Generalizations of Classical Recursive Digital Filters and Their Design with the Aid of a Remez-Type Algorithm

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Summary

- The class of classical recursive digital filters is generalized such they are still implementable using low-sensitivity and low-noise structures.
- The generalized filters can have several passband and stopband regions and arbitrary weightings can be used.
- Some transmission and attenuation zeros can be fixed.
- An efficient algorithm is constructed for filter design.

Generalized Filter Class

A complementary filter pair

$$H(z) = \frac{E(z)}{D(z)} = \sum_{n=0}^{N} e(n)z^{-n}/(1 - \sum_{n=1}^{N} d(n)z^{-n})$$

and

$$G(z) = \frac{F(z)}{D(z)} = \sum_{n=0}^{N} f(n)z^{-n}/(1 - \sum_{n=1}^{N} d(n)z^{-n})$$

satisfying

$$H(z)H(1/z) + G(z)G(1/z) = 1.$$

Type A Filters:

$$e(N-n) = e(n), \quad f(N-n) = -f(n).$$

Type B Filters:

$$e(N-n) = e(n), \quad f(N-n) = f(n).$$

Factorization of the Numerators

$$E(z) = A(z)\widehat{E}(z)$$

$$F(z) = B(z)\widehat{F}(z)$$

$$\widehat{E}(z) = \sum_{n=0}^{N_E} \widehat{e}(n) z^{-n}, \quad \widehat{e}(N_E - n) = \widehat{e}(n)$$

and

$$\widehat{F}(z) = \sum_{n=0}^{N_F} \widehat{f}(n) z^{-n}, \quad \widehat{f}(N_F - n) = \widehat{f}(n)$$

Type A Filters:

$$A(z) = \begin{cases} 1 & \text{for } N \text{ even} \\ (1 + z^{-1})/2 & \text{for } N \text{ odd,} \end{cases}$$

$$B(z) = \begin{cases} (1 - z^{-2})/2 & \text{for } N \text{ even} \\ (1 - z^{-1})/2 & N \text{ odd,} \end{cases}$$

$$N_E = \begin{cases} N & \text{for } N \text{ even} \\ N - 1 & \text{for } N \text{ odd,} \end{cases}$$

and

$$N_F = \begin{cases} N - 2 & \text{for } N \text{ even} \\ N - 1 & \text{for } N \text{ odd.} \end{cases}$$

Type B Filters:

$$N_E = N_F \equiv N$$

and

$$A(z) = B(z) \equiv 1.$$

Squared-Magnitude Functions

$$|H(e^{j\omega})|^2 = \frac{1}{1+[\Phi(\omega)]^2}, \quad |G(e^{j\omega})|^2 = \frac{1}{1+[1/\Phi(\omega)]^2},$$

where

$$\Phi(\omega) = \kappa \Gamma(\omega) \Upsilon(\omega)$$

with

$$\Gamma(\omega) = \begin{cases} \sin\omega & \text{for } N \text{ even} \\ \dfrac{\sin(\omega/2)}{\cos(\omega/2)} & \text{for } N \text{ odd} \end{cases}$$

for Type A and

$$\Gamma(\omega) \equiv 1$$

for Type B, whereas

$$\kappa = f(0)/e(0)$$

$$\Upsilon(\omega) = \frac{\cos N_F\omega + \sum_{n=0}^{N_F-1} \mu(n)\cos n\omega}{\cos N_E\omega + \sum_{n=0}^{N_E-1} \nu(n)\cos n\omega}$$

Here, $\mu(0) = f(N_F)/[2f(0)]$, $\nu(0) = e(N_E)/[2e(0)]$ and $\mu(n) = f(N_F - n)/f(0)$ for $n = 1, 2, \ldots, N_F - 1$ and $\nu(n) = e(N_F - n)/f(0)$ for $n = 1, 2, \ldots, N_E - 1$.

Classical Filters

Type A:

$$\Phi(\omega) = \begin{cases} \kappa \dfrac{\sin(\omega/2) \displaystyle\prod_{k=1}^{(N-1)/2} (\cos\omega - \cos\widehat{\omega}_k)}{\cos(\omega/2) \displaystyle\prod_{k=1}^{(N-1)/2} (\cos\omega - \cos\omega_k)} & \text{for } N \text{ odd} \\[2ex] \kappa \dfrac{\sin(\omega) \displaystyle\prod_{k=1}^{(N-2)/2} (\cos\omega - \cos\widehat{\omega}_k)}{\displaystyle\prod_{k=1}^{N/2} (\cos\omega - \cos\omega_k)} & \text{for } N \text{ even} \end{cases}$$

Type B:

