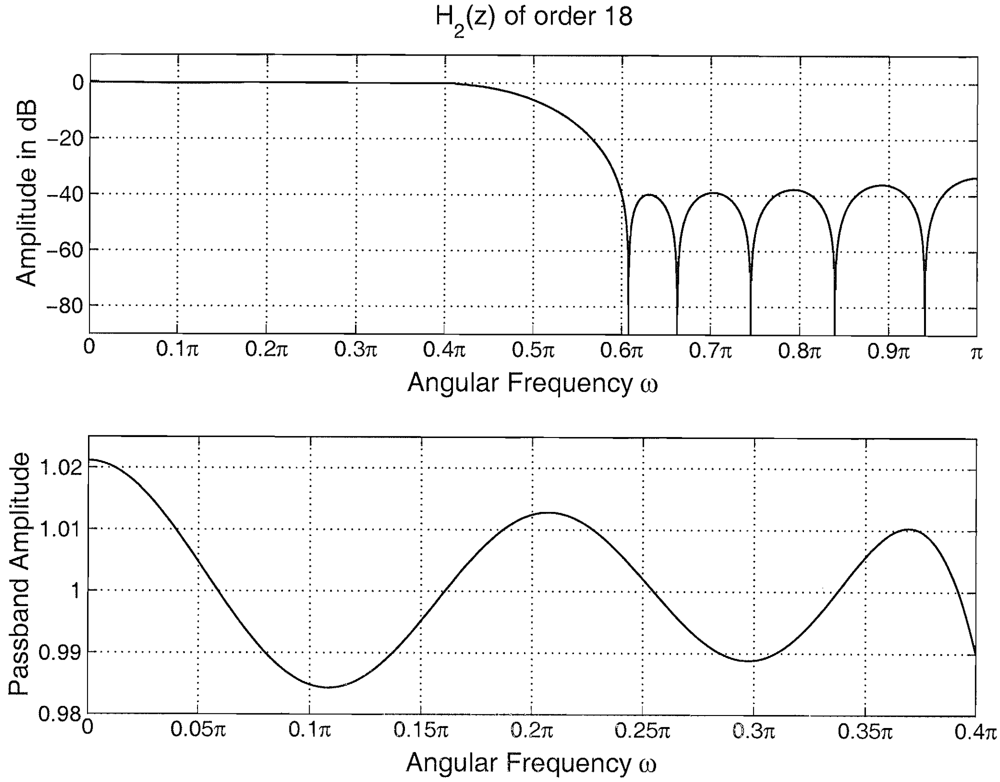
three-stage designs are $32/8 = 4$, $6/4 + 5/8 = 2.125$, and $2/2 + 2/4 + 5/8 = 2.125$, respectively.

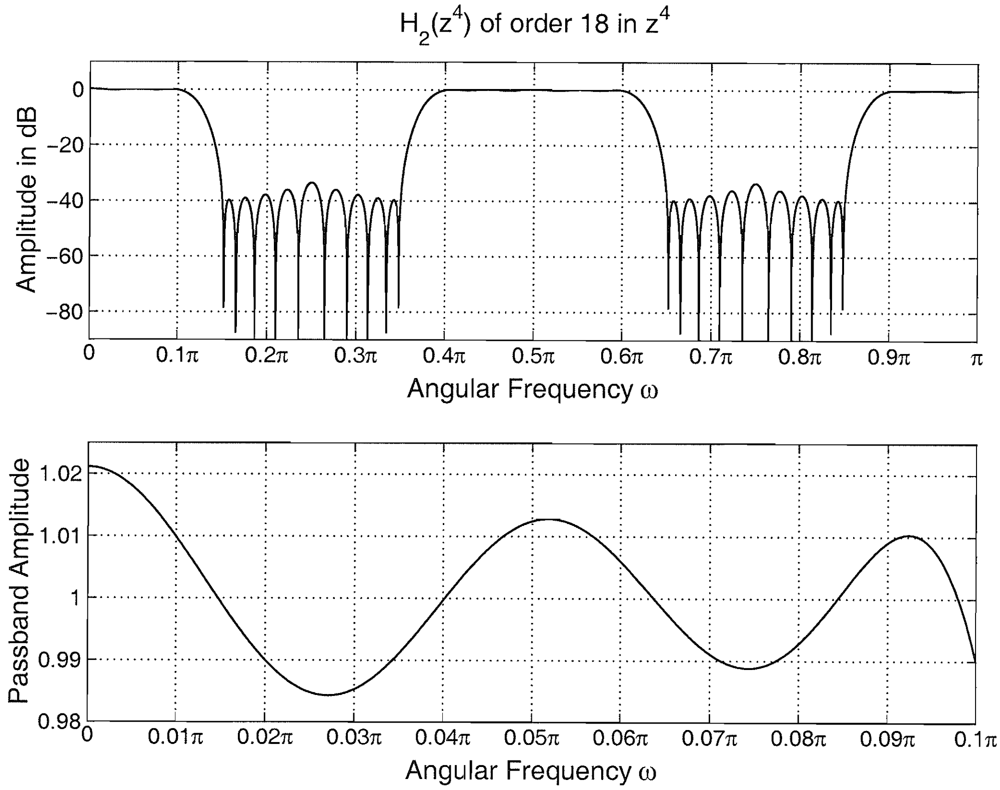
Least-squared design with $K = 3$: In the following, there are 6 pages illustrating the cracteristics of the least-squared three-stage filter with the same subfilter orders as for the above minimax design.

Nonseparable Two-Stage Minimax Eighth-Band ($N = 8$) FIR filter

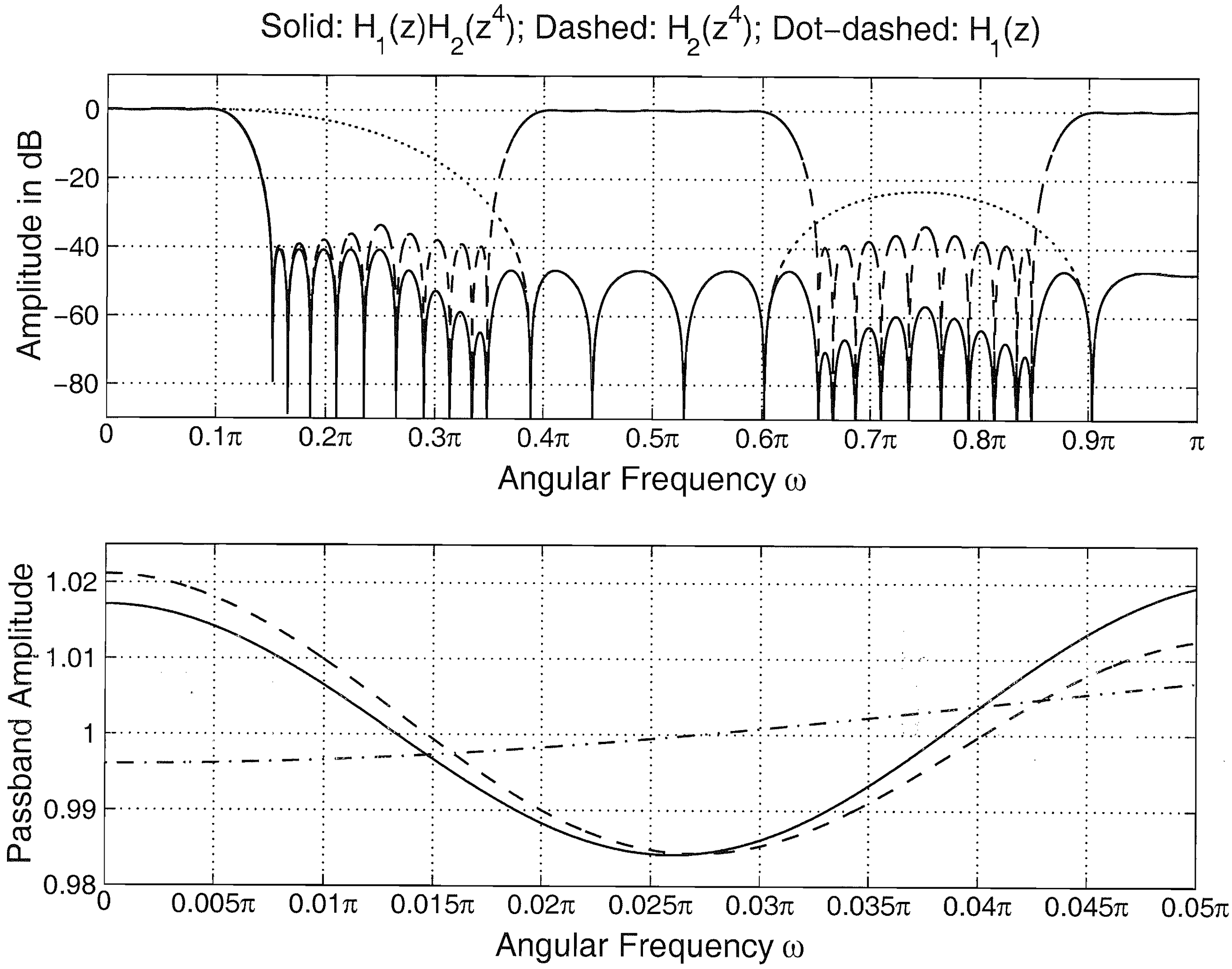


Nonseparable Two-Stage Minimax Eighth-Band ($N = 8$) FIR filter

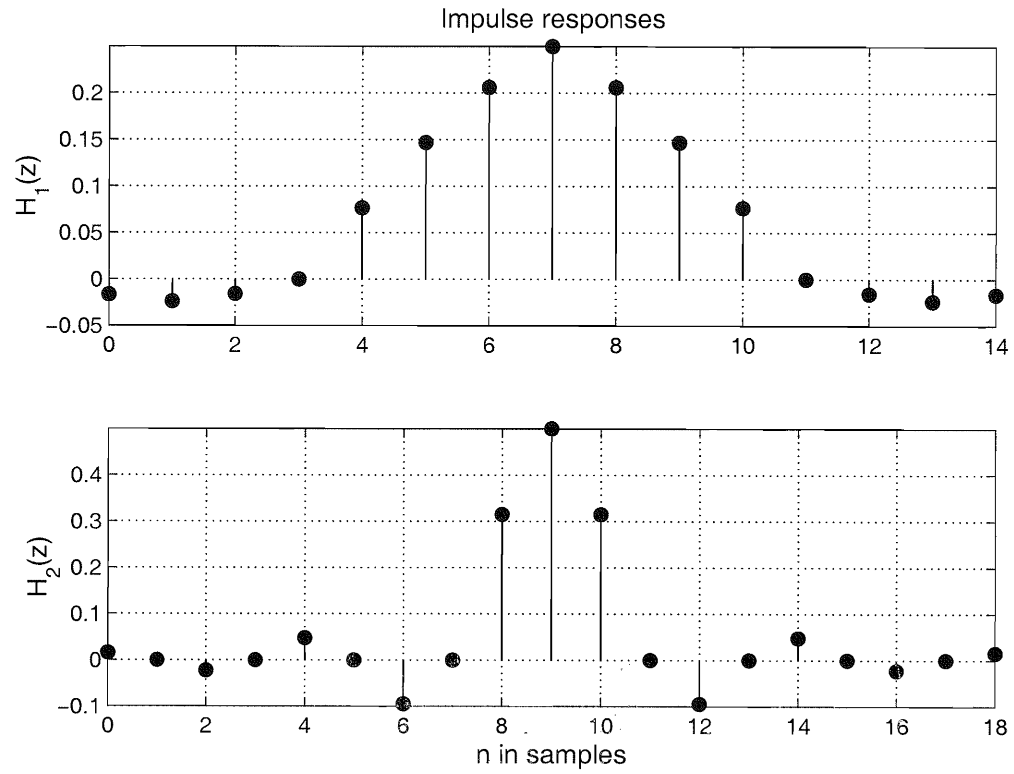


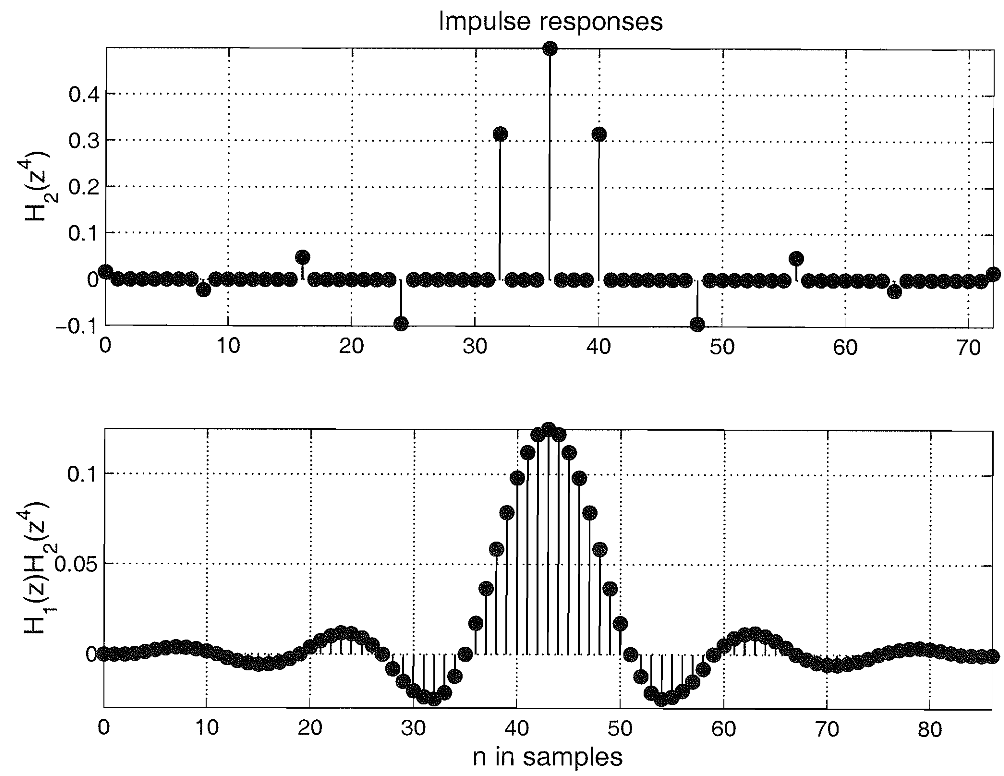


Nonseparable Two-Stage Minimax Eighth-Band ($N = 8$) FIR filter

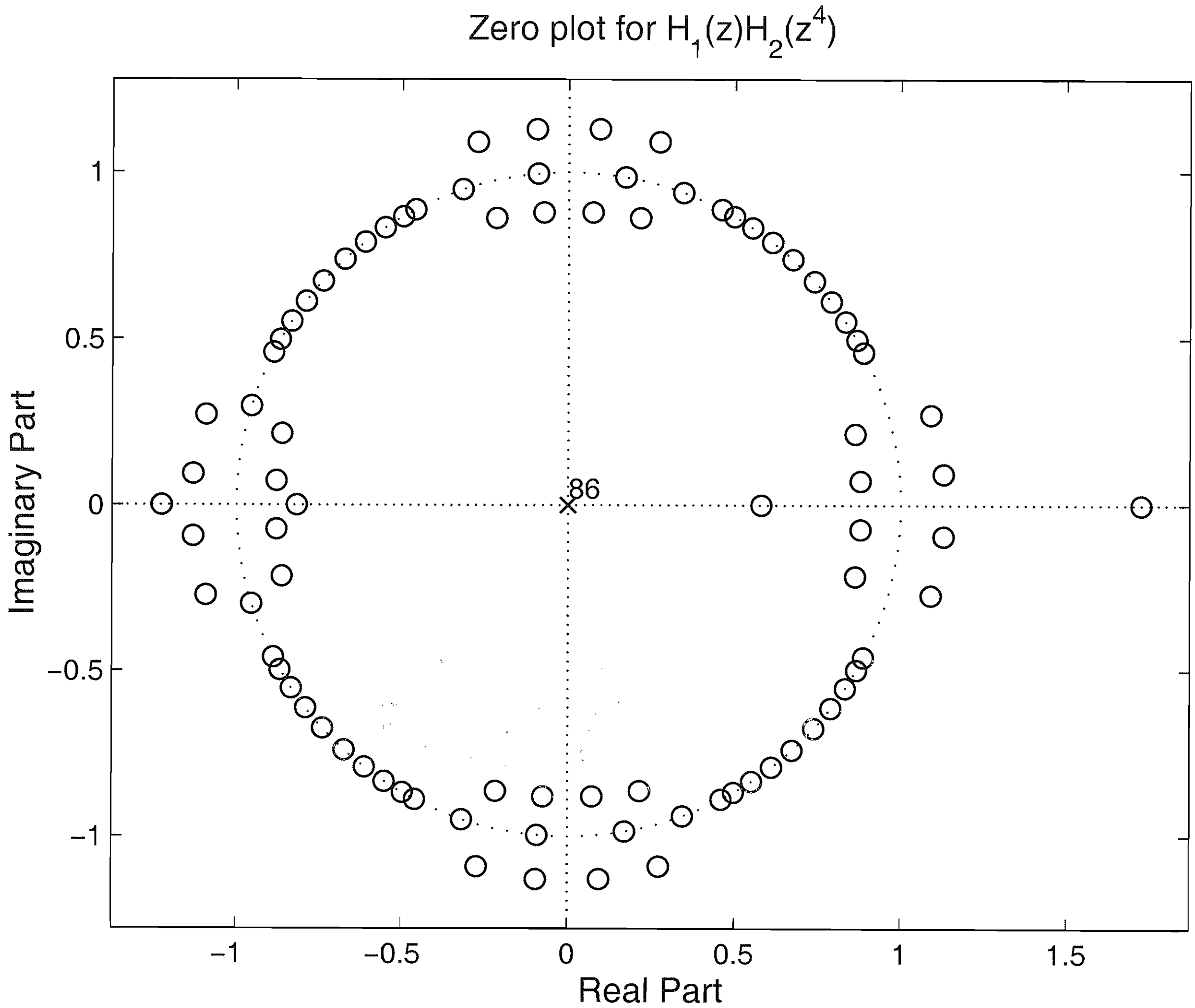


Nonseparable Two-Stage Minimax Eighth-Band ($N = 8$) FIR filter

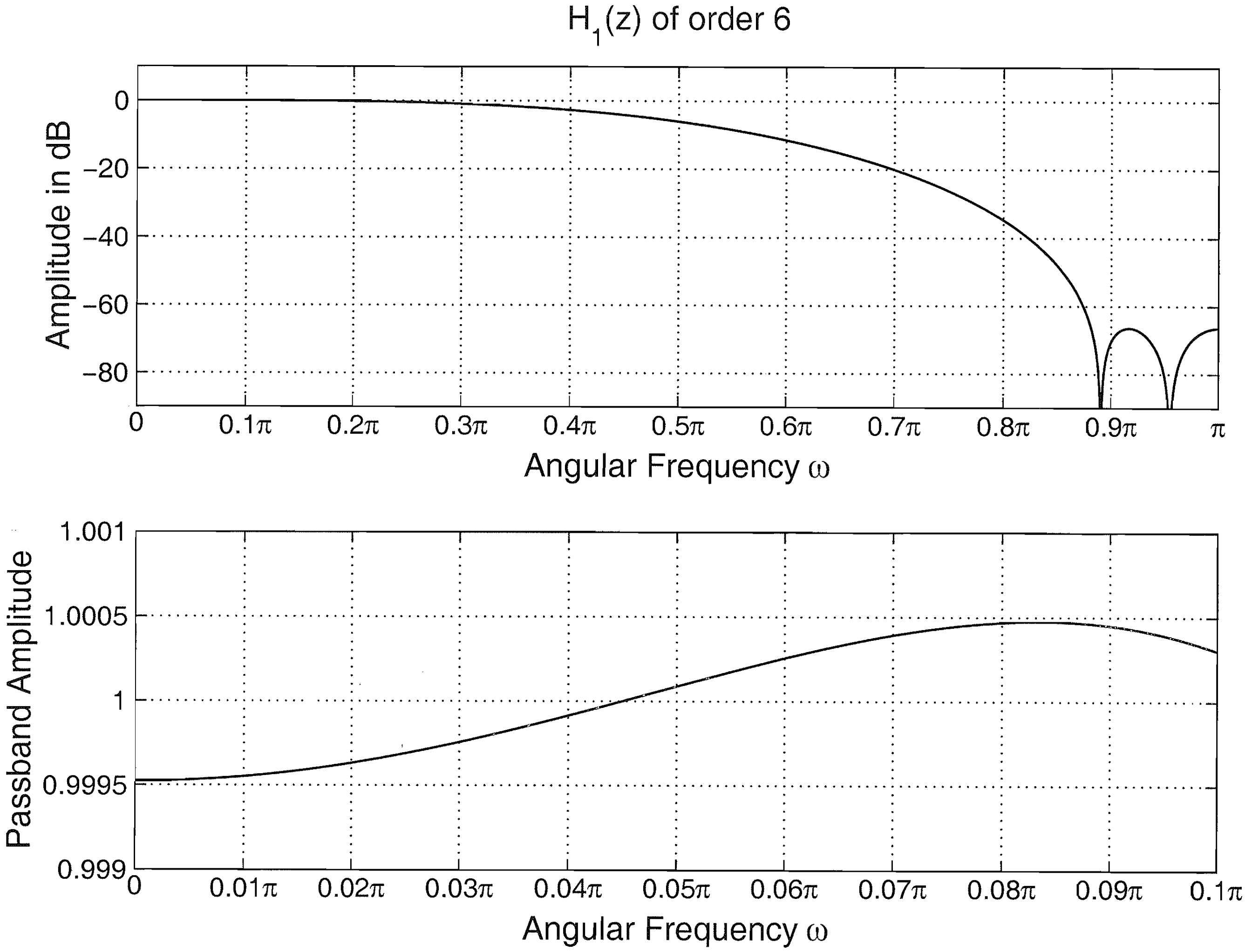




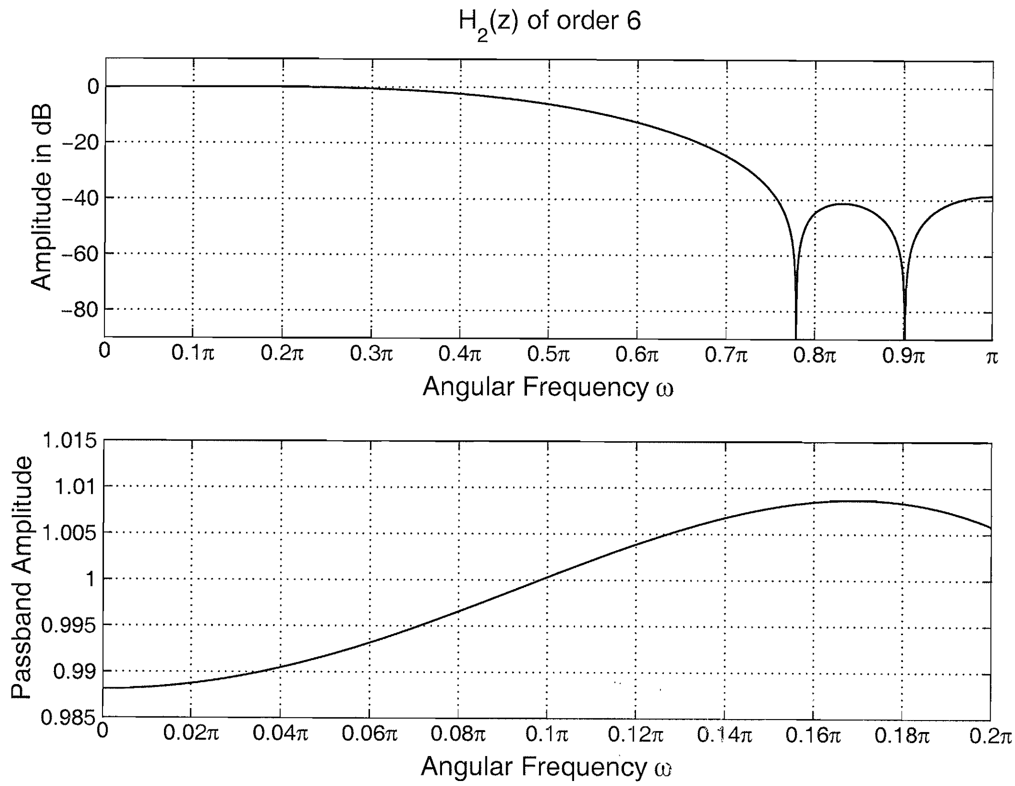
Nonseparable Two-Stage Minimax Eighth-Band ($N = 8$) FIR filter

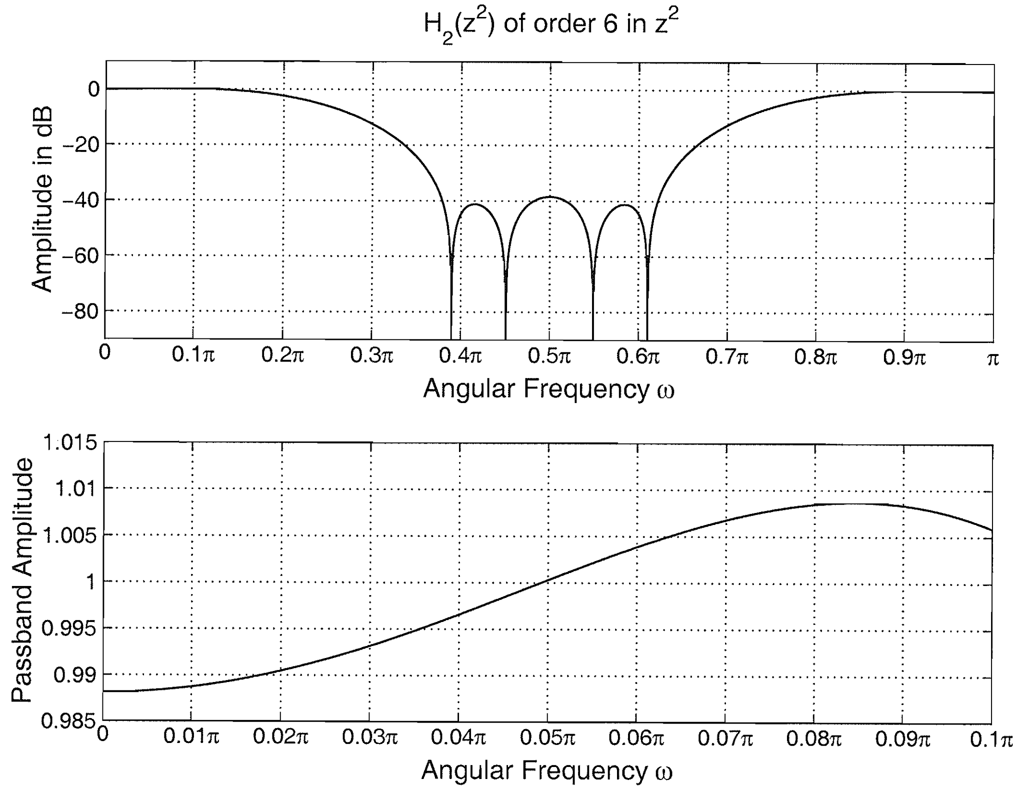


Nonseparable Three-Stage Minimax Eighth-Band ($N = 8$) FIR filter

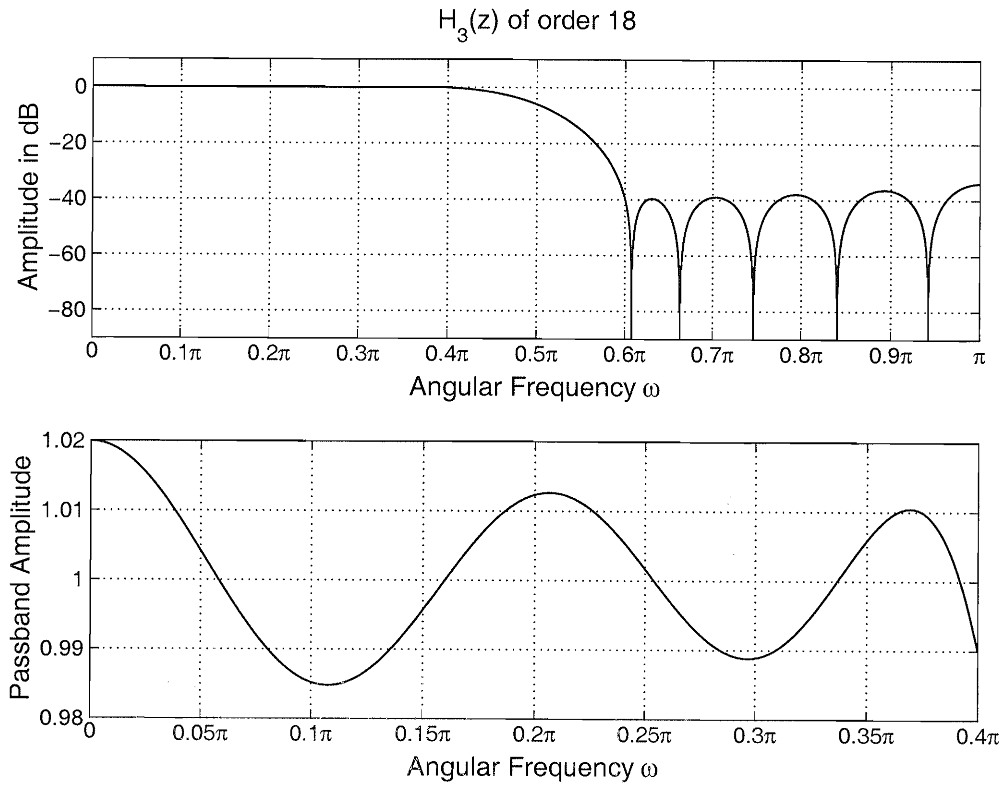


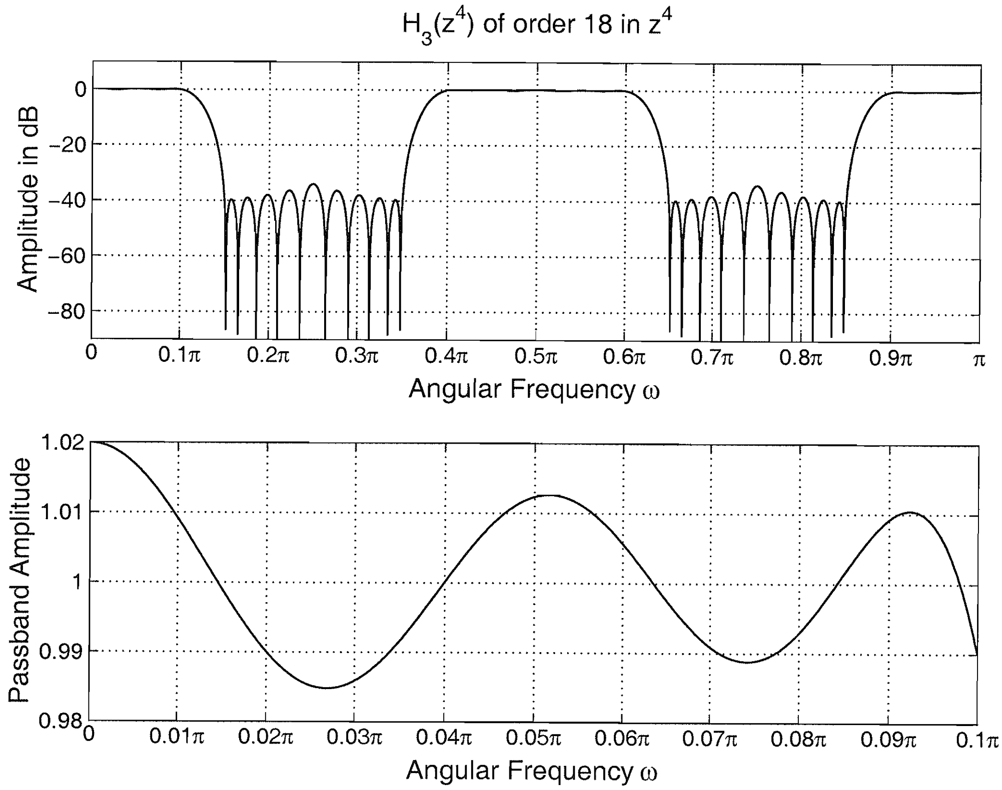
Nonseparable Three-Stage Minimax Eighth-Band ($N = 8$) FIR filter



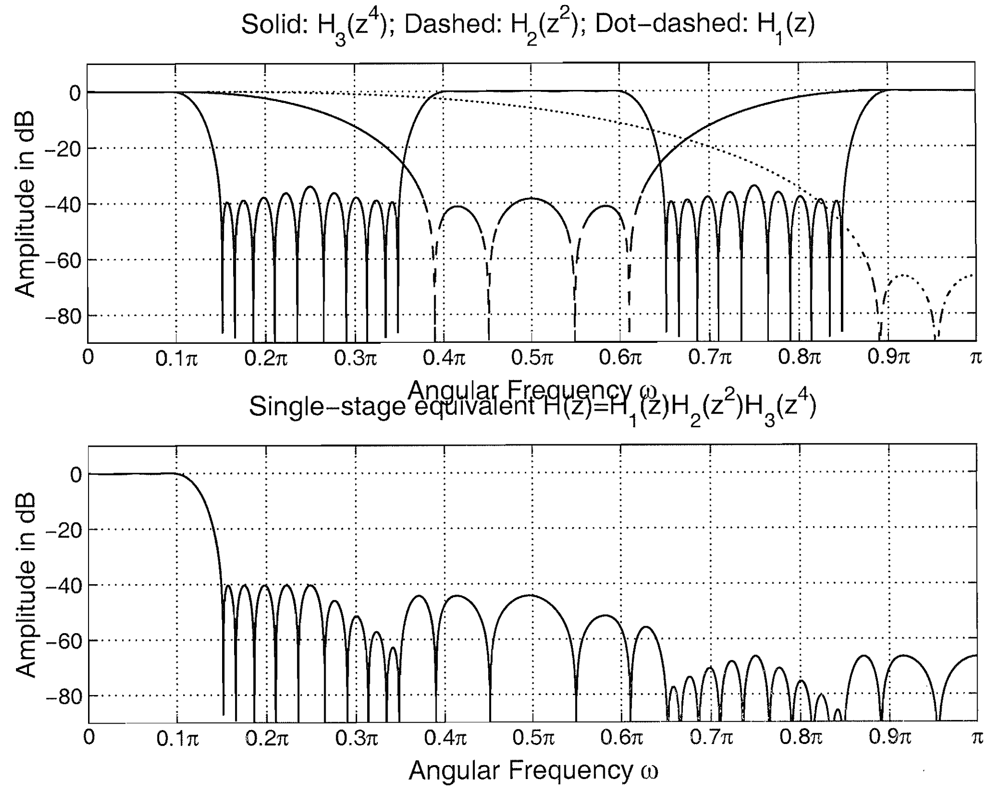


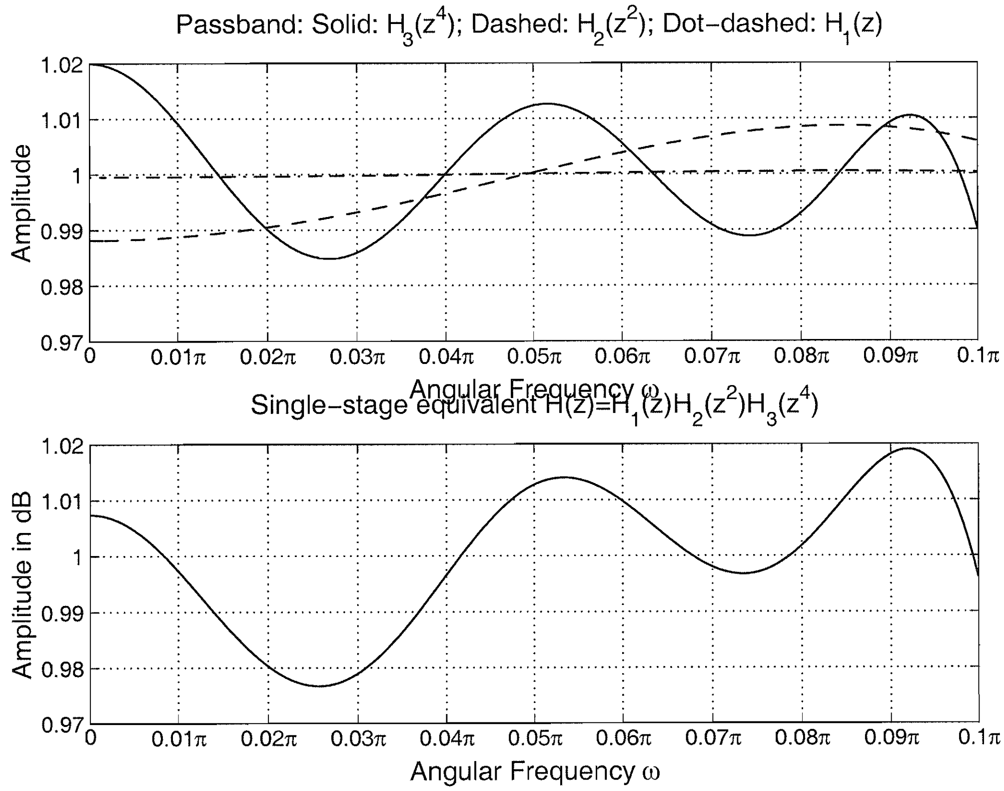
Nonseparable Three-Stage Minimax Eighth-Band ($N = 8$) FIR filter



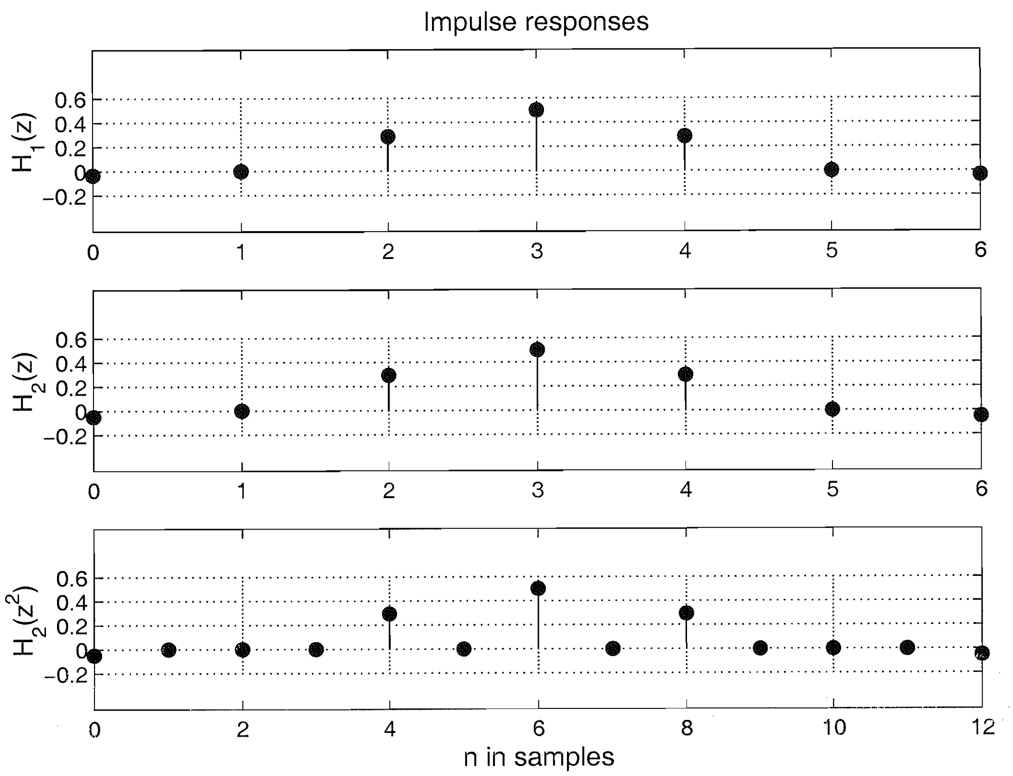


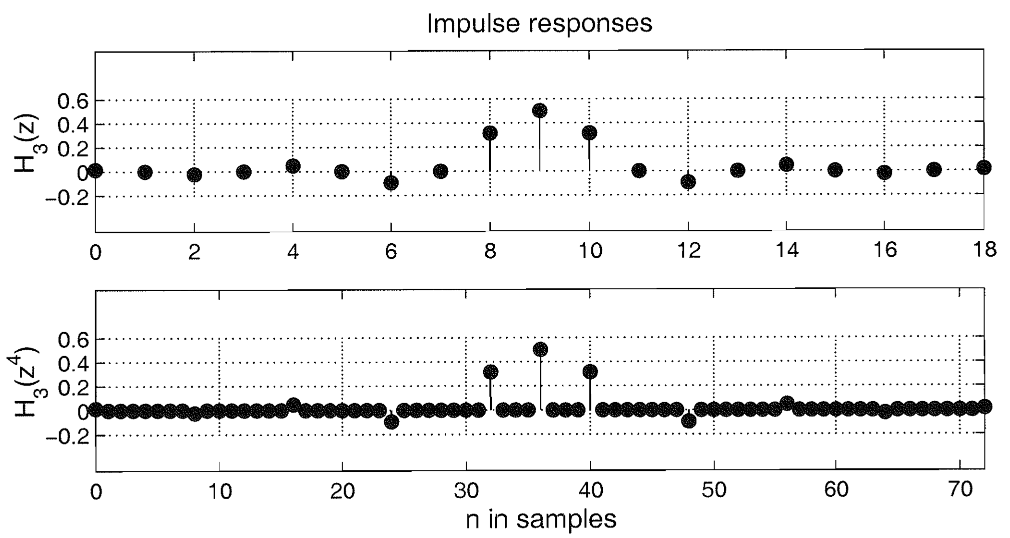
Nonseparable Three-Stage Minimax Eighth-Band ($N = 8$) FIR filter



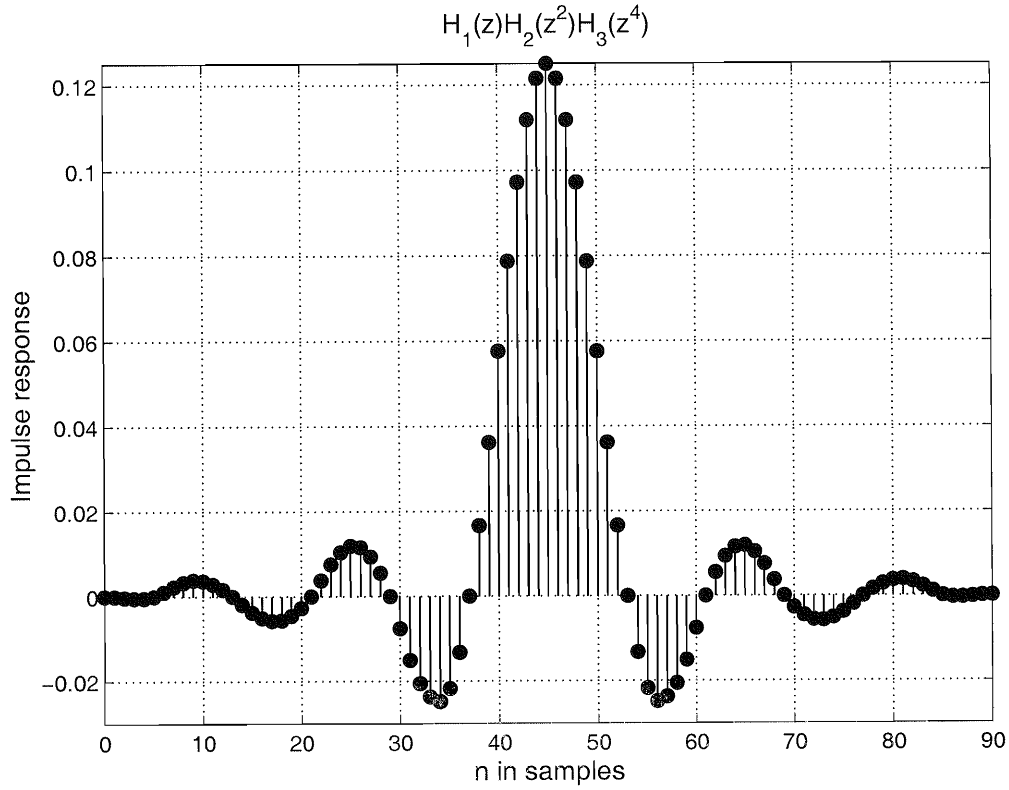


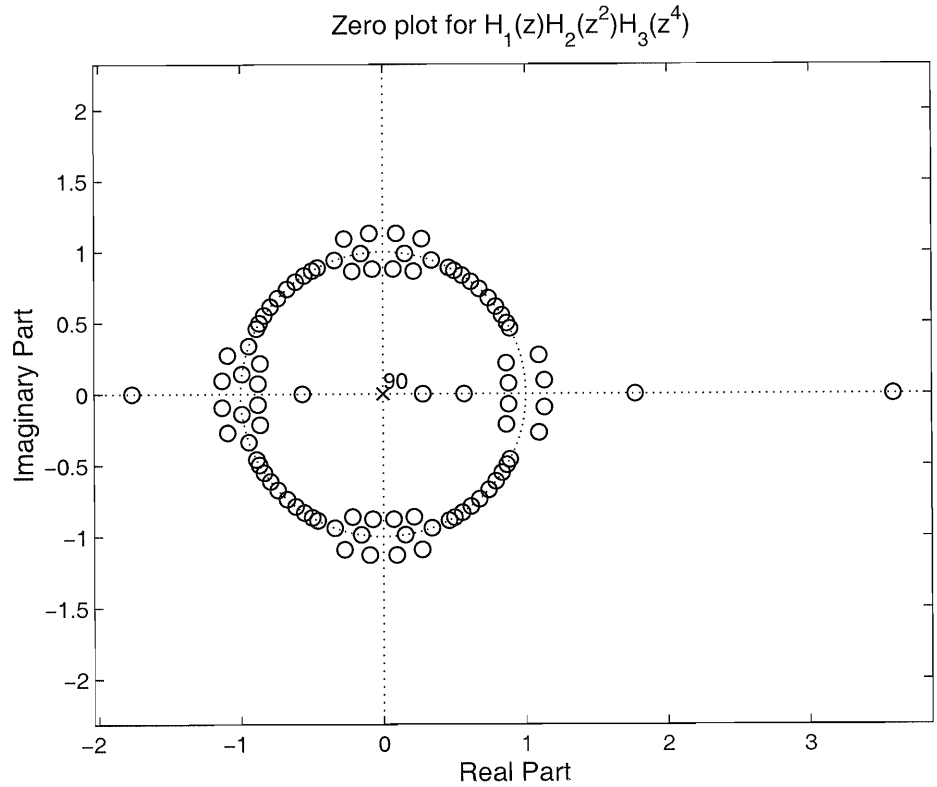
Nonseparable Three-Stage Minimax Eighth-Band ($N = 8$) FIR filter



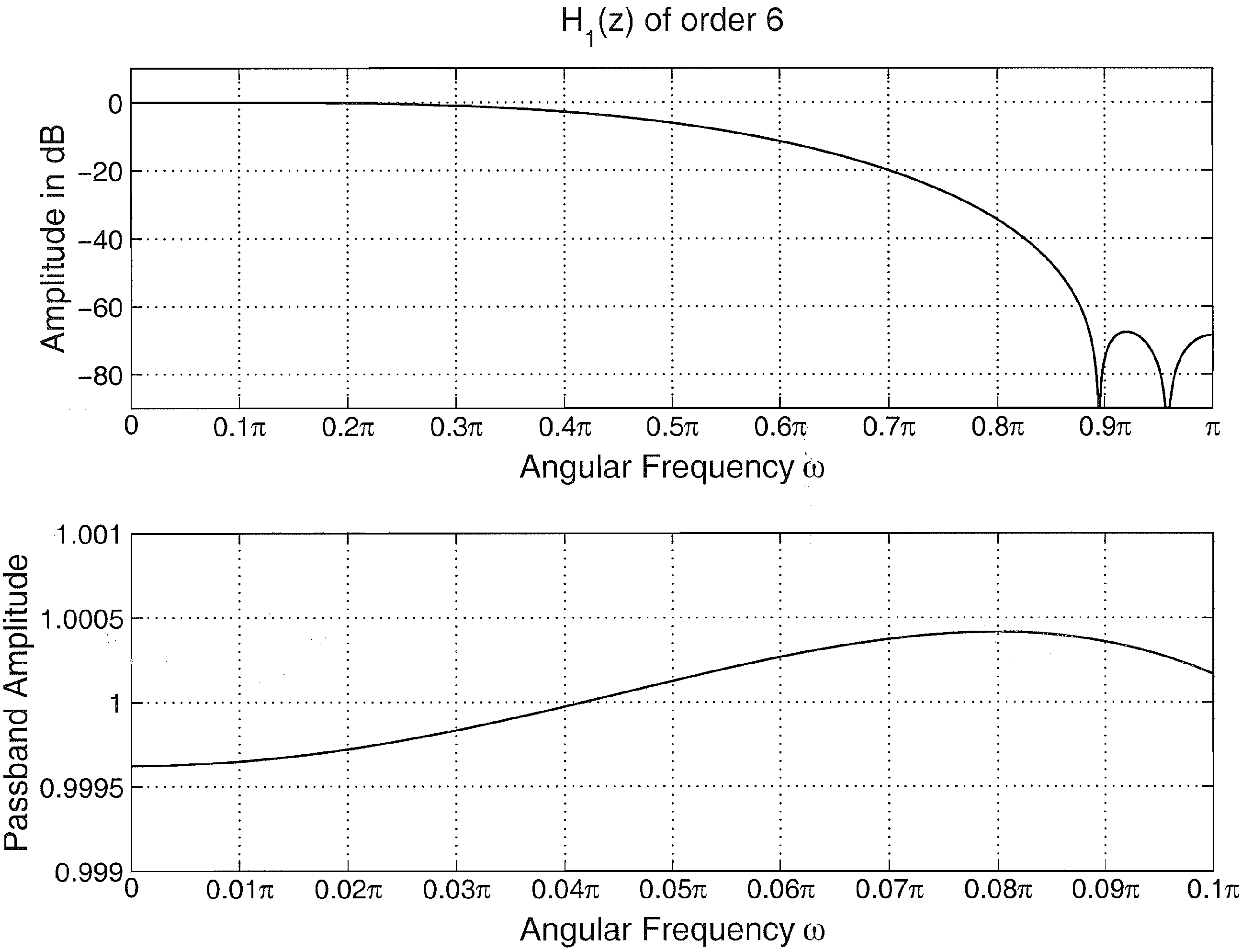


Nonseparable Three-Stage Minimax Eighth-Band $(N = 8)$ FIR filter

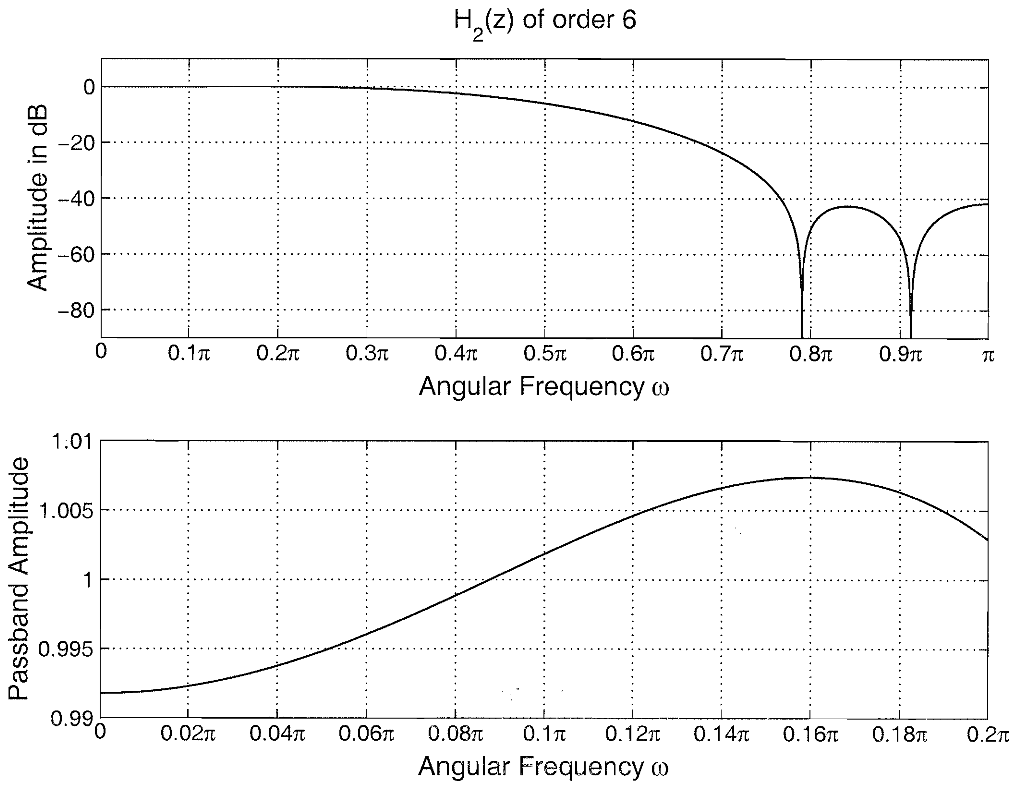


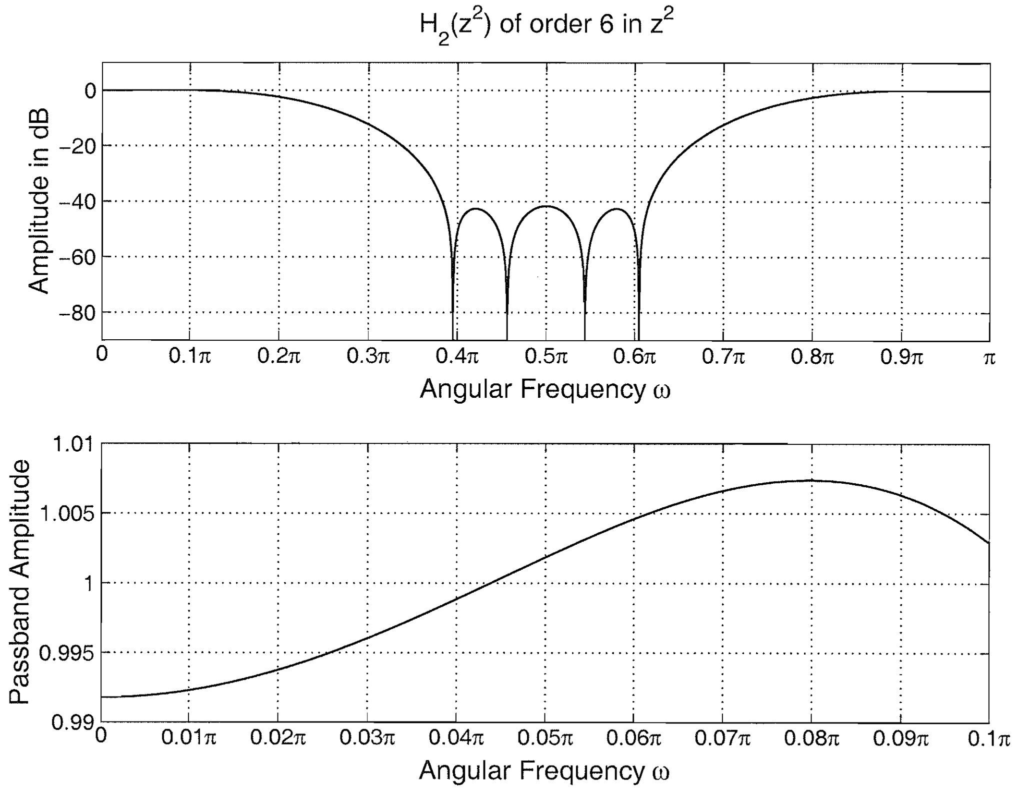


Nonseparable Three-Stage Least-Squared Eighth-Band ($N = 8$) FIR filter

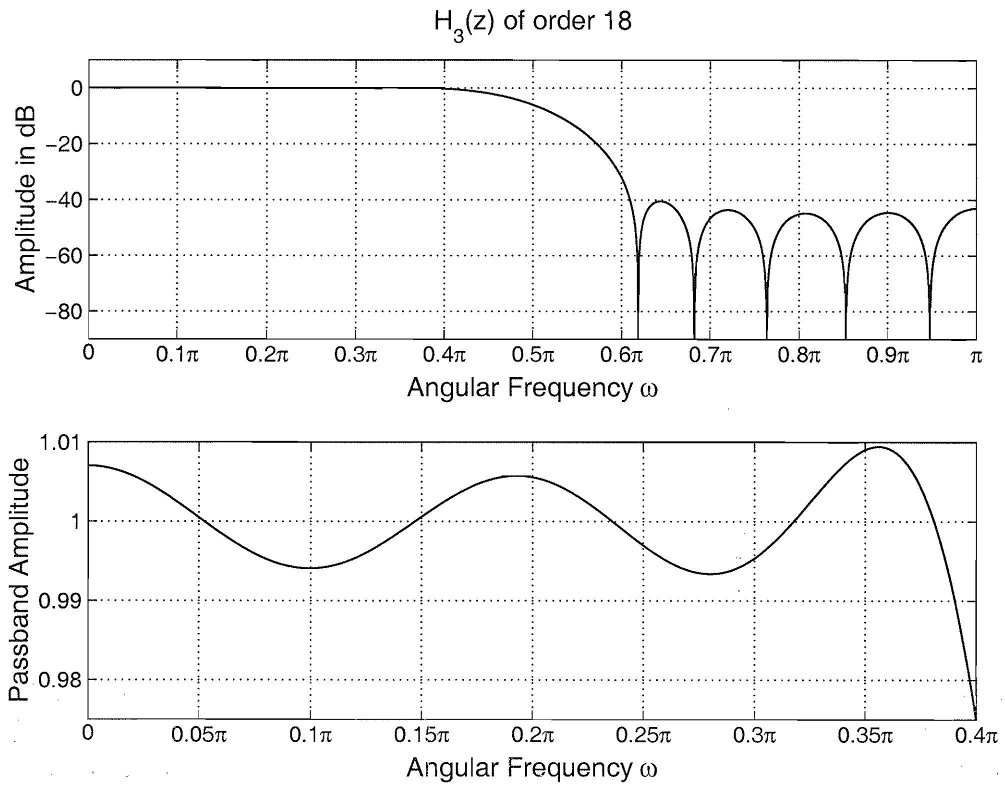


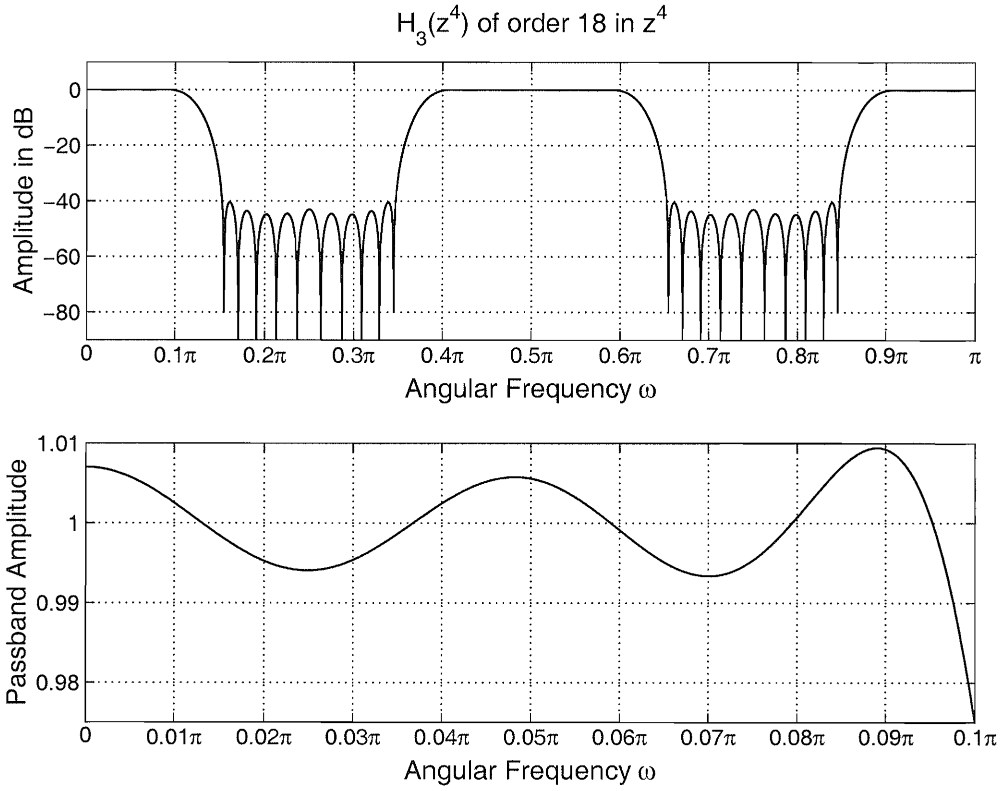
Nonseparable Three-Stage Least-Squared Eighth-Band ($N = 8$) FIR filter



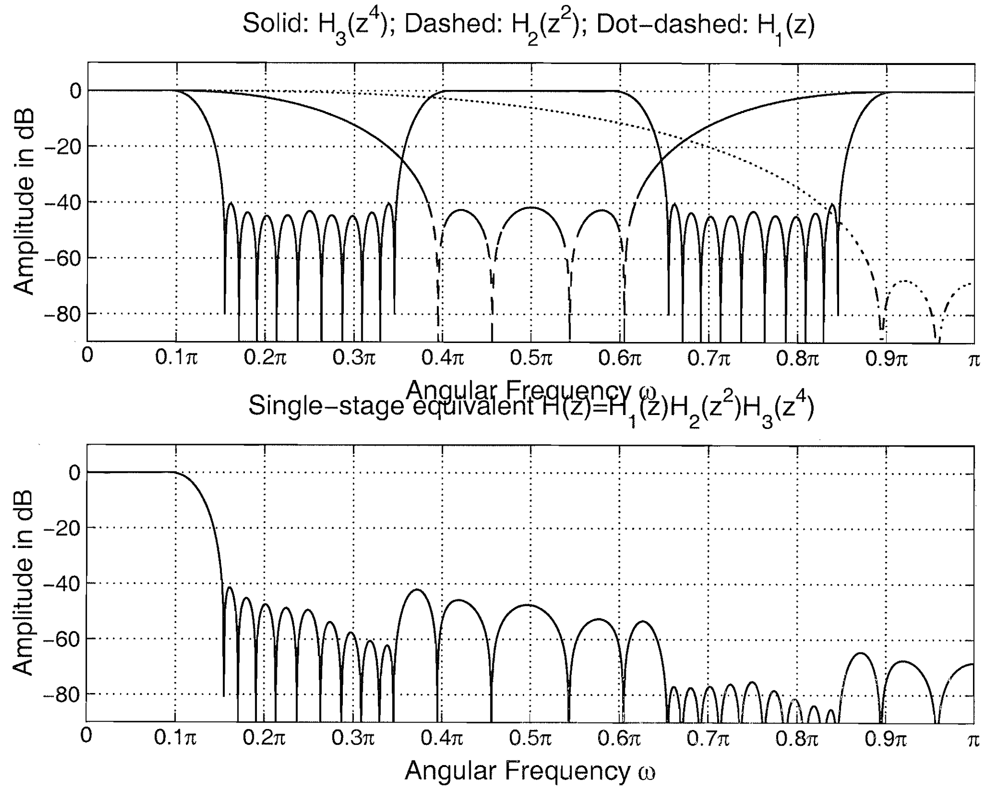


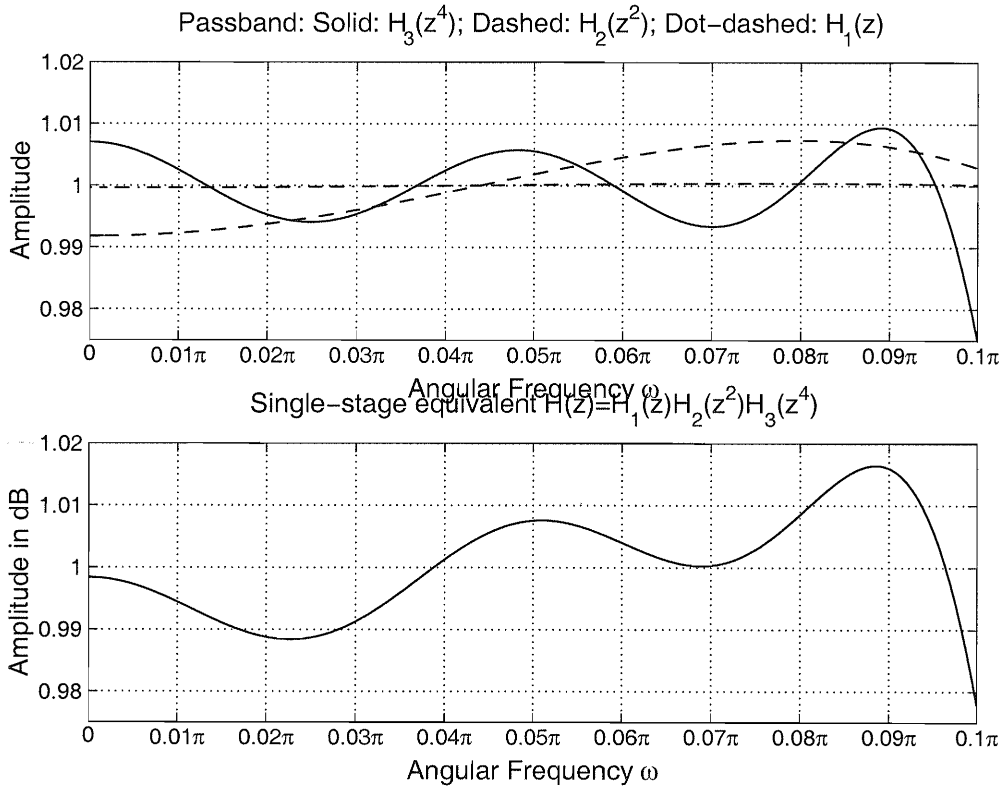
Nonseparable Three-Stage Least-Squared Eighth-Band ($N = 8$) FIR filter



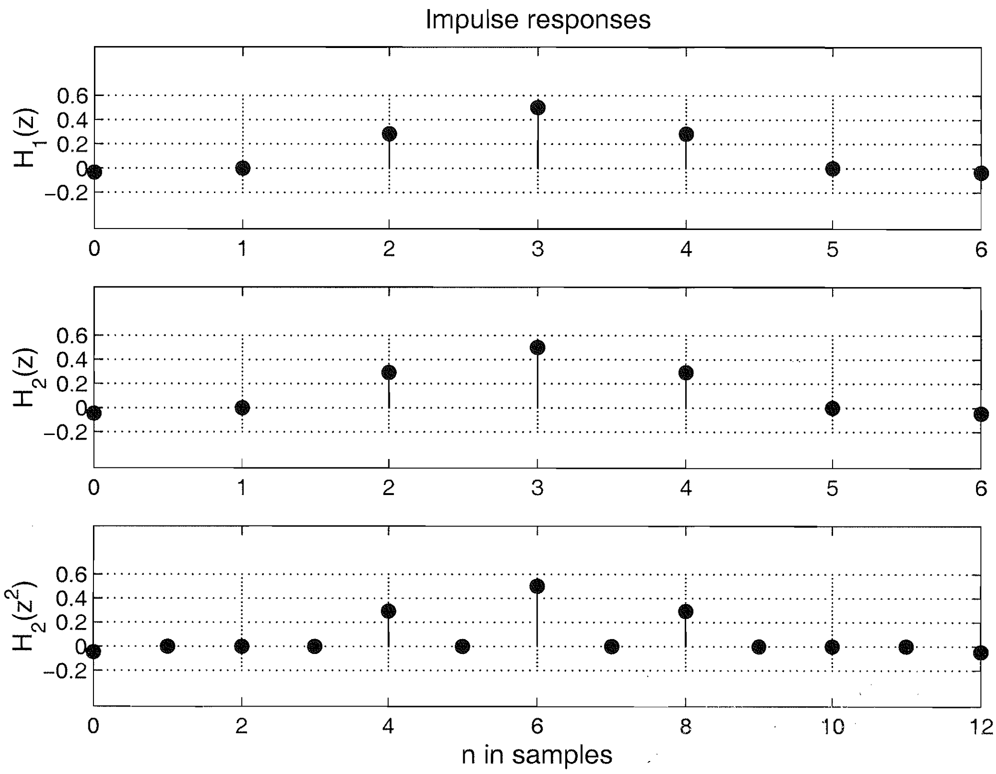


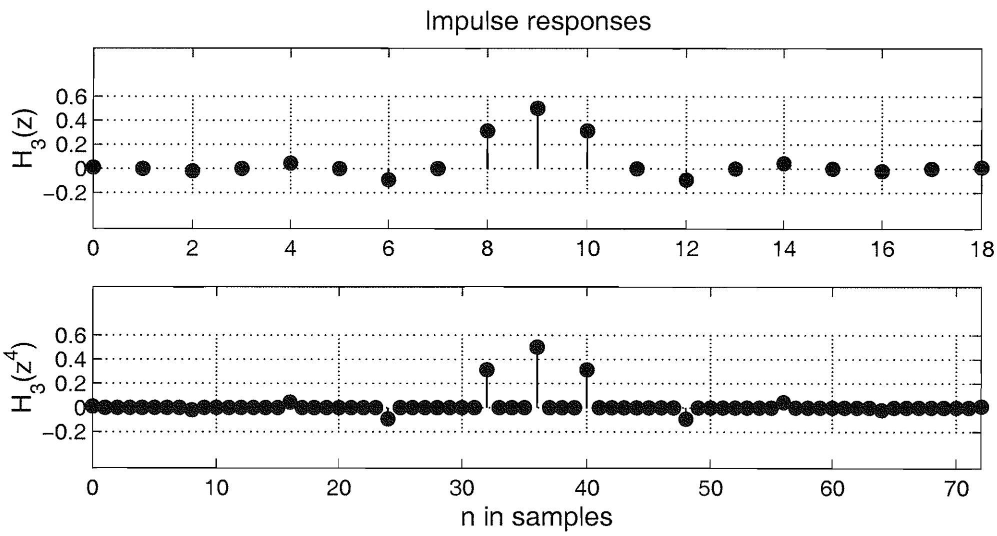
Nonseparable Three-Stage Least-Squared Eighth-Band ($N = 8$) FIR filter



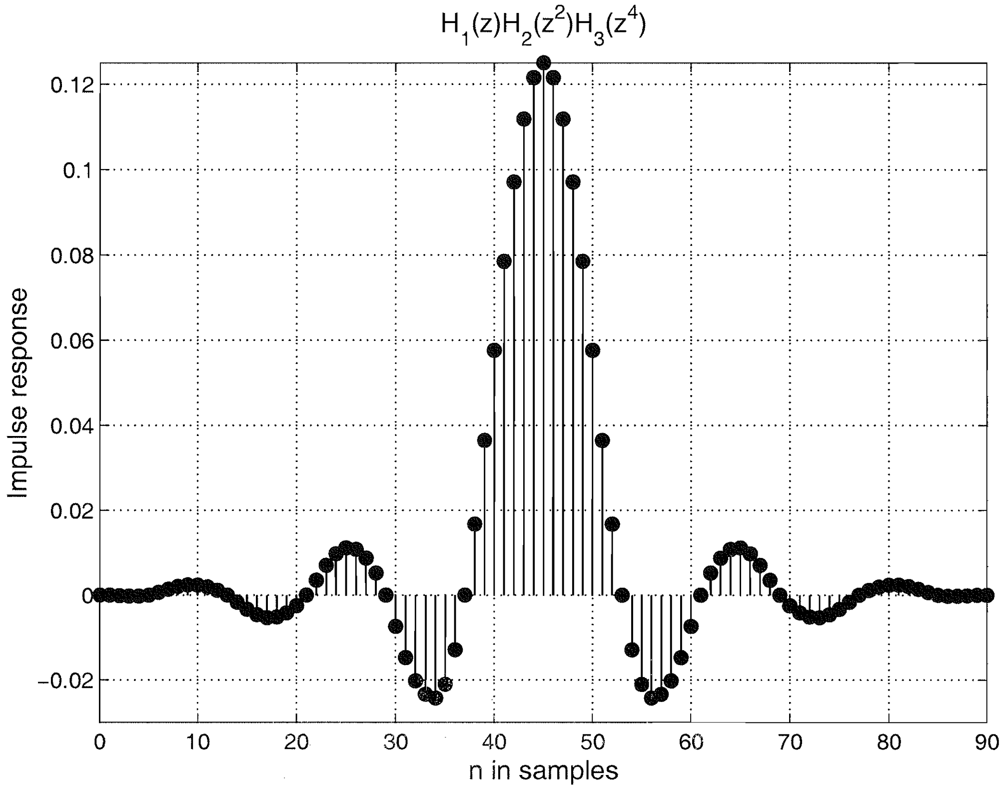


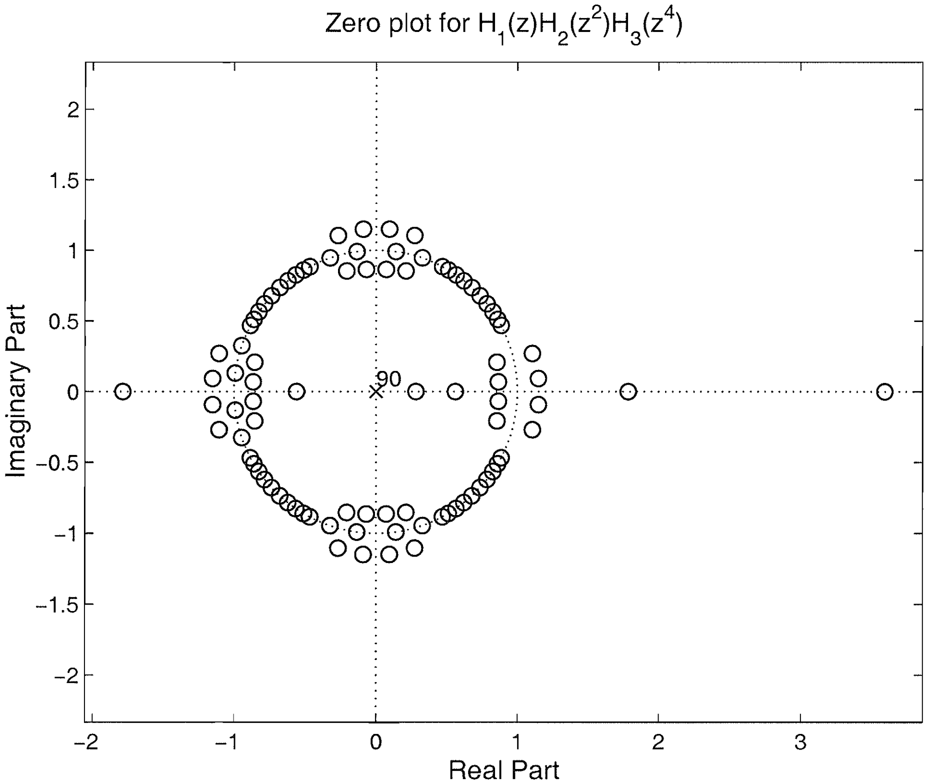
Nonseparable Three-Stage Least-Squared Eighth-Band ($N = 8$) FIR filter





Nonseparable Three-Stage Least-Squared Eighth-Band ($N = 8$) FIR filter





Separable Multistage Nth-Band Linear-Phase FIR Filters

• In pulse shaping in telecommunication applications, it is desired that the overall Nth-band (Nyquist) filter is factorizable as

$$H(z) = T(z)R(z) \tag{45a}$$

where

$$T(z) = \prod_{k=1}^{K} T_k(z^{L_k}), \quad R(z) = \prod_{k=1}^{K} R_k(z^{L_k}). \tag{45b}$$

• Here, the half-Nyquist filters $T_k(z)$ and $R_k(z)$ are obtained by factorizing $H_k(z)$ as

$$H_k(z) = T_k(z)R_k(z) \tag{46}$$

where $T_k(z)$ and $R_k(z)$ have the same magnitude responses and their impulse responses are time-reversed versions of each other, that is, $R_k(z) = z^{-M_k}T_k(z^{-1})$, where M_k is half the order of $H_k(z)$.

• In this case, it is required that the zero-phase frequency responses $H_k(\omega)$ for $k = 1, 2, \cdots, K$ are nonnegative on $[0, \ \pi]$ in order to make $H_k(z)$ factorizable in the desired manner.

• In communication theory, $T(z)$ and $R(z)$ are referred to as a matched filter pair and they are used as transmitter and reciever filters, respectively.

• In practice, the multistage $T(z)$ $[R(z)]$ is implemented using the interpolation (decimation) structure of Page

173 with $H_k(z) = T_k(z)$ and $[H_k(z) = R_k(z)]$ for $k = 1, 2, \cdots, K$.

• The maximum deviation of $T(z)$ and $R(z)$ from zero becomes less than or equal to δ_s on $[(1+\rho)\pi/N, \ \pi]$ if each **non-negative** $H_k(\omega)$ is designed in such a way that

$$\epsilon_\infty^{(k)} = \max_{\omega \in \Omega_s^{(k)}} |W_k(\omega) H_k(\omega)|, \tag{47a}$$

where

$$W_k(\omega) = \prod_{\substack{l=1 \\ l \neq k}}^{K} H_l(\widehat{N}_l \omega / \widehat{N}_k), \tag{47b}$$

and $\Omega_s^{(k)}$ is given by Eqs. (43c) and (43d), becomes less than or equal to $(\delta_s)^2$.

• The overall filter can be designed in the minimax sense in a manner similar to the nonseparable case.

• The main difference is that each $H_k(\omega)$ has to be non-negative.

• In the least-mean-square case, the basic differences are that each $H_k(\omega)$ has to be non-negative and the quantity to be minimized is

$$\epsilon_2^{(k)} = \int_{\Omega_s^{(k)}} [W_k(\omega) H_k(\omega)] d\omega. \tag{48}$$

Example: $\rho = 0.2$ and $N = 8$

Minimax design with $K = 2$: The minimum stopband attenuation of both $T(z)$ and $R(z)$ is at least 40 dB, that is, δ_s is less than or equal to 0.01. For $H(z) = T(z)R(z)$, the minimum attenuation is 80 dB.

- The given criteria are met by $N_1 = 4$, $N_2 = 2$, $2M_1 = 38$, and $2M_2 = 50$.
- In the following, there are 7 pages illustrating the characteristics of this design.
- The orders of $T_1(z)$ and $R_1(z)$ are thus 19, whereas the orders of $T_2(z)$ and $R_2(z)$ are 25. The implementation of both $T(z) = T_1(z)T_2(z^4)$ and $R(z) = R_1(z)R_2(z^4)$ requires $20 + 26 = 46$ multipliers.
- The overall order of $T(z) = T_1(z)T_2(z^4)$ is $(M_1 + N_1M_2) = 119$.
- The corresponding one-stage design $H(z) = T(z)R(z)$ requires $2M = 202$ so that the implementation of both $T(z)$ and $R(z)$ requires 102 multipliers and 101 delays.

Minimax design with $K = 3$: The criteria are met by $N_1 = N_2 = N_3 = 2$, $2M_1 = 10$, $2M_2 = 18$, and $2M_3 = 50$.

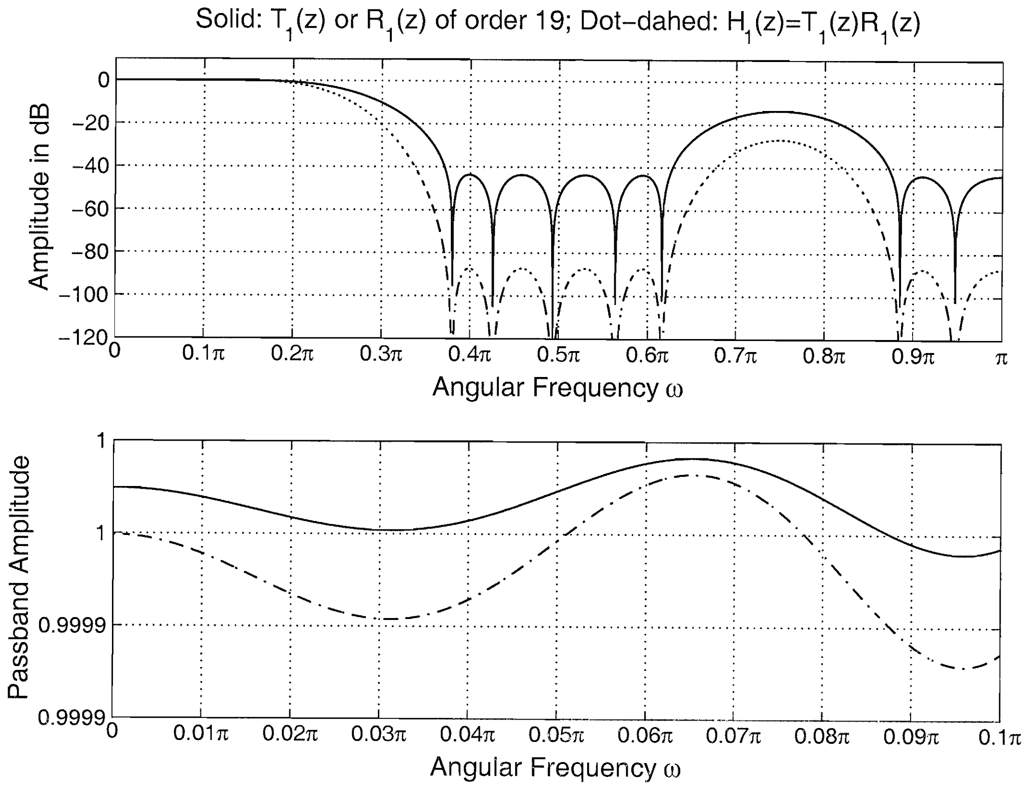
- In the following, there are 9 pages illustrating the characteristics of this design.
- The number of multipliers for both $T(z)$ and $R(z)$ is only $6 + 10 + 26 = 42$ and the overall number of delays

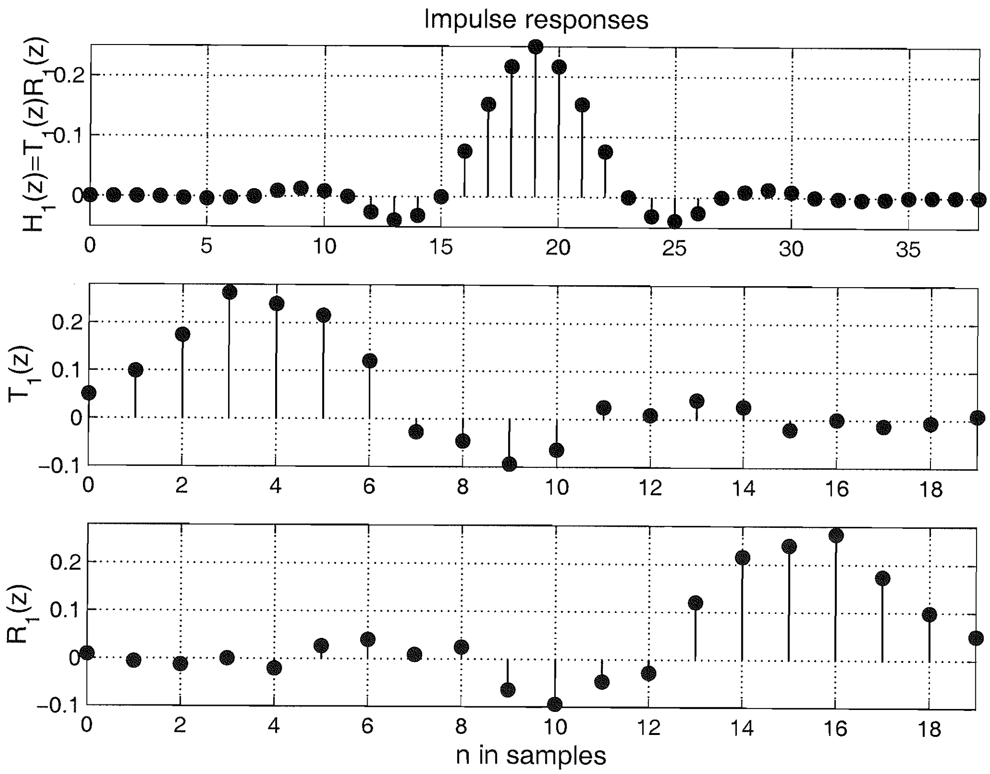
is 143.

• When used for decimation (interpolation) by 8, then the number of multiplications per input sample (per output sample) are for the one-stage, two-stage, and three-stage designs are $102/8 = 12.75$, $20/4 + 26/8 = 8.25$, and $6/2 + 10/4 + 26/8 = 8.75$, respectively.

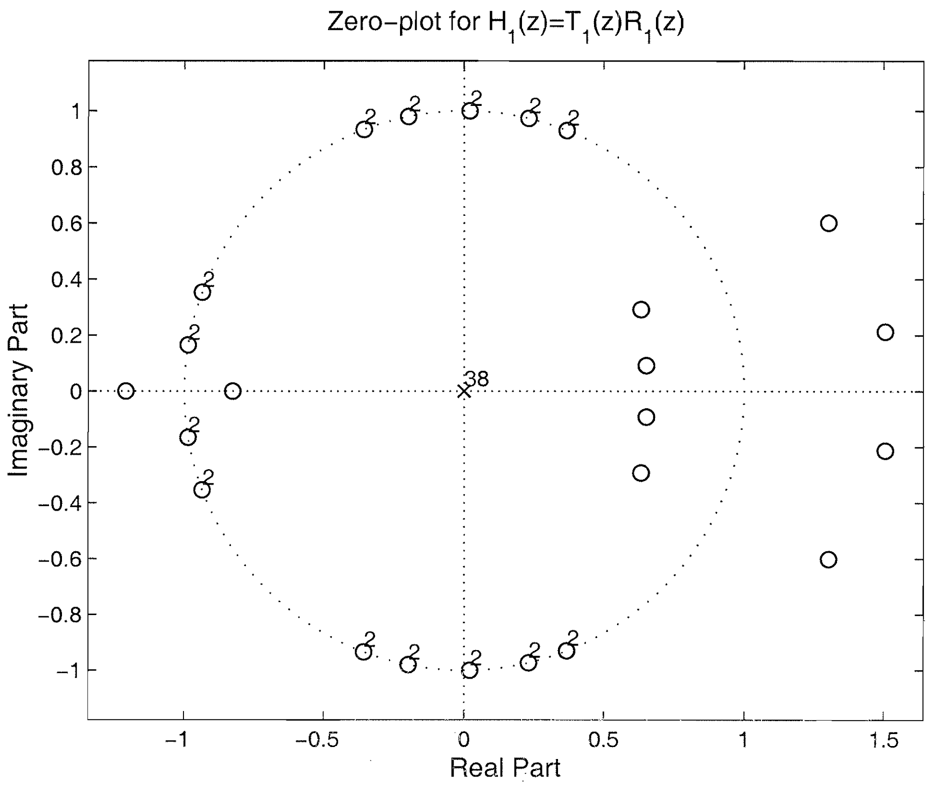
Least-squared design with $K = 3$: In the following, there are 9 pages illustrating the characteristics of the least-squared three-stage filter with the same subfilter orders as for the above minimax design.