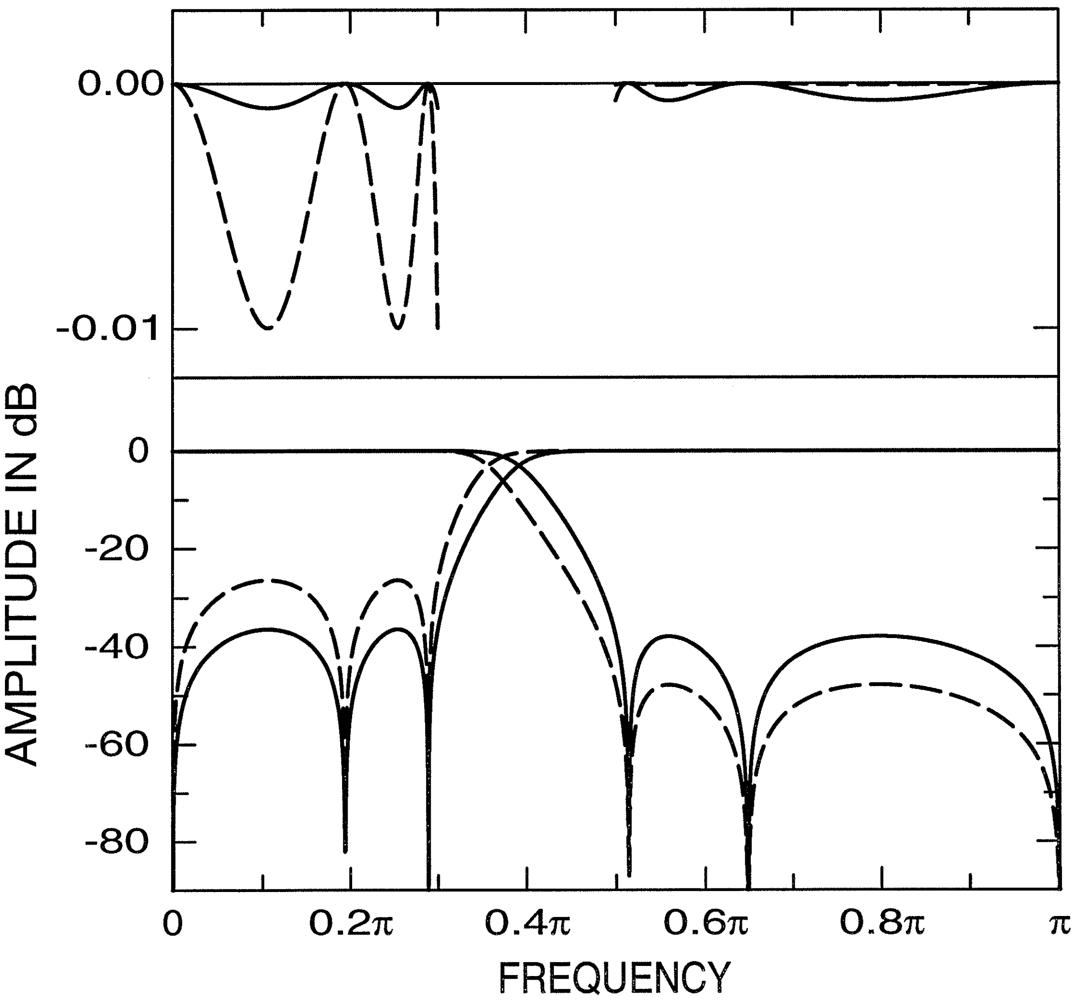
$$\Phi(\omega) = \kappa \frac{\displaystyle\prod_{k=1}^{N/2} (\cos\omega - \cos\widehat{\omega}_k)}{\displaystyle\prod_{k=1}^{N/2} (\cos\omega - \cos\omega_k)}$$

- The zeros of both $H(z)$ and $G(z)$ are on the unit circle.

Example Classical Filter Pair

Fifth-order elliptic filter pair (Type A) with the passband edge of $H(z)$ at $\omega_p = 0.3\pi$ and the stop-band edge at $\omega_s = 0.5\pi$.