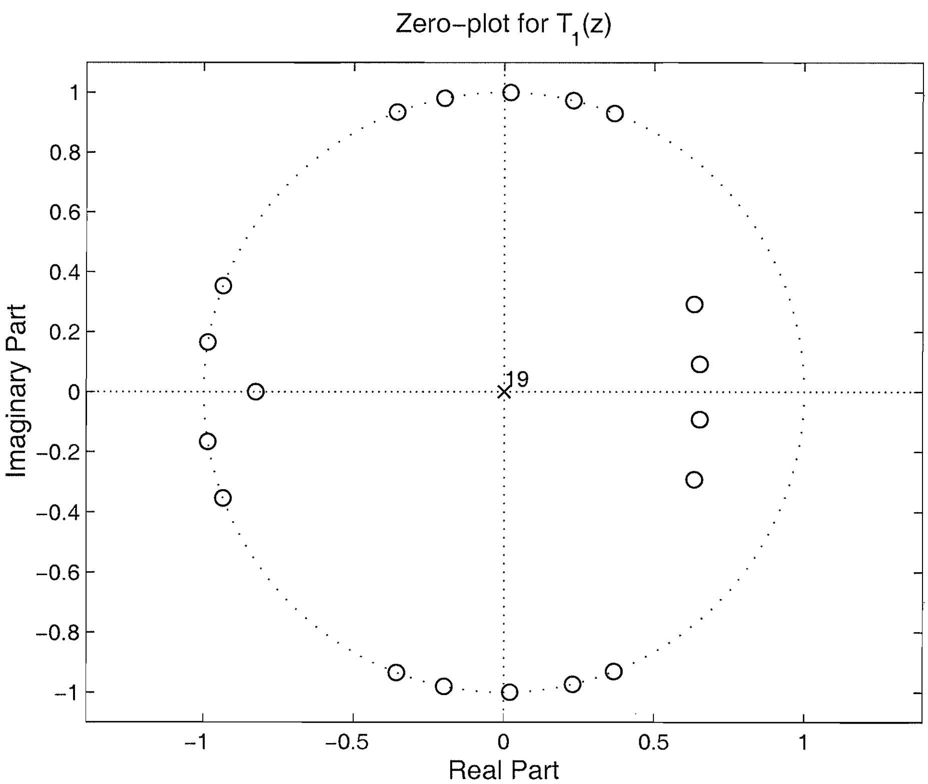
Separable Two-Stage Minimax Eighth-Band ($N = 8$) FIR filter



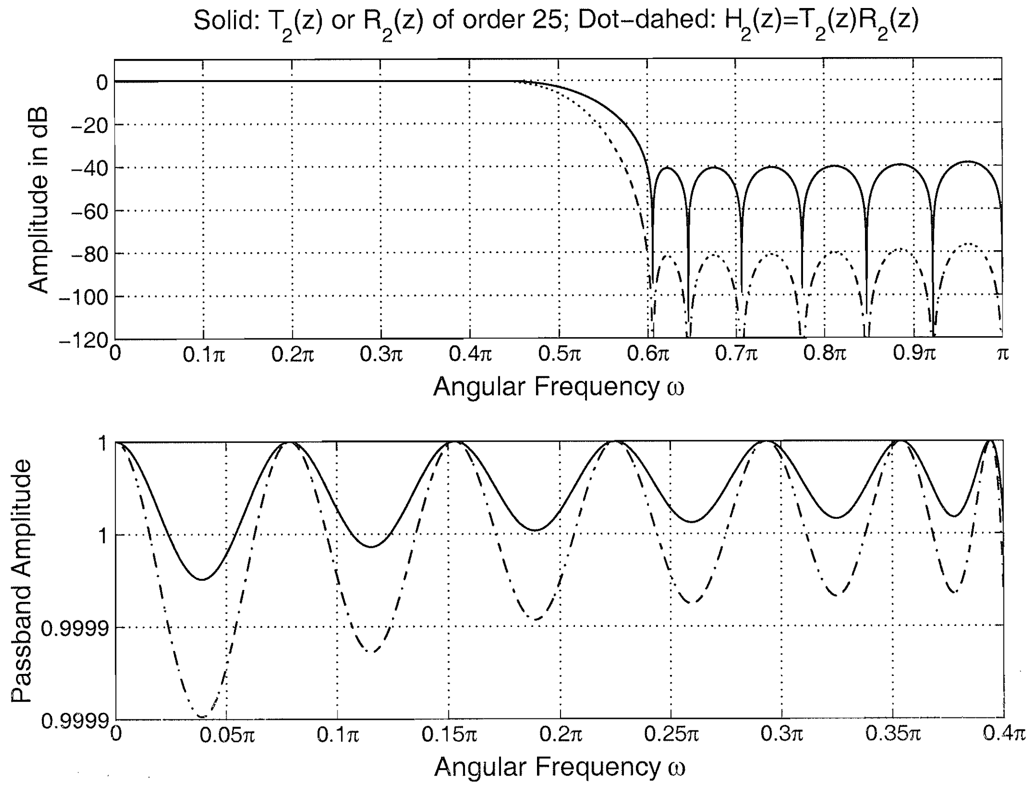


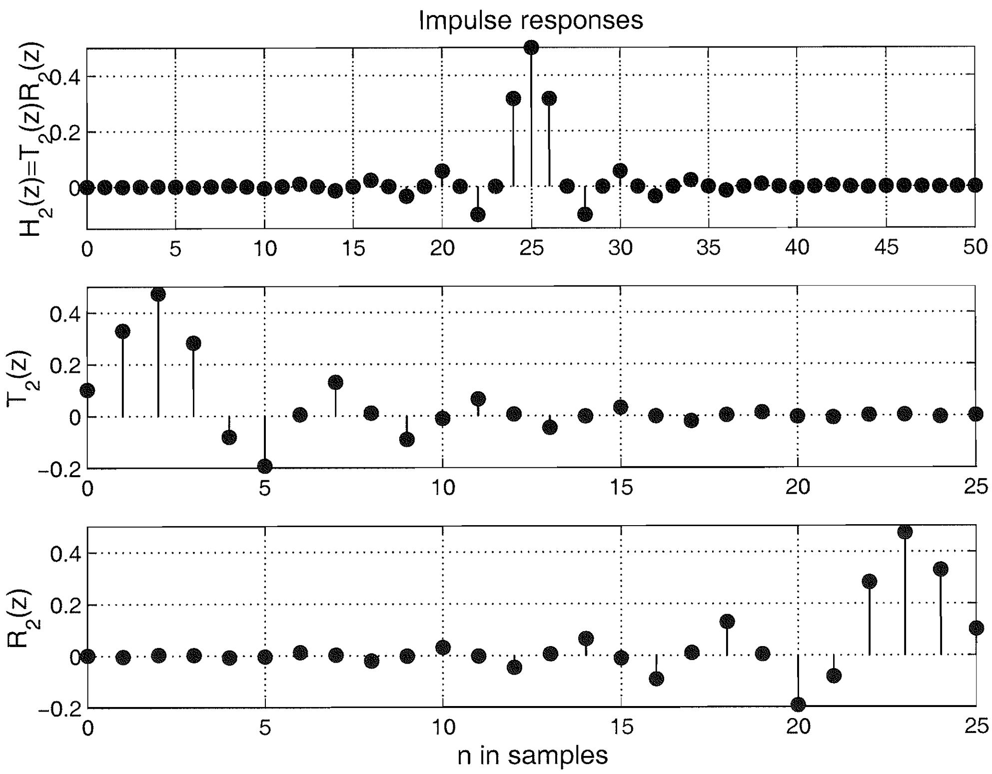
Separable Two-Stage Minimax Eighth-Band ($N = 8$) FIR filter





Separable Two-Stage Minimax Eighth-Band ($N = 8$) FIR filter





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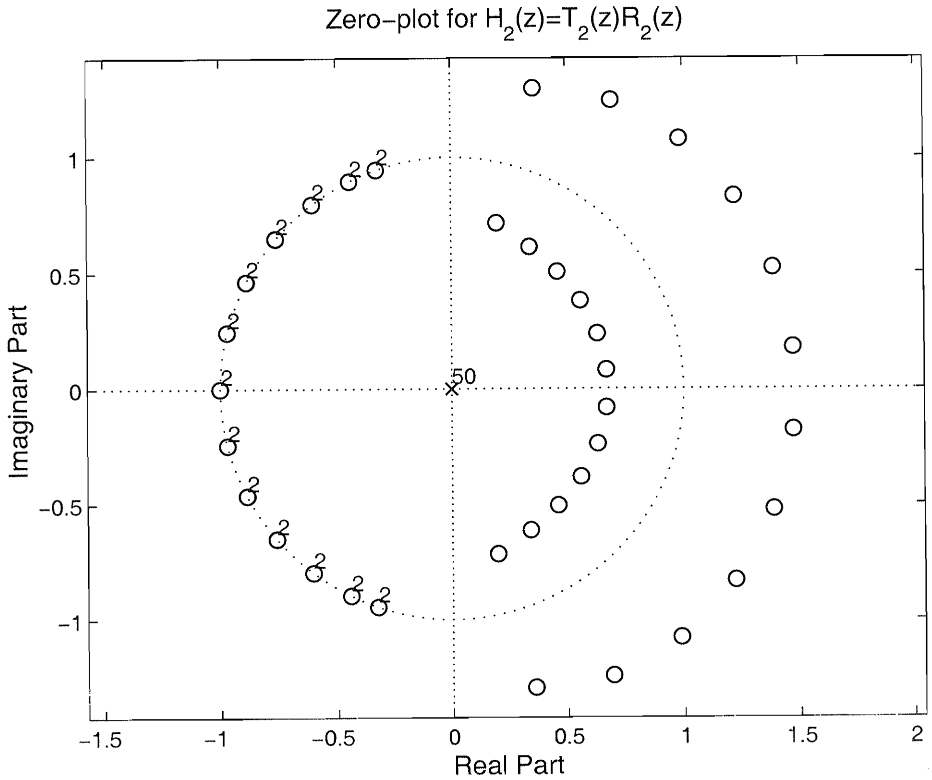

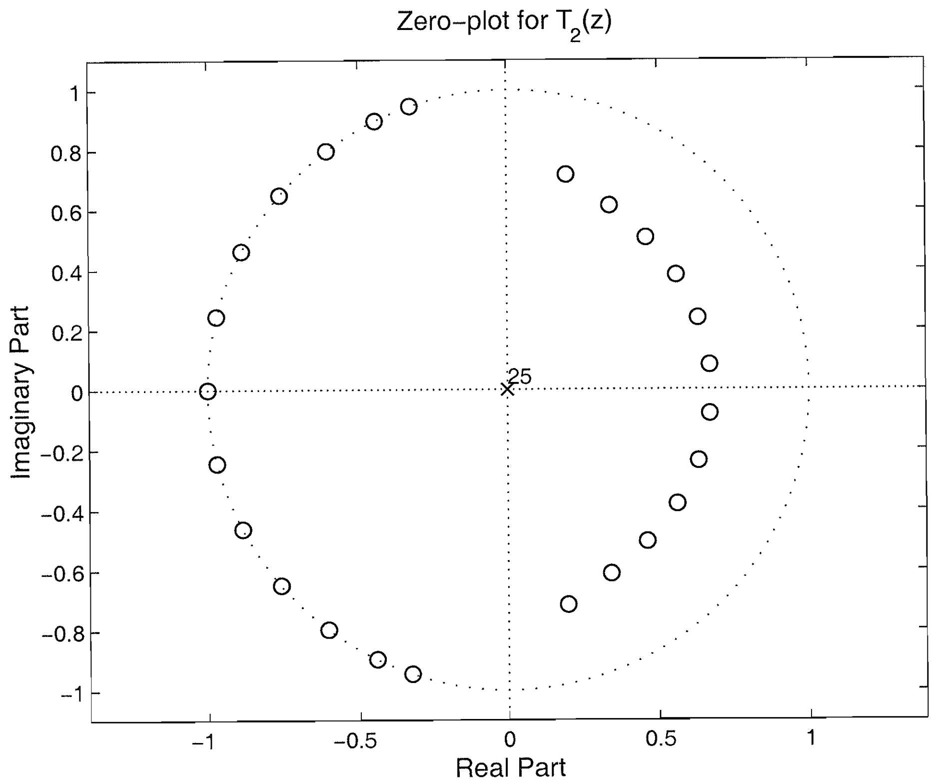

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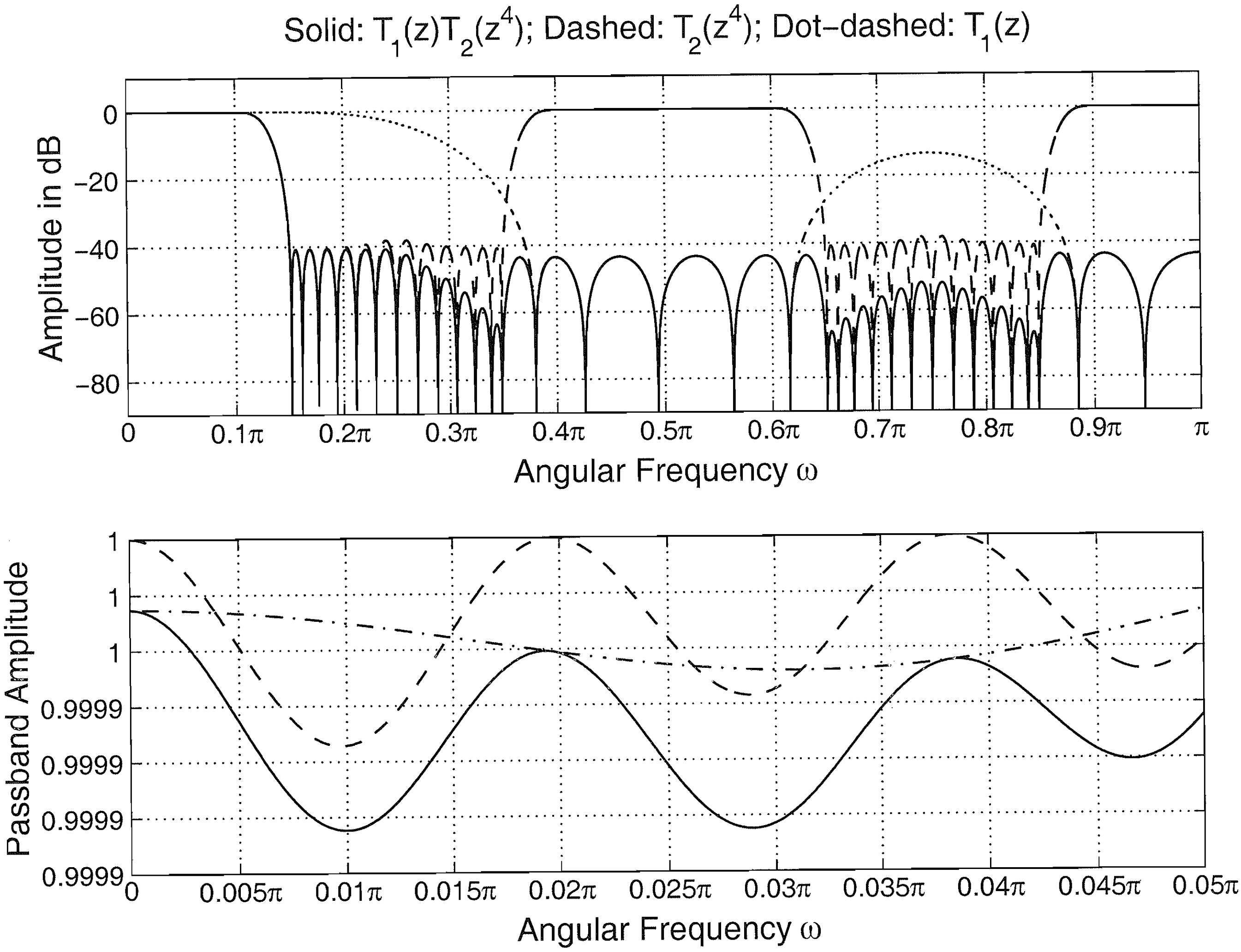

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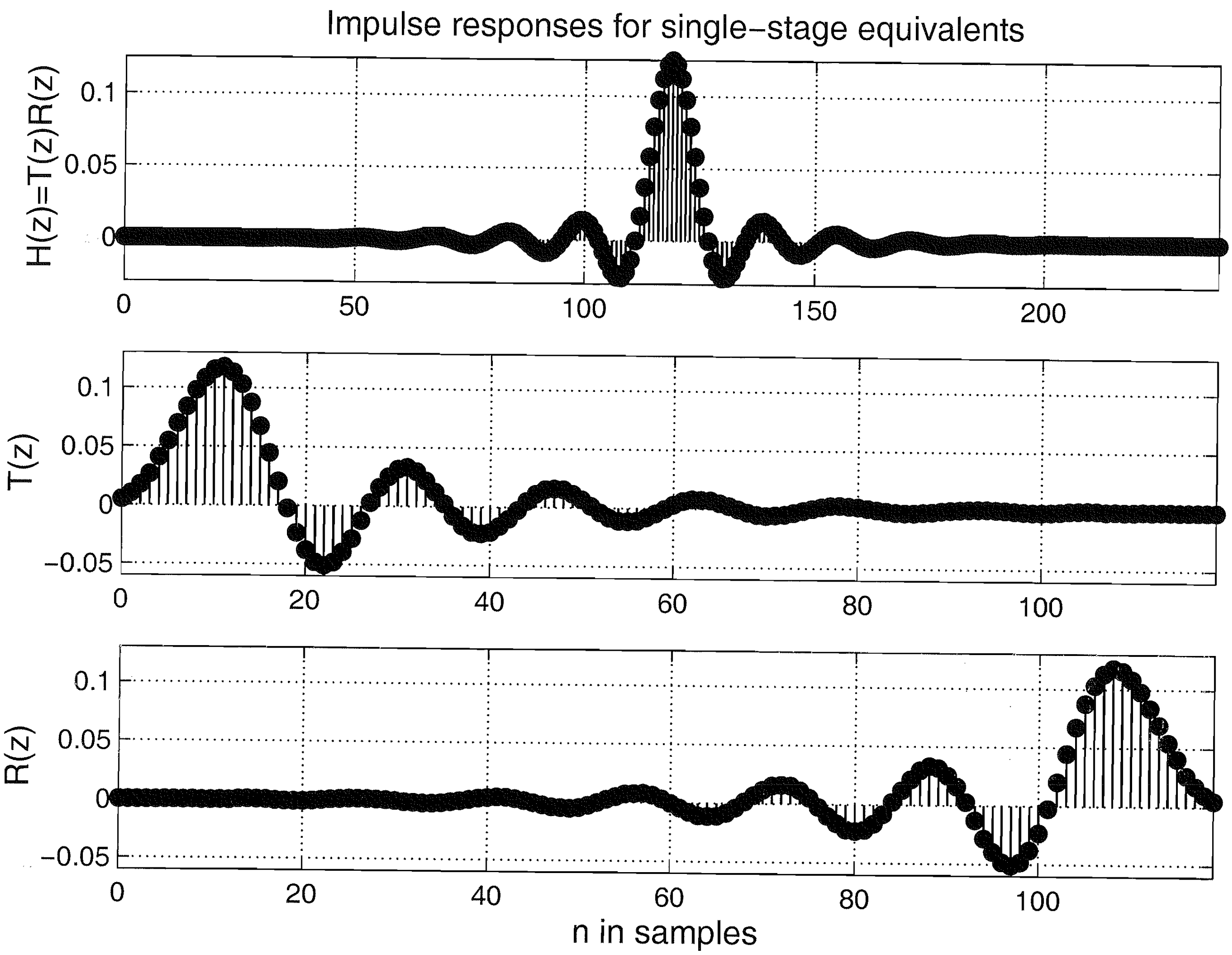

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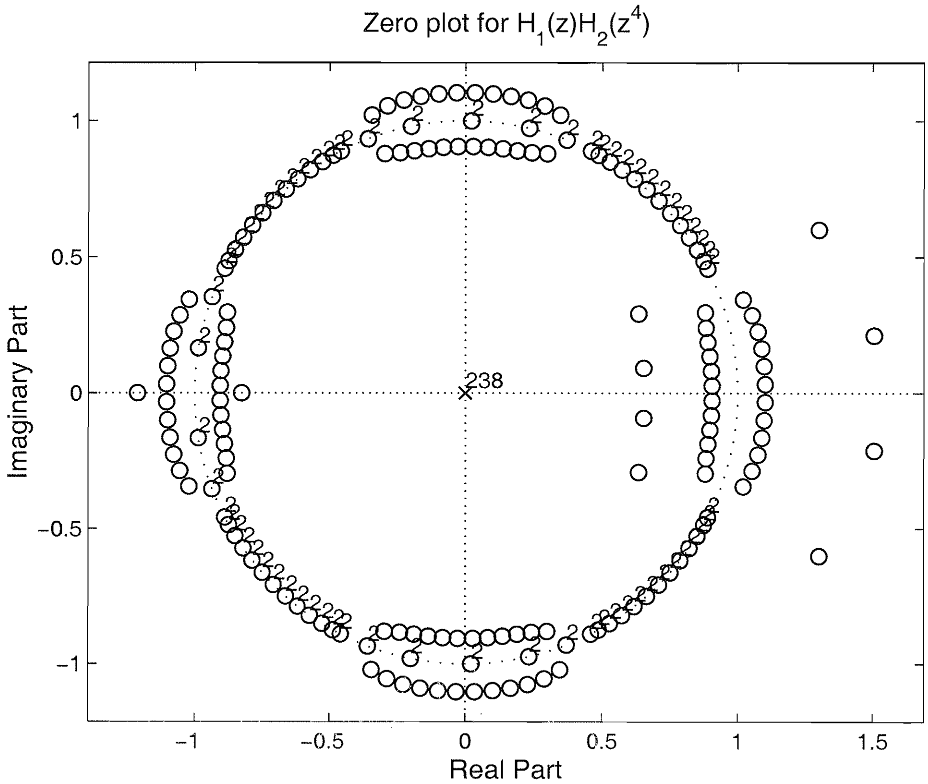
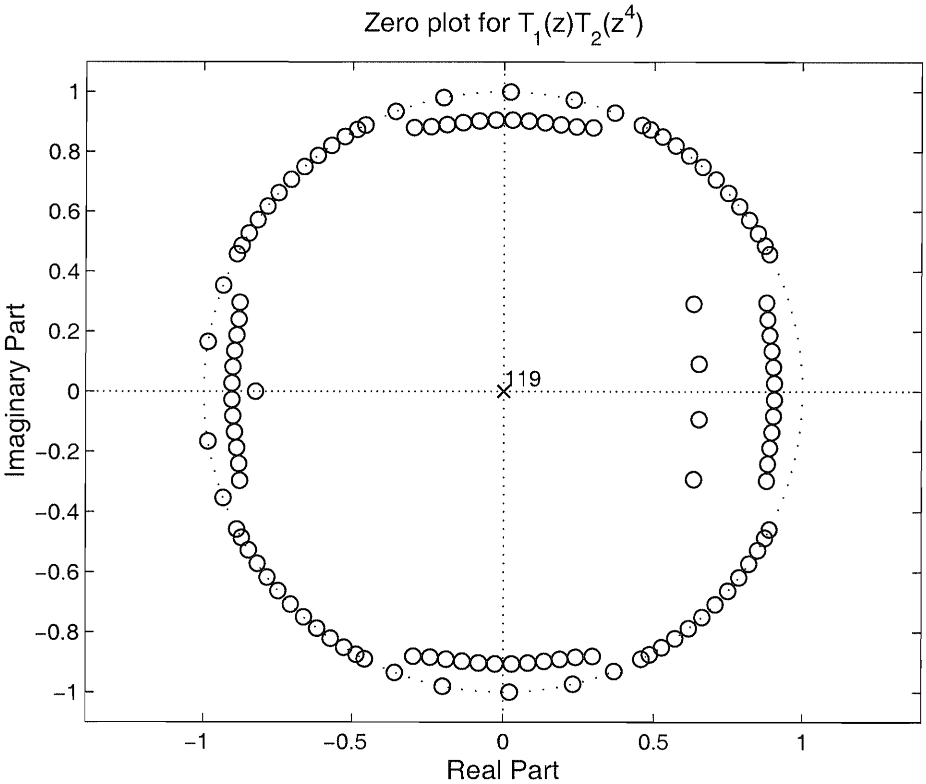
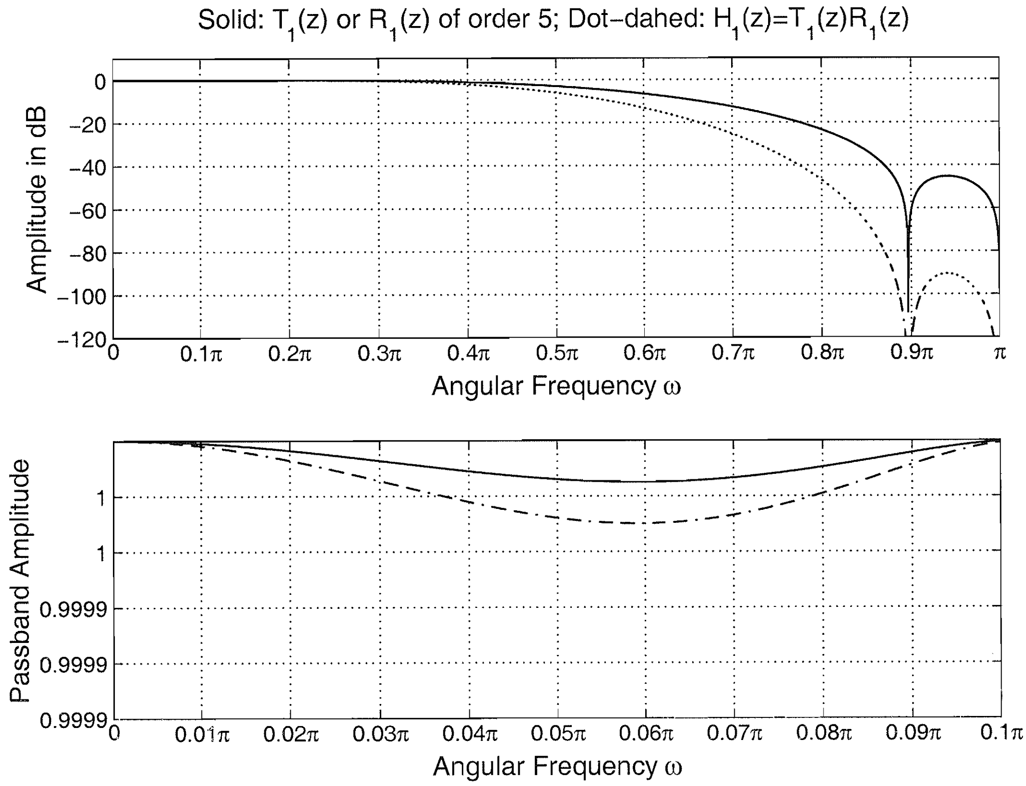
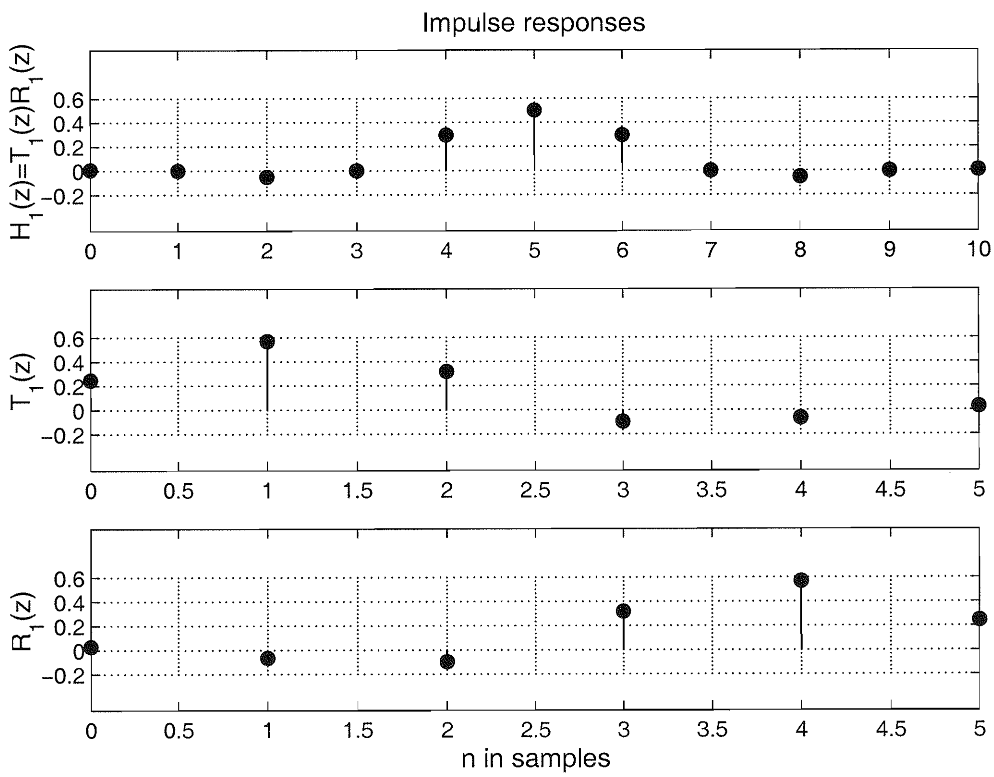




Separable Three-Stage Minimax Eighth-Band ($N = 8$) FIR filter





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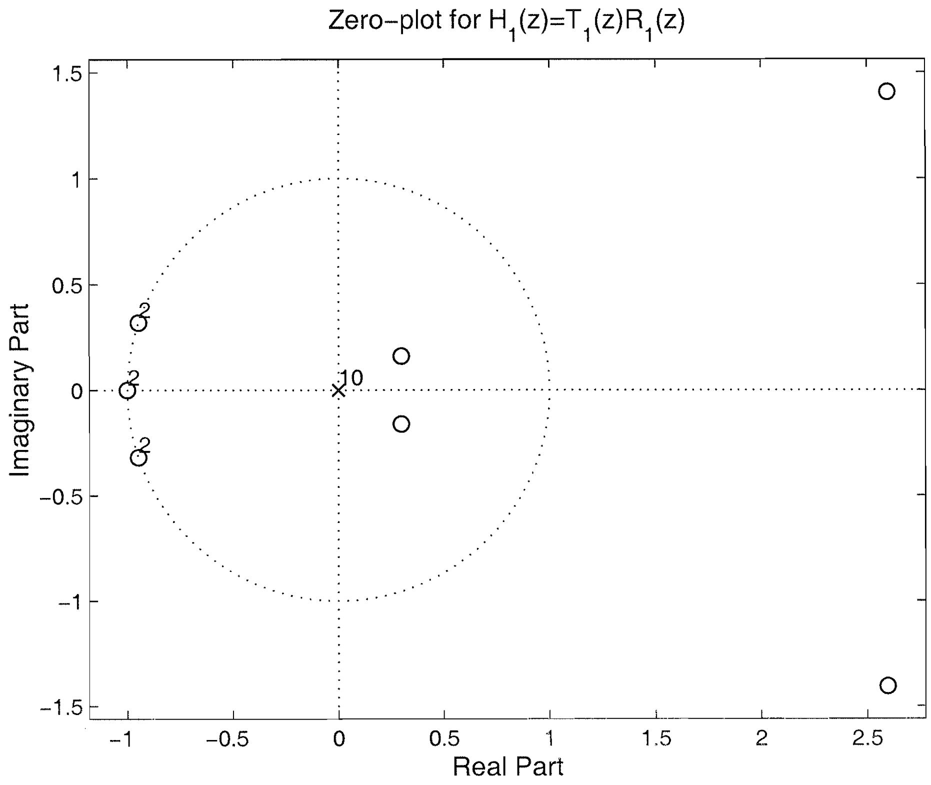
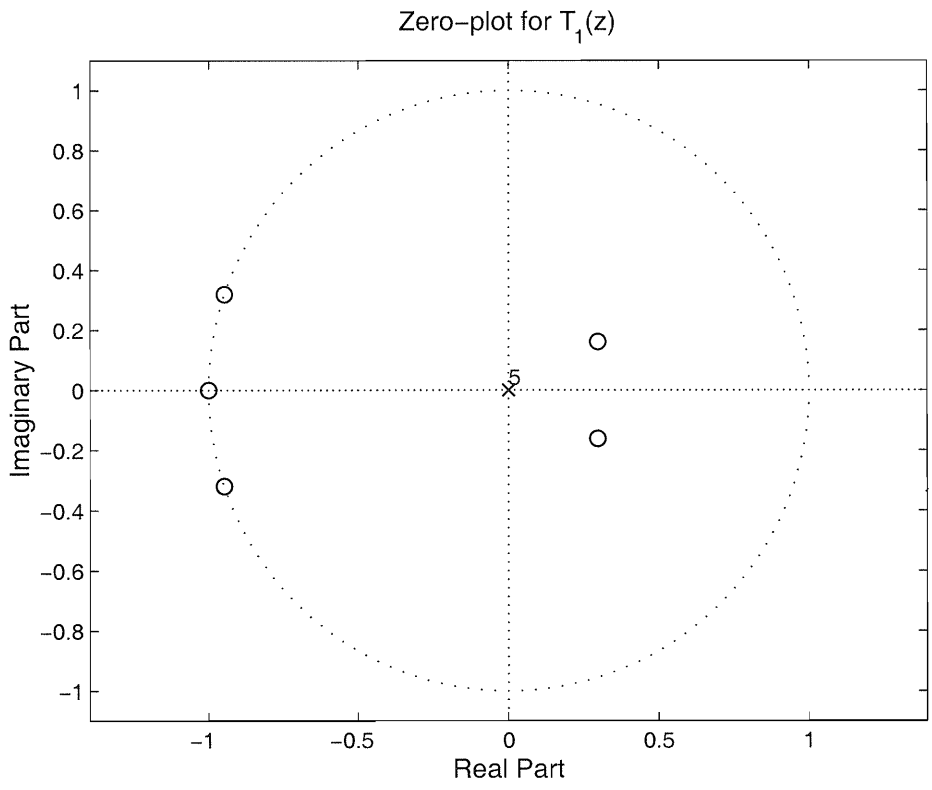
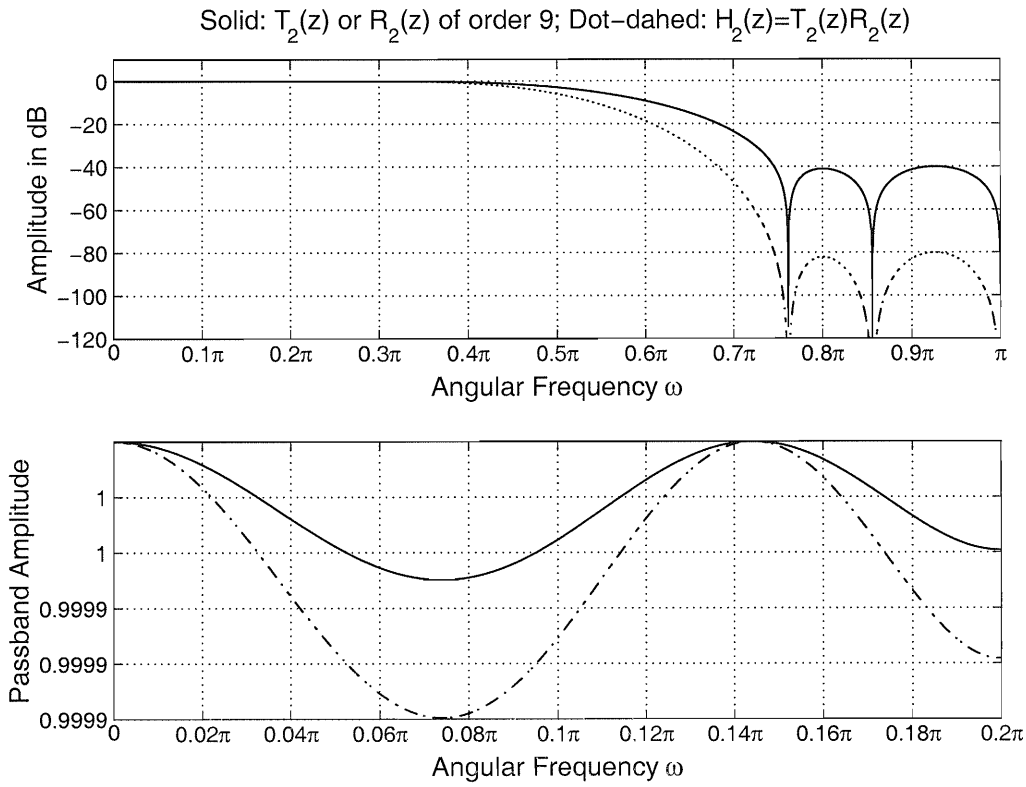
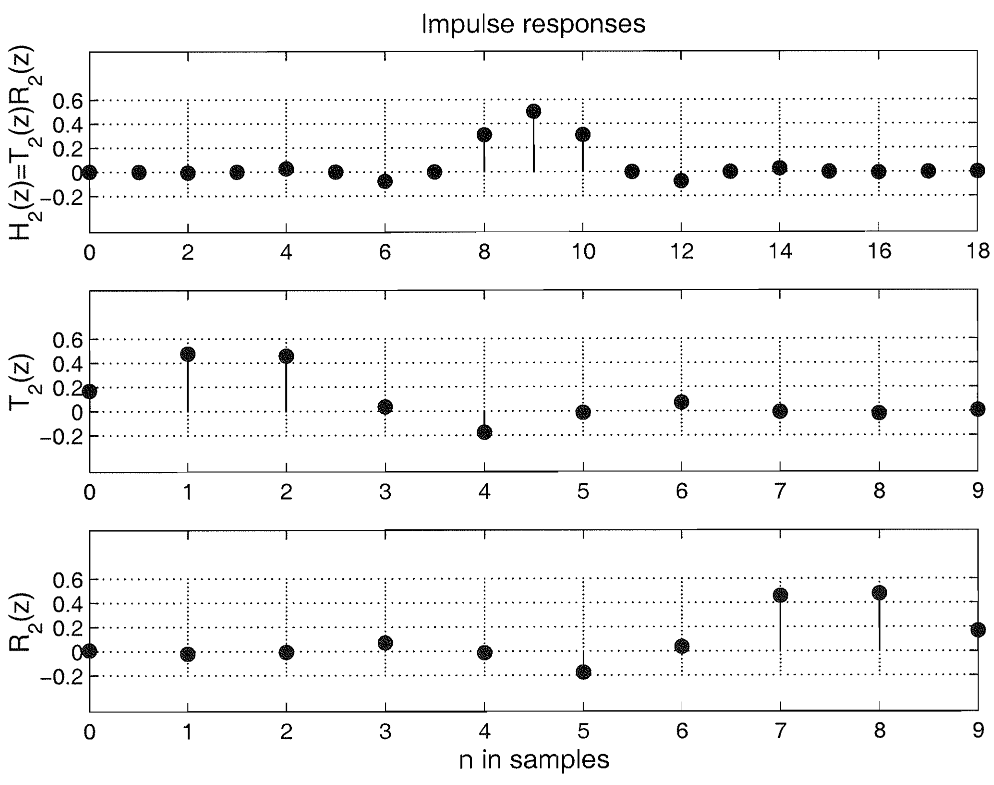




Separable Three-Stage Minimax Eighth-Band $(N = 8)$ FIR filter





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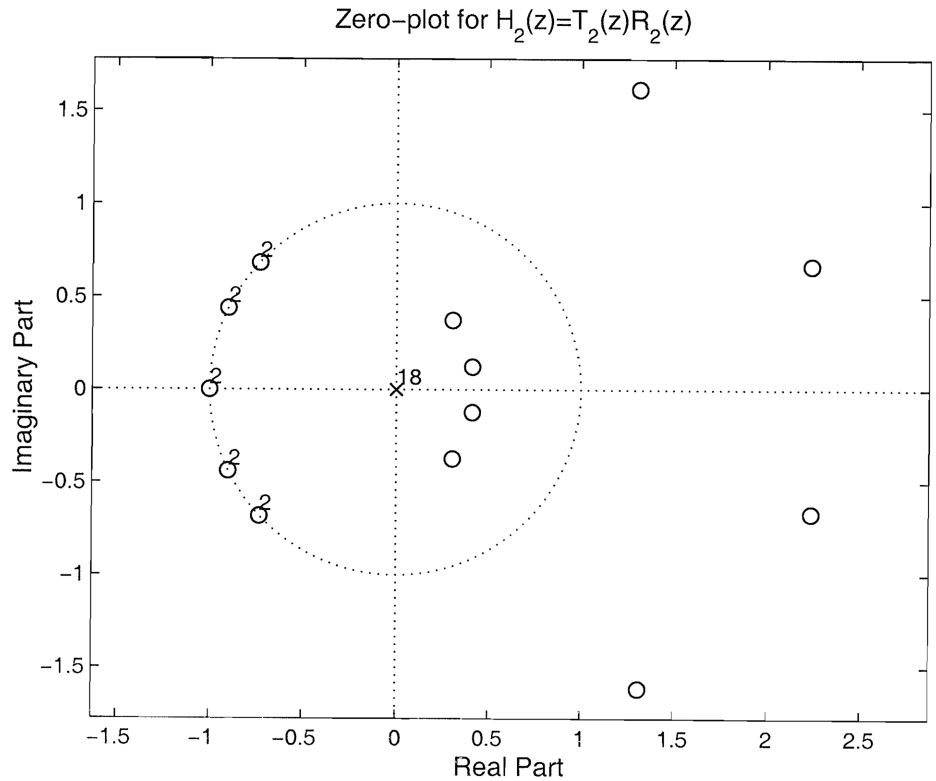

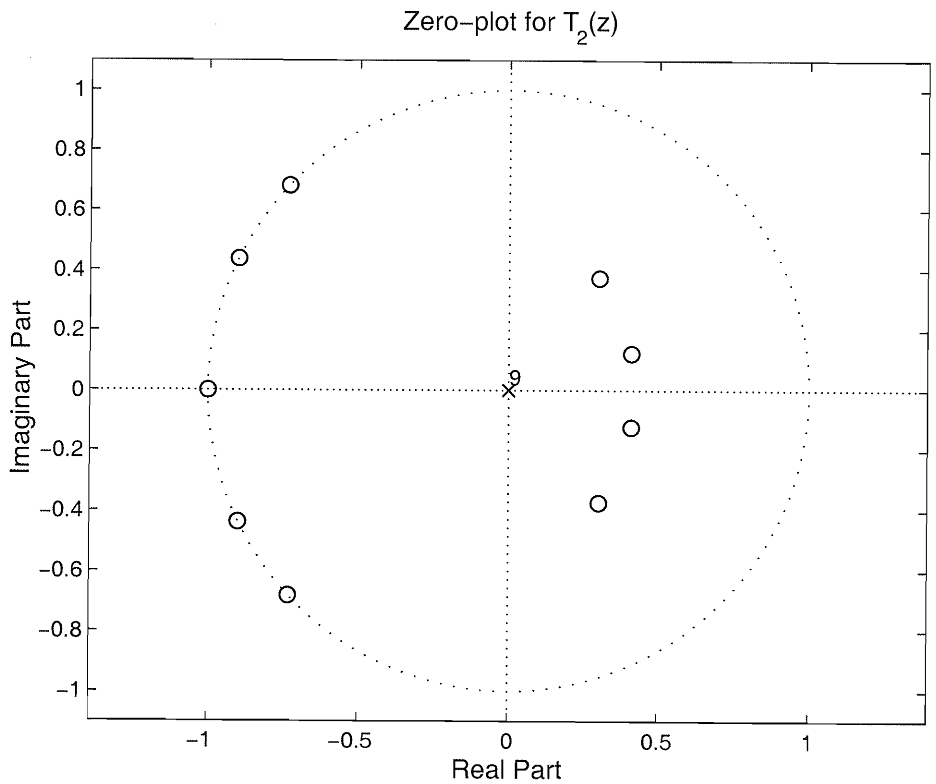

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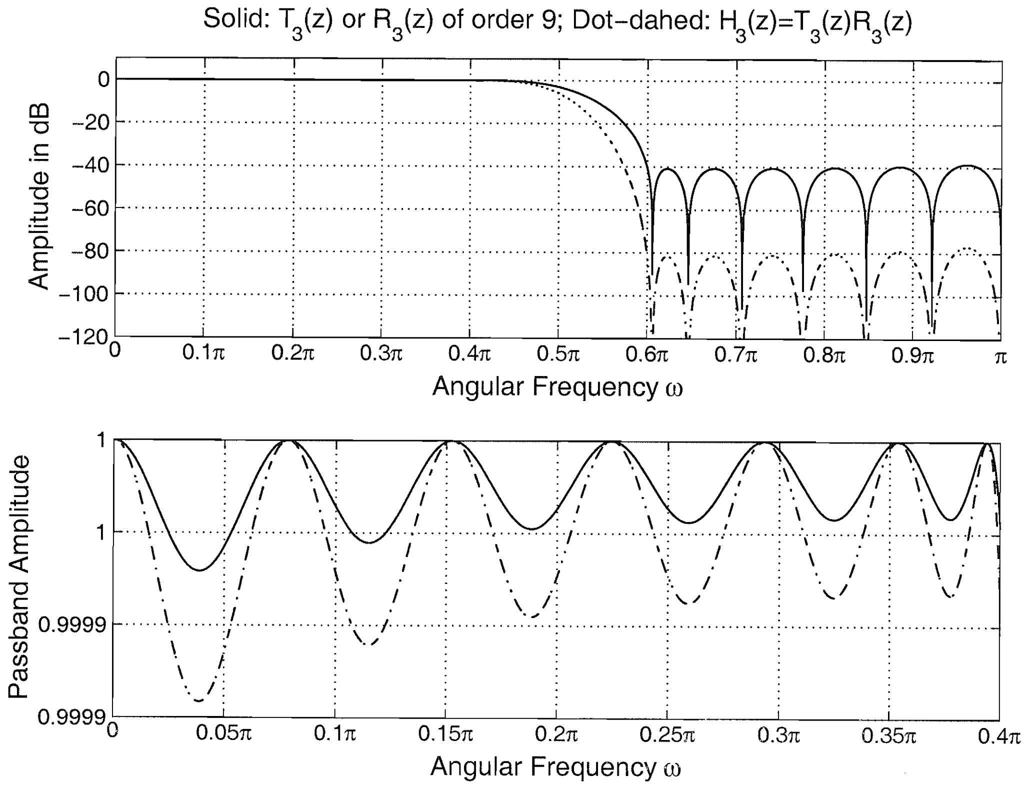