- $\omega_1 = 0.5138\pi$, $\omega_2 = 0.6483\pi$, $\widehat{\omega}_1 = 0.1937\pi$, and $\widehat{\omega}_2 = 0.2890\pi$.
- Solid line: 0.001 dB passband ripple: $\kappa = 0.04950$
- Dashed line: 0.01 dB passband ripple: $\kappa = 0.4950$



Generalized Filters

- Several passband and stopband regions.
- Arbitrary weightings in the passband and stopband regions.
- Some zeros of $H(z)$ and $G(z)$ can be fixed and the remaining ones are optimized.
- $H(z)$ and $G(z)$ can have reciprocal zero pairs at $z = r$ and $z = 1/r$ or zero quadruplets at $z = re^{\pm j\phi}$ and $z = (1/r)e^{\pm j\phi}$.
- Because the symmetries of the numerators of $H(z)$ and $G(z)$, the generalized filter pairs can be implemented using low-sensitivity and low-noise structures originally developed for classical filters:

  a) Wave digital and LDI ladder filters

  b) Wave digital lattice filters

  c) Filters using a complex allpass section as a building block

Design algorithm

- It is based on iteratively determining the numerator and denominator of

$$\Upsilon(\omega) = \frac{\cos N_F\omega + \sum_{n=0}^{N_F-1} \mu(n)\cos n\omega}{\cos N_E\omega + \sum_{n=0}^{N_E-1} \nu(n)\cos n\omega}$$

to achieve the desired passband and stopband shapes.

- At each step, a low-order polynomial is determined with the aid of a Remez-type algorithm.

- The basic routine has been generated by modifying **standard FIR filter design programs** (see the proceedings for the details).

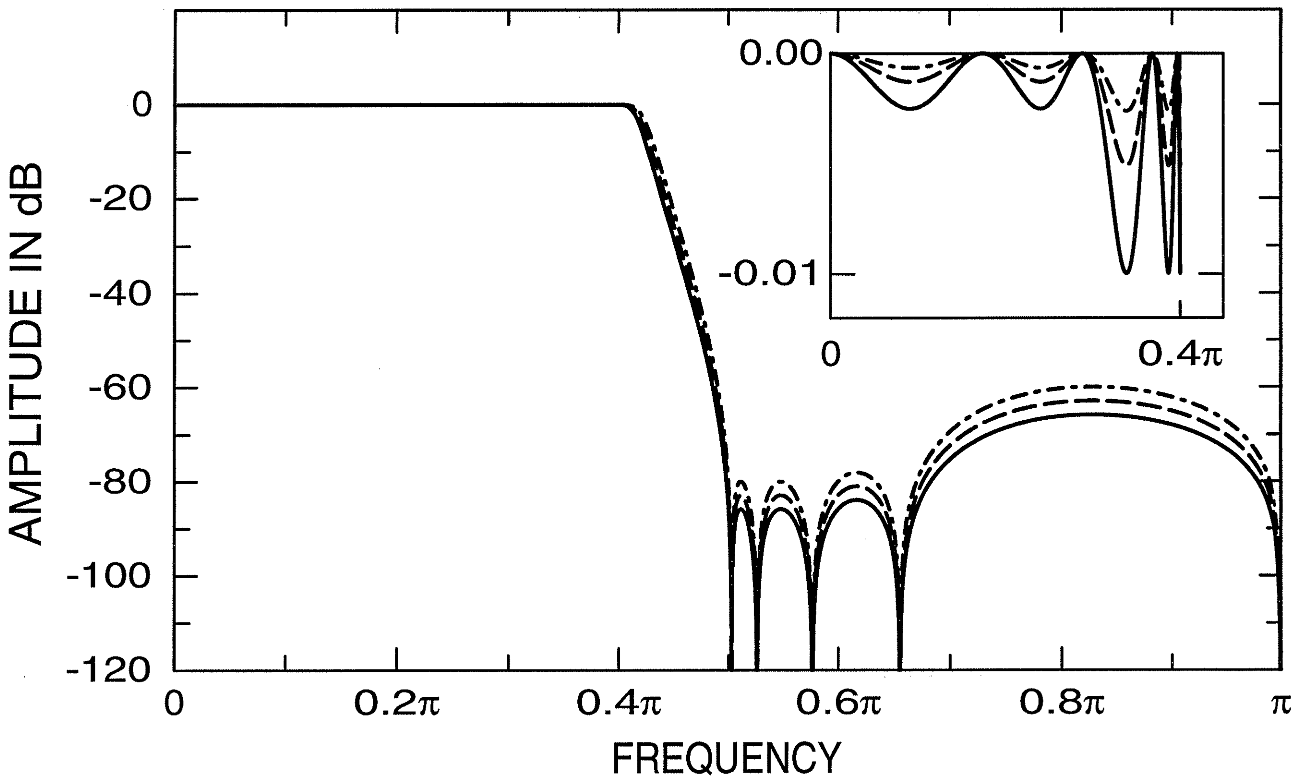
- Only three to five iterations are required.

Example 1

- $\omega_p = 0.4\pi$ and $\omega_s = 0.5\pi$.
- Required passband ripple on $[0,\ 0.3\pi]$ is 0.0025 dB and on $[0.3\pi,\ 0.4\pi]$ 0.01 dB.
- Minimum stopband attenuation on $[0.5\pi,\ 0.6\pi]$ is 80 dB and on $[0.6\pi,\ \pi]$ 60 dB.
- Minimum filter order for Type A filters is nine.

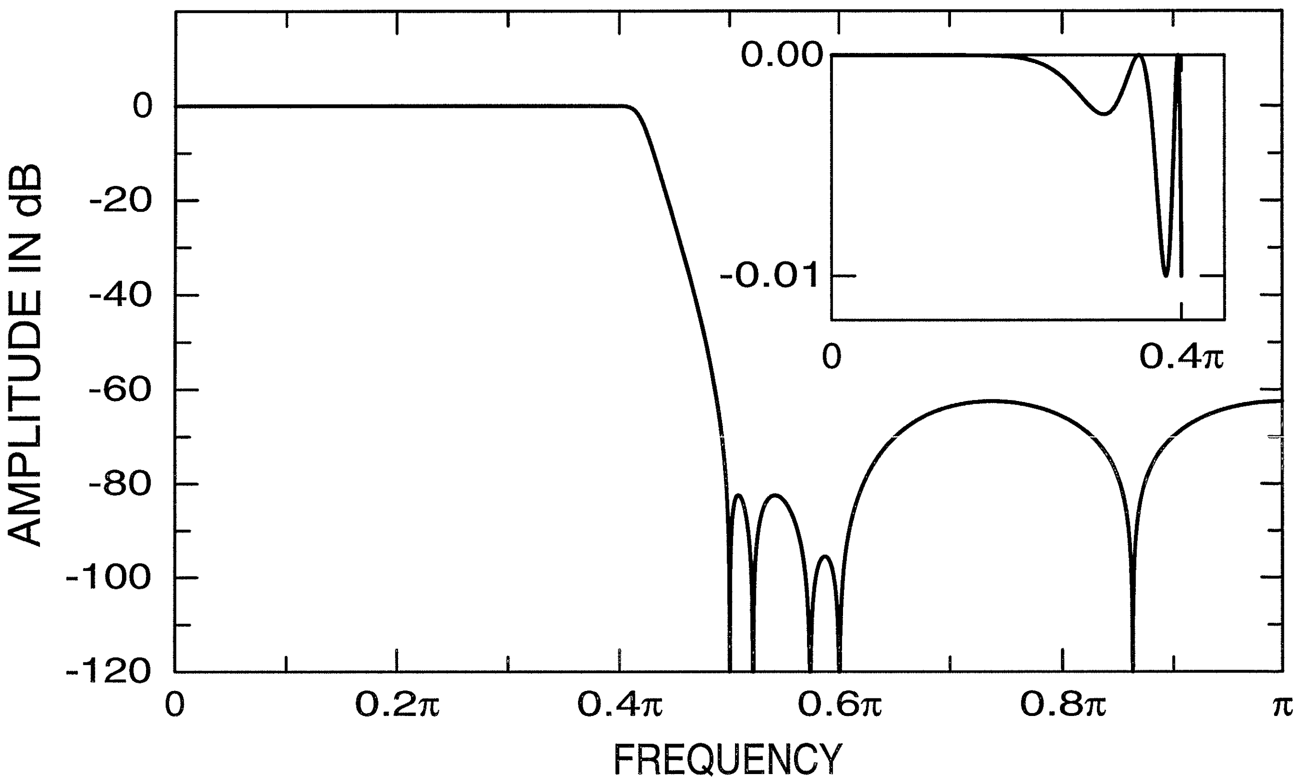
Example 1: Responses

- Solid line: Passband criteria are just met.
- Dot-dashed line: Stopband criteria are just met.
- Dashed line: Both criteria are well met.



Example 2

- Specifications of Example 1
- $H(z)$ has fixed transmission zeros at $\omega = \pm 0.5\pi$ and at $\omega = \pm 0.6\pi$.
- Degree of maximal flatness is 6 at the zero frequency.
- Type B filter of order 10 just meeting the passband criteria.



Example 3

- Type B filter pair of order 32.
- Two passband regions of $H(z)$: $[0,\ 0.08\pi]$ and $[0.52\pi,\ 0.68\pi]$.
- Two stopband regions: $[0.12\pi,\ 0.48\pi]$ and $[0.72\pi,\ \pi]$.
- Degree of maximal flatness of $H(z)$ is 4 at both the zero frequency and at $\omega = 0.6\pi$.
- Degree of maximal flatness of $G(z)$ is 4 at $\omega = 0.3\pi$ and at $\omega = \pi$.