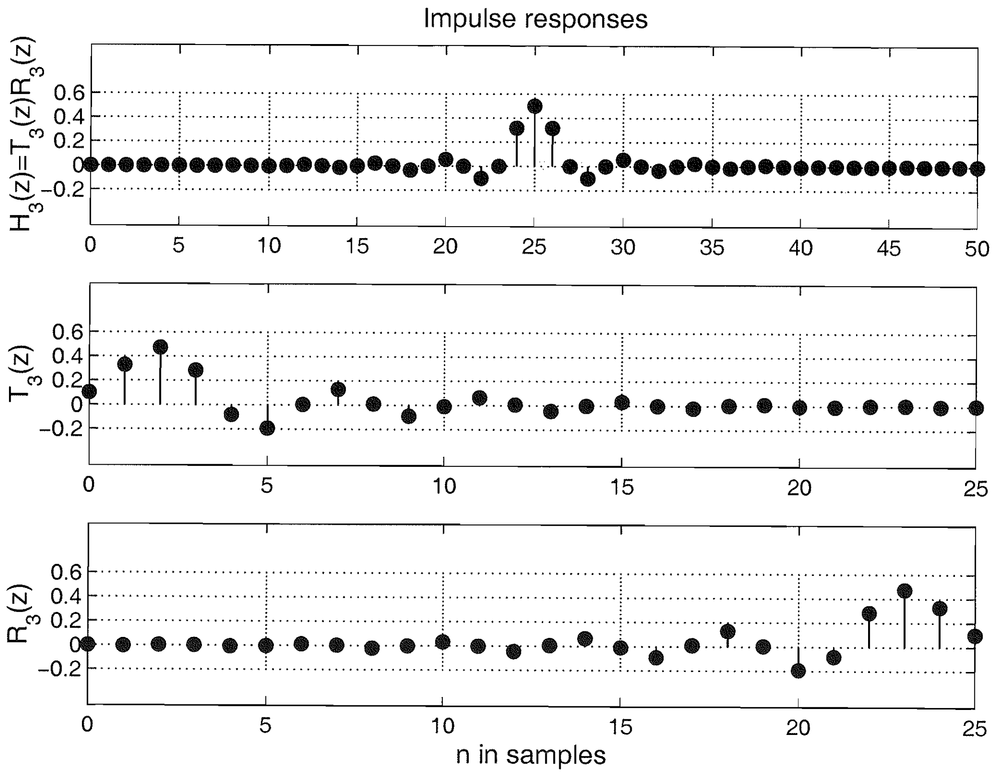

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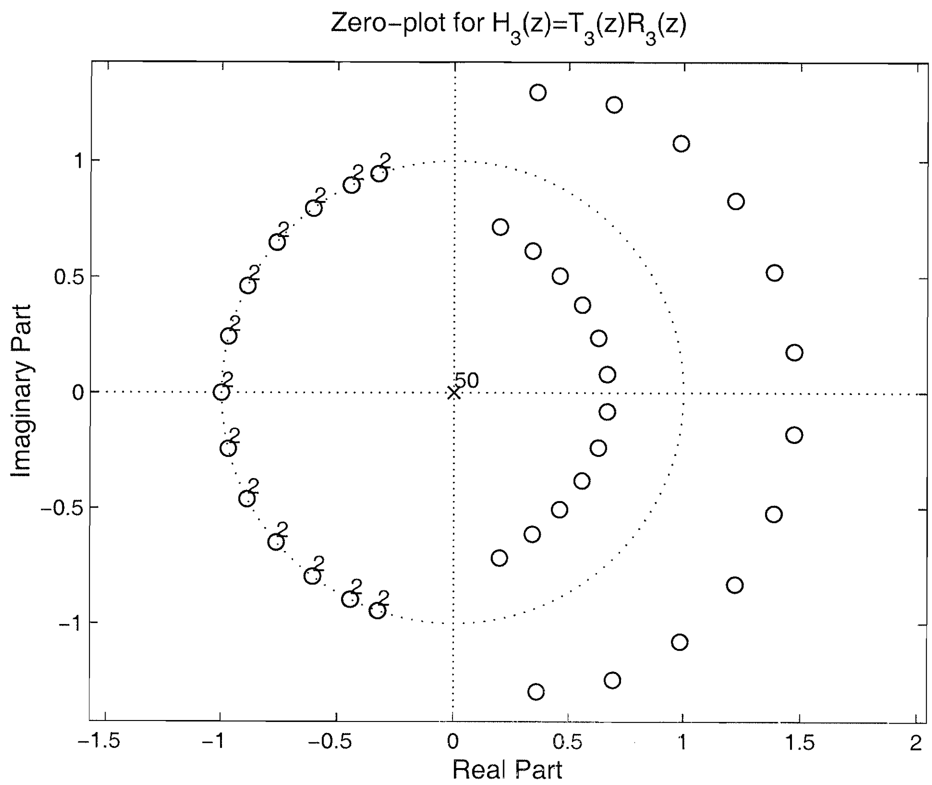

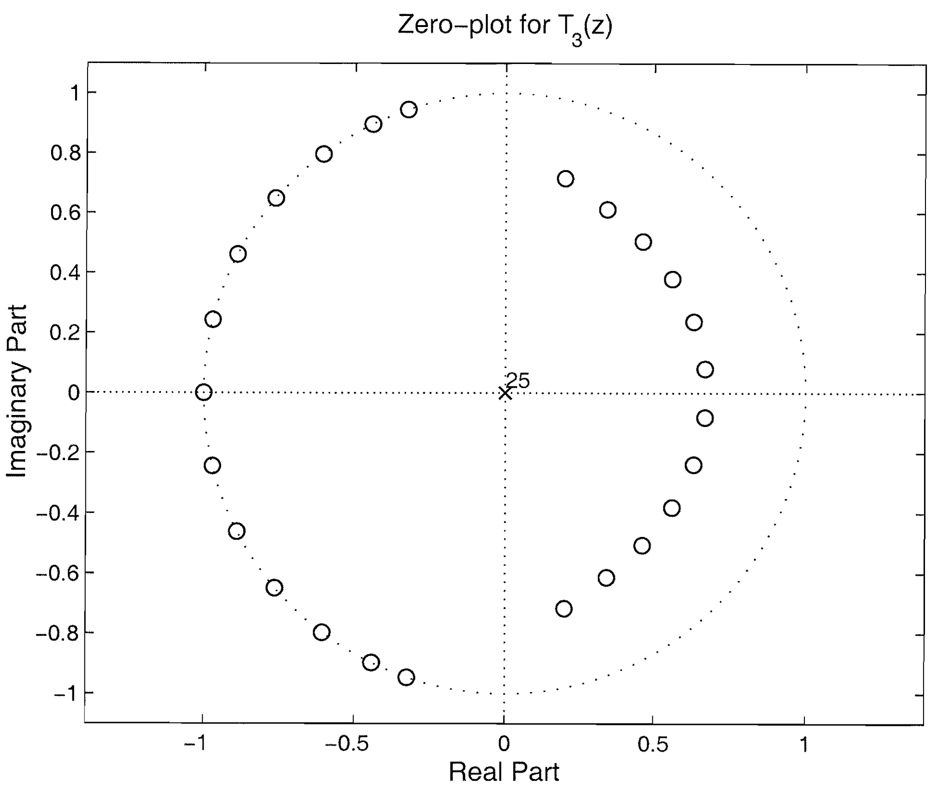

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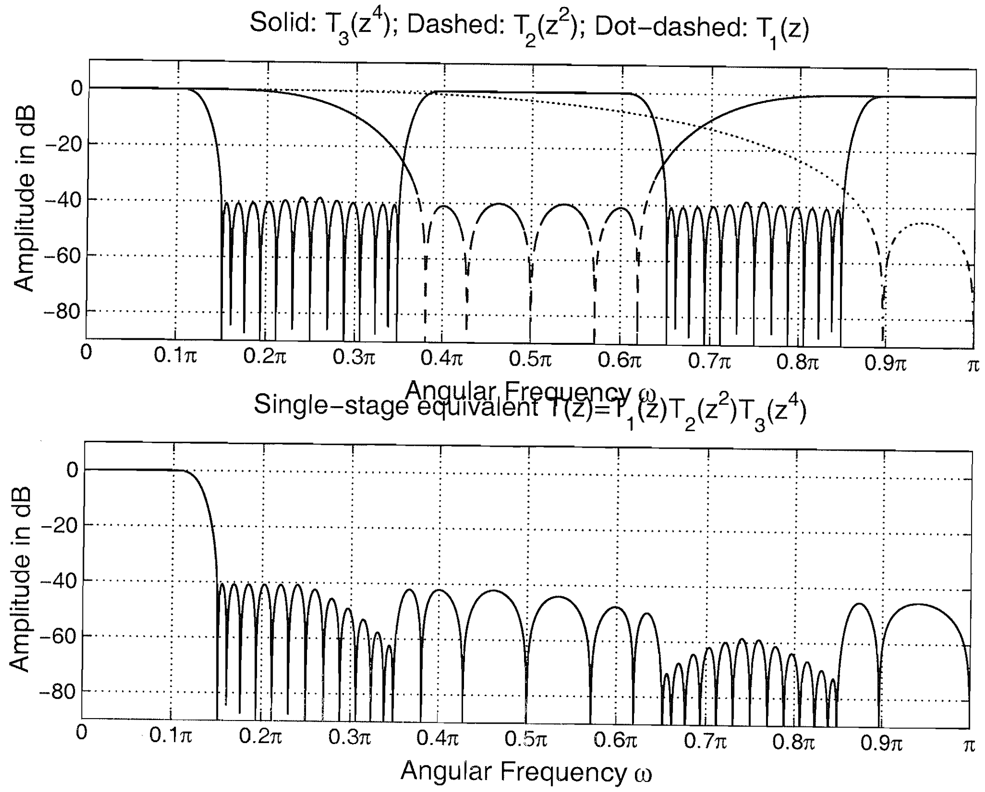

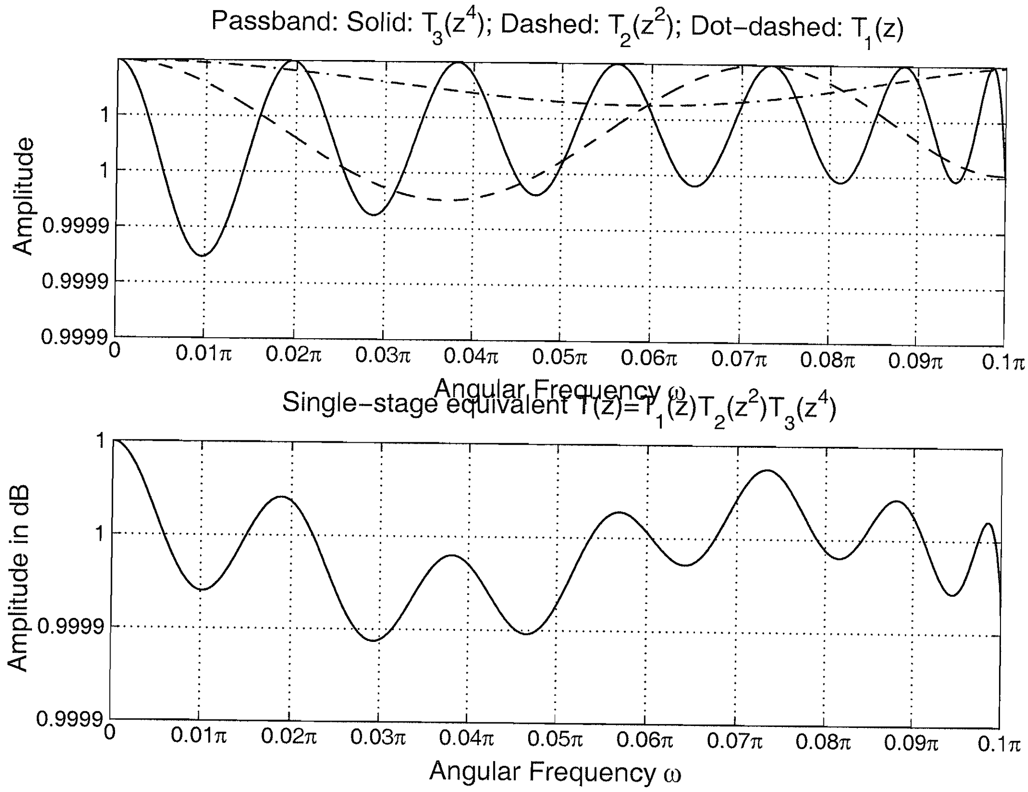

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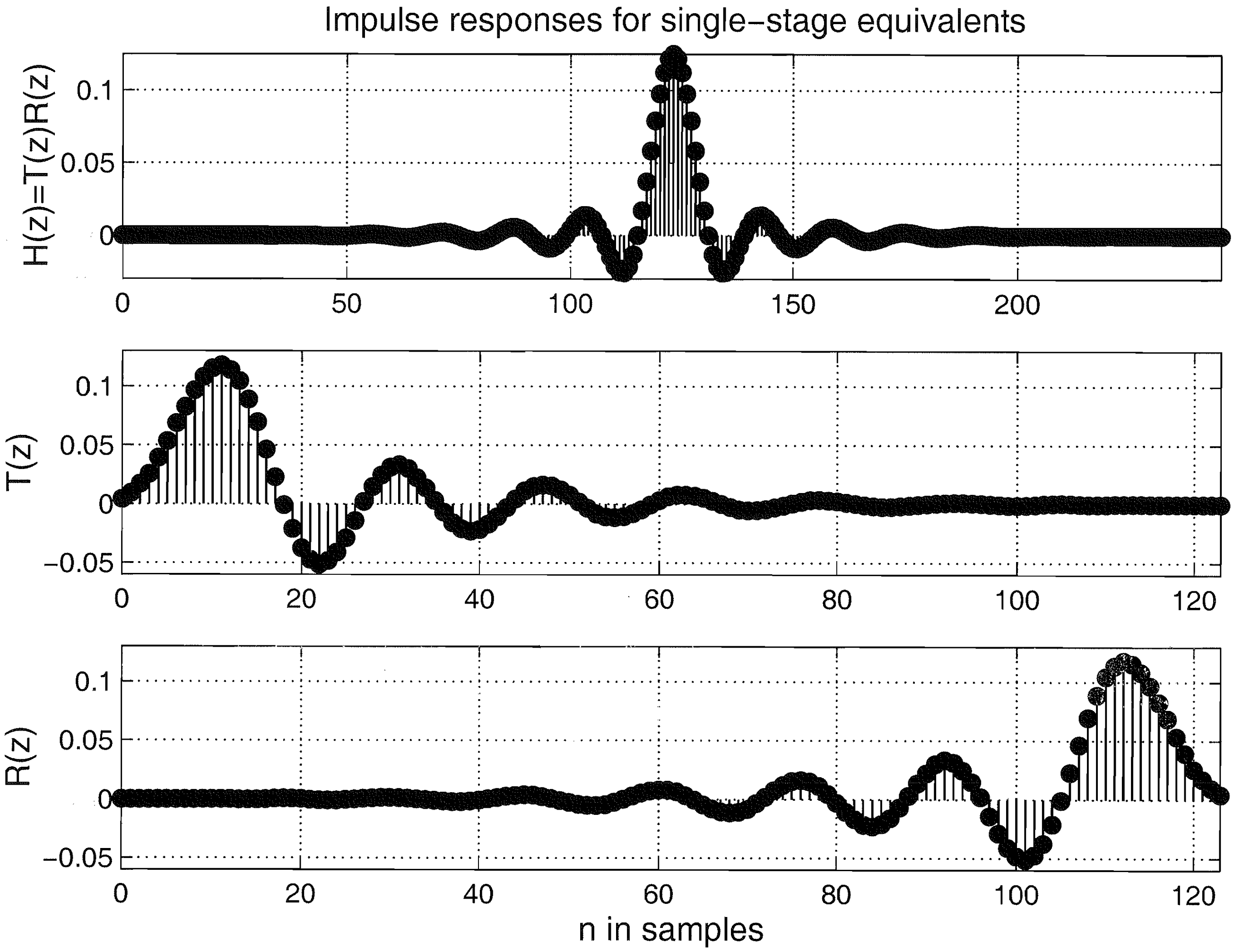

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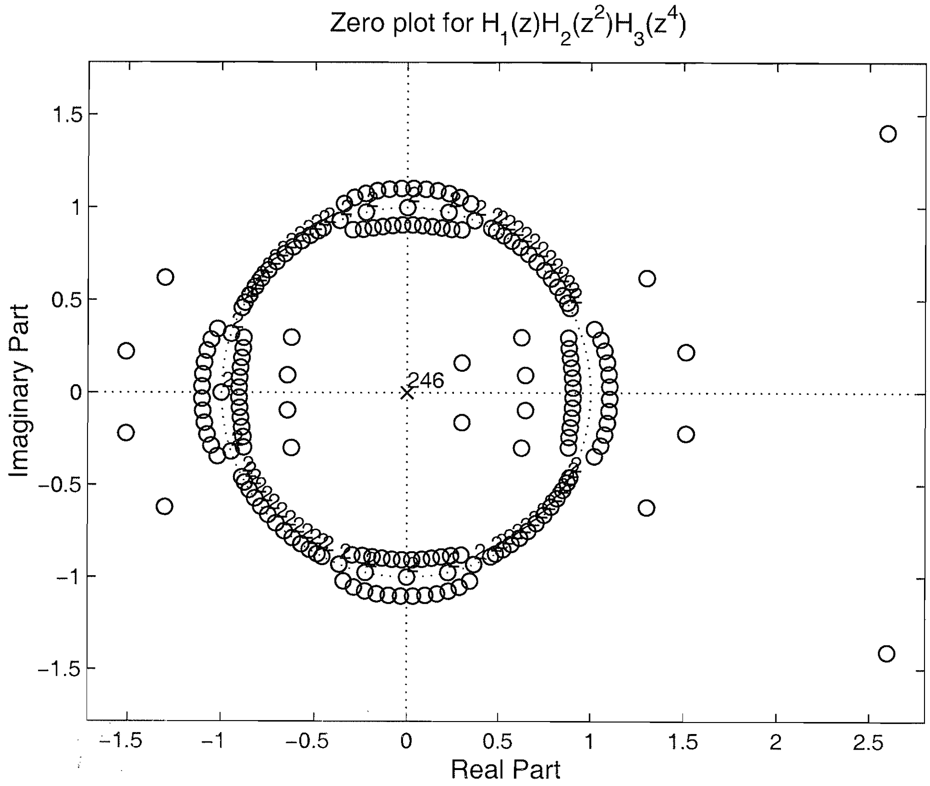
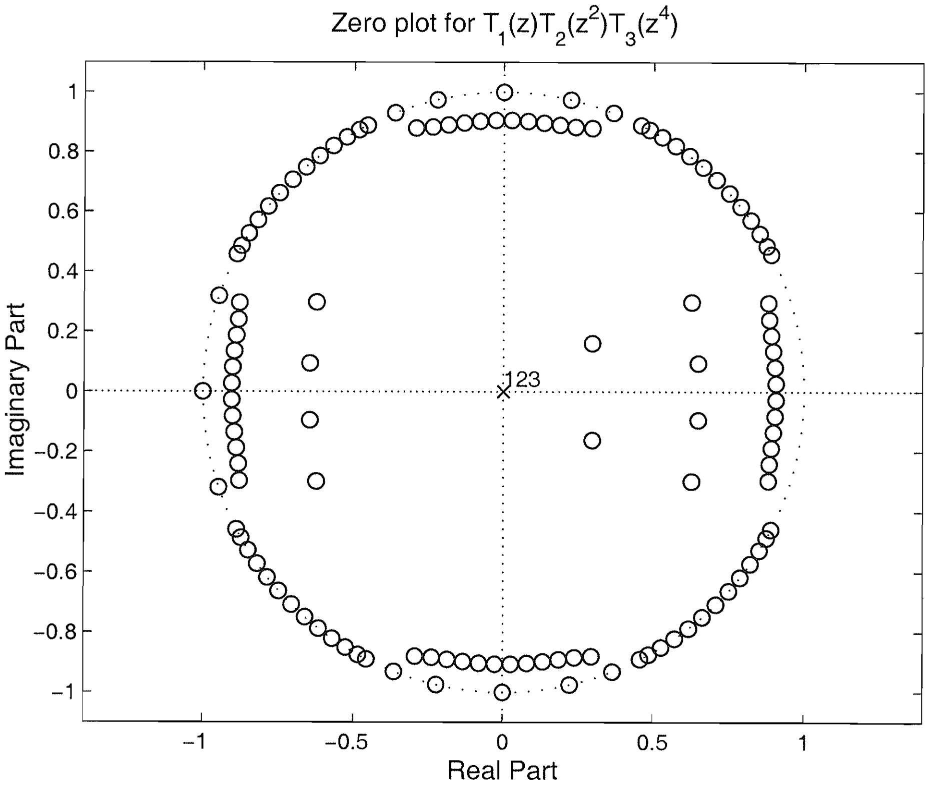
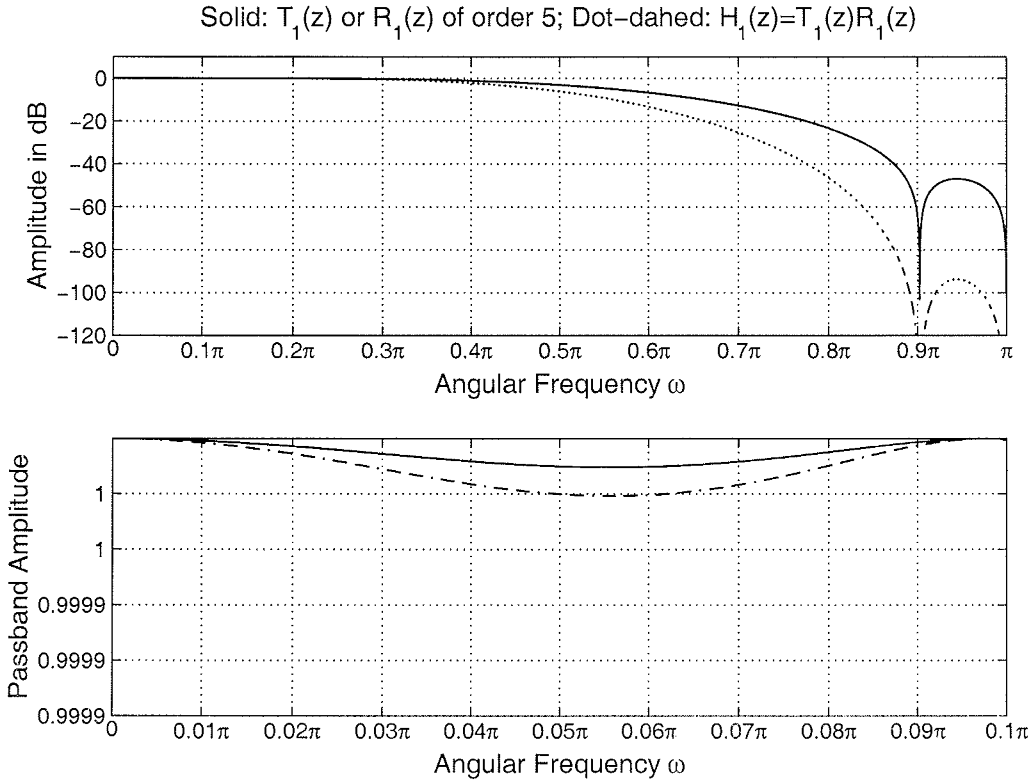
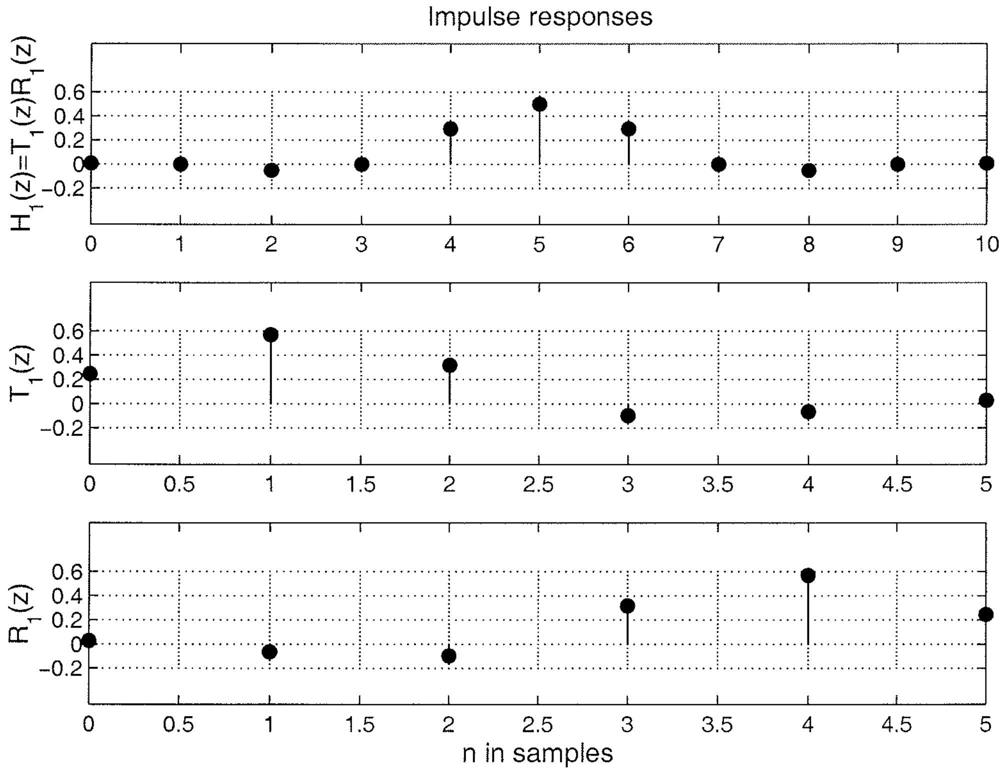




Separable Three-Stage Least-Squared Eighth-Band ($N = 8$) FIR filter





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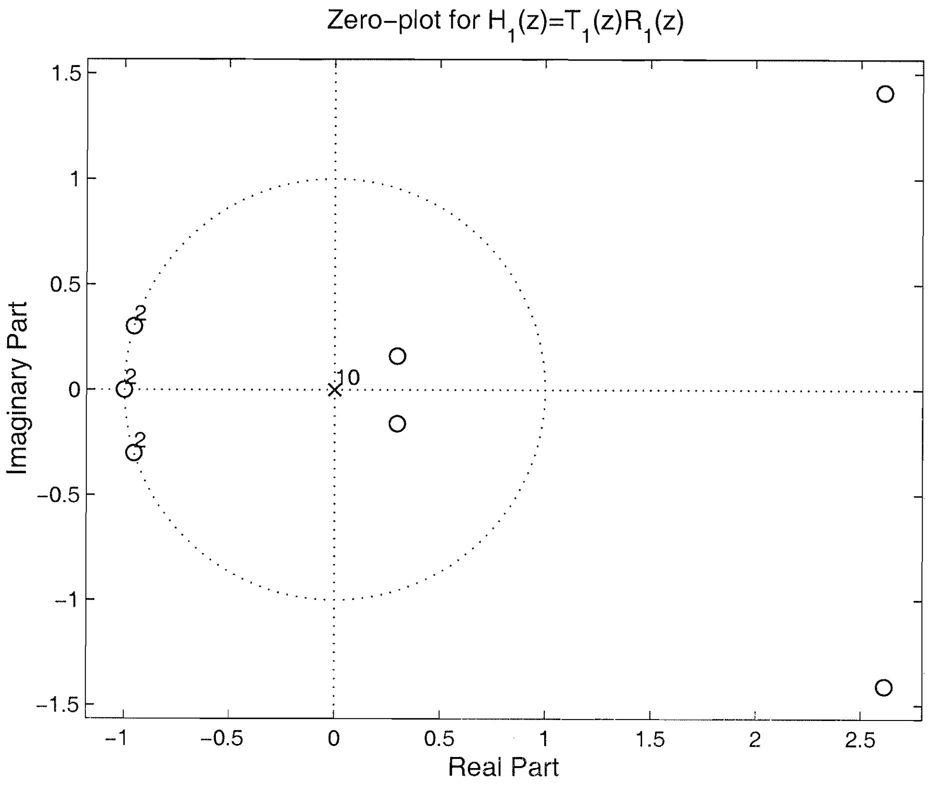

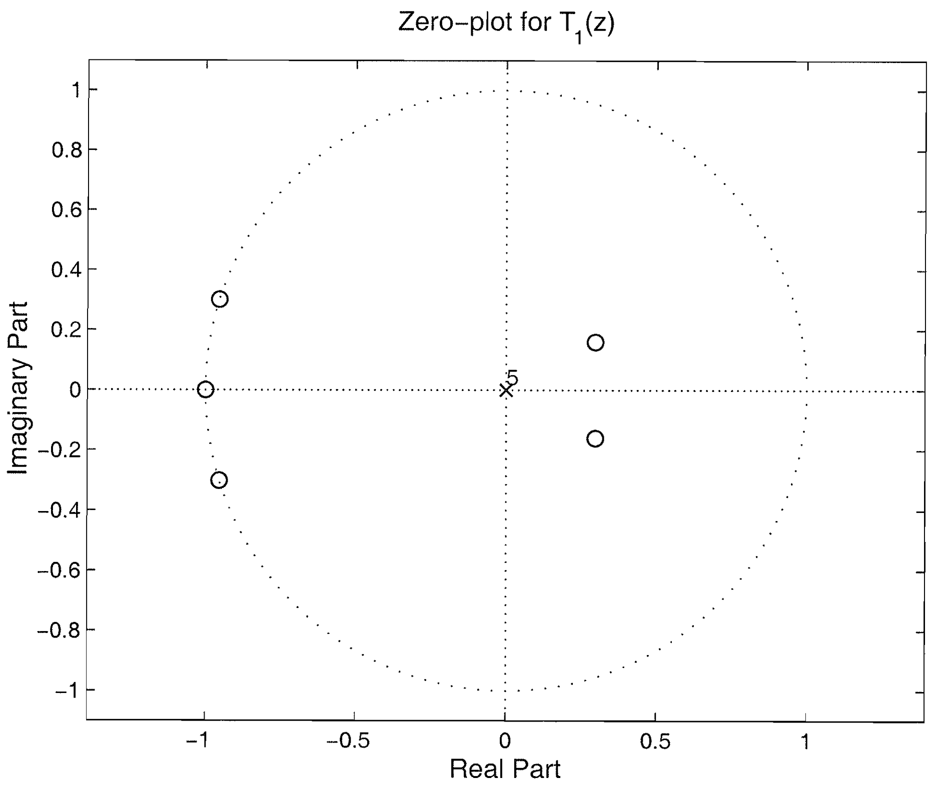

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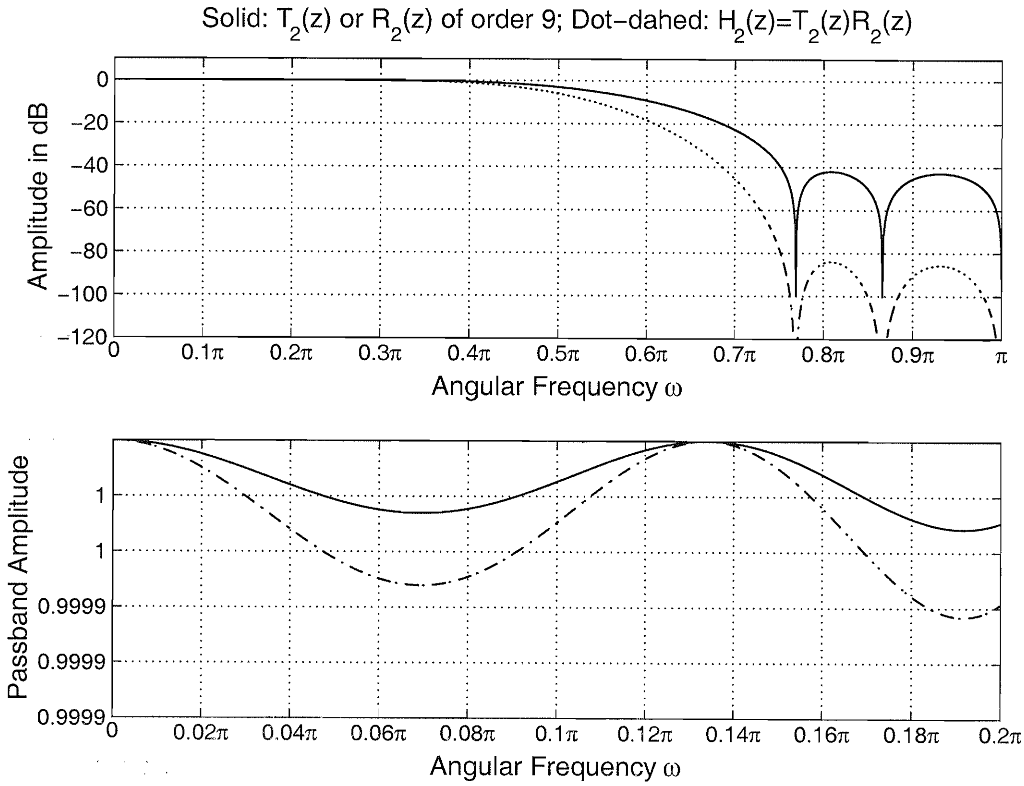

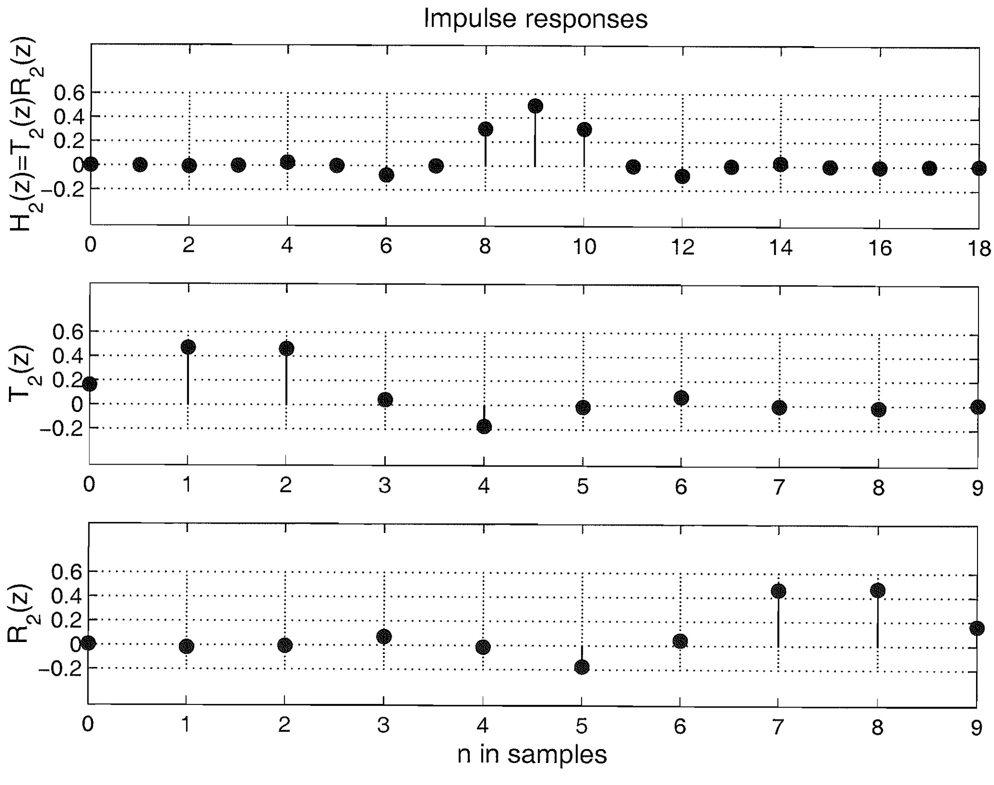

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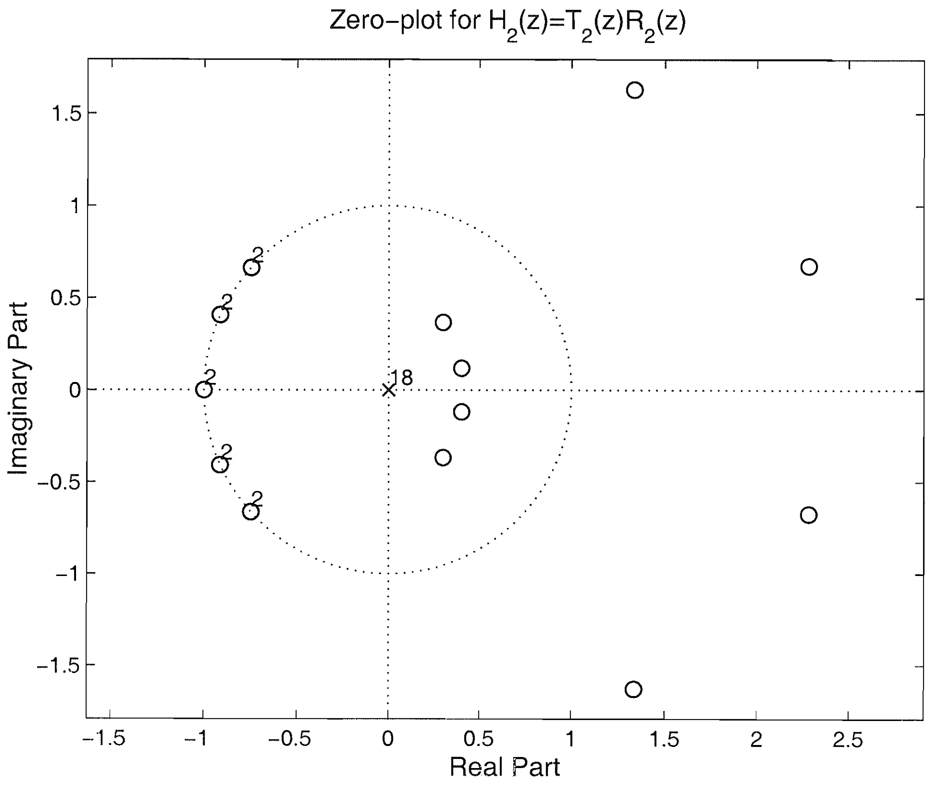
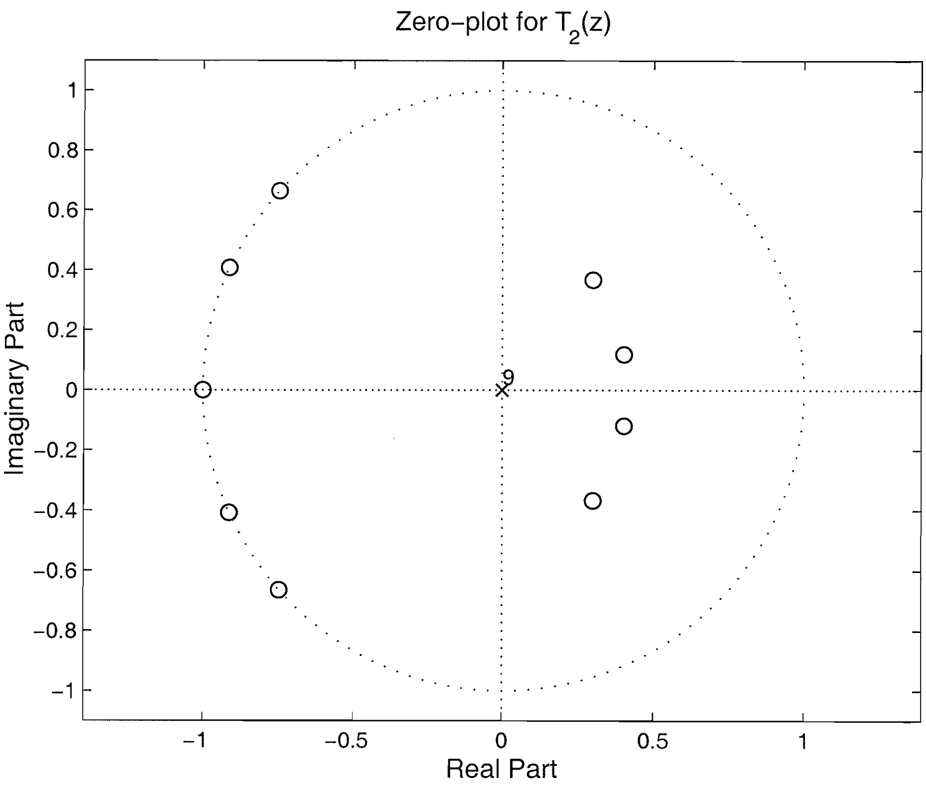
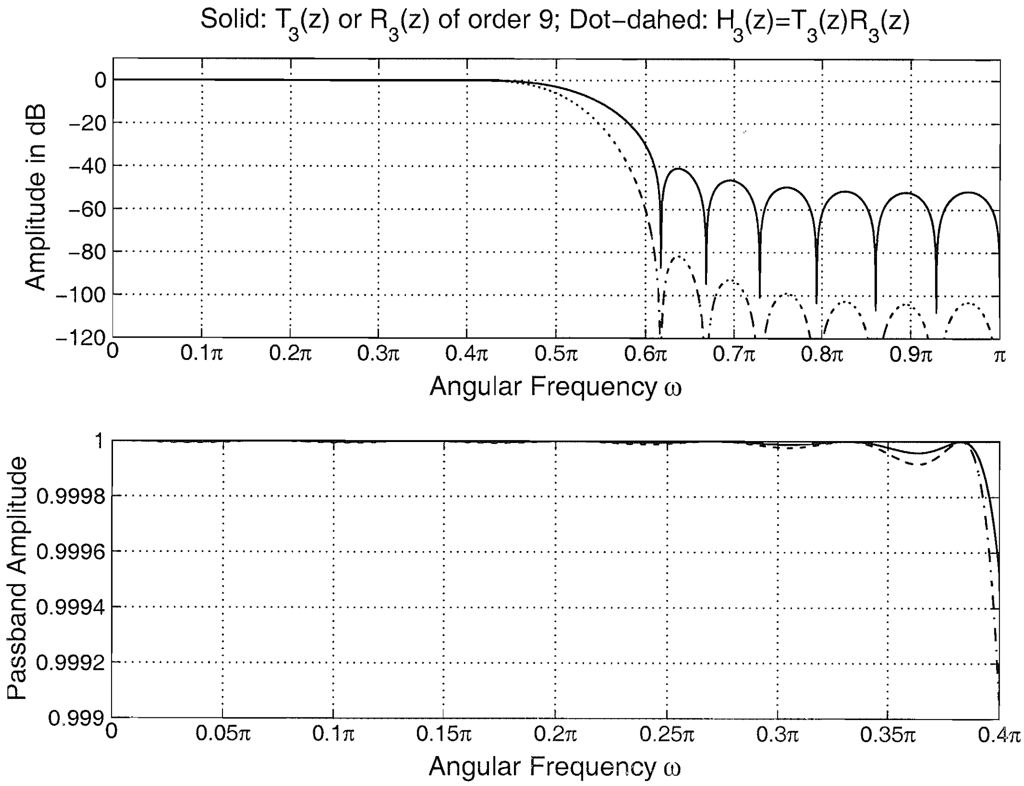
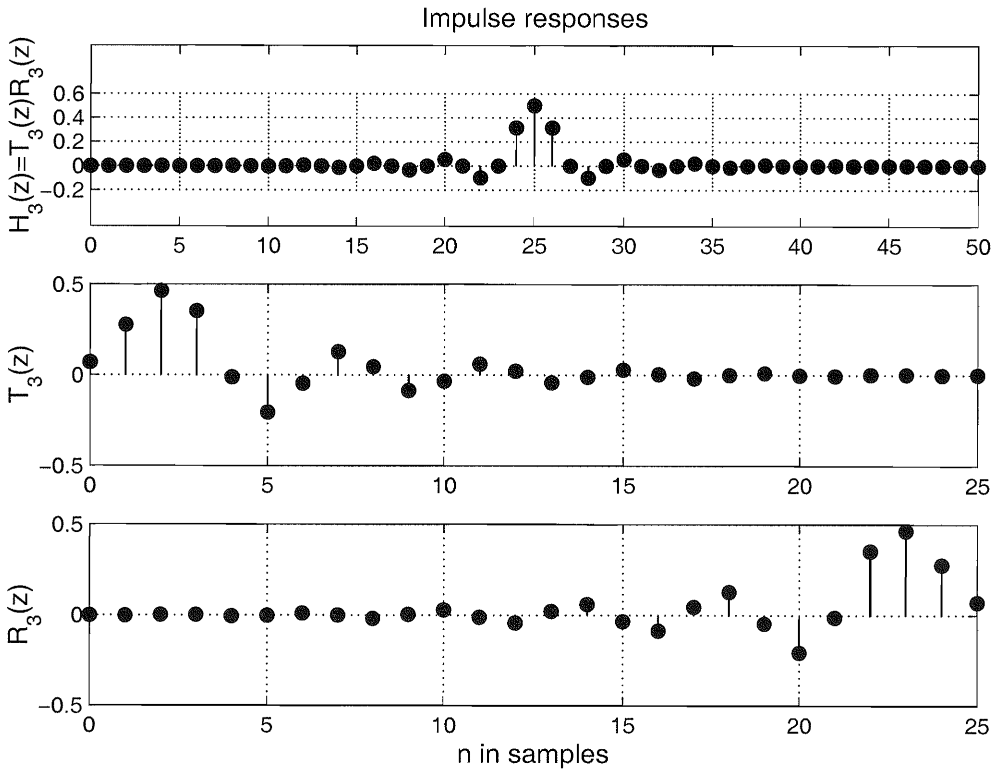


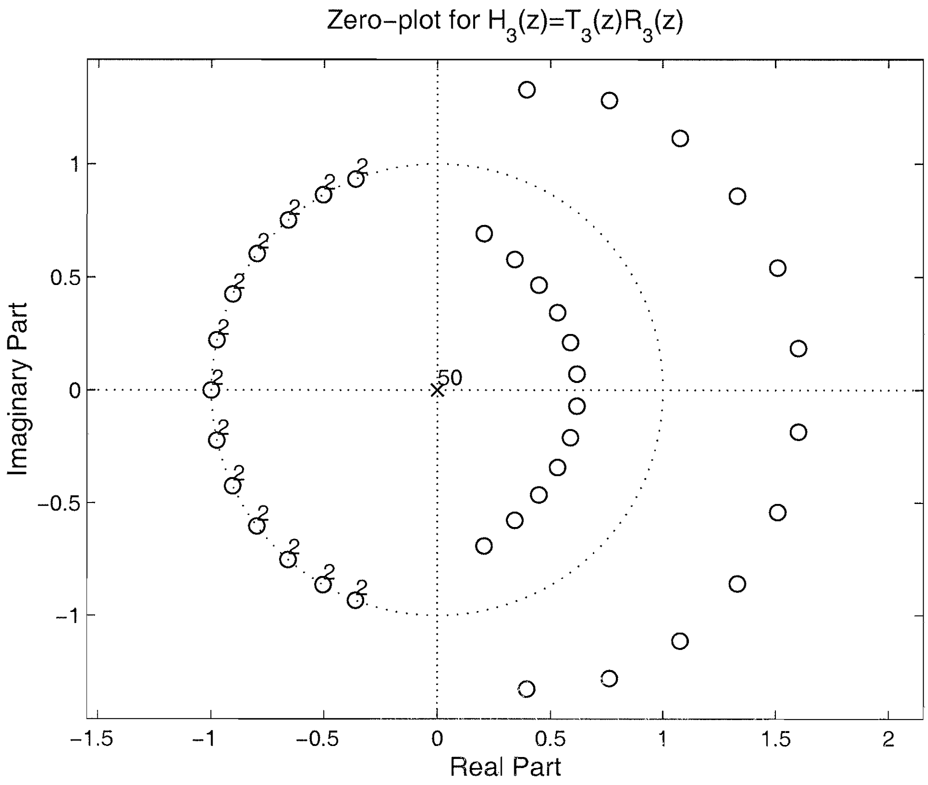


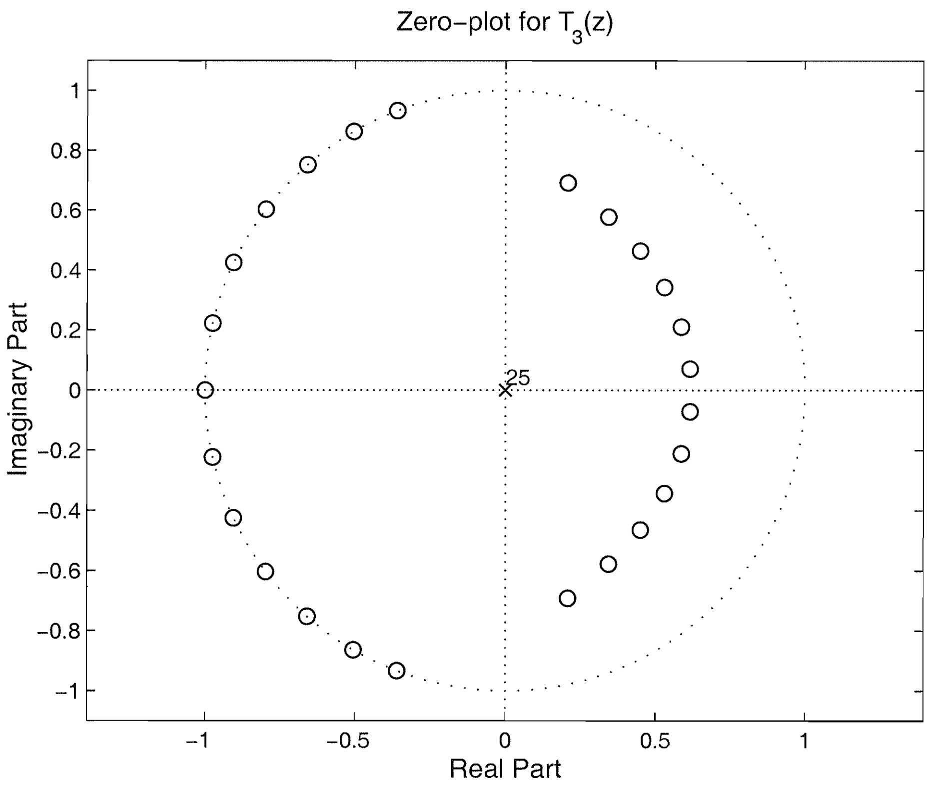
Separable Three-Stage Least-Squared Eighth-Band ($N = 8$) FIR filter



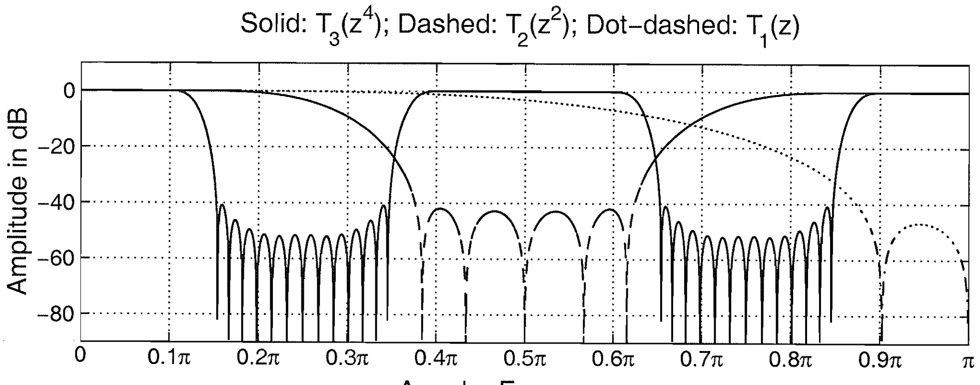


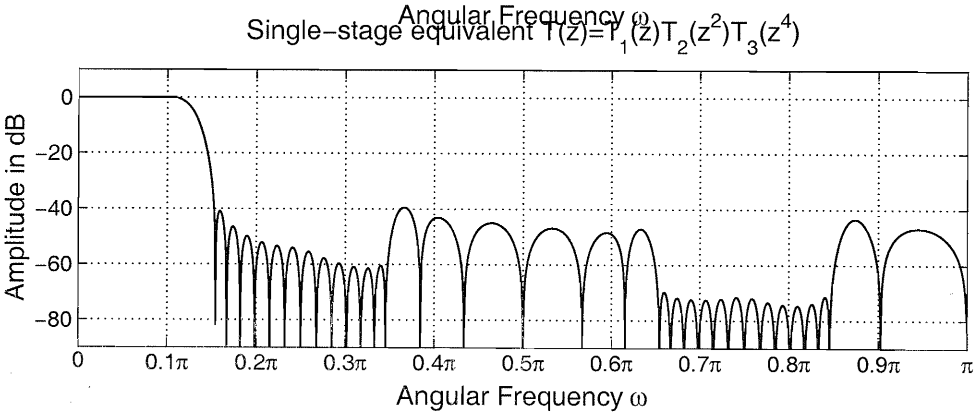
Separable Three-Stage Least-Squared Eighth-Band ($N = 8$) FIR filter

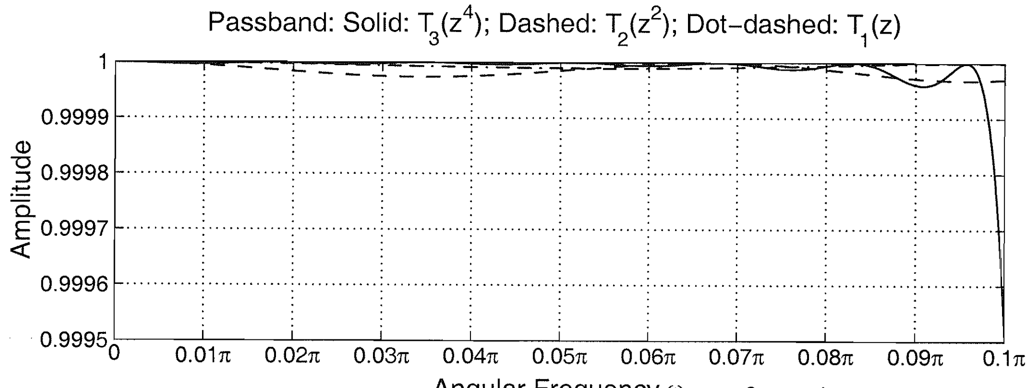


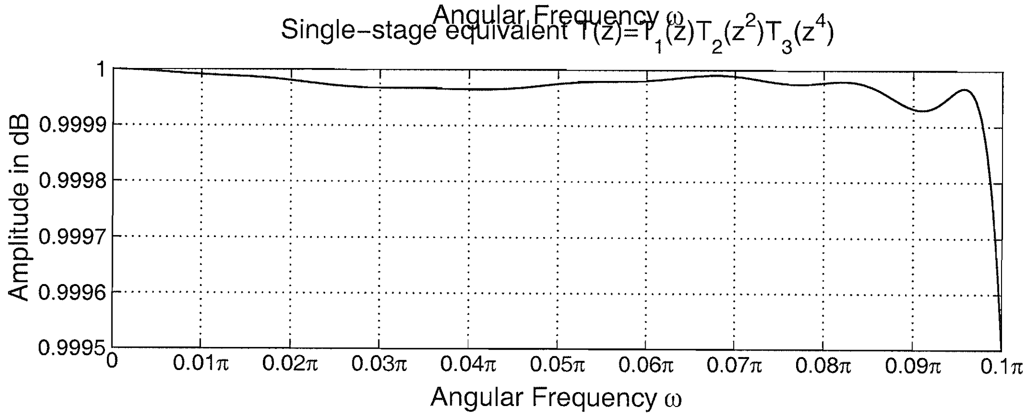


Separable Three-Stage Least-Squared Eighth-Band ($N = 8$) FIR filter

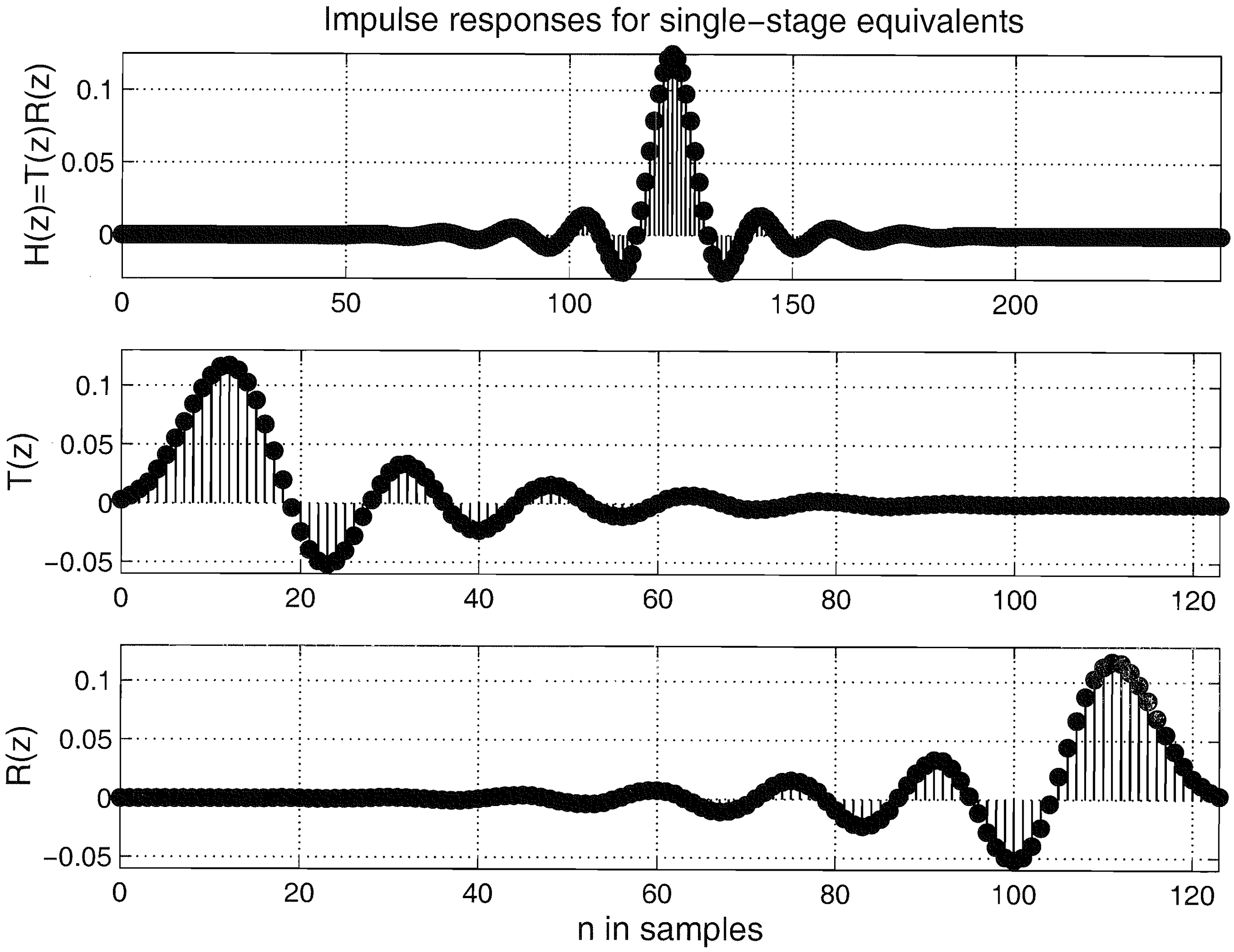




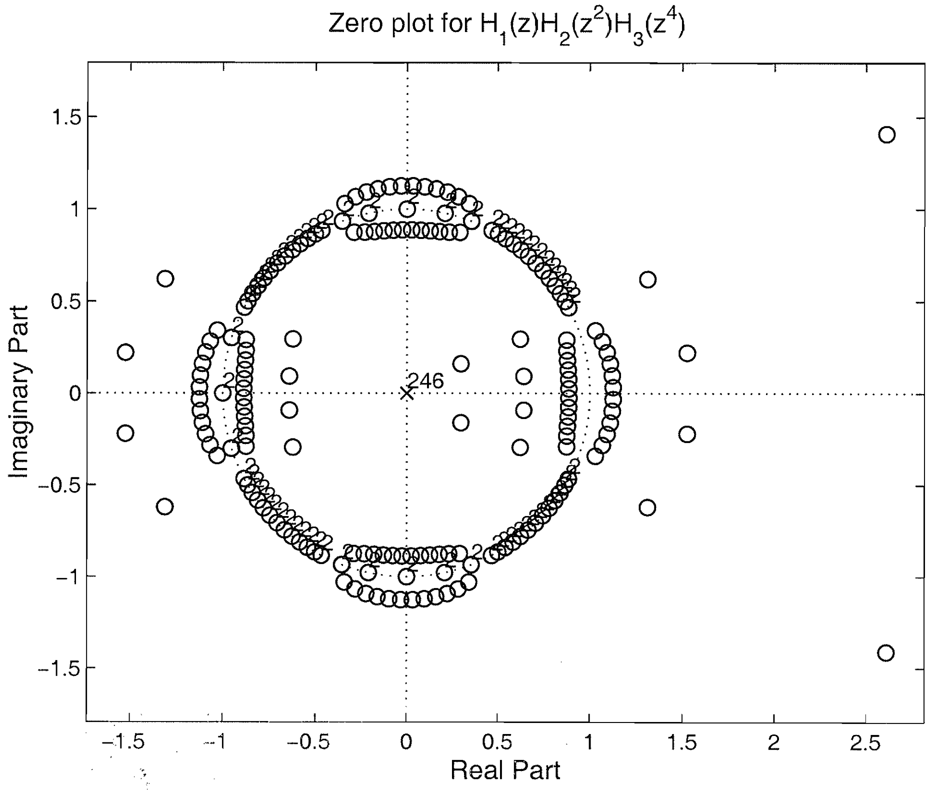


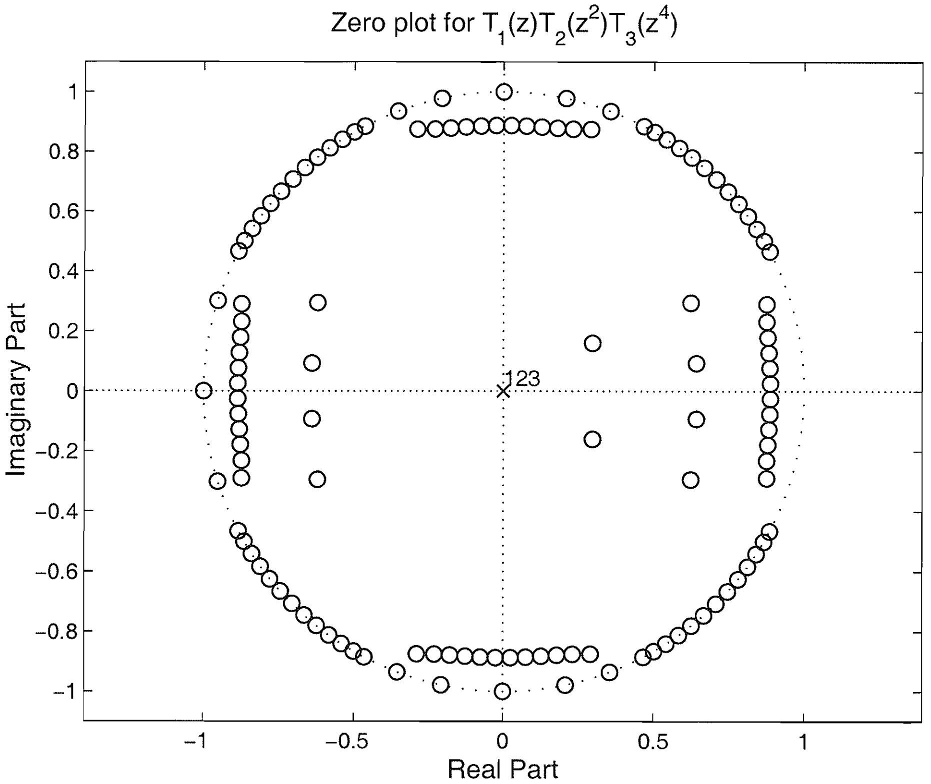


Separable Three-Stage Least-Squared Eighth-Band ($N = 8$) FIR filter



Separable Three-Stage Least-Squared Eighth-Band ($N = 8$) FIR filter





II.F: Half-Band FIR Filters

• For a half-band linear-phase FIR filter, the transfer function is of the form

$$H(z) = \sum_{n=0}^{2M} h[n]z^{-n}, \quad h[2M-n] = h[n], \tag{49}$$

where M is odd.

• For these filters,

$$h[M] = 1/2 \tag{50a}$$

$$h[M+2r] = 0 \quad \text{for} \quad r = \pm 1, \pm 2, \ldots, \pm(M-1)/2. \tag{50b}$$

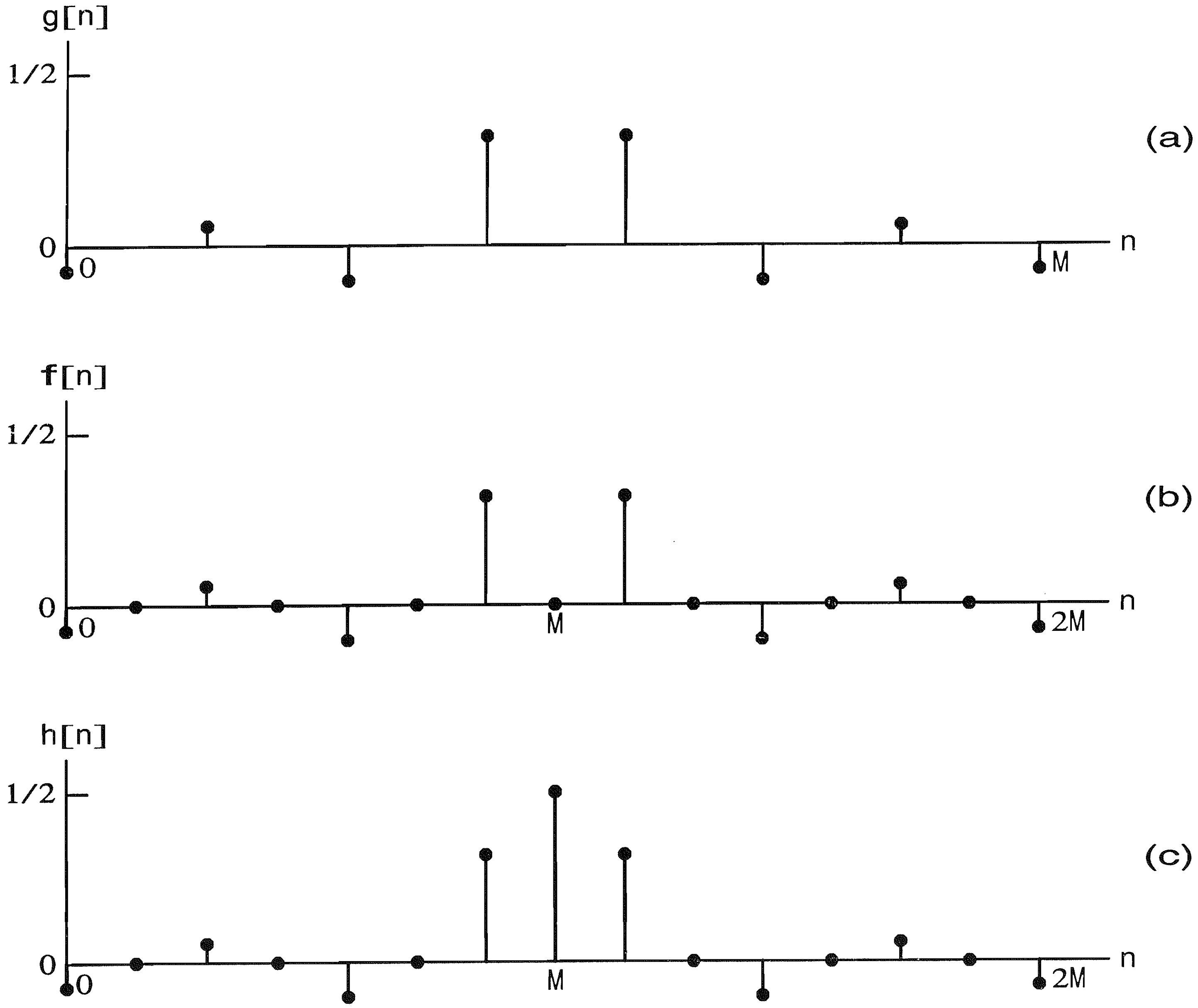
• A filter satisfying these conditions can be generated in two steps by starting with a Type II (M is odd) transfer function

$$G(z) = \sum_{n=0}^{M} g[n]z^{-n}, \quad g[n] = g[M-n]. \tag{51}$$

• In the first step, zero-valued impulse-response values are inserted between the $g[n]$'s [see Figures (a) and (b) on the next page], giving the following Type I transfer function of order $2M$:

$$F(z) = \sum_{n=0}^{2M} f[n]z^{-n} = G(z^2) = \sum_{n=0}^{M} g[n]z^{-2n}. \tag{52}$$

Generation of the Impulse Response of a Half-Band Filter



• The second step is then to replace the zero-valued impulse-response value at $n = M$ by $1/2$ [see Figure (c) on the previous page], resulting in the desired transfer function

$$\begin{aligned} H(z) =& \sum_{n=0}^{2M} h[n] z^{-n} = \frac{1}{2} z^{-M} + F(z) \\ =& \frac{1}{2} z^{-M} + \sum_{n=0}^{M} g[n] z^{-2n}. \end{aligned} \tag{53}$$

• This gives $h[M] = 1/2$, $h[n] = g[n/2]$ for n even, and $h[n] = 0$ for n odd and $n \neq M$, as is desired.

The zero-phase frequency responses of $H(z)$, $F(z)$, and $G(z)$ are related through

$$H(\omega) = 1/2 + F(\omega) = 1/2 + G(2\omega) \qquad (54a)$$

and

$$F(\omega) = G(2\omega) \qquad (54b)$$

where

$$H(\omega) = h[M] + 2\sum_{k=1}^{M} h[M-k]\cos k\omega, \qquad (54c)$$

$$F(\omega) = f[M] + 2\sum_{k=1}^{M} f[M-k]\cos k\omega, \qquad (54d)$$

$$G(\omega) = 2\sum_{k=1}^{(M-1)/2} g[M-k]\cos[(2k-1)\omega/2], \qquad (54e)$$

and

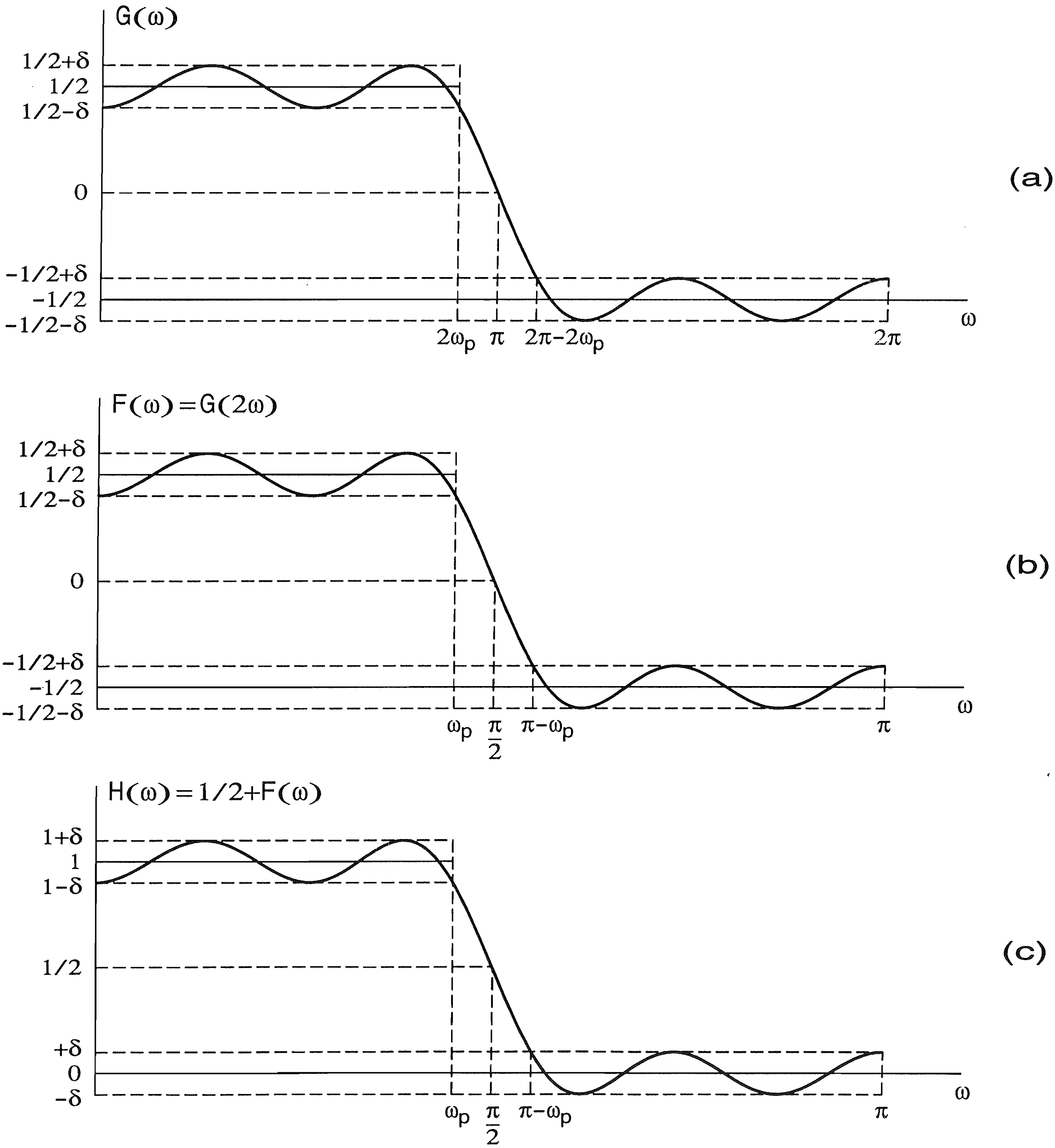
$$G(2\omega) = 2\sum_{k=1}^{(M-1)/2} g[M-k]\cos[2(2k-1)\omega]. \qquad (54f)$$

• Note that the actual frequency responses of filters with transfer functions $H(z)$, $F(z)$, and $G(z^2)$ are obtained by multiplying the zero-phase frequency responses $H(\omega)$, $F(\omega)$, and $G(2\omega)$ by the common phase term $e^{jM\omega}$. In the case of $G(z)$, $G(\omega)$ is multiplied by $e^{jM\omega/2}$.

• Based on these relations, the design of a lowpass half-band filter with passband edge at ω_p and passband ripple of δ can be accomplished by determining $G(z)$ such that $G(\omega)$ oscillates within $1/2 \pm \delta$ on $[0, 2\omega_p]$ [see Figure (a) on the next page].

- Since $G(z)$ is a Type II transfer function, it has one fixed zero at $z = -1$ ($\omega = \pi$).
- $G(z)$ can be designed directly with the aid of the Remez algorithm using only one band $[0, 2\omega_p]$, $D(\omega) = 1/2$, and $W(\omega) = 1$.
- Since $G(z)$ has a single zero at $z = -1$, $G(\omega)$ is odd about $\omega = \pi$.
- Hence, $G(2\pi - \omega) = -G(\omega)$ and $G(\omega)$ oscillates within $-1/2 \pm \delta$ on $[2\pi - 2\omega_p, 2\pi]$.

Design of A Lowpass Half-Band Filter



• The corresponding $F(\omega) = G(2\omega)$ stays within $1/2 \pm \delta$ on $[0, \omega_p]$ and within $-1/2 \pm \delta$ on $[\pi - \omega_p, \pi]$ [see Figure (b) on the previous page].

• Finally, $H(\omega) = 1/2 + F(\omega)$ approximates unity on $[0, \omega_p]$ with tolerance δ and zero on $[\pi - \omega_p, \pi]$ with the same tolerance δ [see Figure (c) on the previous page].

• For the resulting $H(\omega)$, the passband and stopband ripples are thus the same and the passband and stopband edges are related through $\omega_s = \pi - \omega_p$.

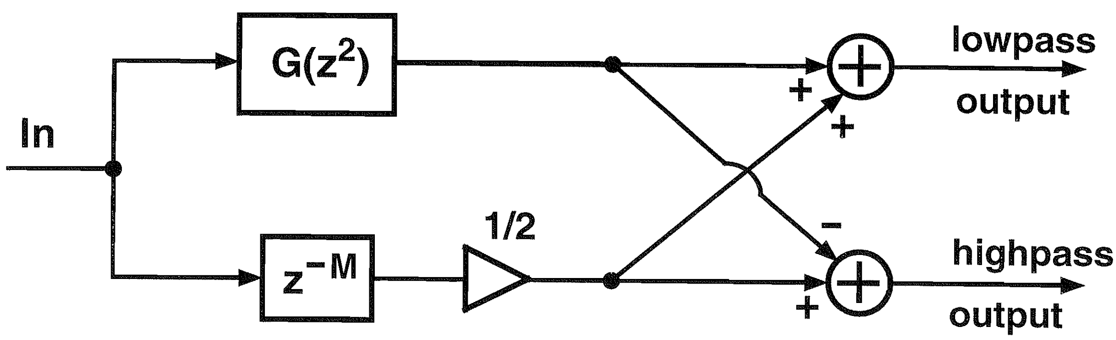
• In general, $H(\omega)$ satisfies

$$H(\omega) + H(\pi - \omega) = 1. \tag{55}$$

• This makes $H(\omega)$ symmetric about the point $\omega = \pi/2$ such that the sum of the values $H(\omega)$ at $\omega = \omega_0 < \pi/2$ and at $\omega = \pi - \omega_0 > \pi/2$ is equal to unity [see Figure (c) on the previous page].

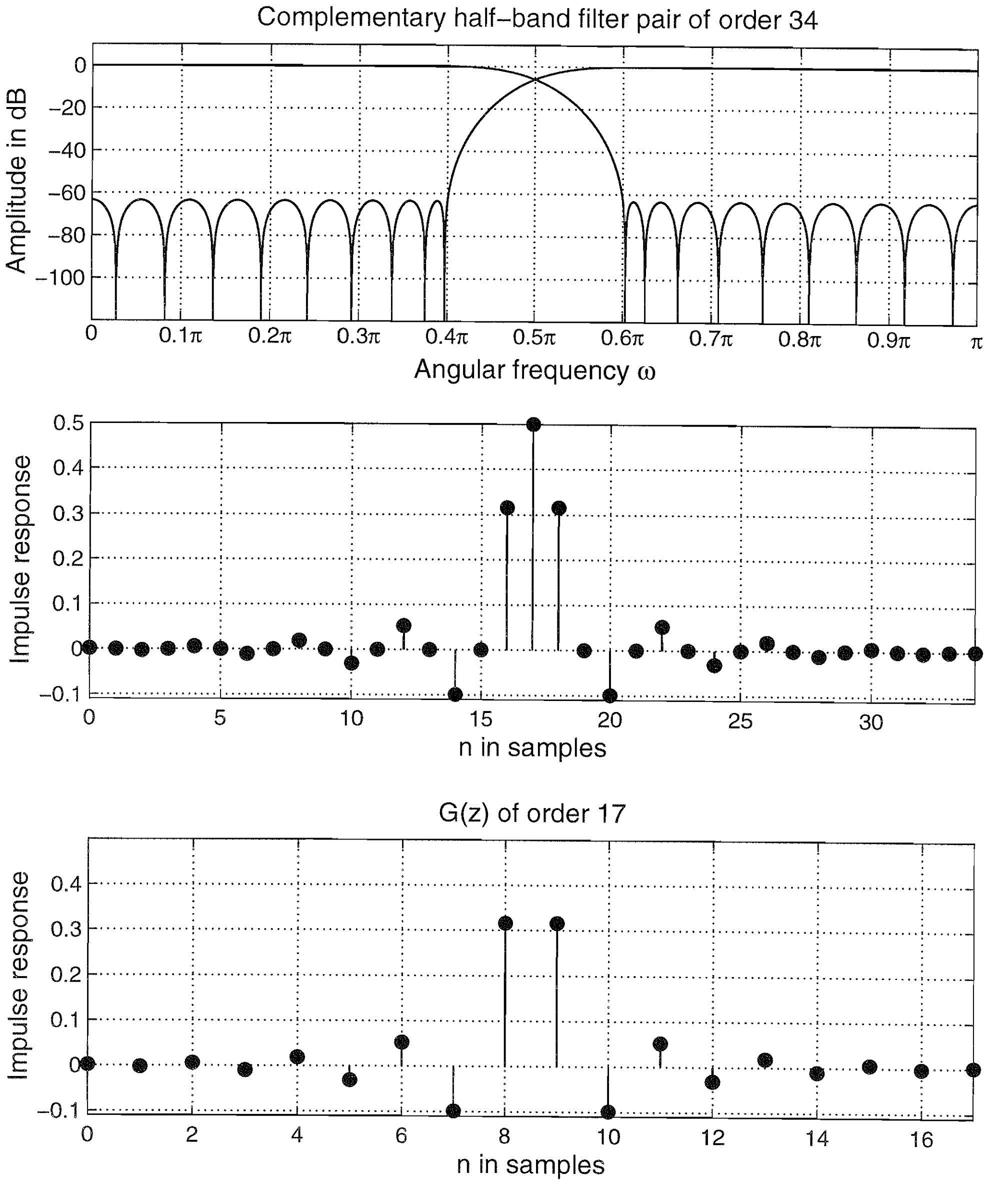
Efficient Implementation of a Half-Band Filter

- An implementation for the half-band filter as a parallel connection of $G(z^2)$ and $(1/2)z^{-M}$ is shown below.
- This implementation is very attractive as in this case the complementary highpass output having the zero-phase frequency response $1 - H(\omega)$ is obtained directly by subtracting $G(z^2)$ from $(1/2)z^{-M}$.
- The term z^{-M} can be shared with $G(z^2)$.
- The number of non-zero coefficients in $G(z^2)$ is $M + 1$. By exploiting the symmetry in these coefficients, only $(M + 1)/2$ multipliers (M is odd) are needed to implement a lowpass–highpass filter pair of order $2M$.



Responses for a Complementary Half-Band Filter Pair of order 34 for $\omega_p = 0.4\pi$.

- The implementation of this filter pair requires only nine multipliers.



Efficient Implementations for Half-Band FIR Decimators and Interpolators

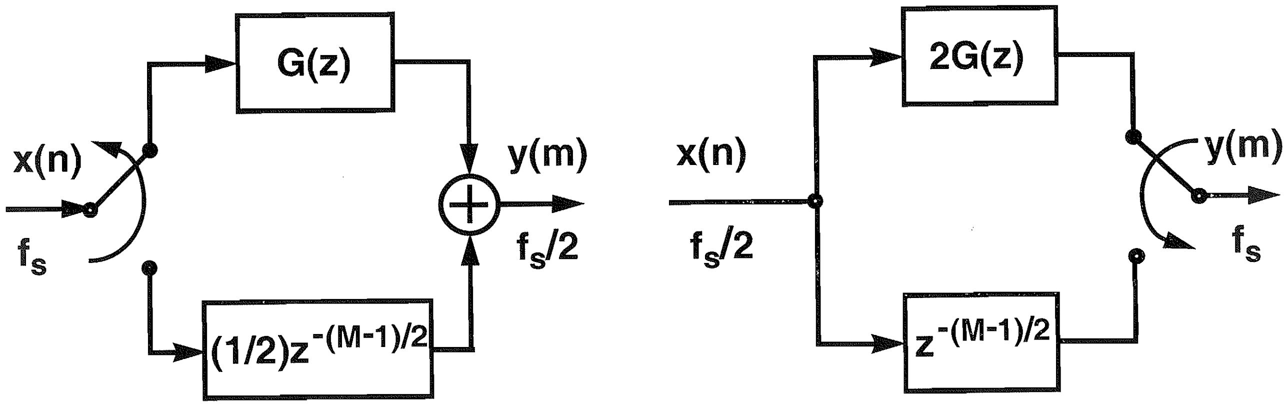
• The transfer function of the half-band filter is expressible as the following polyphase form:

$$H(z) = G_0(z^2) + z^{-1}G_1(z^2), \qquad (56a)$$

where

$$G_0(z) = G(z) \quad \text{and} \quad G_1(z) = (1/2)z^{-(M-1)/2}. \qquad (56b)$$

• Based on these relations, the decimation and interpolation filters for sampling rate alteration by a factor of two can be effectively implemented using the structures shown below.



Design of Factorizable Half-Band Filters

• Given an odd M as well as the stopband edge $\omega_s > \pi/2$, there exists a very efficient procedure for designing a half-band filter of order $2M$ that is separable as $H(z) = T(z)R(z)$, where the impulse responses of $T(z)$ and $R(z)$ are time-reversed versions of each other.

• This procedure illustrated for $\omega_s = 0.6\pi$, $\omega_p = \pi - \omega_s = 0.4\pi$ and $M = 39$ and is carried out in the following three steps:

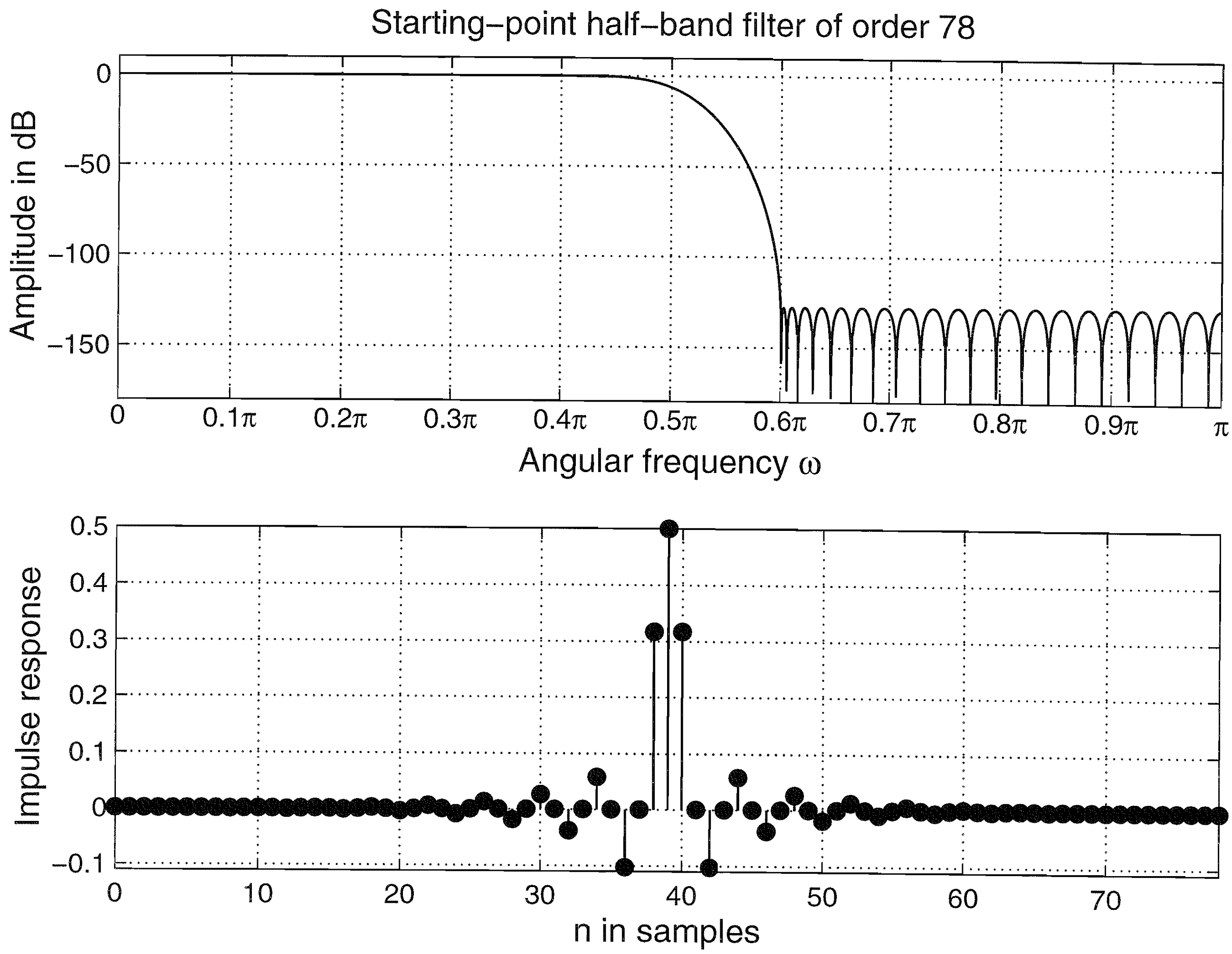
Step 1: Design a half-band filter $\widetilde{E}(z) = \sum_{n=0}^{2M} \widetilde{e}(n)z^{-n}$ ($\widetilde{e}(M) = 1/2$, $\widetilde{e}(M \pm 2r) = 0$ for $r = 1, 2, \cdots, (M-1)/2$) with the stopband edge at ω_s.

• This can be performed with the aid of the Remez algorithm according to the procedure described previously using $\omega_p = \pi - \omega_s$.

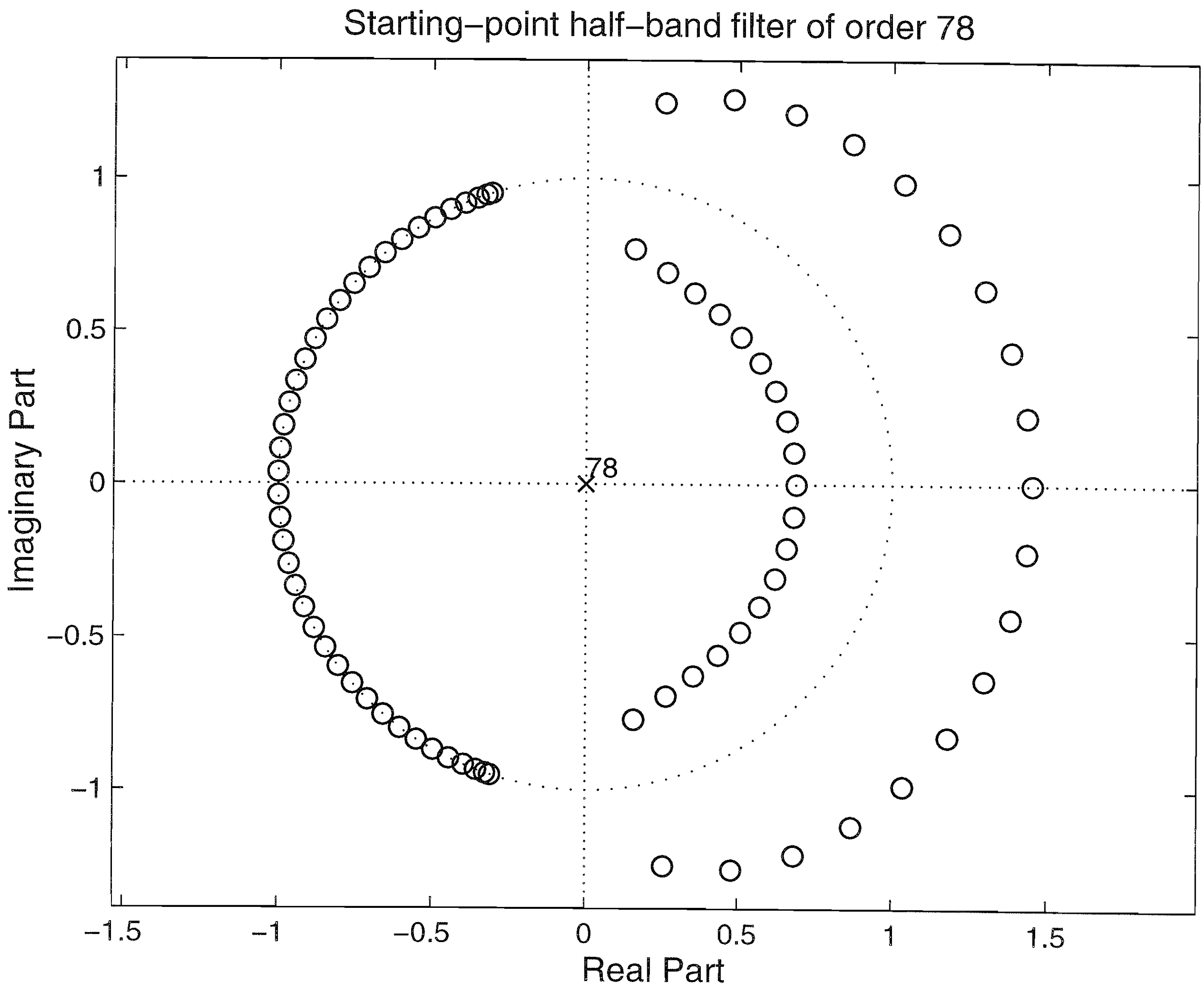
• For $\omega_s = 0.6\pi$ and $M = 39$, the following pages show the overall amplitude response, the zero-plot, the impulse response as well as the zero-phase frequency response in both the passband and stopband. In this case, the passband and stopband ripples are $\delta = 4.11 \cdot 10^{-7}$.

• It is seen that the filter has single zeros on the unit circle. Therefore, the desired factorization is not possible.

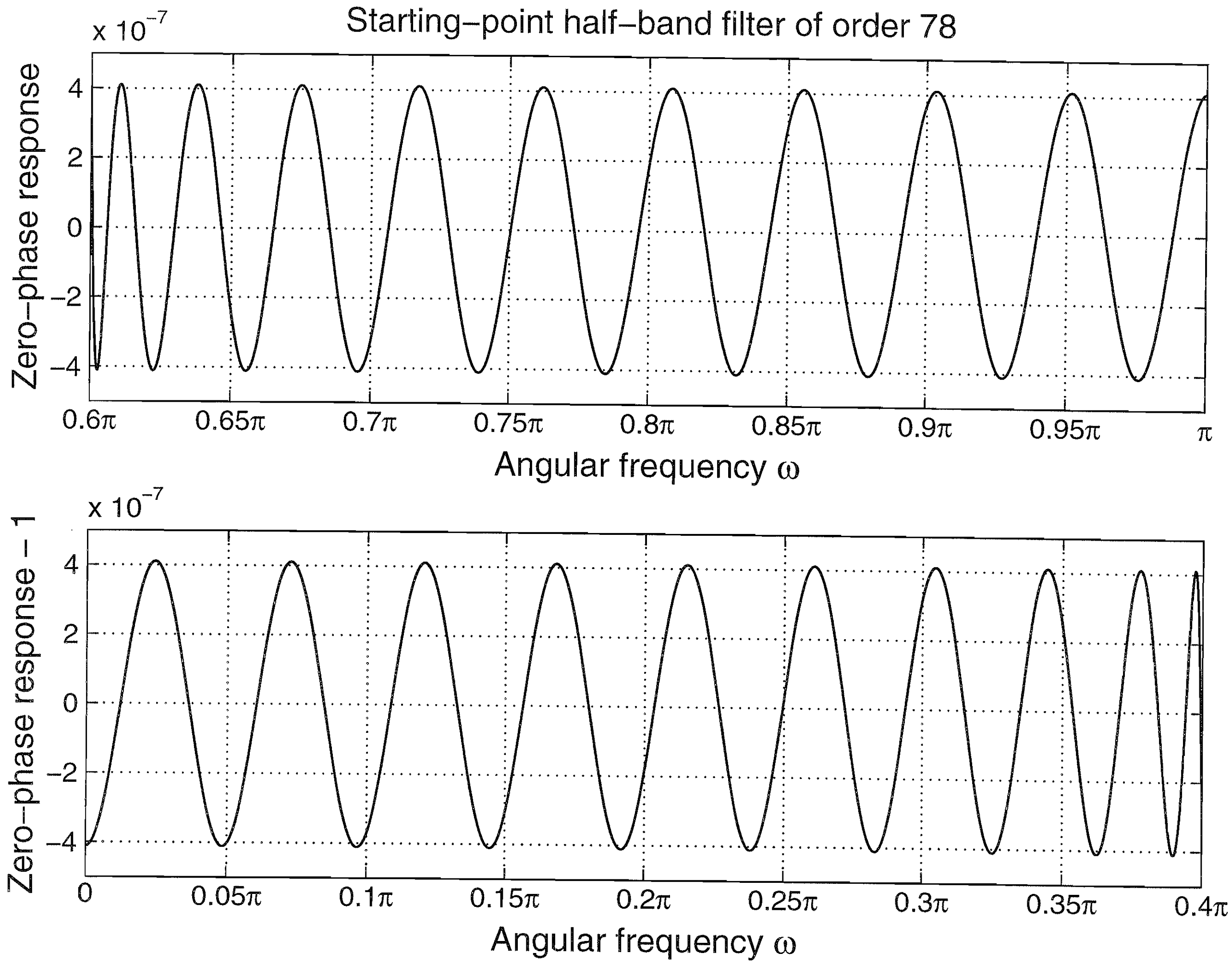
Step 1: Overall amplitude and impulse responses



Step 1: Zero plot



Step 1: Zero-phase frequency response in the passband and stopband regions

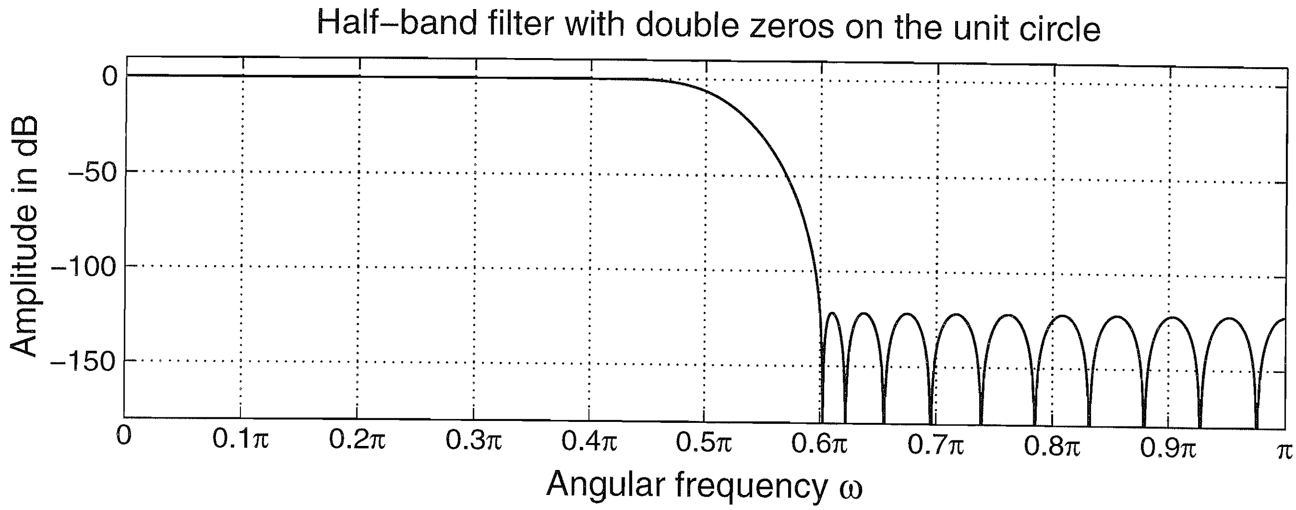


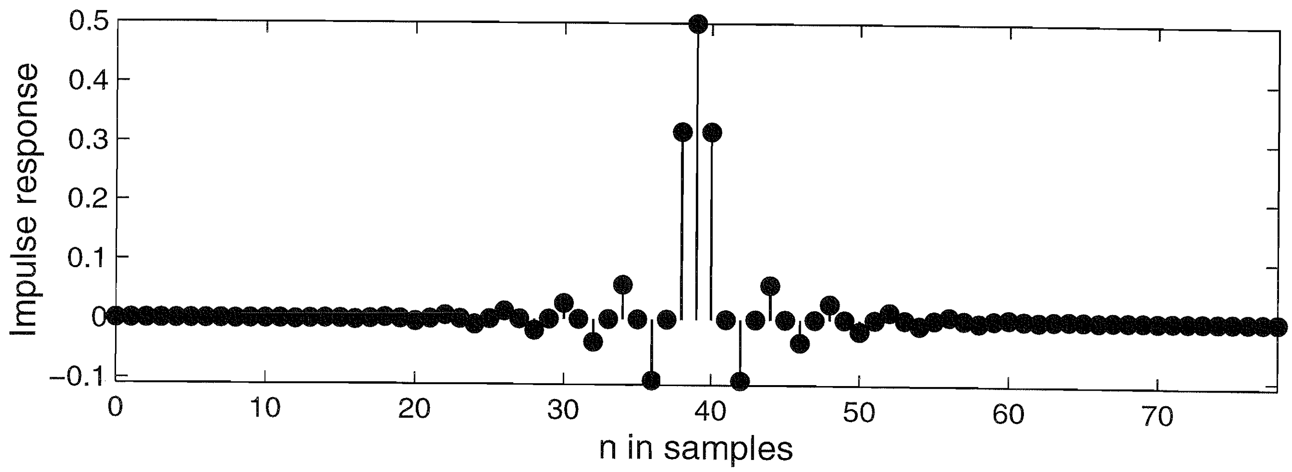
Generation of a Half-Band Filter Having Double Zeros on the Unit Circle

Step 2: $E(z) = \sum_{n=0}^{2M} e(n)z^{-n}$ having double zeros on the unit circle is then obtained by selecting $e(n) = 0.5\widetilde{e}(n)/(0.5+\delta)$ for $n \neq M$ and $e(M) = 1/2$.

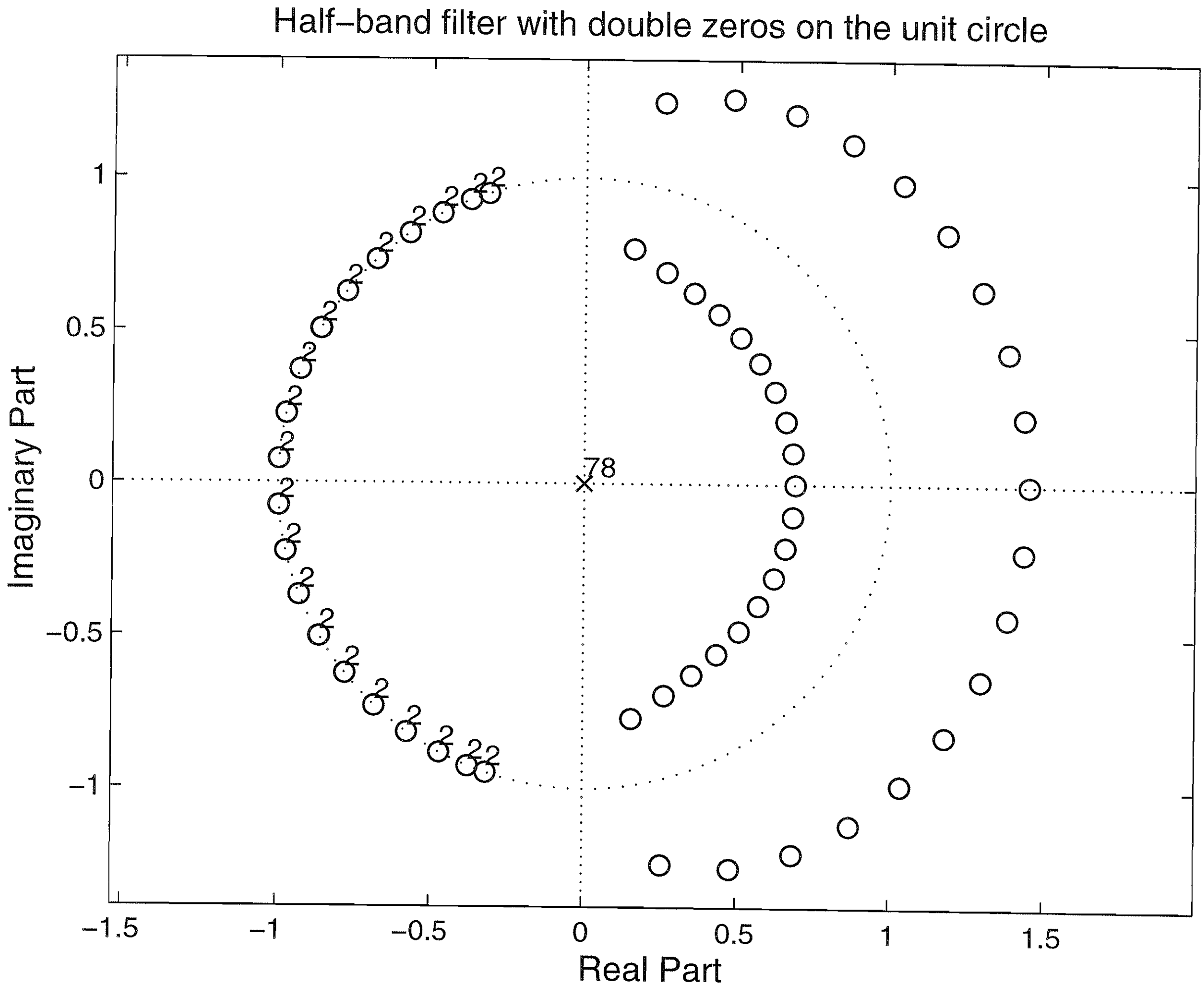
- Like in designing minimum- and maximum-phase FIR filters in the minimax sense, we add to the center impulse response value $\widetilde{e}(M)$ δ ($-\delta$ is the stopband minimum), that is, the new transfer function is $\delta z^{-M} + \widetilde{E}(z)$. This transfer function has double zeros on the unit circle and is factorizable in the desired manner.
- The resulting transfer function has the central coefficient of value $1/2+\delta \neq 1/2$. Therefore, all the impulse response values must be divided by $(1/2+\delta)/(1/2)$ to give the value of $1/2$ for $e(M)$. The resulting transfer function is $E(z)$ as given above.
- For $\omega_s = 0.6\pi$ and $M = 39$, the following pages show the overall amplitude response, the zero-plot, the impulse response as well as the zero-phase frequency response in both the passband and stopband.

Step 2: Overall amplitude and impulse responses

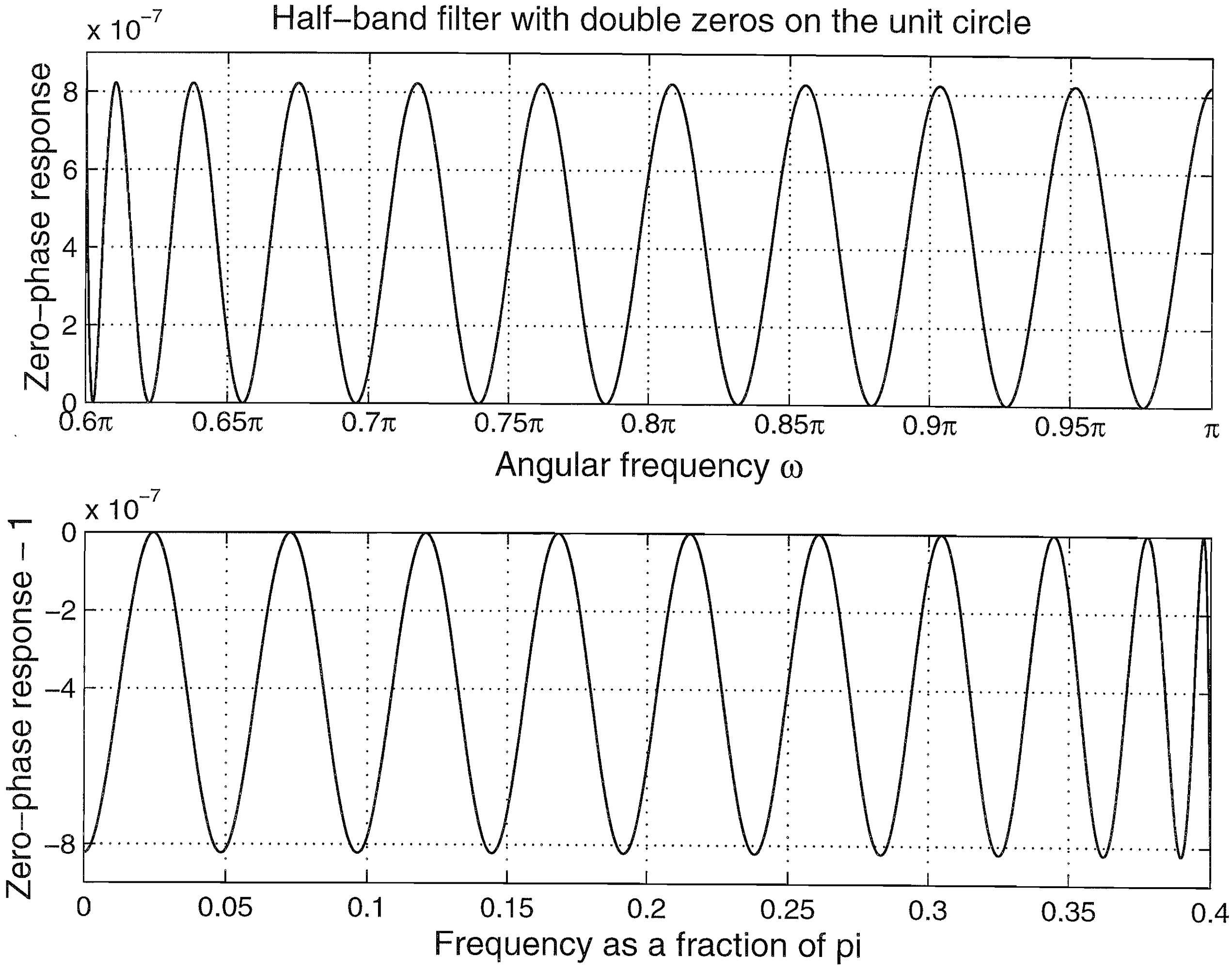




Step 2: Zero plot



Step 2: Zero-phase frequency response in the passband and stopband regions



Factorization into Minimum- and Maximum-Phase Terms $T(z)$ and $R(z)$

Step 3: Form $T(z) = \sum_{n=0}^{M} t(n)z^{-n}$ by selecting the zeros of $E(z)$ inside the unit circle and each of the double zeros on the unit circle. $T(z)$ is scaled such that $|T(1)| = \sqrt{|E(1)|}$ ($z = 1$ corresponds to the zero frequency).

• The resulting $T(z)$ is a minimum-phase lowpass FIR filter satisfying

$$|T(e^{j\omega})| = \sqrt{E(\omega)} = \sqrt{|E(e^{j\omega})|}. \tag{57}$$

• Form $R(z) = \sum_{n=0}^{M} r(n)z^{-n}$ such that $r(n) = t(M-n)$. $R(z)$ has the same amplitude response as $T(z)$, but the off-the-unit-circle zeros are outside the unit circle and its impulse response is the time-reversed version of that of $T(z)$.

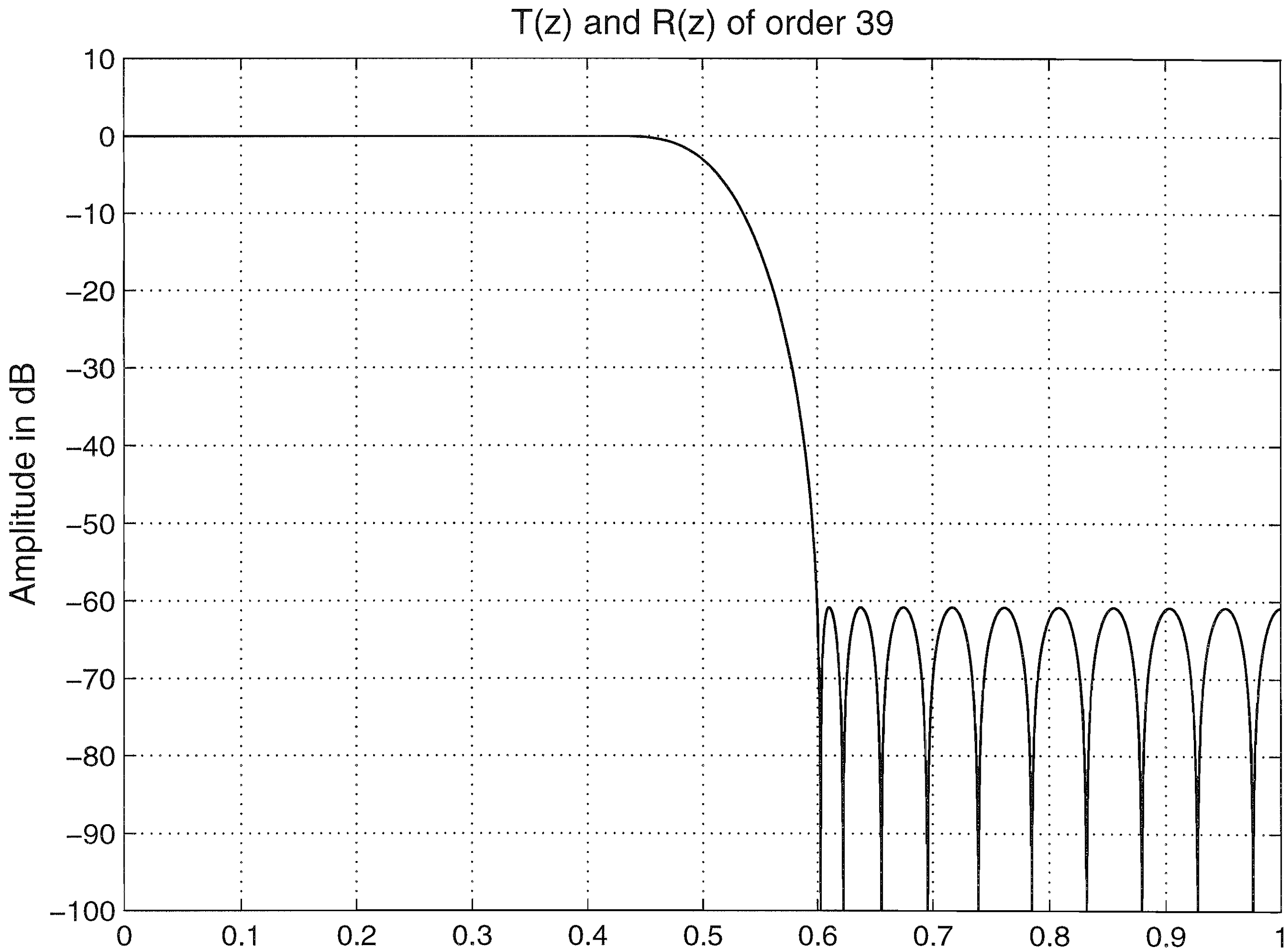
• Furthermore,

$$|T(e^{j\omega})|^2 = |R(e^{j\omega})|^2 = E(\omega) = |E(e^{j\omega})|. \tag{58}$$

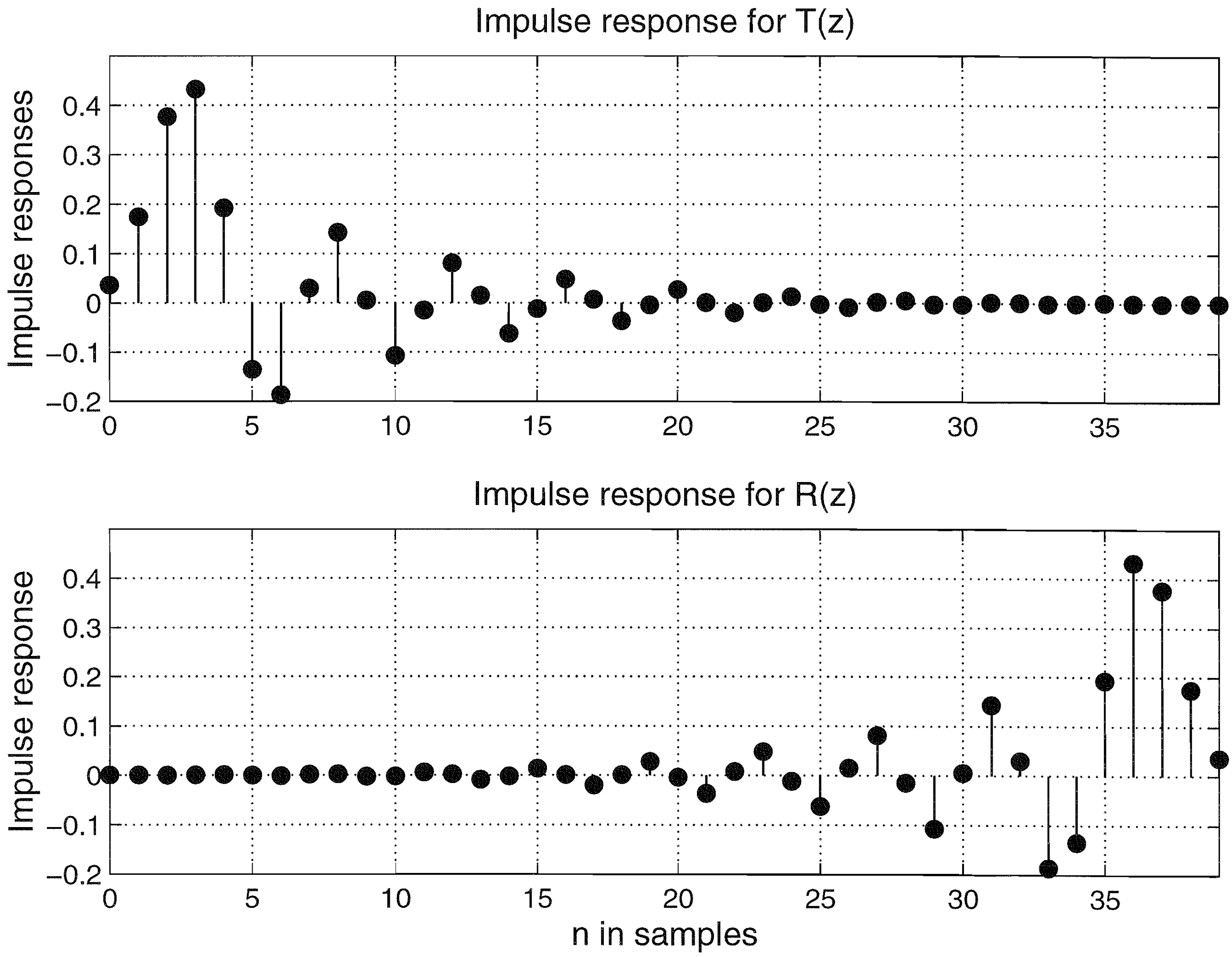
• Note that the the off-the-unit-circle zeros of $E(z)$ can be shared in various other ways, resulting in mixed-phase designs.

• For $\omega_s = 0.6\pi$ and $M = 39$, the following pages show the overall amplitude response, the zero-plot, the impulse response as well as the zero-phase frequency response in both the passband and stopband.

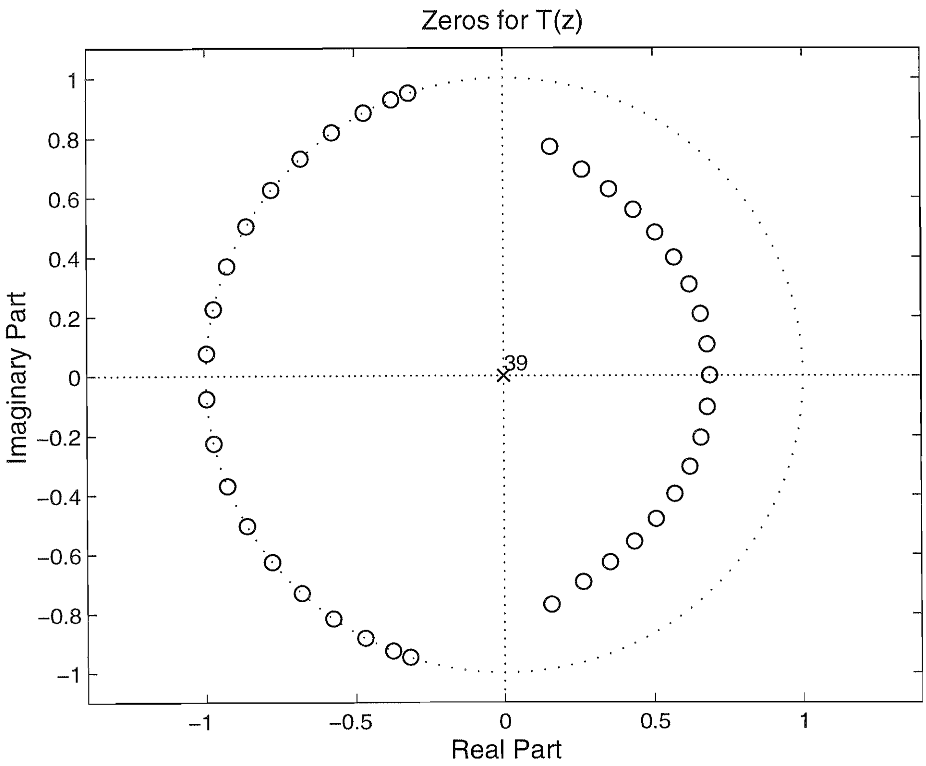
Common amplitude response for $T(z)$ and $R(z)$

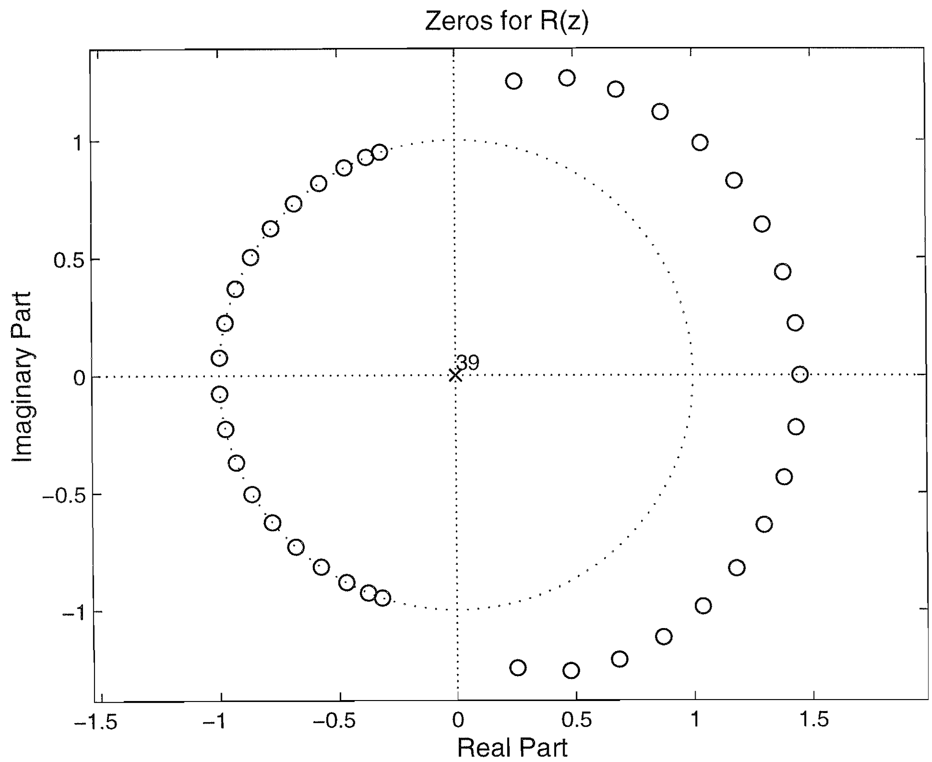


Impulse responses for $T(z)$ and $R(z)$



Zero plots for $T(z)$ and $R(z)$





II.G: Half-Band IIR Filters

- There exist two kinds of half-band IIR filters:

I: Conventional half-band IIR filters without phase constraints.

II: Approximately linear-phase half-band IIR filters.

Conventional Half-Band IIR Filters

- The transfer function is expressible as

$$H(z) = (1/2)[A_0(z^2) + z^{-1}A_1(z^2)], \qquad (59a)$$

where

$$A_0(z) = \prod_{k=1}^{K_0} \frac{a_k^{(0)} + z^{-1}}{1 + a_k^{(0)} z^{-1}} \qquad (59b)$$

and

$$A_1(z) = \prod_{k=1}^{K_1} \frac{a_k^{(1)} + z^{-1}}{1 + a_k^{(1)} z^{-1}} \qquad (59c)$$

are allpass filters consisting of first-order sections.

- All the $a_k^{(0)}$'s and $a_k^{(1)}$'s are negative so that the poles of the overall filter are located at the imaginary axis. One pole is at the origin.
- The overall order of this filter is $2(K_0 + K_1) + 1$.
- Here, $K_0 = K_1$ or $K_0 = K_1 + 1$ for a lowpass filter.
- For a power-complementary highpass filter $(|H(e^{j\omega})|^2 + |G(e^{j\omega})|^2 = 1)$, the transfer function is simply given by

$$G(z) = (1/2)[A_0(z^2) - z^{-1}A_1(z^2)]. \qquad (60)$$

Example and Properties of Conventional Half-Band IIR Filters

• Consider a power-complementary filter pair with $K_0 = K_1 = 2$, $a_1^{(0)} = 0.07986644637$, $a_2^{(0)} = 0.5453236405$, $a_1^{(1)} = 0.2838293419$, and $a_2^{(1)} = 0.8344118932$.

• In the following, there are two pages illustrating the performance of both $H(z) = (1/2)[A_0(z^2) + z^{-1}A_1(z^2)]$ and $G(z) = (1/2)[A_0(z^2) - z^{-1}A_1(z^2)]$.

• $H(z)$ is a special elliptic filter of order $2(K_0 + K_1) + 1 = 9$ designed to have the stopband edge at $\omega_s = 0.6\pi$:

1) The passband edge is located at $\omega_p = \pi - \omega_s = 0.4\pi$.

2) $|H(e^{j\omega_0})|^2 + |H(e^{j(\pi-\omega_0)})|^2 = 1$ for any $\omega_0 \in [0, \ , \pi]$.

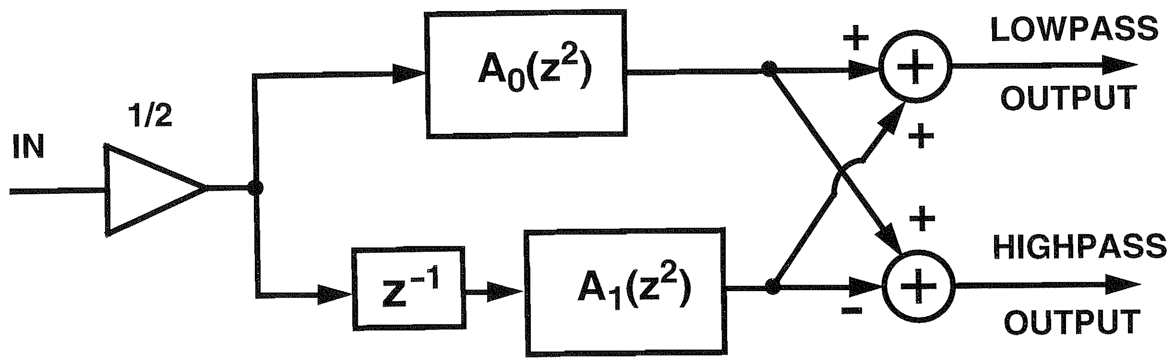
3) $|G(e^{j\omega})|^2 = 1 - |H(e^{j(\pi-\omega)})|^2$.

4) If the maximum deviation of $|H(e^{j\omega})|^2$ from zero in the stopband is δ, then the maximum deviation from unity in the passband (the maximum value is unity) is also δ. Hence, the passband variation is very small.

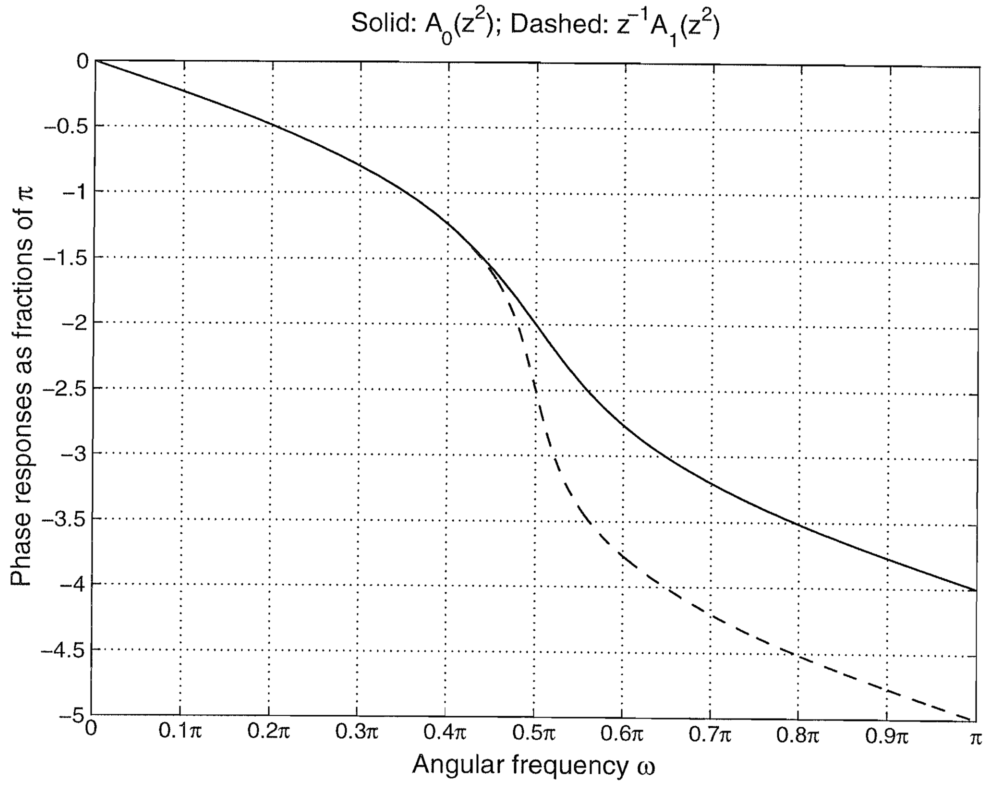
5) All the poles of $H(z)$ and $G(z)$ are located on the imaginary axis.

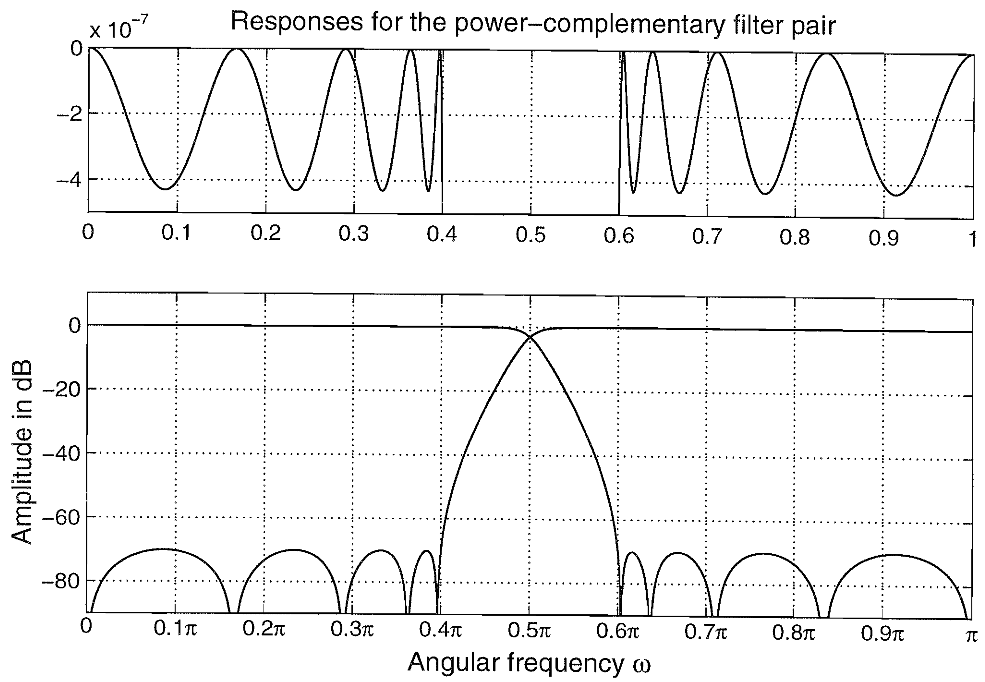
6) Only $K_0 + K_1 = 4$ multipliers are required to implement an elliptic filter of order $2(K_0 + K_1) + 1 = 9$ when using the lattice wave digital filter structures.

• An efficient structure for simultaneously implementing $H(z)$ and $G(z)$ is shown below.

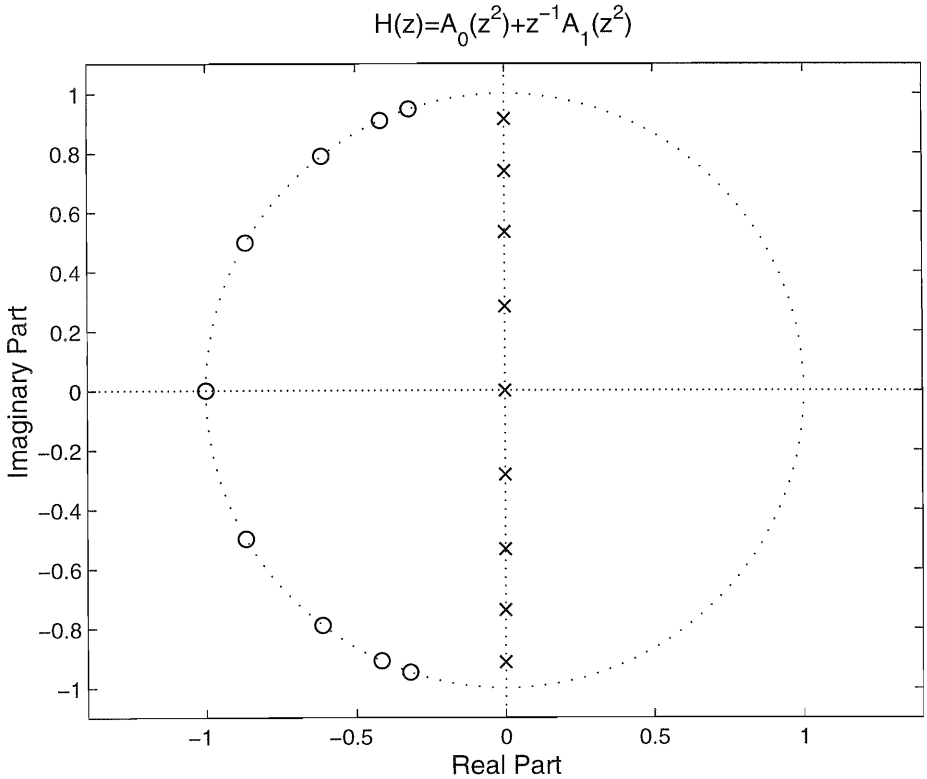


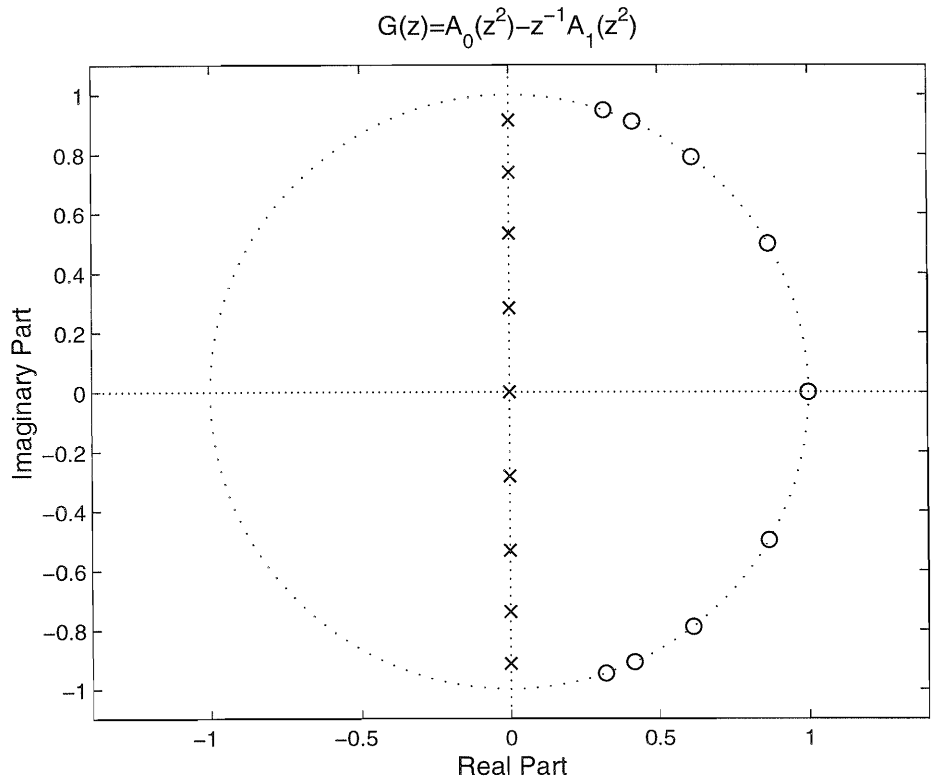
Example Conventional Half-Band IIR Filter





Example Conventional Half-Band IIR Filter





Approximately Linear-Phase Half-Band IIR Filters

- The transfer function is expressible as

$$H(z) = (1/2)[A_0(z^2) + z^{-K_1}], \tag{61a}$$

where

$$A_0(z) = \prod_{k=1}^{K_0^{(1)}} \frac{a_k + z^{-1}}{1 + a_k z^{-1}} \prod_{k=1}^{K_0^{(2)}} \frac{c_k + b_k z^{-1} + z^{-2}}{1 + b_k z^{-1} + c_k z^{-1}} \tag{61b}$$

is an allpass filters consisting of first-order and second-order sections.

- This filter is a special case where the second allpass filter is a pure delay term.
- The overall order of this filter is $K_0 + 2(K_1 + 2K_1)$.
- Here, $K_1 = 2(K_1^{(1)} + 2K_1^{(2)}) - 1$.
- For a power-complementary highpass filter ($|H(e^{j\omega})|^2 + |G(e^{j\omega})|^2 = 1$), the transfer function is simply given by

$$G(z) = (1/2)[A_0(z^2) - z^{-K_1}]. \tag{62}$$

Example and Properties of Approximately Linear-Phase Half-Band IIR Filters

• Consider a power-complementary filter pair with $K_1 = 13$, $K_0^{(1)} = 1$, $K_0^{(2)} = 3$, $a_1 = 0.7977649139$, $b_1 = -0.6270566870$, $c_1 = 0.1207727840$, $b_2 = -0.1697974960$, $c_2 = 0.1300746053$, $b_3 = 0.4897869030$, and $c_3 = 0.1663758012$.

• In the following, there are three pages illustrating the performance of both $H(z) = (1/2)[A_0(z^2) + z^{-13}]$ and $G(z) = (1/2)[A_0(z^2) - z^{-13}]$.

• $H(z)$ has the stopband edge at $\omega_s = 0.6\pi$ and the required stopband attenuation is 60 dB:

1) The passband edge is located at $\omega_p = \pi - \omega_s = 0.4\pi$.

2) $|H(e^{j\omega_0})|^2 + |H(e^{j(\pi-\omega_0)})|^2 = 1$ for any $\omega_0 \in [0, \ , \pi]$.

3) $|G(e^{j\omega})|^2 = 1 - |H(e^{j(\pi-\omega)})|^2$.

4) If the maximum deviation of $|H(e^{j\omega})|^2$ from zero in the stopband is δ, then the maximum deviation from unity in the passband (the maximum value is unity) is also δ. Hence, the passband variation is very small.

5) The poles of $H(z)$ and $G(z)$ are located either on the imaginary axis are they are located symmtrically with respect to the imaginary axis. $K_1 = 13$ poles are at the origin.

6) Only $K_0^{(1)} + 2K_0^{(2)} = 7$ multipliers are required to implement this filter of order $K_1 + 2(K_0^{(1)} + 2K_0^{(2)}) = 23$

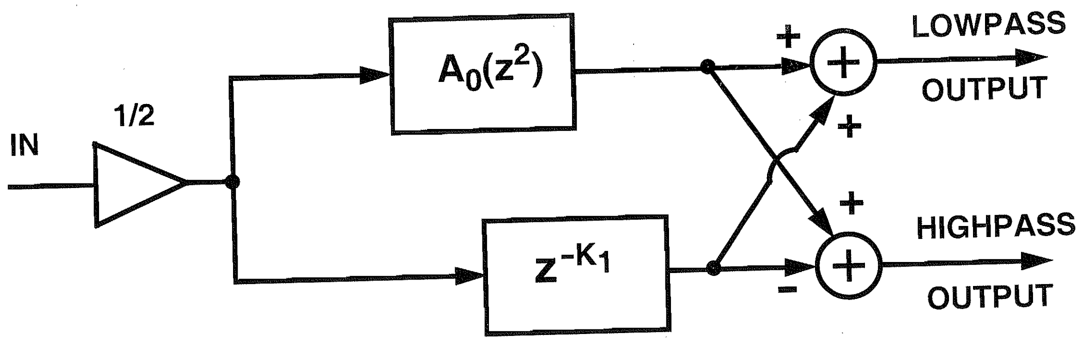
when using the lattice wave digital filter structures.

7) The passband phase deviations of both $H(z)$ and $G(z)$ from $-K_1\omega = -13\omega$ are very small.

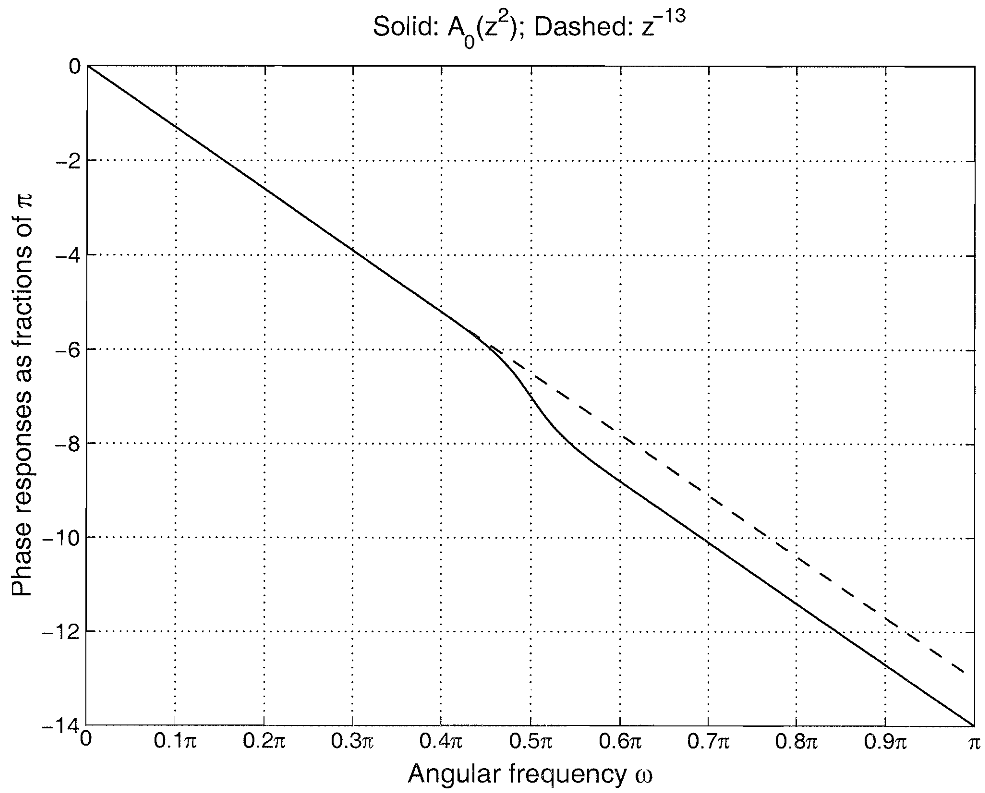
8) The impulse responses of both $H(z)$ and $G(z)$ achieve the value of half at $n = K_1 = 13$. At other odd values of n, the impulse-response values are zero.

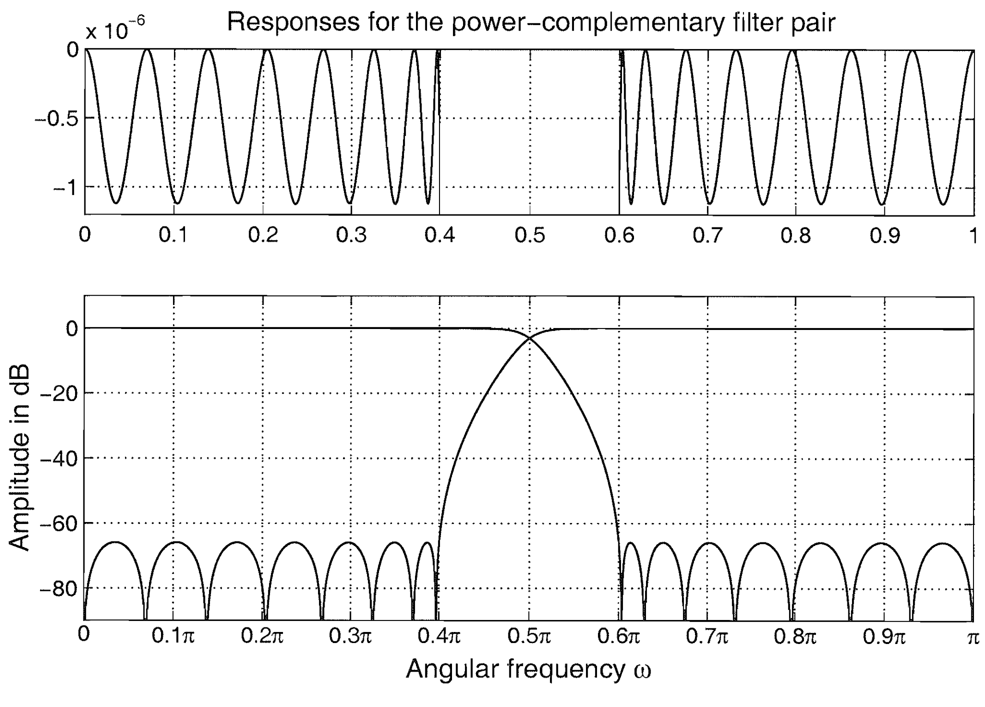
9) At even values of n, the impulse-response values of $H(z)$ and $G(z)$ have opposite signs.

- An efficient structure for simultaneously implementing $H(z)$ and $G(z)$ is shown below.

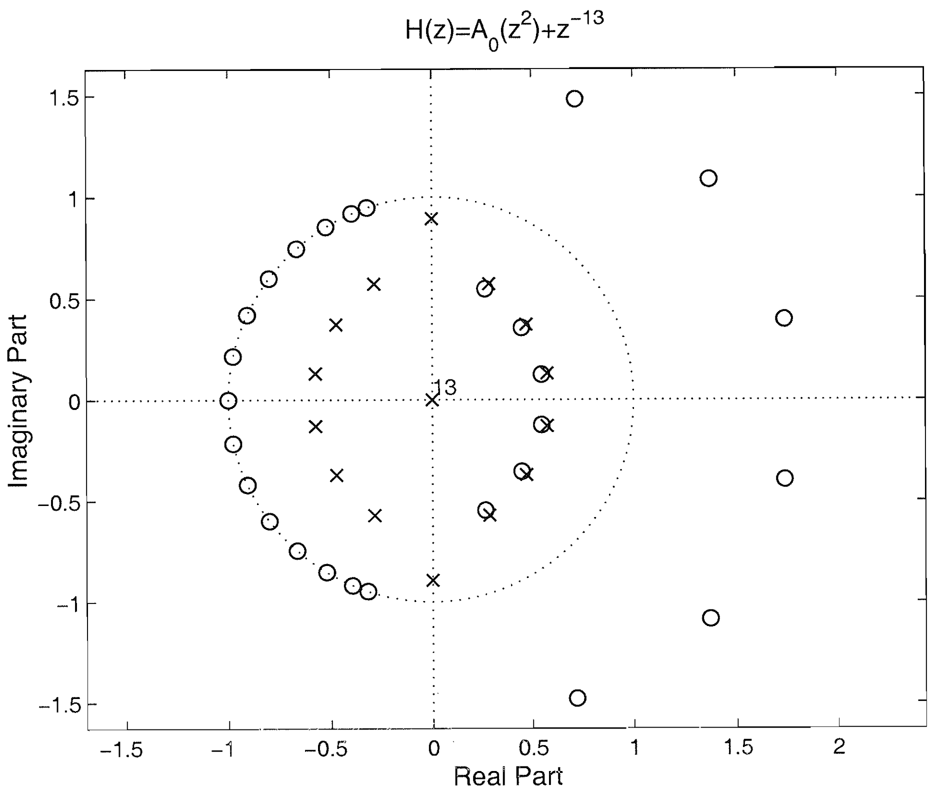


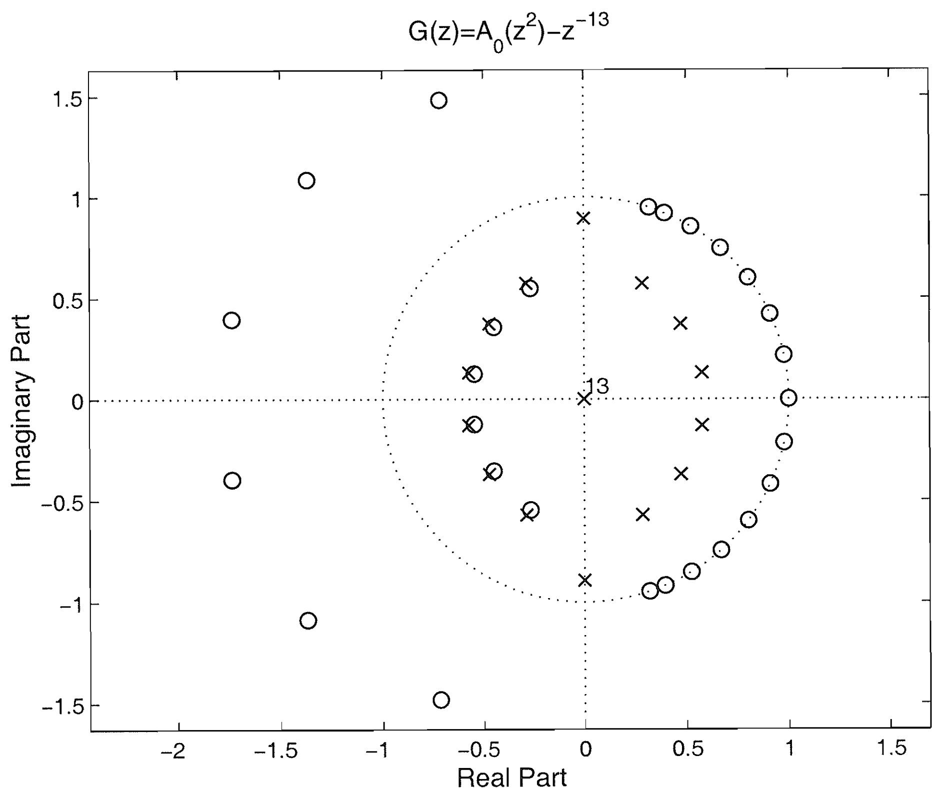
Example Approximately Linear-Phase Half-Band IIR Filter



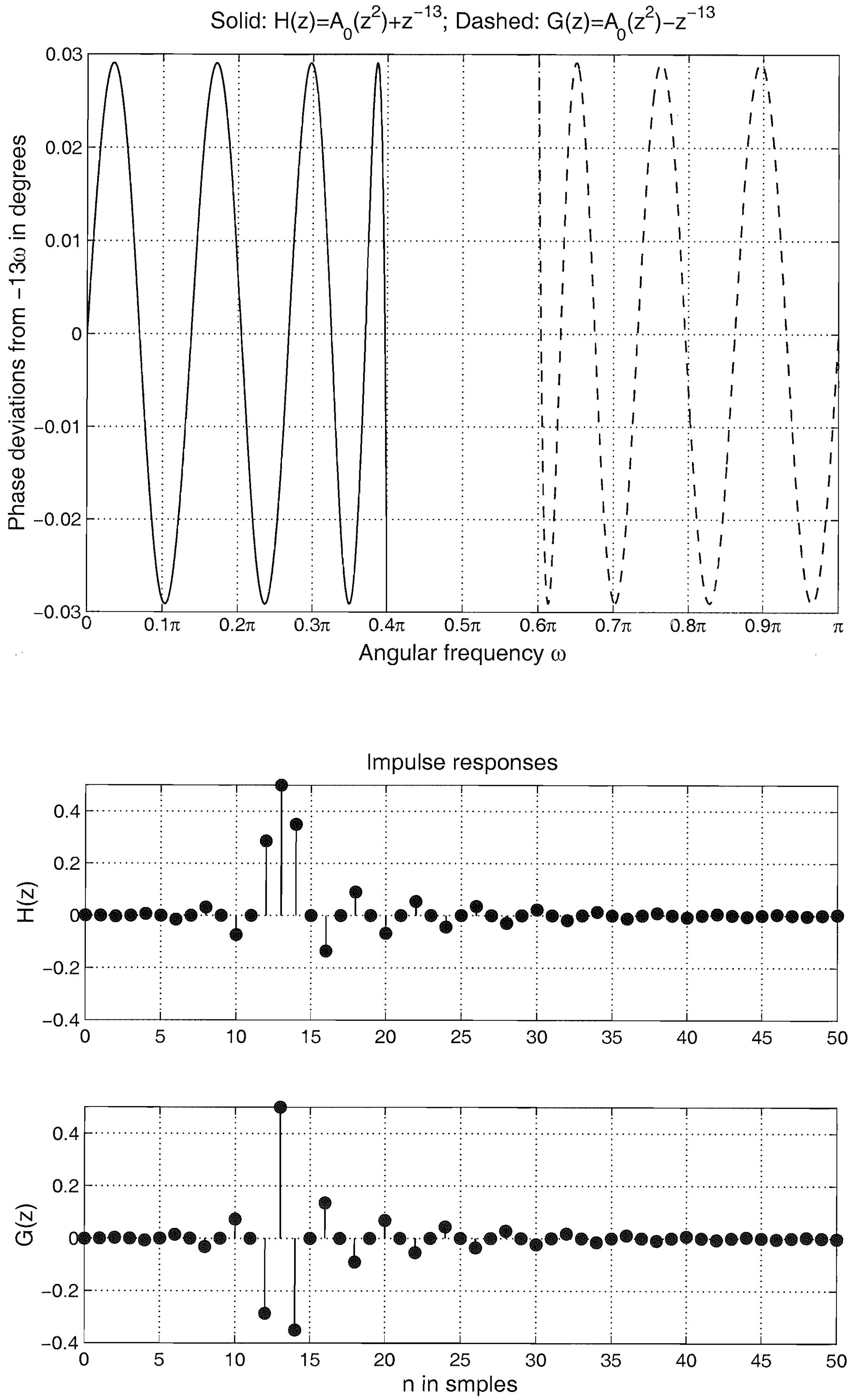


Example Approximately Linear-Phase Half-Band IIR Filter



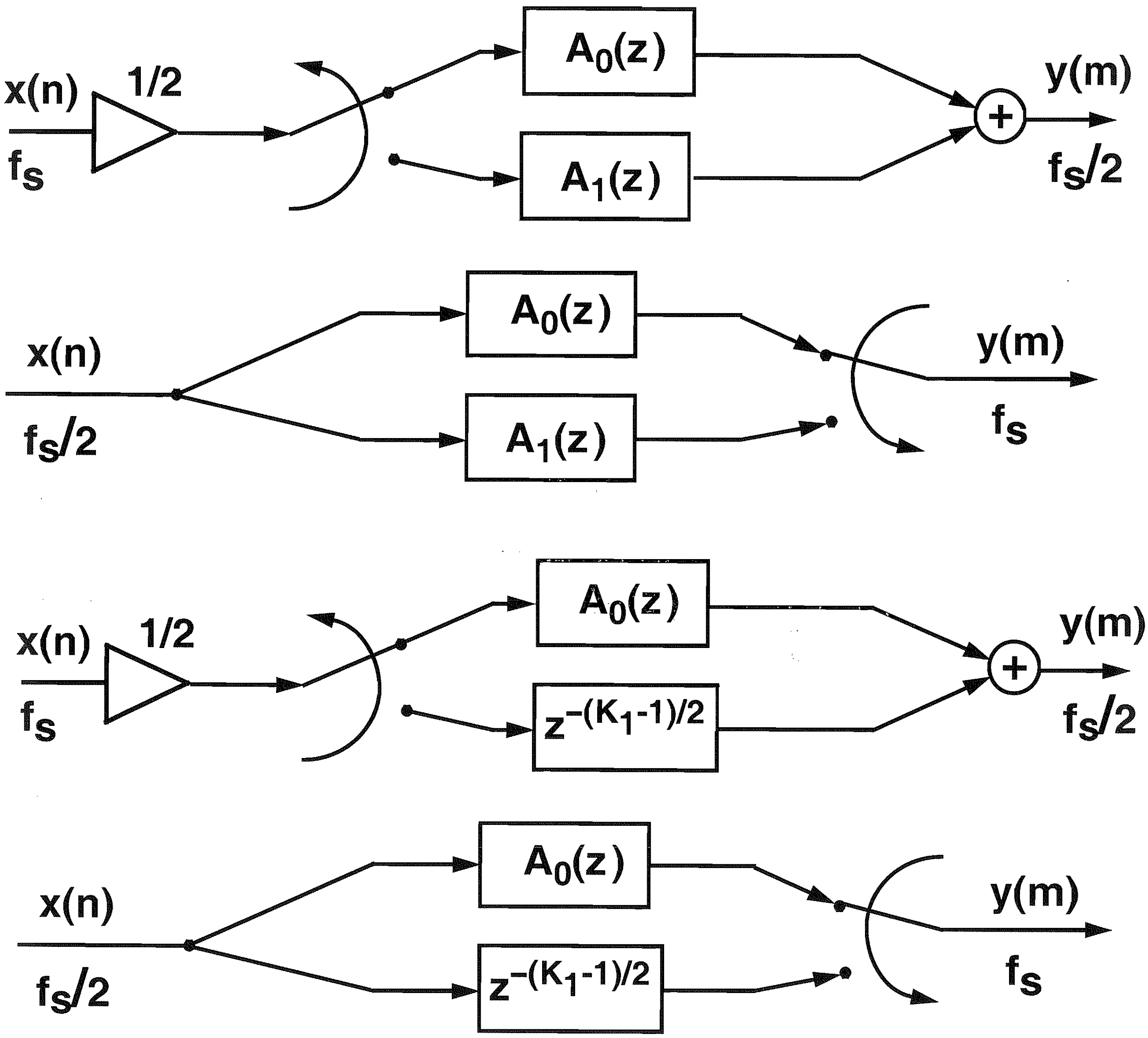


Example Approximately Linear-Phase Half-Band IIR Filter



Decimator and Interpolator Implementations of Half-Band IIR Filters

• The figure below shows the decimator and interpolator implementations for $H(z) = (1/2)[A_0(z^2) + z^{-1}A_1(z^2)]$ and $H(z) = (1/2)[A_0(z^2) + z^{-K_1}]$.



Decimator and Interpolator Example

- It is desired to design a decimator (interpolator) for $N = 8$ using three stages, each decimating by 2.
- Case C specifications are used with the passband and stopband edges at $0.8\pi/8 = 0.1\pi$ and $1.2\pi/8 = 0.15\pi$. The required stopband attenuation is 60 dB.

Conventional half-band filters: The given criteria are met as follows:

- First stage $H_1(z)$: $K_0 = K_1 = 1$. (see Eq. (59))
- Second stage $H_2(z)$: $K_0 = K_1 = 1$.
- Third stage $H_3(z)$: $K_0 = K_1 = 2$.
- In the following, there are two pages illustrating the characteristics of this design. For the single-stage equivalent the transfer function is $H(z) = H_1(z)H_2(z^2)H_3(z^4)$.
- Notice the extremely small passband amplitude variation.
- For the overall three-stage decimator, the number of multipliers is $2 + 2 + 4 = 6$ and the number of multiplications per input sample is $2/2 + 2/4 + 4/8 = 2.0$.

Approximately linear-phase half-band filters: The given criteria are met as follows:

- First stage $H_1(z)$: $K_1 = 3$, $K_0^{(1)} = 2$, $K_0^{(2)} = 0$. (see Eq. (61))
- Second stage $H_2(z)$: $K_1 = 5$, $K_0^{(1)} = 1$, $K_0^{(2)} = 1$.
- Third stage $H_3(z)$: $K_1 = 13$, $K_0^{(1)} = 1$, $K_0^{(2)} = 3$.

• In the following, there are three pages illustrating the characteristics of this design. For the single-stage equivalent the transfer function is $H(z) = H_1(z)H_2(z^2)H_3(z^4)$.

• Notice the extremely small passband amplitude variation as well as a very small phase error from the constant phase response -63ω.

• Notice also that the impulse response of the single-stage equivalent with transfer function $H(z) = H_1(z)H_2(z^2)H_3(z^4)$ achieves the value of 1/8 at $n = 63$ and for other odd values of n the impulse-response values are zero.

• For the overall three-stage decimator, the number of multipliers is $2+3+7 = 12$ and the number of multiplications per input sample is $2/2 + 3/4 + 7/8 = 2.625$.

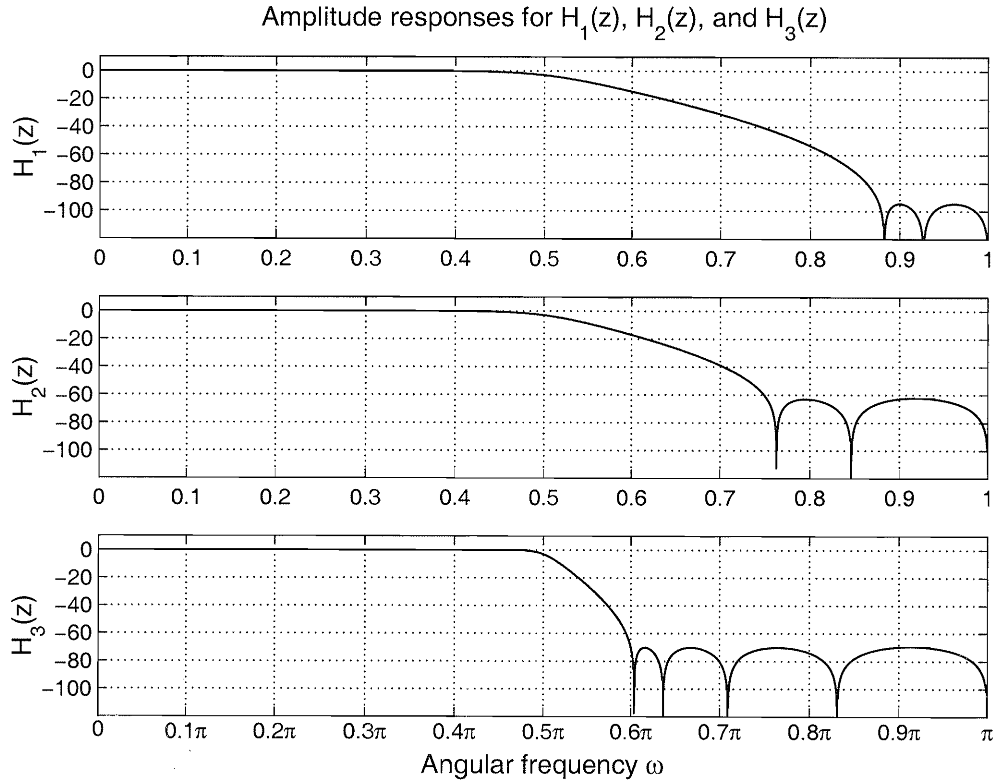
• If linear-phase half-band filters are used, the minimum orders to meet the criteria are $2M_1 = 6$, $2M_1 = 14$, and $2M_1 = 34$.

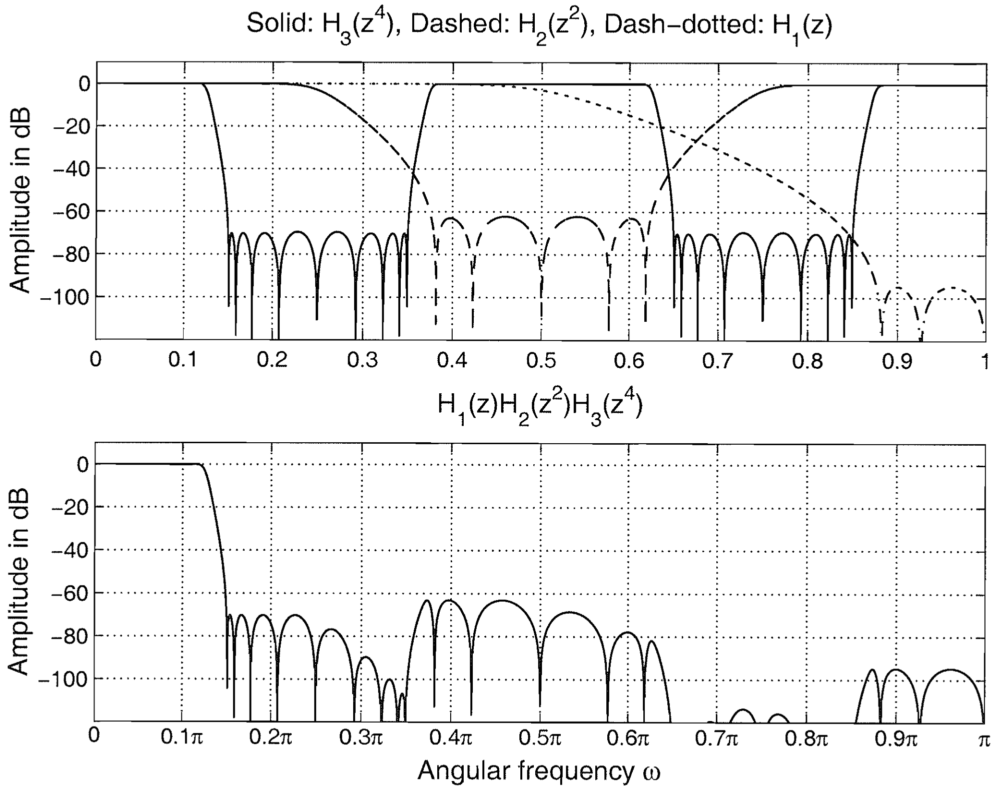
• Assuming the the central coefficients of value 1/2 require no multipliers, the overall number of multipliers for this design is $2+4+9 = 15$ and the number of multiplications per input sample is $2/2 + 4/4 + 9/8 = 3.125$.

• The delay caused to the input signal by the decimator built using approximately linear-phase half-band IIR filters is 63 input samples, whereas the corresponding delay is 85 samples when using linear-phase half-band

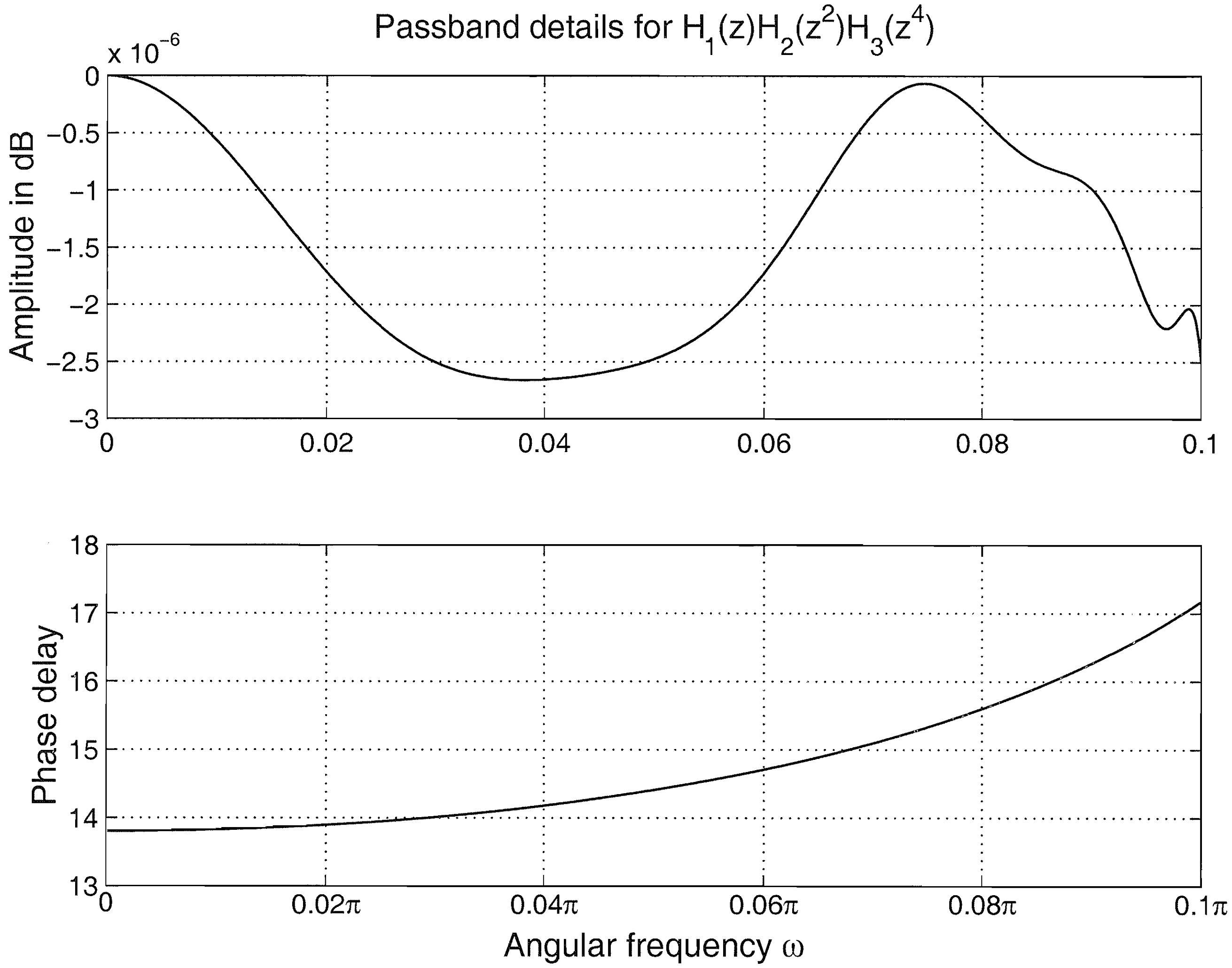
FIR filters.

Three-Stage Decimator for $N = 8$ using Three Conventional Half-Band IIR Filters

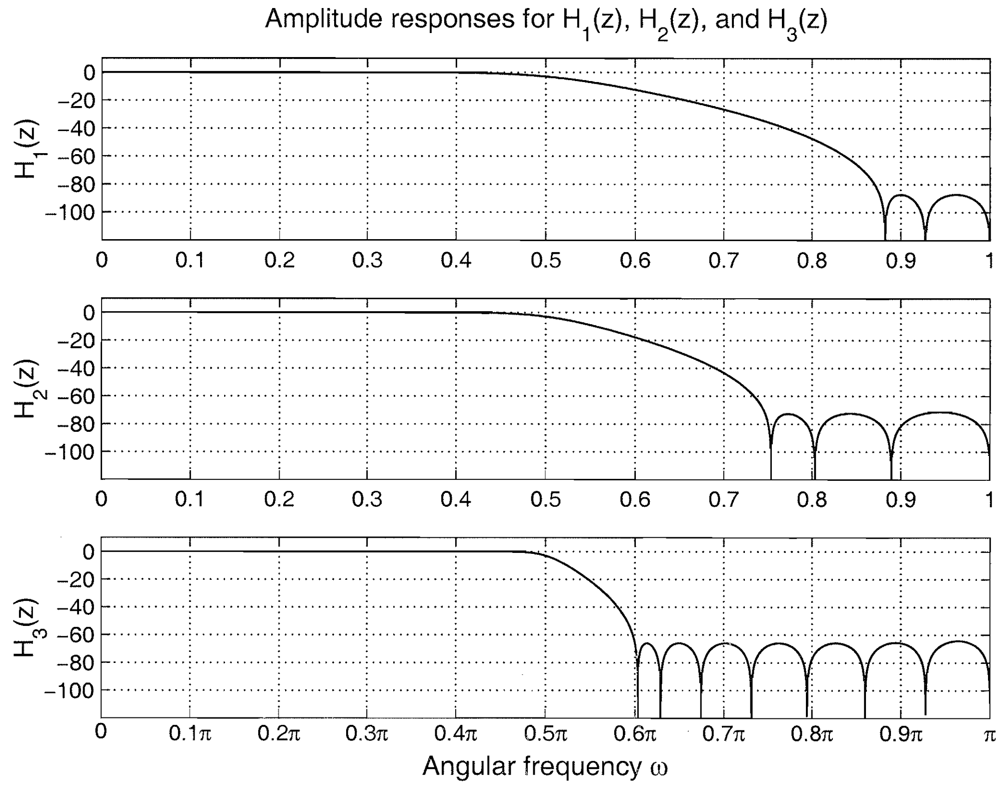


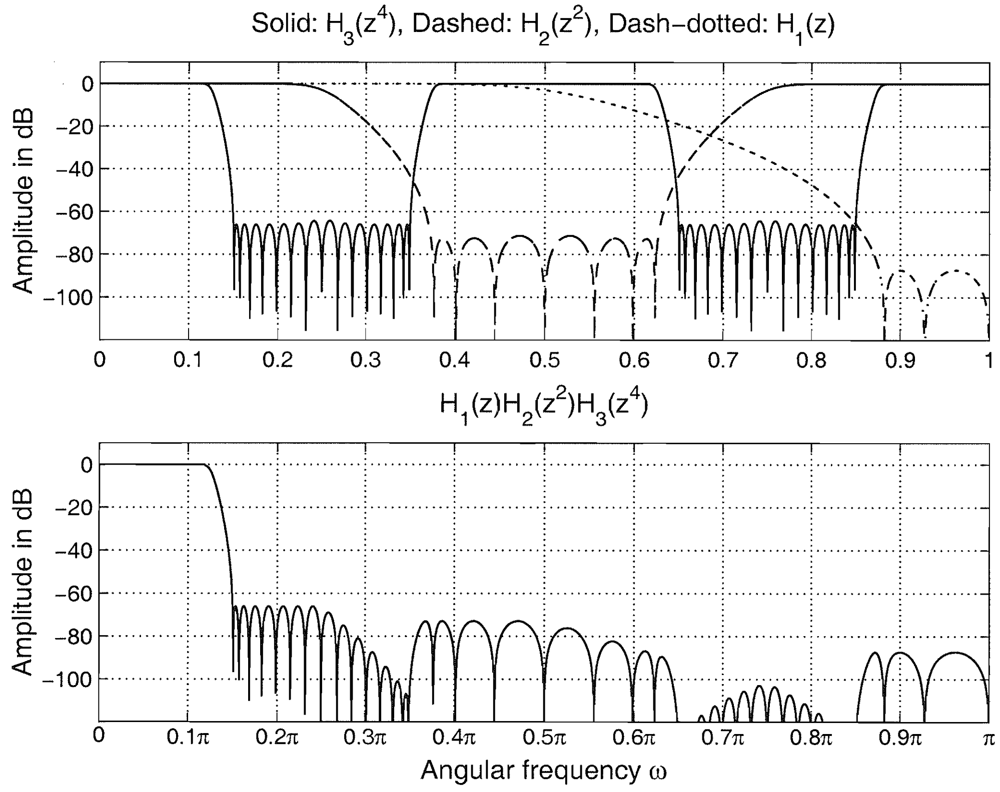


Three-Stage Decimator for $N = 8$ using Three Conventional Half-Band IIR Filters

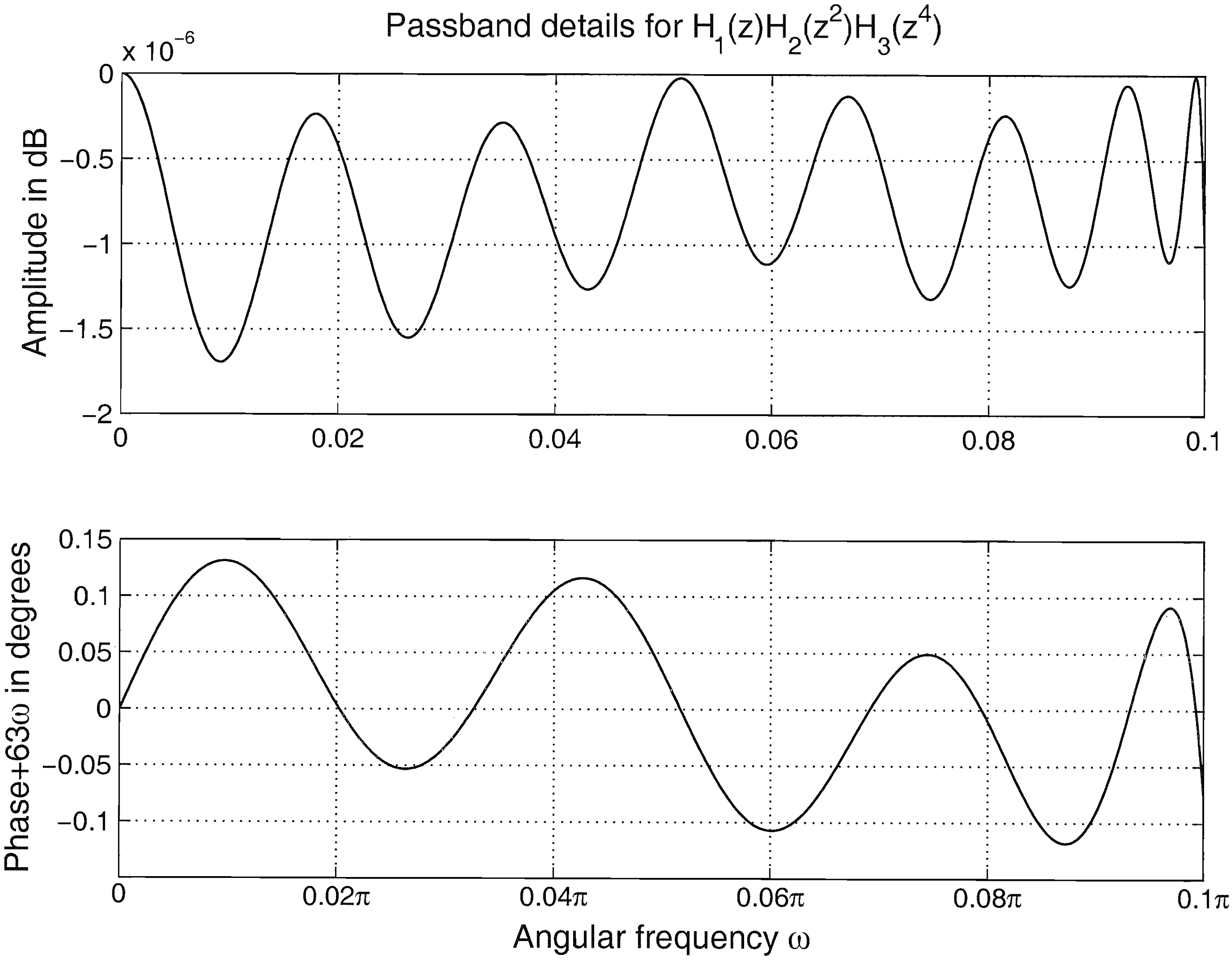


Three-Stage Decimator for $N = 8$ using Three Approximately Linear-Phase Half-Band IIR Filters

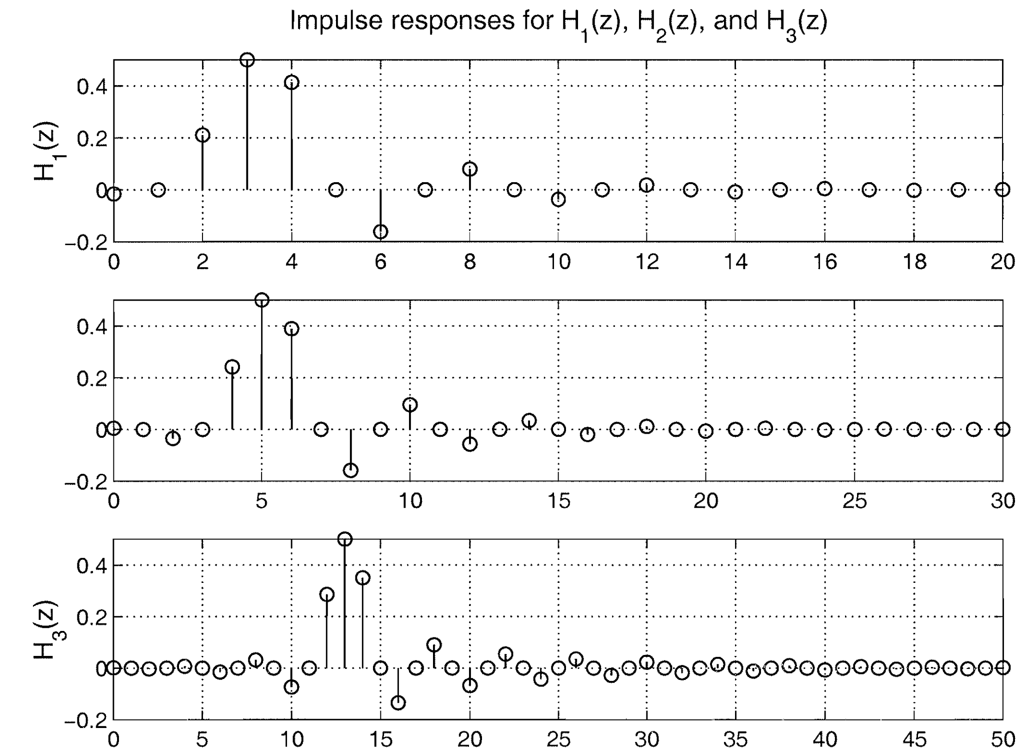


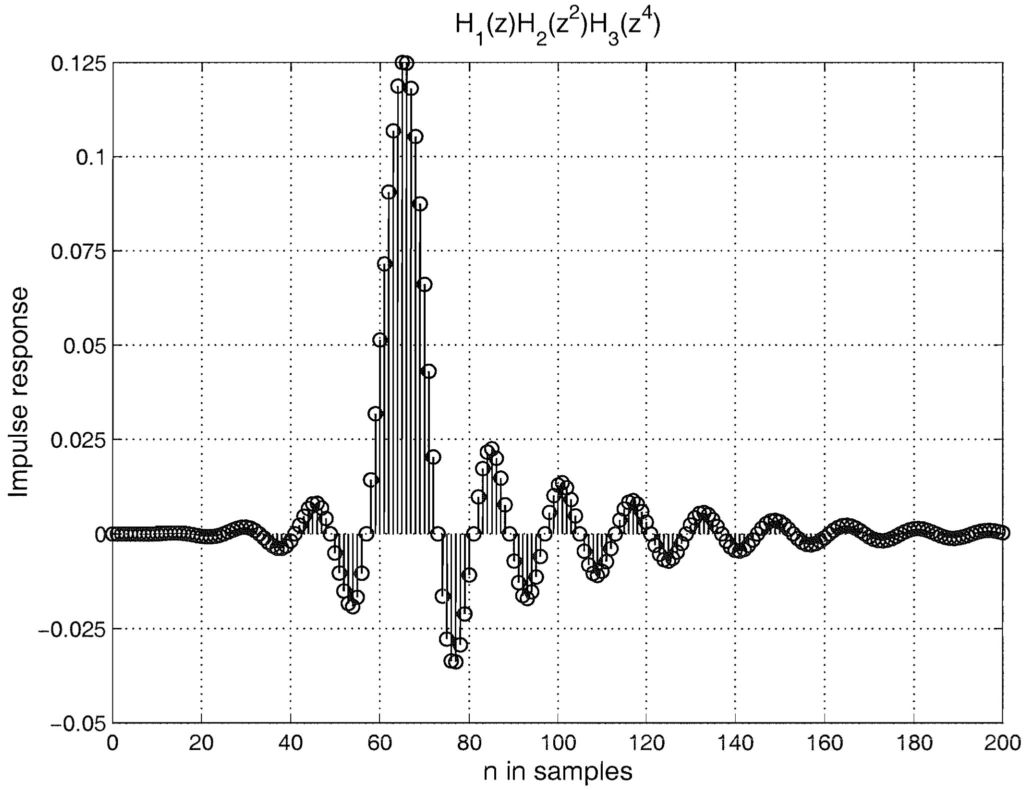


Three-Stage Decimator for $N = 8$ using Three Approximately Linear-Phase Half-Band IIR Filters



Three-Stage Decimator for $N = 8$ using Three Approximately Linear-Phase Half-Band IIR Filters





Part II.H: Use of Conventional and Modified Comb (Running Sum) Structures as a First Stage for Multistage Decimator Implementations

- This part shows how to use a cascade of comb filters (or running sum or sinc filters) as a first decimation or last interpolation stage in multistage implementations.
- Also a modified structure intoduced by Saramäki and Ritoniemi is considered.
- Moreover, practical examples are included.

What is a Comb Filter (a Running Sum or a Sinc Filter)?

• The transfer function of a comb or running sum filter of order $K-1$ (length K) is given by

$$E(z) = 2^{-P} G(z), \tag{63a}$$

where

$$G(z) = \sum_{n=0}^{K-1} z^{-n} = \frac{1 - z^{-K}}{1 - z^{-1}} \tag{63b}$$

and P is integer satisfying

$$2^{-P} \leq 1/K. \tag{63c}$$

• Page 270 shows efficient implementations for the above transfer function.

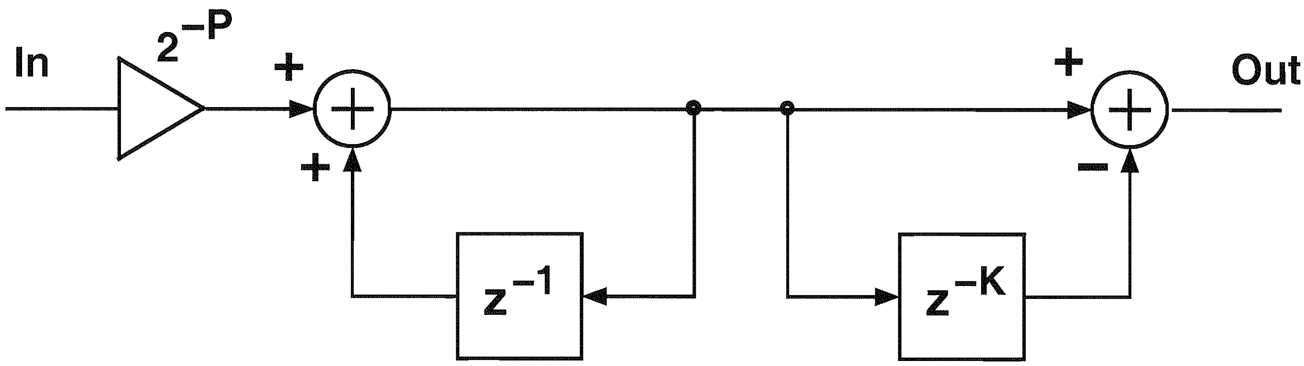
• Both implementations require no multipliers and only two adders. The first structure has the following attractive property:

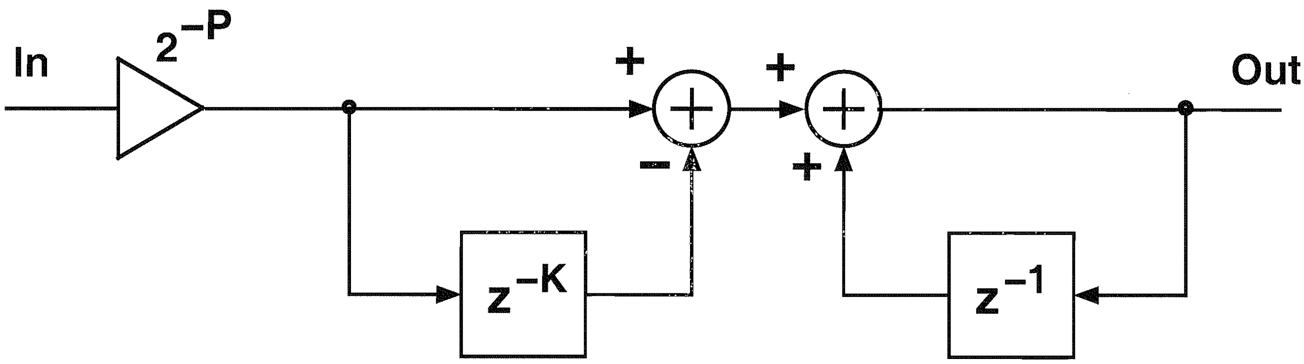
• If modulo arithmetic (e.g., 1's or 2's complement arithmetic) and the worst-case scaling (corresponds to peak scaling in this case) are used ($2^{-P} \leq 1/K$), the output of $E(z)$ is correct even though internal overflows may occur.

• This implementation is very attractive as, in this case, the system does not need initial resetting and the effect of temporary miscalculations vanishes automatically from the output in a finite time.

- For the second structure, resetting is needed and temporary miscalculations are not allowed.

Efficient Structures for Implementing a Comb Filter





Cascaded Comb Filter for Decimation or Interpolation by an Integer Factor K

• By cascading M comb filters, we end up with the following transfer function:

$$E(z) = 2^{-P}[G(z)]^M, \qquad (64a)$$

where

$$G(z) = \frac{1 - z^{-K}}{1 - z^{-1}} \qquad (64b)$$

and P is integer satisfying

$$2^{-P} \leq (1/K)^M. \qquad (64c)$$

• Pages 273 and 274 show efficient decimator and interpolator implementations for this transfer function.

• The second implementation form for the decimator has the attractive property that only $2M$ delay elements and adders are required regardless of the value of K.

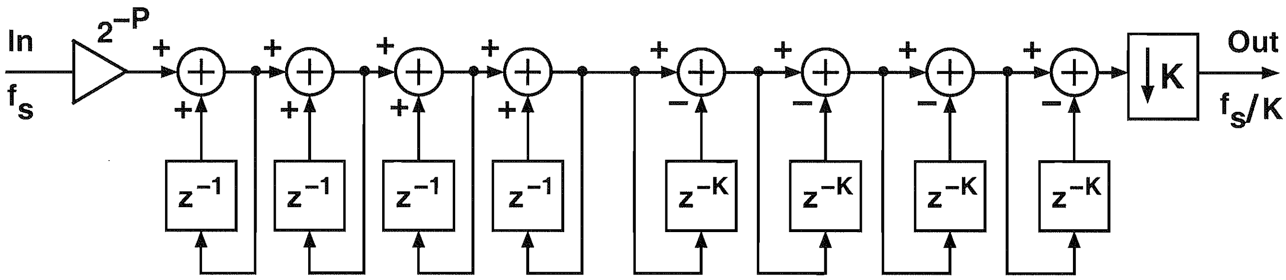
• In the third implementation form for the interpolator the digital zero-order hold increases the sampling rate by K in such a way that it repeats the input sample K times, thus increasing the sampling rate by K.

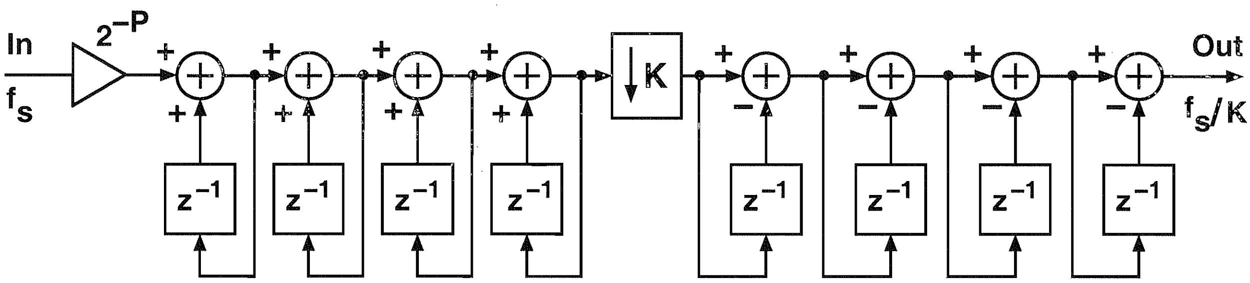
• In this case, the number of feedforward and feedback loops is decreased by one. Therefore, only $2(M-1)$ delay elements and adders are required regardless of the value of K.

• For the interpolator, it is required that $2^{-P} \leq (1/K)^{M-1}$ since the transfer function must be multiplied by K in order to keep the signal energy the same.

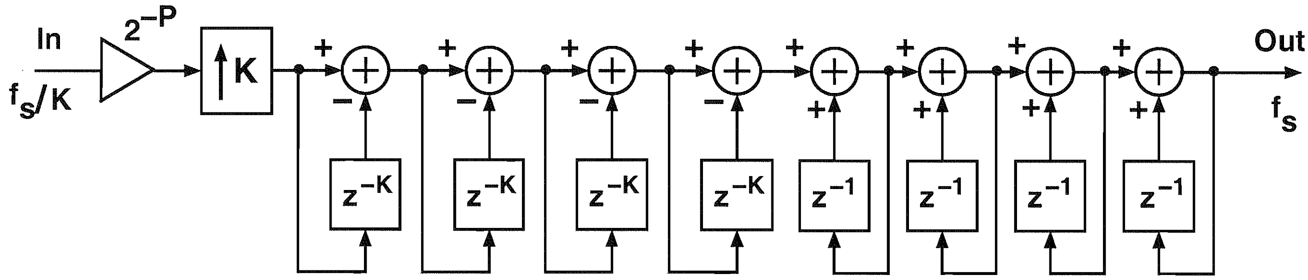
- For the interpolator, resetting in necessary and no miscalculations re allowed. Therefore, after a thunderstorm, resetting is needed. Otherwise, the output of the overall system may be meaningless!
- If it is desired that the output noise due to the multipcation roundoff errors correspond to rounding at the output of the overall filter, P extra bits are required for internal calculations for both the decimator and interpolator.

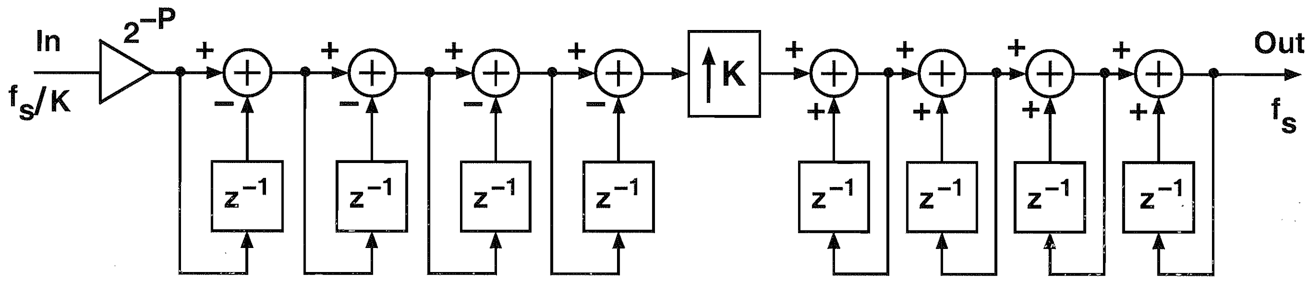
Structures for Cascaded Comb Filters for Decimation by an Integer Factor K

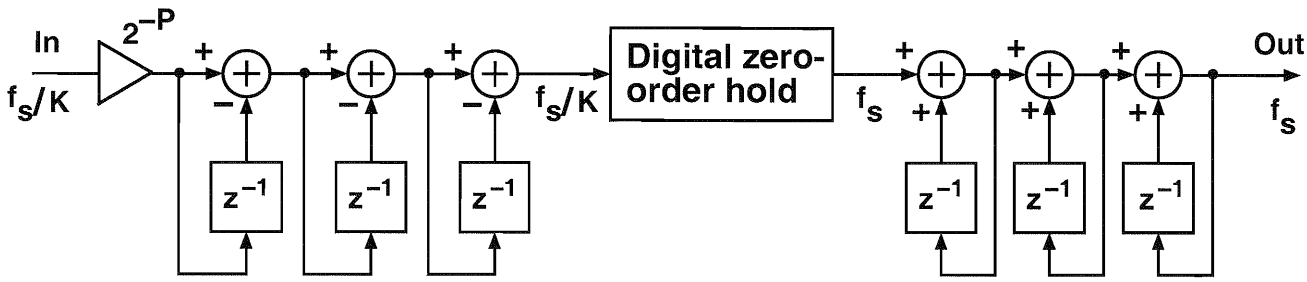




Structures for Cascaded Comb Filters for Interpolation by an Integer Factor K







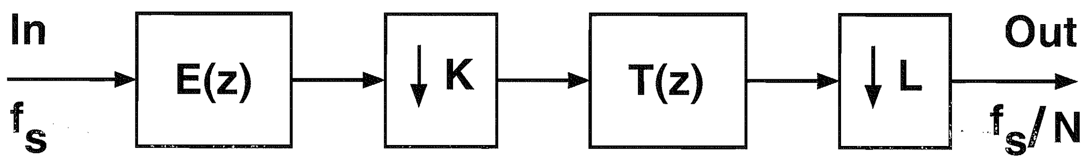
When to Use a Cascaded Comb Filter for Decimation or Interpolation Purposes?

- $E(z)$ can be used as a first stage (as a last stage) when the overall decimation (interpolation) ratio N is factorizable as

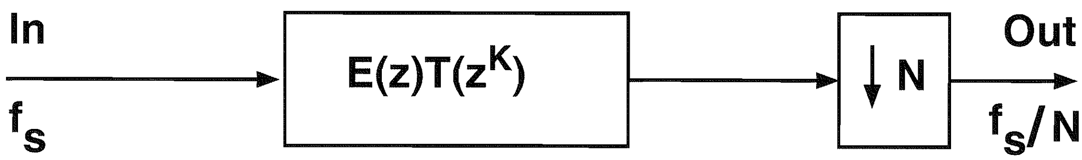
$$N = K \cdot L. \tag{65}$$

- The overall implementations as well as the corresponding single-stage equivalents are shown below.

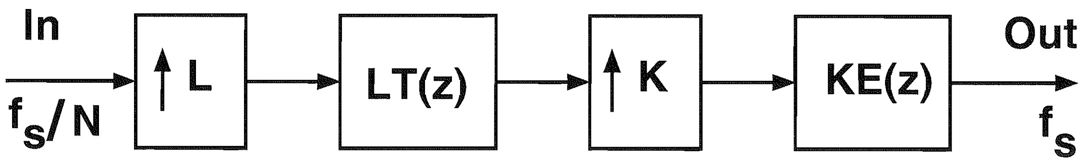
Decimator



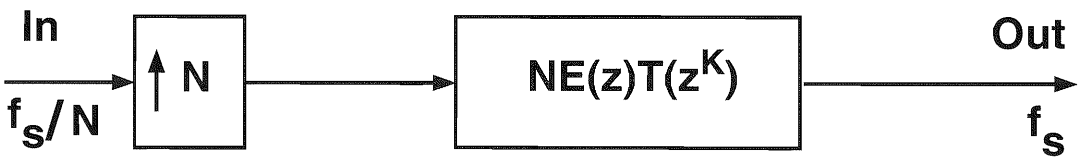
Single-stage equivalent



Interpolator



Single-stage equivalent



• If the transfer function for the second decimation stage (the first interpolation stage) is denoted by $T(z)$, then the transfer functions for the single-stage equivalents in the decimation and interpolation cases are given by

$$H(z) = T(z^k)E(z) \tag{66a}$$

and

$$H(z) = NT(z^k)E(z), \tag{66b}$$

respectively.

How to Design the Overall Filter?

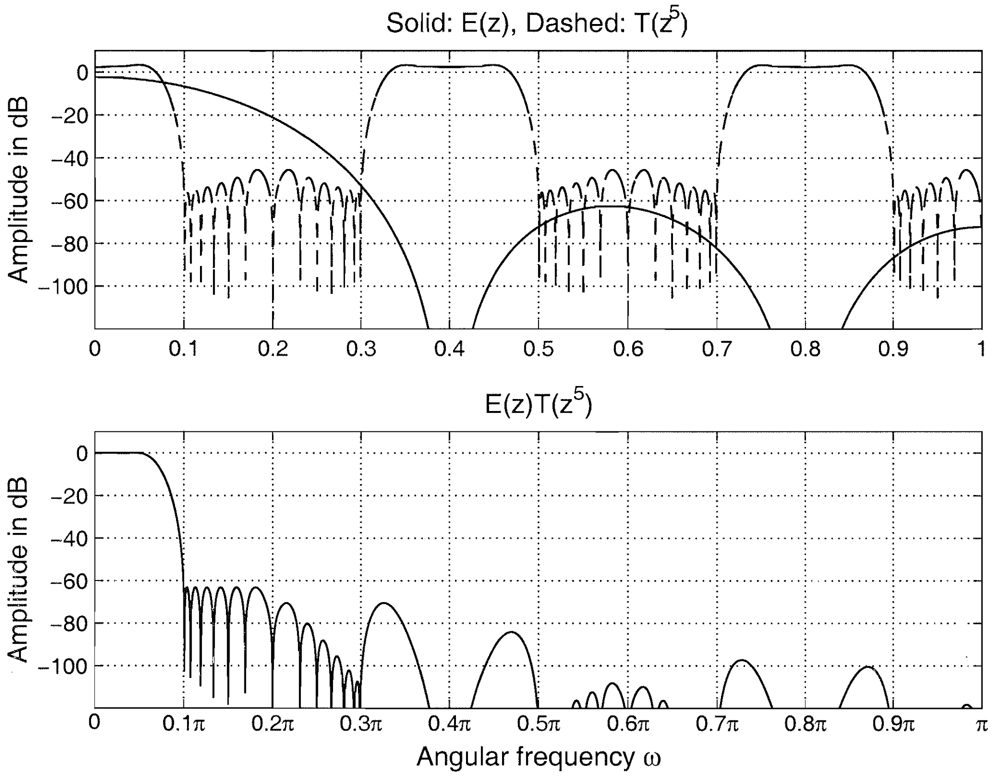
- The zero-phase frequency response of $E(z)$ is given by

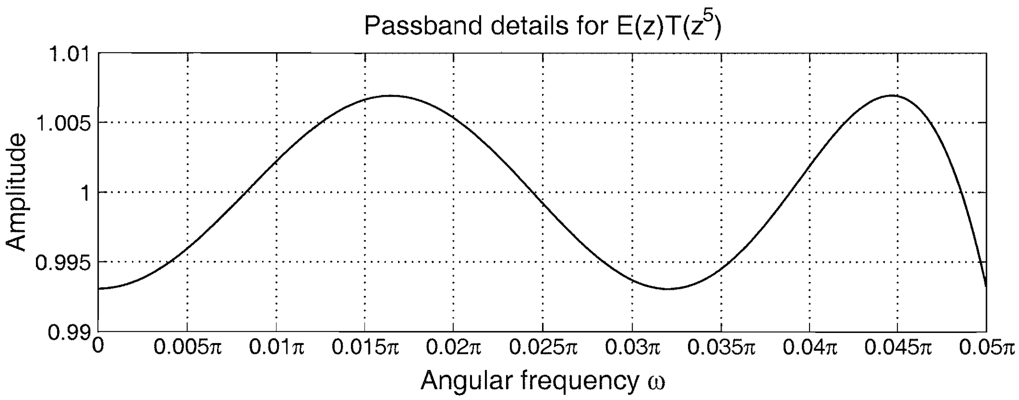
$$E(\omega) = 2^{-P}[\frac{\sin(N\omega/2)}{\sin(\omega/2)}]^{M} \tag{67}$$

and provides M zero pairs at $\omega = 2k\pi/K$ for $k = 1, 2, \cdots, \lfloor K/2 \rfloor$.

- To illustrate the design of the overall decimator, we consider the following criteria: $N = 10$, $\omega_p = 0.05\pi$, $\omega_s = 0.1\pi$, $\delta_p = 0.01$, $\delta_s = 0.001$.
- As shown on the next page, the given critria are met by $K = 5$, $M = 5$, $P = 12$, $L = 2$ and $T(z)$ of order 21.
- In this case, $T(z^5)$ takes care of the overall frequency response in the range $0 \leq \omega \leq \pi/5$, whereas $E(z)$ attenuates the unwanted extra transition bands and stopbands of $T(z^5)$ around $2\pi/5$ and $4\pi/5$ by providing 5 zero pairs at these frequencies.
- The only adjustable parameter for $E(z)$ is M and $M = 5$ is the minimum value of M required to attenuate the extra transition bands and stopbands to the desired level of 60 dB.

Responses for an Example Two-stage Decimator with the First Stage Being a Cacade of $M = 5$ Comb Filters of Length $K = 5$





Modified Comb Filter

• The cascaded comb filter structure suffers from the drawback that the only adjustable parameter is M, the number of zeros at the centers of the extra unwanted passbands of $T(z^K)$.

• In order to get around this problem, Saramäki and Ritoniemi have introduced a modified comb filter structure. For this structure, the overall transfer function is given by

$$E(z) = 2^{-P} E_1(z) E_2(z), \tag{66a}$$

where

$$E_1(z) = [z^{-1}\frac{1 - z^{-K}}{1 - z^{-1}}]^M \tag{66b}$$

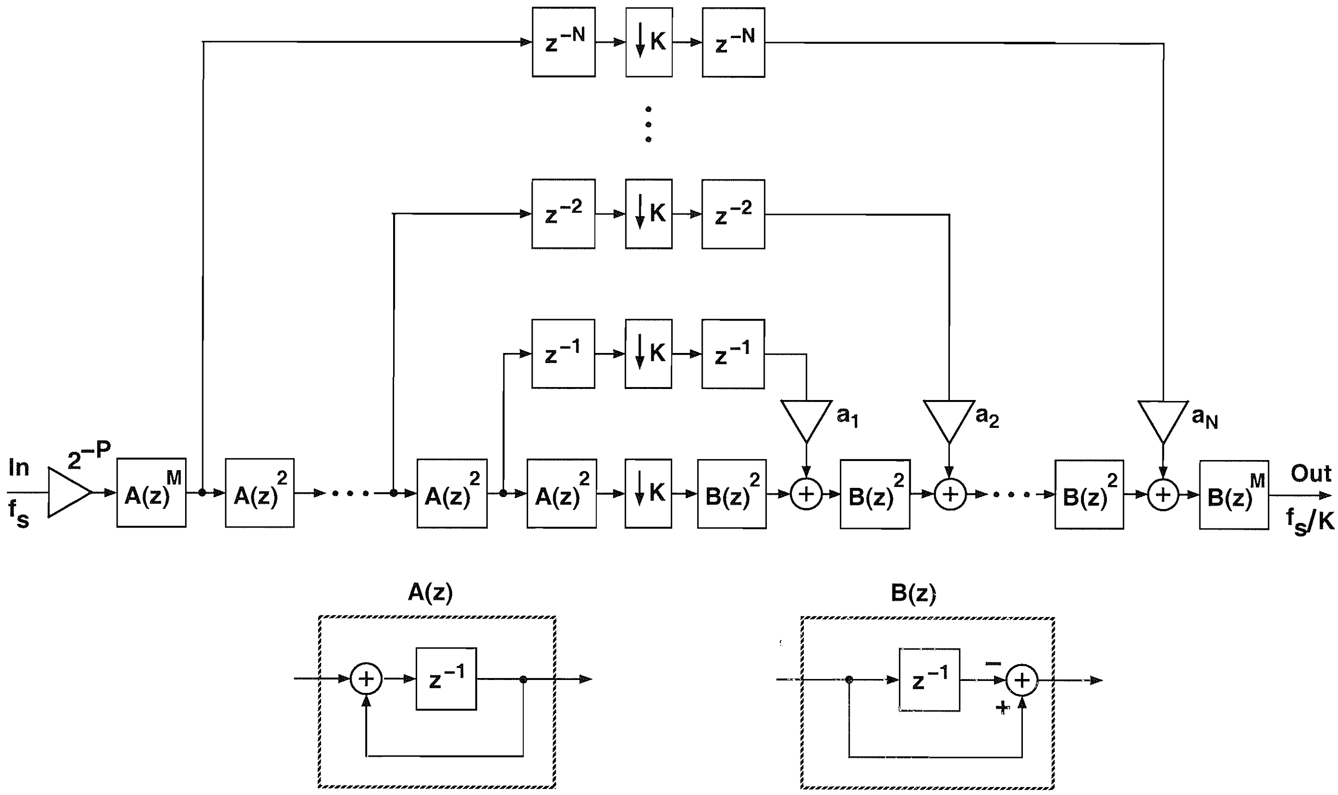
and

$$E_2(z) = [z^{-1}\frac{1 - z^{-K}}{1 - z^{-1}}]^{2N} + \sum_{r=1}^{N} a_r z^{-r(K+1)} [z^{-1}\frac{1 - z^{-K}}{1 - z^{-1}}]^{2(N-r)} . \tag{66c}$$

• The next page shows an efficient implementation for decimating by an integer factor K in the case where a_k's are integers.

• Also in this case, the output of this implementation is correct for 2's complement artitmetic even though internal overflows may occur, provided that $2^{-P} \leq /[E_1(1)E_2(1)]$.

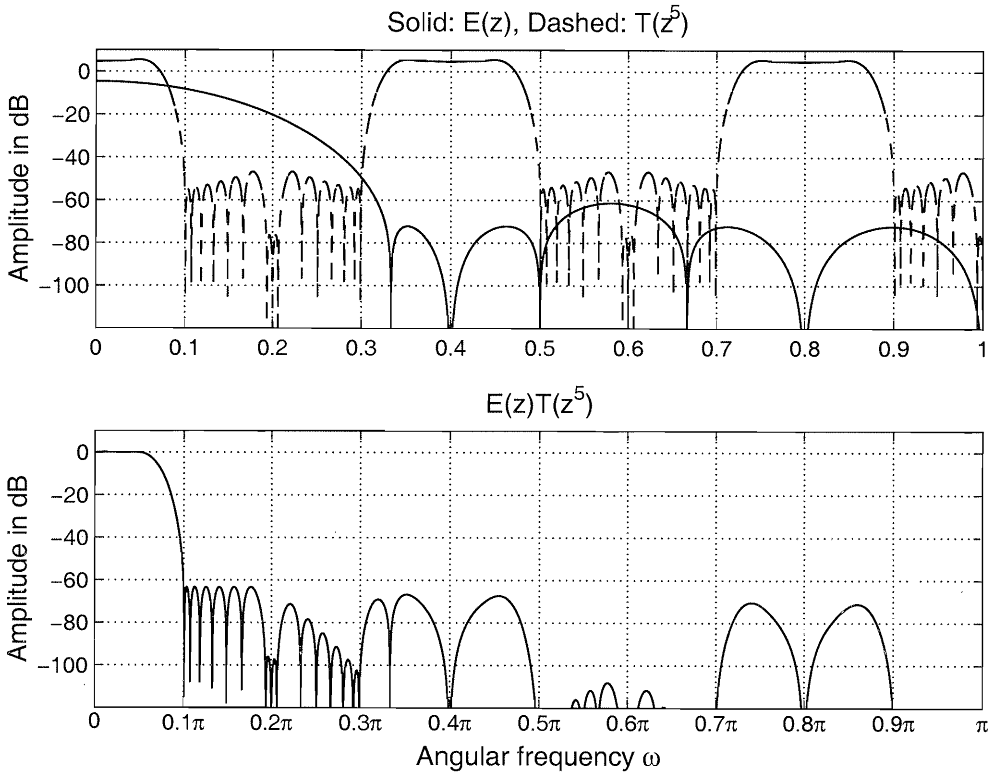
Implementation of the Modified Comb Filter for Decimation by K in the Case where the a_k's are Integers

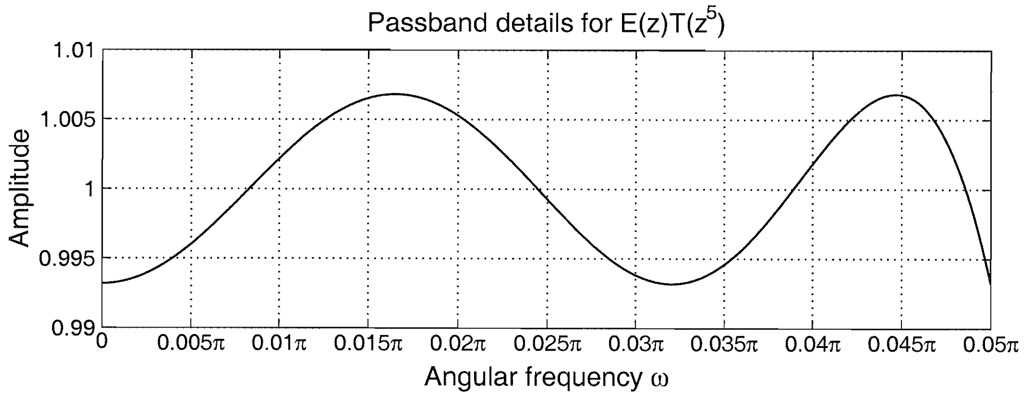


Example

- We consider the same criteria as for the cascaded comb filter structure, that is, $N = 10$, $\omega_p = 0.05\pi$, $\omega_s = 0.1\pi$, $\delta_p = 0.01$, and $\delta_s = 0.001$.
- As shown on the next page, the given critria are met by $K = 5$, $M = 2$, $N = 1$, $P = 10$, $a_1 = 1$, and $T(z)$ of order 21.
- It is interesting to observe that the modified comb filter provides a zero pair before and after $\omega = 2\pi/5$.
- This explains why the overall number of feedback and feedforward loops reduces from 5 to 4 compared to the cascaded comb filter structure considered earlier.
- Also the number of additional bits from internal calculations reduces from $P = 12$ to $P = 10$.

Responses for an Example Two-stage Decimator with the First Stage Being a Modified Comb Filter with $K = 5$, $M = 2$, and $N = 1$

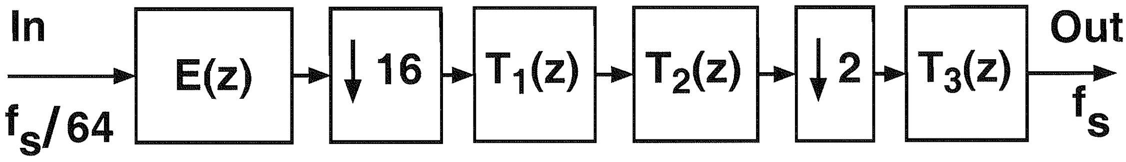




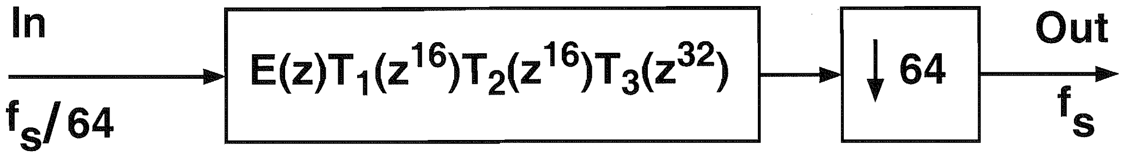
Illustrative Example

- It is desired to design a highly selective decimator after a sigma-delta modulator working at the samling rate being 64 times the output sampling rate.
- The final output sampling rate is $F_s = 44.1$ kMz.
- The passband edge is 20 kHz$= 0.4535F_s$ and it is desired that the passband ripple for the amplitude is less than or equal to 0.0001 and the components aliasing into the passband are attenuated at least 120 dB.
- Aliasing is allowed to the frequency range between 20 kHz and $F_s/2 = 22.05$ kHz from the range between $F_s/2 = 22.05$ kHz and $F_s - 20$ kHz=24.1 kHz. A normal human being is not able to hear this aliasing.
- In terms of the angular frequency the criteria are thus: $N = 64$, $\omega_p = 0.907\pi/64$, $\omega_s = 1.093\pi/64$, $\delta_p = 0.0001$, and $\delta_s = 10^{-6}$.

• The overall filter has been synthesized using the following structure:



Single-stage equivalent



• As shown above, the transfer function for the single-stage equivalent is given by

$$H(z) = E(z)T_1(z^{16})T_2(z^{16})T_3(z^{32}). \tag{67}$$

• The given criteria are met by designing $E(z)$ to be a modified comb filter with $K = 16$, $M = N = 2$, $a_1 = -2^3$, $a_2 = 2^4$, and $P = 24$.

• $T_1(z)$ is a linear-phase FIR filter of order 4, whereas $T_2(z)$ and $T_3(z)$ are half-band filters of orders 26 and 162, respectively.

• Various responses for the overall design are depicted on Pages 287, 288, and 289.

• The roles of the subfilters are the following:

1) $T_2(z^{16})$ and $T_3(z^{32})$ take care of providing the desired attenuation for the overall transfer function, as given by Eq. (67), from $1.093F_s/2$ to $2F_s$.

2) $E(z)T_1(z^{16})$ attenuates the extra passbands and transition bands of $T_2(z^{16})T_3(z^{32})$ around $4kF_s$ for $k = 1, 2, \cdots, 8$.

• Since $T_2(z)$ and $T_3(z)$ are half-band filters, their passband ripples are very small and they cannot be used for compensating the passband distortion caused by $E(z)$.

• The roles of $E(z)$ and $T_1(z^{16})$ in generating $E(z)T_1(z^{16})$ are the following:

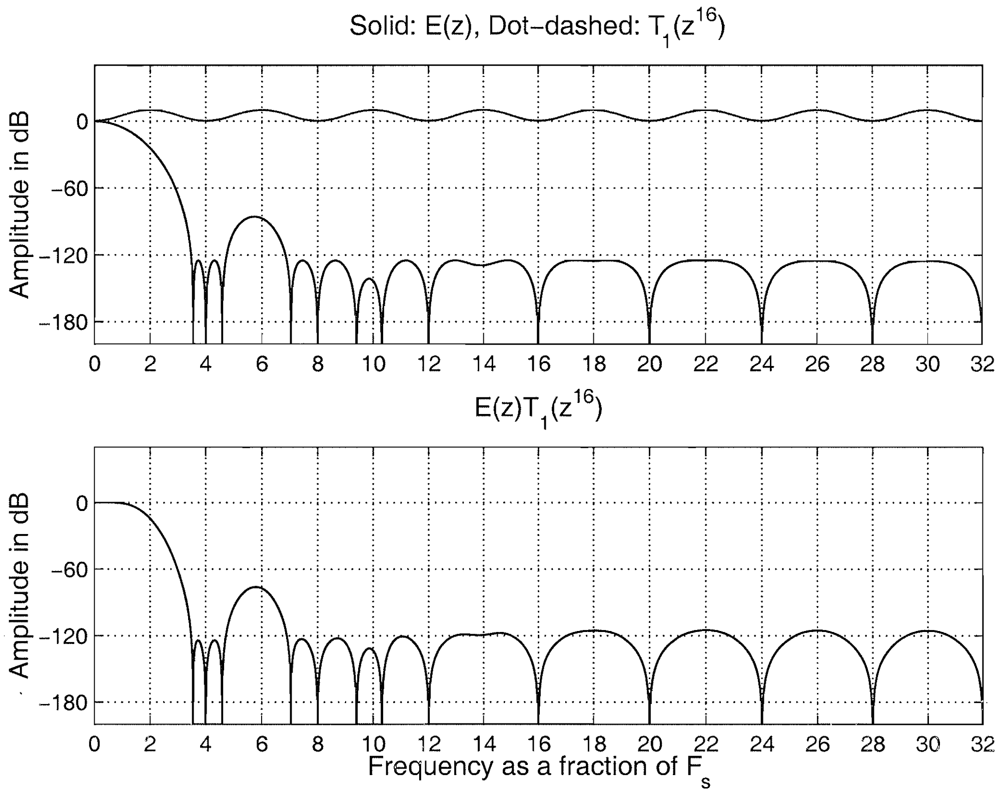
1) $E1(z)$ provides concentrates on attenuating the extra passbands and transition bands of $T_2(z^{16})T_3(z^{32})$.

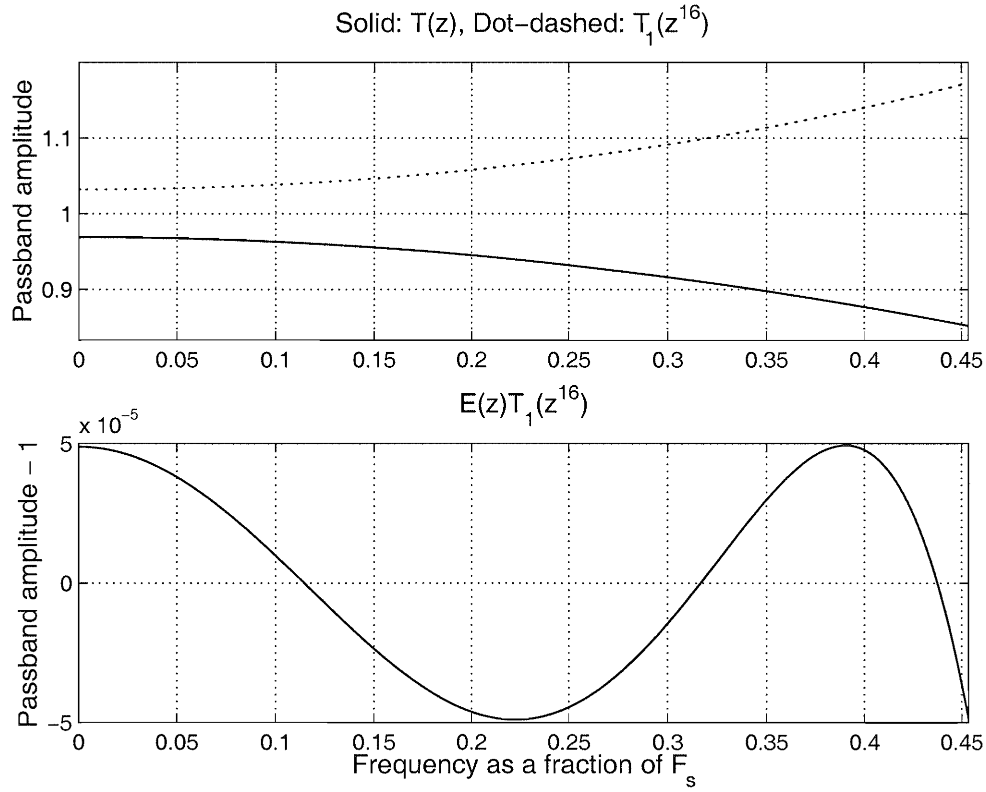
2) $T_1(z^{16})$ compensates the passband distortion caused by $E(z)$. Because of periodicity, it only slightly decreses the stopband attenuation provided by $G(z)$.

• For the modified comb filter, the number of feedback and feedforward loops is 6 and $P = 24$, whereas for the corresponding direct cascade, the corresponding figures are 8 and 32.

• The modified comb filter has been used as a first stage for decimating a one-bit stream from a sigma-delta modulator, and the estimated saving in the overall silicon area provided by this filter over its cascaded counterpart is 50 %.

Responses for the Example Multistage Decimator





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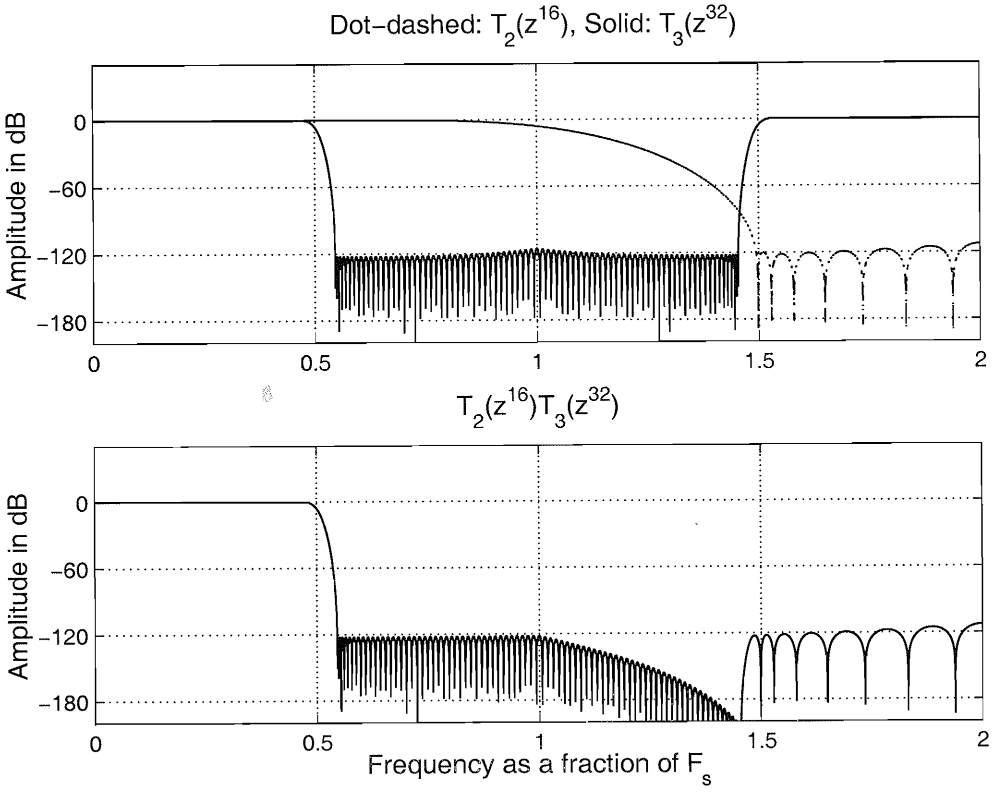

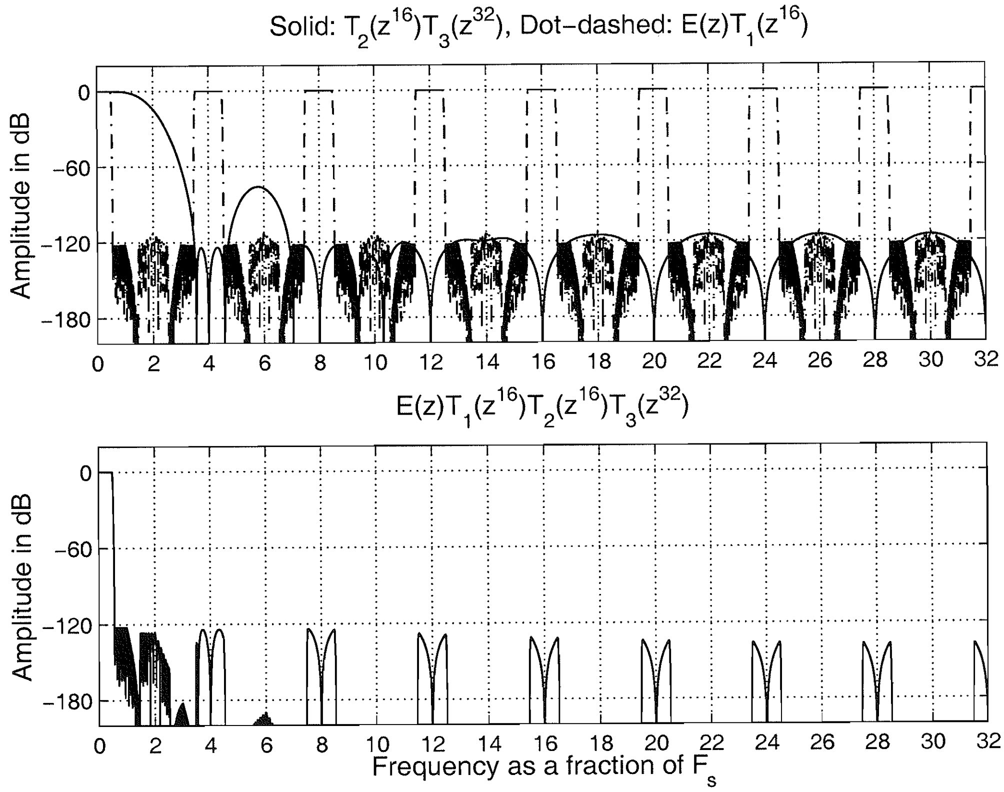

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